Lyapunov Function Constructions for Slowly Time-Varying Systems

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REVIEW of MODEL and LITERATURE

Goals: For large constants $\alpha > 0$, prove input-to-state stability (ISS) for

$$\dot{x} = f(x, t, t/\alpha) + g(x, t, t/\alpha)u, \quad x(t_0) = x_o \qquad (\Sigma)$$

and construct explicit corresponding ISS Lyapunov functions.

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Literature: Uses exponential-like stability of $\dot{x} = f(x, t, \tau)$ aka (Σ_{fro}) for all relevant values of the scalar τ to show stability for $u \equiv 0$ but does not lead to explicit Lyapunov functions for (Σ) (Peuteman-Aeyels, Solo).

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Our Contributions: We explicitly construct Lyapunov functions for (Σ) in terms of given Lyapunov functions for (Σ_{fro}) without assuming any exponential-like stability of (Σ_{fro}) and we allow τ to be a vector.

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Significance: Lyapunov functions for (Σ_{fro}) are often readily available. Explicit Lyapunov functions and slowly time-varying models are important in control engineering e.g. control of friction, pendulums, etc.

MAIN ASSUMPTION and MAIN THEOREM

We first assume our (sufficiently regular) system (Σ) has the form

$$\dot{x} = f(x, t, p(t/\alpha)) \qquad (\Sigma_p)$$

where $p : \mathbb{R} \to \mathbb{R}^d$ is bounded and $\bar{p} := \sup\{|p'(r)| : r \in \mathbb{R}\} < \infty$.

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Assume: $\exists \delta_1, \delta_2 \in \mathcal{K}_\infty$; constants $c_a, c_b, T > 0$; a continuous function $q : \mathbb{R}^d \to \mathbb{R}$; and a C^1 $V : \mathbb{R}^n \times [0, \infty) \times \mathbb{R}^d \to [0, \infty)$ s.t. $\forall\, x \in \mathbb{R}^n$, $t \geq 0$, and $\tau \in \mathcal{R}(p) := \{p(t) : t \in \mathbb{R}\}$: (i) $|V_\tau(x, t, \tau)| \leq c_a V(x, t, \tau)$, (ii) $\delta_1(|x|) \leq V(x, t, \tau) \leq \delta_2(|x|)$, (iii) $\int_{t-T}^{t} q(p(s))\mathrm{d}s \geq c_b$, and (iv) $V_t(x, t, \tau) + V_x(x, t, \tau) f(x, t, \tau) \leq -q(\tau) V(x, t, \tau)$ all hold.

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Theorem A: For each constant $\alpha > 2T c_a \bar{p}/c_b$, (Σ_p) is UGAS and

$$V_\alpha^\sharp(x, t) := e^{\frac{\alpha}{T} \int_{\frac{t}{\alpha} - T}^{\frac{t}{\alpha}} \int_s^{\frac{t}{\alpha}} q(p(l)) \mathrm{d}l \, \mathrm{d}s} V(x, t, p(t/\alpha))$$

is a Lyapunov function for (Σ_p). [UGAS: $|\phi(t; t_o, x_o)| \leq \beta(|x_o|, t - t_o)$]

SKETCH of PROOF of THEOREM A

Step 1: Set $\hat{V}(x,t) := V(x,t,p(t/\alpha))$. Along trajectories of (Σ_p),

$$\frac{d}{dt}\hat{V} \leq \left[-q(p(t/\alpha)) + \frac{c_a \bar{p}}{\alpha}\right] \hat{V}(x,t). \qquad (\star)$$

Important: The term involving α in $(\star)$ vanishes if $V_\tau \equiv 0$.

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Step 2: Substitute $(\star)$ into

$$\begin{aligned} \dot{V}_\alpha^\sharp &= E(t,\alpha)\left[\frac{d}{dt}\hat{V} + \left\{q(p(t/\alpha)) - \frac{1}{T}\int_{\frac{t}{\alpha}-T}^{\frac{t}{\alpha}} q(p(l))\mathrm{d}l\right\}\hat{V}\right] \\ &\leq E(t,\alpha)\left[\frac{c_a\bar{p}}{\alpha} - \frac{c_b}{T}\right]\hat{V}(x,t), \end{aligned}$$

where $V_\alpha^\sharp(x,t) = E(t,\alpha)\hat{V}(x,t)$ and

$$E(t,\alpha) := e^{\frac{\alpha}{T}\int_{\frac{t}{\alpha}-T}^{\frac{t}{\alpha}}\left[\int_s^{\frac{t}{\alpha}} q(p(l))\mathrm{d}l\right]\mathrm{d}s}.$$

EXAMPLE 1: STABILITY for ALL PARAMETER VALUES

The assumptions of Theorem A hold for

$$\dot{x} = f(x, t, \cos^2(t/\alpha)) := \frac{x}{\sqrt{1+x^2}} \left[1 - 90\cos^2\left(\frac{t}{\alpha}\right)\right]$$

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This follows from the estimates

$$\nabla \bar{V}(x) f(x, t, \tau) \;\leq\; \left[\frac{2e^{\sqrt{2}}}{e-1} - 45\tau\right] \bar{V}(x)$$

$$\int_{t-\pi}^{t} \left[45\cos^2(s) - \frac{2e^{\sqrt{2}}}{e-1}\right] \mathrm{d}s \;=\; \pi\left(\frac{45}{2} - \frac{2e^{\sqrt{2}}}{e-1}\right) \;>\; 0$$

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so for all $\alpha > 0$ we get UGAS and the Lyapunov function

$$V_{\alpha}^{\sharp}(x,t) \ := \ e^{\frac{\alpha}{\pi}\int_{\frac{t}{\alpha}-\pi}^{\frac{t}{\alpha}} \left[\int_{s}^{\frac{t}{\alpha}} \left[45\cos^2(l) - \frac{2e^{\sqrt{2}}}{e-1}\right] \mathrm{d}l\right] \mathrm{d}s} \bar{V}(x)$$

$$= \ e^{45\frac{\alpha}{4}\left[\sin\left(\frac{2t}{\alpha}\right)+\pi-\frac{4\pi e^{\sqrt{2}}}{45(e-1)}\right]} [e^{\sqrt{1+x^2}} - e]$$

EXAMPLE 2: MECHANICAL SYSTEM with FRICTION

Model: Dynamics for x_1=mass position and x_2=velocity:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sigma_1(t/\alpha)x_2 - k(t)x_1 + u \\ &\quad - \left\{\sigma_2(t/\alpha) + \sigma_3(t/\alpha)e^{-\beta_1\mu(x_2)}\right\} \mathrm{sat}(x_2) \end{aligned} \qquad \text{(MSF)}$$

σ_i are positive friction-related coefficients; β_1 is a positive constant corresponding to Stribeck effect; $\mu \in \mathcal{PD}$ is related to Stribeck effect; k is a positive time-varying spring stiffness-related coefficient; and $\mathrm{sat}(x_2) = \tanh(\beta_2 x_2)$, where β_2 is a large positive constant. $\alpha > 1$.

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Assumptions: (a) $\sigma_i \in C^1$,valued in $(0, 1]$, σ_i' bounded; (b) $\exists$ constants $c_b, T > 0$ such that $\int_{t-T}^{t} \sigma_1(r)dr \;\geq\; c_b$ $\forall t \geq 0$; (c) $k \in C^1$, k' bounded, $\exists k_o, \bar{k} > 0$ s.t. $k_o \leq k(t) \leq \bar{k}$ and $k'(t) \leq 0$ $\forall t \geq 0$.

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We apply our theorem to (MSF) with $p(t) = (\sigma_1(t), \sigma_2(t), \sigma_3(t))$.

EXAMPLE 2: MECHANICAL SYSTEM with FRICTION (cont'd)

The function

$$V(x,t,\tau) = A(k(t)x_1^2 + x_2^2) + \tau_1 x_1 x_2, \quad A = 1 + \frac{k_o}{2} + \frac{(1+2\beta_2)^2}{k_o}$$

satisfies the following for the corresponding frozen dynamics:

$$\tfrac{1}{2}(x_1^2 + x_2^2) \ \le\ V(x,t,\tau) \ \le\ A^2\bar{k}(|x_1| + |x_2|)^2 \ \le\ 2A^2\bar{k}|x|^2$$

$$V_t(x,t,\tau) + V_x(x,t,\tau)f(x,t,\tau) \ \le\ -\tfrac{\tau_1 k_o}{4A^2\bar{k}}V(x,t,\tau)$$

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Corollary: There exists a constant $\alpha_o > 0$ such that for all $\alpha > \alpha_o$, (MSF) is UGAS and admits the Lyapunov function

$$V_\alpha(t,x) := V(x,t,p(t/\alpha))\, e^{\frac{\alpha\bar{b}}{T}\int_{\frac{t}{\alpha}-T}^{\frac{t}{\alpha}}\int_s^{\frac{t}{\alpha}} \sigma_1(l)\mathrm{d}l\,\mathrm{d}s}$$

where V is above, $\bar{b} = k_o/(4A^2\bar{k})$, and $p(t) = (\sigma_1(t), \sigma_2(t), \sigma_3(t))$.

INPUT-TO-STATE STABILITY (ISS)

Additional Assumptions: To show ISS for

$$\dot{x} = \mathcal{F}(x, t, u, \alpha) := f(x, t, p(t/\alpha)) + g(x, t, p(t/\alpha))u \qquad (\Sigma_u)$$

for large constants $\alpha > 0$ and f, g, and p as before, we also assume: For all $t \geq 0$, $\alpha > 0$, and $x \in \mathbb{R}^n$, (v) $|V_x(x, t, p(t/\alpha))| \leq c_a \sqrt{\delta_1(|x|)}$, and (vi) $|g(x, t, p(t/\alpha))| \leq c_a \{1 + \sqrt[4]{\delta_1(|x|)}\}$.

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ISS: $\exists\, \beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ s.t. $|\phi(t; t_o, x_o, \mathbf{u})| \leq \beta(|x_o|, t - t_o) + \gamma(|\mathbf{u}|_\infty)$.

ISS-CLF: $\exists \mu_1, \mu_2, \chi \in \mathcal{K}_\infty$, $\mu_3 \in \mathcal{PD}$ s.t. $\mu_1(|x|) \leq W(x, t) \leq \mu_2(|x|)$ and $|u| \leq \chi(|x|) \Rightarrow W_t(x, t) + W_x(x, t)\mathcal{F}(x, t, u, \alpha) \leq -\mu_3(|x|)$.

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ISS: $\exists\, \beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ s.t. $|\phi(t;t_o,x_o,\mathbf{u})| \leq \beta(|x_o|, t-t_o) + \gamma(|\mathbf{u}|_\infty)$. ISS-CLF: $\exists \mu_1, \mu_2, \chi \in \mathcal{K}_\infty$, $\mu_3 \in \mathcal{PD}$ s.t. $\mu_1(|x|) \leq W(x,t) \leq \mu_2(|x|)$ and $|u| \leq \chi(|x|) \Rightarrow W_t(x,t) + W_x(x,t)\mathcal{F}(x,t,u,\alpha) \leq -\mu_3(|x|)$.

Theorem B: $\forall$ constants $\alpha > 4Tc_a\bar{p}/c_b$, the dynamics (Σ_u) are ISS and

$$V_\alpha^\sharp(x,t) := e^{\frac{\alpha}{T}\int_{\frac{t}{\alpha}-T}^{\frac{t}{\alpha}}\int_s^{\frac{t}{\alpha}} q(p(l))\mathrm{d}l\,\mathrm{d}s} V(x,t,p(t/\alpha))$$

is an ISS-CLF for (Σ_u).

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