Dynamic Event-Triggered Control under Input and State Delays using Interval Observers

Frédéric Mazenc Michael Malisoff

***Abstract*—We prove global exponential stability estimates for a class of nonlinear control systems that contain uncertain time-varying input delays and uncertain state delays. We use new dynamic event-triggered controls that ensure that Zeno's phenomenon does not occur. Our analysis uses new synergies of interval observers and Halanay's inequality. We illustrate our approach in a marine robotic dynamics that contains uncertain nonlinear terms.**

***Index Terms*—Stabilization, event-triggered, delay**

I. Introduction

DYNAMIC event-triggered control is an alternative to standard event-triggered controls (e.g., [1]) or traditional controls, where instead of changing control values at times that are independent of the state, the times when the control values change are determined by an interconnected dynamic extension. This calls for finding (a) the control, whose values only change at event triggering times and (b) the dynamic extension, to choose triggering times. This differs from static event triggers, where there is a supremum that is used to determine the trigger times instead of using a dynamic extension [2], [3]. Dynamic event-triggered control has been shown to reduce the numbers of trigger times, as compared with static event-triggers [2], [3]. However, by reducing the use of scarce communication resources by only changing control values when the system under consideration requires attention, dynamic and static event-triggered controls have both inspired significant theoretic research and applications [4]–[14].

One recent event-triggered approach combines interval observers with matrices of absolute values [2], [15]–[17]. These techniques were shown to significantly reduce the average numbers of trigger times on intervals of fixed length, and to beneficially increase the lower bounds on the inter-execution times (which are the differences between consecutive trigger times) compared with traditional event-triggered controls like [1] (and so ensure less triggering), when applied to an underwater robotic model [2]. Interval observers yield componentwise upper and lower bounds for unknown functions [18].

Interval observers appeared in several works, leading to solutions of stabilization problems. However, we believe that this letter is the first to apply this approach to dynamic event-triggered control with unknown input delays and unknown nonlinearities, using dynamical extensions that construct the control and new triggering rules which have not appeared before. Compared with [2], [16], [19], this letter has several key novel features. First, our dynamic extensions and trigger rules use suprema of available state measurements, contrasting with the linear cases of [2] which required a supremum of an uncertainty bound. Second, the uncertainty in [2] was restricted to being an added uncertainty on the right side of the dynamics. Third, our novel use of Halanay's inequality here allows us to overcome the significant challenge of extending the results [2], [16], [19] (which were confined to linear systems) to the more challenging nonlinear systems presented here, where there can also be uncertain coefficients in a linear part of the system including unknown control gains, as well as unknown state and input delays with known bounds. Fourth, our work also contrasts significantly with delay-compensating chain predictor approaches [16] which made the assumption that the input delay was known and constant, which is not assumed in this letter. This letter is strongly motivated by the prevalence and potentially destabilizing influences of uncertain coefficients, delays, and nonlinearities [6], which call for our analysis of delayed event-triggered nonlinear systems.

We use standard notation, where $\mathbb{R}^n$ (resp., $\mathbb{R}^{p\times q}$) is the set of all vectors of real n-tuples (resp., $p \times q$ matrices) and the dimensions of our Euclidean spaces are arbitrary, unless indicated otherwise. We set $\mathbb{Z}_0 = \{0, 1, 2, ...\}$, and $||\cdot||$ denotes the Euclidean $2-$norm, and $||\cdot||_S$ is the supremum in this norm over sets S. For a matrix $G = [g_{ij}] \in \mathbb{R}^{r\times s}$, we set $|G| = [|g_{ij}|]$, so the entries of $|G|$ are the absolute values of the corresponding entries g_{ij} of G, and then the supremum $|\cdot|_S$ over sets S is also defined entrywise. We let G^+ denote the matrix whose entries are $\max\{0, g_{ij}\}$ and $G^- = G^+ - G$, so $|G| = G^+ + G^-$. For matrices $D = [d_{ij}]$ and $E = [e_{ij}]$ of the same size, we write $D < E$ (resp., $D \leq E$) provided $d_{ij} < e_{ij}$ (resp., $d_{ij} \leq e_{ij}$) for all i and j. We adopt similar notation for vectors. We call a matrix S positive (resp., nonnegative) provided $0 < S$ (resp., $0 \leq S$), where 0 is the zero matrix, and I is the identity matrix. For square matrices A, we let D_A denote the diagonal matrix whose diagonal entries are those of A, $R_A = D_A + (A - D_A)^+$ and $N_A = (A - D_A)^+ - (A - D_A)$. For constants $a > 0$, we let $C_{\text{in}}(a)$ denote the set of all absolutely continuous functions $\phi : [-a, 0] \to \mathbb{R}^n$, and x_t is defined by $x_t(\ell) = x(t + \ell)$ for all $\ell \in [-a, 0]$, all functions x, and all $t \geq 0$ such that $t + \ell$ is in the domain of x. A real square matrix is called Metzler provided all of its off-diagonal entries are nonnegative. We assume for simplicity that the initial times of our solutions $x(t)$ are $t_0 = 0$, and that the initial functions are constant, so $x(\ell) = x(0)$ for all $\ell \leq 0$.

F. Mazenc is with INRIA EPI DISCO, L2S-CNRS-CentraleSupélec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: frederic.mazenc@l2s.centralesupelec.fr).

M. Malisoff is with Department of Mathematics, 303 Lockett Hall, 175 Field House Drive, Louisiana State University, Baton Rouge, LA 70803, USA (e-mail: malisoff@lsu.edu).

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II. Main result

A. Studied system

We consider the system

$$\begin{aligned}\dot{x}(t) &= (A+\Delta_A(t))x(t)+(B+\Delta_B(t))u(t-\tau(t)) \\ &\quad +f(t,x_t)\end{aligned} \tag{1}$$

where x is valued in $\mathbb{R}^n$, the control u is valued in $\mathbb{R}^p$ and will be specified below, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are known matrices, the piecewise continuous functions $\Delta_A : [0,+\infty) \to \mathbb{R}^{n\times n}$ and $\Delta_B : [0,+\infty) \to \mathbb{R}^{n\times p}$ represent model uncertainty, and $f : [0,+\infty)\times C_{\rm in}(h) \to \mathbb{R}^n$ for a known value $h>0$ represents unknown state delayed terms. Our first assumption is as follows, where $|\phi|_{[-h,0]}$ is the supremum of ϕ over the set $S=[-h,0]$ as defined in Section I (but see Remark 3, for ways to relax our assumptions on f):

Assumption 1: There is a matrix $K\in\mathbb{R}^{p\times n}$ such that, with

$$H = A+BK, \tag{2}$$

the matrix R_H+N_H is Hurwitz. Also, $\tau(t)$ is piecewise constant, and there are a known constant $\overline{\tau}\geq 0$ and known matrices $\overline{\Delta}_A\in\mathbb{R}^{n\times n}$ and $\overline{\Delta}_B\in\mathbb{R}^{n\times p}$ such that

$$0\leq\tau(t)\leq\overline{\tau},\;\; |\Delta_A(t)|\leq\overline{\Delta}_A,\;\; \text{and } |\Delta_B(t)|\leq\overline{\Delta}_B \tag{3}$$

hold for all $t\geq 0$. Finally, f is continuous in its first variable t and locally Lipschitz in its second variable x_t, and there is a constant $f_0>0$ such that

$$|f(t,\phi)|\leq f_0|\phi|_{[-h,0]} \tag{4}$$

holds for all $t\geq 0$ and $\phi\in C_{\rm in}(h)$. □

According to [20, Theorem 2.11, p. 38], Assumption 1 provides a vector $V>0$ and a real value $c>0$ such that

$$V^\top(R_H+N_H)\leq -cV^\top, \tag{5}$$

since R_H+N_H is Metzler and Hurwitz. We also use a matrix $\Gamma>0$ in $\mathbb{R}^{n\times n}$ such that there is a constant $r>0$ such that

$$-cV^\top+V^\top|BK|\Gamma\leq -rV^\top, \tag{6}$$

which holds when Γ has small enough entries. We also use

$$\Omega(s)=e^{As}+\textstyle\int_0^s e^{(s-m)A}{\rm d}mBK. \tag{7}$$

Since $\Gamma>0$ and $\Omega(0)=I$, there is a constant $\nu>0$ such that $\Omega(t)$ is nonsingular for each $t\in[0,\nu]$ and such that

$$|BK(\Omega^{-1}(s)-I)|\leq|BK|\Gamma \text{ for all } s\in[0,\nu]. \tag{8}$$

We fix $\overline{\tau}$, c, r, h, A, B, V, f_0, Γ, K, $\overline{\Delta}_A$, $\overline{\Delta}_B$, and ν satisfying the preceding requirements in the sequel, and then we set

$$\begin{aligned}R &= \left(I+\left|BK\Omega^{-1}\right|_{[0,\nu]}\textstyle\int_0^\nu\left|e^{A\ell}\right|{\rm d}\ell\right)B^\sharp, \text{ where} \\ B^\sharp &= (I+\overline{\tau}|BK|)(f_0I+\overline{\Delta}_A+\overline{\Delta}_B|K|) \\ &\quad +\overline{\tau}(|BKA|+|(BK)^2|).\end{aligned} \tag{9}$$

We then fix a constant $p_*>0$ such that

$$V^\top R\leq p_*V^\top \tag{10}$$

and then our last assumption is the following, which can be viewed as a smallness condition on $\overline{\tau}$, $\overline{\Delta}_A$, $\overline{\Delta}_B$, or f_0:

Assumption 2: The inequality

$$p_*<r \tag{11}$$

is satisfied. □

Assumption 2 implies that for each constant $T>\nu$, there is a unique constant $R_*>0$ such that

$$R_*=r-p_*e^{R_*(2T+2\overline{\tau}+h+\nu)}. \tag{12}$$

We fix constants $T>\nu$ and $R_*>0$ satisfying (12). Our R_* will be our exponential convergence rate in our application of Halanay's inequality (as stated in [21, Lemma 4.2]) in our proof of our theorem, in which $V_i>0$ will denote the ith entry of the positive vector V from (5) for $i=1,\ldots n$. Halanay's inequality is called for here to handle the delays, uncertainties f, and unknown coefficients that were not present in [2].

Remark 1: As in [2, Remark 3.1], our Hurwitzness assumption on R_H+N_H is unrestrictive, because it can be satisfied for all controllable pairs (A,B) after a change of coordinates. Another key point is that f can be uncertain, as long as we know constants h and f_0 satisfying (4). See Section III, where we illustrate how our assumptions allow significant uncertain input and state delays, and large entries for the uncertainty Δ_B, as compared with the entries of B, and so are not too restrictive, and where we also illustrate a benefit of using the new inequality in (8) instead of the inequality $|\Omega^{-1}(s)-I|\leq\Gamma$ that was used in its place in [2], [16], and [19]. □

B. Event-Triggered Control

Fix a diagonal matrix D whose main diagonal entries are all positive, which is a tuning matrix, and any positive vector $z_0\in\mathbb{R}^n$. In terms of our fixed constant $T>\nu$, consider

$$\left\{\begin{aligned}\dot{x}(t) &= (A+\Delta_A(t))x(t) \\ &\quad +(B+\Delta_B(t))Kx(\mu(t)-\tau(t))+f(t,x_t) \\ \dot{z}(t) &= (R_H+N_H)z(t)-B^\star(t)+\lambda(\zeta(t,x_t)) \\ e(t) &= x(t_i)-x(t) \text{ for all } t\in[t_i,t_{i+1}) \\ t_{i+1} &= \sup\{t\in[t_i,t_i+T]: z(t) \\ &\quad -DB^\star(t)+D\lambda(\zeta(t,x_t))>0\} \\ \mu(t) &= t_i \text{ for all } t\in[t_i,t_{i+1})\end{aligned}\right. \tag{13}$$

with $z(0)=z_0$ and the function λ defined by

$$\lambda(\zeta(t,x_t))=\left|BK\Omega^{-1}\right|_{[0,\nu]}\textstyle\int_{t-\nu}^t\left|e^{A(t-\ell)}\right|\zeta(\ell,x_\ell){\rm d}\ell, \tag{14}$$

and where

$$\begin{aligned}B^\star(t) &= |BKe(t)|-|BK|\Gamma|x(t)| \text{ and} \\ \zeta(t,x_t) &= \overline{\tau}|BKA||x|_{[\mu(t)-\overline{\tau},\mu(t)]} \\ &\quad +\overline{\tau}|(BK)^2||x|_{[\mu(t)-2\overline{\tau}-T,\mu(t)]} \\ &\quad +f_0(I+|BK|\overline{\tau})|x|_{[t-h-T-\overline{\tau},t]} \\ &\quad +(I+\overline{\tau}|BK|)\left(\overline{\Delta}_A+\overline{\Delta}_B|K|\right)|x|_{[t-2(T+\overline{\tau}),t]}\end{aligned} \tag{15}$$

and where $t_0=0$. Our main result is then as follows:

Theorem 1: Let Assumptions 1-2 hold. Then, all solutions $x:[0,+\infty)\to\mathbb{R}^n$ of (13) are such that for all $t\geq 0$, we have

$$||x(t)||\leq\tfrac{n\max\{V_i:1\leq i\leq n\}}{\min\{V_i:1\leq i\leq n\}}e^{-R_*t}\left(2||x(0)||+||z_0||\right). \tag{16}$$

Also, $t_{i+1}-t_i\geq\nu$ holds for all $i\in\mathbb{Z}_0$. □

The λ in (14) collects the effects of f, Δ_A, and Δ_B, and τ. The triggered mechanism (13) is reminiscent of those of [3], except we adopt a positive systems approach. System (13) uses the control $u(t) = Kx(\mu(t)-\tau(t))$ (where $\tau(t)$ represents the delayed impact of triggering a new control value at time $\mu(t)$), and whose trigger times t_i are found recursively, as follows. We solve the initial value problems given by the x and z systems in (13) with the initial time 0 and initial states $x(0)$ and $z(0) = z_0$, while monitoring the componentwise strict inequality in the sup in (13) with $i = 0$ (so we do not need to compute the z values on $[t_i, t_{i+1})$ until the measurement at time t_i is obtained). If this strict inequality holds for all $t \in [0, T]$, then $t_1 = T$. Otherwise, this supremum is in $[0, T)$ and is our t_1. We repeat this with $t_0 = 0$ replaced by t_1, by solving the x and z systems in (13) with the initial time t_1 and the initial state $z(t_1)$ obtained by solving the z system in (13) on $[0, t_1]$. This repeats for all $i \in \mathbb{Z}_0$, and gives a continuous solution $x(t)$ for all $t \geq 0$ and trigger time sequence $\{t_i\}$, because our lower bound $t_{i+1} - t_i \geq \nu$ for all i implies that the Zeno phenomenon does not occur (meaning, only finitely many t_i's occur on each finite length time interval).

Remark 2: While more complicated than standard controllers, (13) ensures robustness to unknown τ, Δ_A, Δ_B, and f while only updating u when a new control value is required. By contrast, standard controllers do not offer these benefits. Our proof of Theorem 1 will show that the theorem remains true if we replace the suprema in (15) by upper bounds, e.g., $|x|_{[\mu(t)-\overline{\tau},\mu(t)]} + \delta_1(t)$ instead of $|x|_{[\mu(t)-\overline{\tau},\mu(t)]}$, and similarly for the other suprema in (15), except we conclude exponential input-to-state stability with respect to the added nonnegative uncertainties δ's, instead of (16); this follows from generalized Halanay's inequalities with gain terms [22]. Hence, it suffices to have upper estimates for the suprema. This can help make our controllers easy to implement. □

C. Proof of Theorem 1

The proof has four parts. The first proves that the solution of the z system in (13) satisfies $z(t) > 0$ for all $t \geq 0$. Since this part is the same as the first part of the proof of [2, Theorem 1] except with the λ from [2, Equation (8)] replaced by (14), and with $|BK|(|e(t)|-\Gamma|x(t)|)$ replaced by the function $B^\star(t)$ from (15) and $\delta = 0$, we omit the first part. In the second part, we prove that $t_{i+1} - t_i \geq \nu$ holds for all $i \in \mathbb{Z}_0$. In the third part, we build our interval observer, to prove (16) in the fourth part. While similar in structure to the proof of [2, Theorem 1], our proof is significantly different, because we need to take into account the f, τ, Δ_A, and Δ_B from (1), which were not allowed in [2] and which call for the significantly different new analogs (14) of the λ's from [2, Equation (8)], and for using Halanay's inequality, which was also not required in [2].

Second Part: Ruling Out Zeno's Phenomenon. We have

$$\begin{aligned} \dot{x}(t) &= (A + \Delta_A(t))x(t) + BKx(\mu(t)) \\ &\quad + \Delta_B Kx(\mu(t) - \tau(t)) \\ &\quad + BK[x(\mu(t) - \tau(t)) - x(\mu(t))] + f(t, x_t) \\ &= Ax(t) + BKx(\mu(t)) - BK\int_{\mu(t)-\tau(t)}^{\mu(t)} \dot{x}(m)\mathrm{d}m \\ &\quad + f(t, x_t) + \Delta_A(t)x(t) + \Delta_B Kx(\mu(t) - \tau(t)) \end{aligned} \tag{17}$$

and so also

$$\dot{x}(t) = Ax(t) + BKx(\mu(t)) + \rho(t) \tag{18}$$

for all $t \geq 0$, where

$$\begin{aligned} \rho(t) &= -BKA\int_{\mu(t)-\tau(t)}^{\mu(t)} x(m)\mathrm{d}m + f^\sharp(t, x_t) + \Delta^\sharp(t) \\ &\quad - (BK)^2 \int_{\mu(t)-\tau(t)}^{\mu(t)} x(\mu(m) - \tau(m))\mathrm{d}m, \\ \Delta^\sharp(t) &= -BK \int_{\mu(t)-\tau(t)}^{\mu(t)} \Delta_A(m)x(m)\mathrm{d}m \\ &\quad -BK \int_{\mu(t)-\tau(t)}^{\mu(t)} \Delta_B(m)Kx(\mu(m)-\tau(m))\mathrm{d}m \\ &\quad + \Delta_A(t)x(t) + \Delta_B(t)Kx(\mu(t) - \tau(t)), \text{ and} \\ f^\sharp(t, x_t) &= f(t, x_t) - BK \int_{\mu(t)-\tau(t)}^{\mu(t)} f(m, x_m)\mathrm{d}m. \end{aligned} \tag{19}$$

Fix an $i \in \mathbb{Z}_0$. We next argue by contradiction. Suppose, for the sake of obtaining a contradiction, that $t_{i+1} - t_i < \nu$. By applying the method of variation of parameters to (18), we get

$$x(t) = \Omega(t - t_i)x(t_i) + \int_{t_i}^{t} e^{A(t-m)}\rho(m)\mathrm{d}m \tag{20}$$

for all $t \in [t_i, t_{i+1})$, where Ω is defined in (7). Hence, since $t_{i+1} - t_i < \nu$, we would be able to use $\Omega^{-1}(t - t_i)$ to solve for $x(t_i)$ in (20) and use our choice $e(t) = x(t_i) - x(t)$ from (13) to get a formula for $e(t)$, then left multiply the result by BK and then apply $|\cdot|$ to both sides of the result, to get

$$\begin{aligned} |BKe(t)| &= \Big|BK[\Omega(t - t_i)^{-1} - I]x(t) \\ &\quad - BK\Omega(t - t_i)^{-1} \int_{t_i}^{t} e^{A(t-m)}\rho(m)\mathrm{d}m\Big| \end{aligned} \tag{21}$$

for all $t \in [t_i, t_{i+1})$, so our choice of $B^\star$ from (15) gives

$$\begin{aligned} B^\star(t) &\leq [|BK(\Omega(t - t_i)^{-1} - I)| - |BK|\Gamma]|x(t)| \\ &\quad + \Big|BK\Omega(t - t_i)^{-1} \int_{t_i}^{t} e^{A(t-m)}\rho(m)\mathrm{d}m\Big| \\ &\leq \Big|BK\Omega(t - t_i)^{-1} \int_{t_i}^{t} e^{A(t-m)}\rho(m)\mathrm{d}m\Big| \end{aligned} \tag{22}$$

for all $t \in [t_i, t_{i+1})$, where the first inequality followed by applying the triangle inequality to the right side of (21) and then subtracting $|BK|\Gamma|x(t)|$ from both sides of the result, and the second inequality followed from (8), using $s = t - t_i$.

Using (22) to lower bound the second right side term in our $\dot{z}$ formula in (13), we get

$$\begin{aligned} \dot{z}(t) &\geq (R_H + N_H)z(t) + \lambda(\zeta(t, x_t)) \\ &\quad - \Big|BK\Omega(t - t_i)^{-1} \int_{t_i}^{t} e^{A(t-m)}\rho(m)\mathrm{d}m\Big| \end{aligned} \tag{23}$$

for all $t \in [t_i, t_{i+1})$. We can also use our choices of ρ and λ in (19) and (14) and the triangle inequality to obtain

$$\begin{aligned} &-\Big|BK\Omega(t - t_i)^{-1} \int_{t_i}^{t} e^{A(t-m)}\rho(m)\mathrm{d}m\Big| + \lambda(\zeta(t, x_t)) \\ &\geq -\Big|BK\Omega(t-t_i)^{-1}\int_{t_i}^{t} e^{A(t-m)}\Big[BKA\int_{\mu(m)-\tau(m)}^{\mu(m)} x(s)\mathrm{d}s \\ &\quad +(BK)^2 \int_{\mu(m)-\tau(m)}^{\mu(m)} x(\mu(s) - \tau(s))\mathrm{d}s\Big]\, \mathrm{d}m\Big| \\ &\quad +\Big|BK\Omega^{-1}\Big|_{[0,\nu]} \int_{t-\nu}^{t} \big|e^{A(t-\ell)}\big|\zeta(\ell, x_\ell)\mathrm{d}\ell \\ &\quad -\Big|BK\Omega(t - t_i)^{-1} \int_{t_i}^{t} e^{A(t-m)}[f^\sharp(m, x_m) + \Delta^\sharp(m)]\mathrm{d}m\Big| \end{aligned} \tag{24}$$

for all $t \in [t_i, t_{i+1})$, where $f^\sharp$ was defined in (19). Also,

$$\int_{\mu(m)-\tau(m)}^{\mu(m)} |x(\mu(s)-\tau(s))|\mathrm{d}s \leq \overline{\tau}|x|_{[\mu(m)-2\overline{\tau}-T,\mu(m)]} \tag{25}$$

holds for all $m \geq 0$, since $\mu(m) \geq s \geq \mu(s) \geq \mu(s) - \tau(s) \geq s - T - \overline{\tau} \geq \mu(m) - 2\overline{\tau} - T$ holds for all $s \in [\mu(m) - \tau(m), \mu(m)]$, which follows because of our choice of μ in (13) and because of (3) from Assumption 1.

Also, it follows from (15) and (24) that, with the choice

$$f_0^\sharp = (I + |BK|\overline{\tau}) \left(f_0 I + \overline{\Delta}_A + \overline{\Delta}_B |K|\right), \tag{26}$$

we have

$$\begin{aligned}
&-\left|BK\Omega(t - t_i)^{-1} \int_{t_i}^t e^{A(t-m)}\rho(m)\mathrm{d}m\right| + \lambda(\zeta(t, x_t)) \\
&\geq -\Big|BK\Omega(t-t_i)^{-1}\int_{t_i}^t e^{A(t-m)}\Big[BKA\int_{\mu(m)-\tau(m)}^{\mu(m)} x(s)\mathrm{d}s \\
&+(BK)^2 \int_{\mu(m)-\tau(m)}^{\mu(m)} x(\mu(s) - \tau(s))\mathrm{d}s\Big]\, \mathrm{d}m\Big| \\
&+\left|BK\Omega^{-1}\right|_{[0,\nu]}\int_{t-\nu}^t \left|e^{A(t-\ell)}\right|\big[\overline{\tau}|BKA||x|_{[\mu(\ell)-\overline{\tau},\mu(\ell)]} \\
&+|(BK)^2|\overline{\tau}|x|_{[\mu(\ell)-2\overline{\tau}-T,\mu(\ell)]}\big]\, \mathrm{d}\ell \\
&-\left|BK\Omega(t - t_i)^{-1} \int_{t_i}^t e^{A(t-m)}(f^\sharp(m, x_m) + \Delta^\sharp(m))\mathrm{d}m\right| \\
&+\left|BK\Omega^{-1}\right|_{[0,\nu]} \int_{t-\nu}^t \left|e^{A(t-\ell)}\right| f_0^\sharp|x|_{[\ell-h-2T-2\overline{\tau},\ell]}\mathrm{d}\ell
\end{aligned} \tag{27}$$

for all $t \in [t_i, t_{i+1})$. Since $t_{i+1} - t_i < \nu$, it follows from (25), (27), and our formulas for $f^\sharp$ and $f_0^\sharp$ in (19) and (26) that

$$\begin{aligned}
&-\left|BK\Omega(t - t_i)^{-1} \int_{t_i}^t e^{A(t-m)}\rho(m)\mathrm{d}m\right| + \lambda(\zeta(t, x_t)) \\
&\geq 0
\end{aligned} \tag{28}$$

for all $t \in [t_i, t_{i+1})$, by using the nonnegative right side terms of (27) to dominate the other right side terms of (27). Hence, (23) gives $\dot{z}(t) \geq (R_H + N_H)z(t) \geq D_H z(t)$ for all $t \in [t_i, t_{i+1})$, where the second inequality followed because of the nonnegative valuedness of z from the first part of the proof.

Therefore, with the choice $g = \min\{e^{-|d_i|\nu} : 1 \leq i \leq n\}$, where d_i is the ith main diagonal entry of the diagonal matrix D_H, we can apply a variation of parameters argument to obtain

$$z(t) \geq e^{D_H(t-t_i)}z(t_i) \geq gz(t_i) \text{ for all } t \in [t_i, t_{i+1}). \tag{29}$$

Since $D \geq 0$ is nonnegative, we can left multiply (22) by $-|BK|$ and use (29) and our $B^\star$ from (15) to conclude that

$$\begin{aligned}
&z(t) - DB^\star(t) \\
&+D\left|BK\Omega(t - t_i)^{-1} \int_{t_i}^t e^{A(t-m)}\rho(m)\mathrm{d}m\right| \geq gz(t_i)
\end{aligned} \tag{30}$$

holds for all $t \in [t_i, t_{i+1})$. From (28) and (30), it follows that

$$z(t) - DB^\star(t) + D\lambda(\zeta(t, x_t)) \geq gz(t_i) \tag{31}$$

for all $t \in [t_i, t_{i+1})$. Since $e(t_{i+1}) = 0$, and since x and z are continuous, it follows that (31) holds for all $t \in [t_i, t_{i+1}]$.

Therefore, by the positive valuedness of $z(t)$ from the first part of the proof, we get

$$z(t) - DB^\star(t) + D\lambda(\zeta(t, x_t)) > 0 \tag{32}$$

for all $[t_i, t_{i+1}]$. Since $e(t)$ is right continuous, this gives an $\epsilon_0 > 0$ such that (32) holds for all $t \in [t_i, t_{i+1} + \epsilon_0]$. By (13), we have $t_{i+1} = \sup\{t > t_i : z(t) - DB^\star(t) + D\lambda(\zeta(t, x_t)) > 0\}$, which follows because we supposed that $t_{i+1} - t_i < \nu \leq T$. This is a contradiction, so $t_{i+1} - t_i \geq \nu$ for all $i \in \mathbb{Z}_0$.

Third Part: Interval Observer. Observe that (18) gives $\dot{x}(t) = Hx(t) + BKe(t) + \rho(t)$ for all $t \geq 0$, by (2) and (13). It follows that the x dynamics in (13) can be written as

$$\dot{x}(t) = (R_H - N_H)x(t) + BKe(t) + \rho(t) \tag{33}$$

for all $t \geq 0$. This motivates our use of the interval observer

$$\left\{\begin{array}{rcl}
\dot{\overline{x}}(t) &=& R_H\overline{x}(t) - N_H\underline{x}(t) + (BKe(t))^+ + \rho(t)^+ \\
\dot{\underline{x}}(t) &=& R_H\underline{x}(t) - N_H\overline{x}(t) - (BKe(t))^- - \rho(t)^-
\end{array}\right. \tag{34}$$

with the initial state $\overline{x}(0) = |x(0)|$ and $\underline{x}(0) = -|x(0)|$. Since

$$\begin{bmatrix} R_H & N_H \\ N_H & R_H \end{bmatrix} \tag{35}$$

is Metzler, we can consider the dynamics for the pairs $(\overline{x} - x, x - \underline{x})$ and $(\overline{x}, -\underline{x})$, to conclude from our choices of $\overline{x}(0)$ and $\underline{x}(0)$ that $\overline{x}(t) \geq 0$, $\underline{x}(t) \leq 0$, and $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ hold for all $t \geq 0$, e.g., by [23, Lemma 1]. Hence,

$$|x(t)| \leq \overline{x}(t) - \underline{x}(t) \tag{36}$$

for all $t \geq 0$.

Fourth Part: Stability Analysis. We use the linear Lyapunov function $U(\overline{x}, \underline{x}, z) = V^\top(\overline{x} - \underline{x} + z)$. Then, since (34) gives

$$\begin{aligned}
\dot{\overline{x}}(t) - \dot{\underline{x}}(t) \;\;=\;\; &(R_H + N_H)(\overline{x}(t) - \underline{x}(t)) \\
&+|BKe(t)| + |\rho(t)|
\end{aligned} \tag{37}$$

for all $t \geq 0$, we conclude from (5), (15), and (36) that the time derivative of U along trajectories of (33) and (13) satisfies

$$\begin{aligned}
\dot{U}(t) \;\leq\; &-cV^\top(\overline{x}(t)-\underline{x}(t))+V^\top|BKe(t)|-cV^\top z(t) \\
&+V^\top|\rho(t)| + V^\top\lambda(\zeta(t, x_t)) - V^\top B^\star(t) \\
\leq\; &-cV^\top(\overline{x}(t) - \underline{x}(t) + z(t)) + V^\top\lambda(\zeta(t, x_t)) \\
&+V^\top|BK|\Gamma|x(t)| + V^\top|\rho(t)| \\
\leq\; &-cV^\top(\overline{x}(t) - \underline{x}(t) + z(t)) + V^\top\lambda(\zeta(t, x_t)) \\
&+V^\top|BK|\Gamma(\overline{x}(t) - \underline{x}(t)) + V^\top|\rho(t)|
\end{aligned} \tag{38}$$

for all $t \geq 0$. From (15), (19), and (25), we deduce that

$$\begin{aligned}
\dot{U}(t) \;\leq\; &-cV^\top(\overline{x}(t) - \underline{x}(t) + z(t)) + V^\top\lambda(\zeta(t, x_t)) \\
&+V^\top|BK|\Gamma(\overline{x}(t) - \underline{x}(t)) + V^\top\zeta(t, x_t) \\
\leq\; &(-cV^\top + V^\top|BK|\Gamma)(\overline{x}(t) - \underline{x}(t) + z(t)) \\
&+V^\top\zeta(t, x_t) + V^\top\lambda(\zeta(t, x_t)) \\
\leq\; &-rU(\overline{x}(t), \underline{x}(t), z(t)) + V^\top(\zeta(t, x_t) \\
&+ V^\top\lambda(\zeta(t, x_t))
\end{aligned} \tag{39}$$

for all $t \geq 0$, where the second inequality followed because z is nonnegative valued, and the last inequality used (6) and the fact that $\overline{x} - \underline{x}$ and z are nonnegative valued. Also, since (13) gives $\mu(m) \in [m - T, m]$ for all $m \geq 0$, we observe that

$$\begin{aligned}
\zeta(t, x_t) \;\leq\; &\overline{\tau}|BKA||x|_{[\mu(t)-\overline{\tau},\mu(t)]} + f_0^\sharp|x|_{[t-h-2T-2\overline{\tau},t]} \\
&+\overline{\tau}|(BK)^2||x|_{[t-2T-2\overline{\tau},t]} \\
\leq\; &B^\sharp|x|_{[t-2(T+\overline{\tau})-h,t]}
\end{aligned} \tag{40}$$

for all $t \geq 0$, by our choices of $f_0^\sharp$ and $B^\sharp$ in (26) and (9).

Using (40) and our choice of λ in (14), we deduce that

$$\begin{aligned}
&\lambda(\zeta(t, x_t)) \\
&\leq \left|BK\Omega^{-1}\right|_{[0,\nu]} \int_{t-\nu}^t \left|e^{A(t-\ell)}\right|B^\sharp|x|_{[\ell-2(T+\overline{\tau})-h,\ell]}\mathrm{d}\ell \\
&\leq \left|BK\Omega^{-1}\right|_{[0,\nu]} \int_0^\nu \left|e^{A\ell}\right|\mathrm{d}\ell B^\sharp|x|_{[t-2(T+\overline{\tau})-h-\nu,t]}\mathrm{d}\ell
\end{aligned} \tag{41}$$

for all $t \geq 0$. By using (40) and (41) to upper bound the last two terms in (39) and recalling our choice of R in (9), we get $\dot{U}(t) \leq -rU(\overline{x}(t), \underline{x}(t), z(t)) + V^\top R|\overline{x} - \underline{x} +$

$z|_{[t-2(T+\overline{\tau})-h-\nu,t]}$ for all $t \geq 0$, where we also used (36) and the nonnegative valuedness of $z(t)$. Hence,

$$\dot{U}(t) \leq -rU(\overline{x}(t), \underline{x}(t), z(t)) + p_* \sup_{s\in[t-2(T+\overline{\tau})-h-\nu,t]} U(\overline{x}(s), \underline{x}(s), z(s)) \tag{42}$$

holds for all $t \geq 0$, by our condition (10) on p_*.

It follows from our choice (12) of R_* and from applying Halanay's inequality (e.g., from [21, Section 4.1.2]) that $U(\overline{x}(t), \underline{x}(t), z(t)) \leq U(\overline{x}(0), \underline{x}(0), z_0)e^{-R_* t}$ holds for all $t \geq 0$. This exponential decay estimate implies that

$$\min_{1\leq i\leq n} V_i|x_\ell(t)| \leq \max_{1\leq i\leq n} V_i \sum_{j=1}^{n} (\overline{x}_j(0)-\underline{x}_j(0)+z_{0j})e^{-R_* t} \tag{43}$$

for all $t \geq 0$ and $\ell \in \{1, \ldots, n\}$, where $z_{0j} > 0$ is the jth component of z_0, and where we used (36) and the positive valuedness of z to get the lower bound in (43). Hence, by the Cauchy-Schwarz inequality and our choices of $\overline{x}(0)$ and $\underline{x}(0)$,

$$|x_\ell(t)| \leq \tfrac{\sqrt{n}\max\{V_i:1\leq i\leq n\}}{\min\{V_i:1\leq i\leq n\}} ||2|x(0)| + z_0||e^{-R_* t} \tag{44}$$

holds for all $t \geq 0$ and $\ell \in \{1, \ldots, n\}$. The conclusion (16) follows by squaring both sides of (44), then summing the result over ℓ, then taking square roots of both sides of the result.

Remark 3: If, instead of (4), we require a continuous nondecreasing function $\mathcal{F} : [0, +\infty) \to [0, +\infty)$ such that

$$|f(t, \phi)| \leq \mathcal{F}(|\phi|_{[-h,0]})|\phi|_{[-h,0]} \tag{45}$$

holds for all $t \geq 0$ and $\phi \in C_{\text{in}}(h)$, then we can prove a local analog of Theorem 1, which applies for suitable positive vectors $z_0 \in \mathbb{R}^n$ and constants $\sigma_0 > 0$, and all initial states $x(0)$ such that $||x(0)|| \leq \sigma_0$. This is done by setting

$$B_* = \tfrac{n\max\{V_i:1\leq i\leq n\}}{\min\{V_i:1\leq i\leq n\}} (2\sigma_0 + ||z_0||) \tag{46}$$

and then replacing f_0 by $\mathcal{F}(B_*)$ in the formulas for $B^\sharp$ and ζ in (9) and (15). With this replacement, we can then apply the proof of Theorem 1 from above on the interval $[0, t]$, for each $t \geq 0$ such that $||x(\ell)|| \leq B_*$ for all $\ell \in [0, t]$, to conclude that (16) holds for all $t \in [0, T_*)$, where $T_* = \sup\{t \geq 0 : ||x(\ell)|| \leq B_* \text{ for all } \ell \in [0, t]\}$. One can then show that $T_* = +\infty$, by arguing by contradiction, as follows.

First, note that $T_* > 0$, because (46) gives $||x(0)|| \leq \sigma_0 < B_*$ and because x is continuous. On the other hand, if $T_* < +\infty$, then the continuity of $x(t)$ implies that $||x(T_*)|| = B_*$, so evaluating (16) at $t = T_*$ gives the contradiction $B_* = ||x(T_*)|| \leq e^{-R_* T_*}B_* < B_*$. This contradiction implies that $T_* = +\infty$, so (16) holds for all $t \geq 0$. Our condition (45) allows cases where the dynamics can exhibit quadratic (or higher order) growth, instead of the linear growth in the state from (4). □

III. Illustrations

A. Marine Robotic Example

We first study a dynamics for the control of the depth and pitch degrees-of-freedom of an autonomous underwater vehicle from [2], [15], [17], whose linearization was used in [15], [17] to represent the dynamics of the BlueROV2 vehicle, which is commonly used in environmental surveys. Following [15], [17], we assume that the vehicle has a Doppler Velocity Logger (or DVL) for estimating the velocity of the vehicle. The DVL experiences bottom lock, making it impractical to continuously change the control values and also producing input delays. Hence, we illustrate benefits of our new event-triggered approach, which are beyond the capabilities of [2], [17] or other event-triggered studies that did not allow nonlinearities with unknown delays or did not quantify the effects of uncertain coefficient matrices. This strongly motivates our more complicated control and trigger rule from Theorem 1.

Using [24, Equation (9.28)], and assuming that the vehicle is neutrally buoyant, the vehicle dynamics become

$$\begin{aligned}&(m - X_{\dot{w}(t)})\dot{w}(t) - (mx_g + Z_{\dot{q}})\dot{q}(t) - Z_w w(t)\\ &+f_a(t, w_t, q_t) - (mU(t) + z_q)q(t) = Z_{\gamma_s}u_Z\end{aligned} \tag{47}$$

$$\begin{aligned}&(mx_g + M_{\dot{w}}(t))\dot{w}(t) + (I_{yy} - M_{\dot{q}})\dot{q}(t) - M_w w(t)\\ &+f_b(t, w_t, q_t) + (mx_g U - M_q)q(t) - M_\theta\theta = M_{\gamma_s}u_M\end{aligned} \tag{48}$$

whose parameters were experimentally computed in [24], where $f_a(t, w_t, q_t)$ and $f_b(t, w_t, q_t)$ represent nonlinear and delayed effects. As in [17], we assume that the surge nominal velocity is $U = 0.1$m/s. The states are the depth velocity w and the pitch velocity q, and the controls u_Z and u_M are the force and moment to produce motion.

With the parameter values and controller from [24], the system (47)-(48) takes the form (1) from Theorem 1 with

$$A = \begin{bmatrix} -0.387 & 0 \\ 0 & -1.8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.038 \\ 1.5 \end{bmatrix}, \tag{49}$$

which are the A and B in [15]. Choosing $K = [-0.977852, -0.097546799]$ as in the linearized case in [2] gives the eigenvalues -1.94988 and -0.420595 for

$$H = A + BK = \begin{bmatrix} -0.424158 & -0.00370678 \\ -1.46678 & -1.94632 \end{bmatrix} \tag{50}$$

which satisfies Assumption 1. We then applied Theorem 1 in three scenarios: (i) $f_0 = 0$, with $\overline{\tau} > 0$ and ν as large as possible, (ii) $\overline{\tau} = 0$, with f_0 chosen as large as possible while still satisfying our assumptions with $\nu = 0.12$, and (iii) $f_0 = \overline{\tau} = 0$, $\overline{\Delta}_A \in \mathbb{R}^{n\times n}$ being the zero matrix, and $\overline{\Delta}_B \in \mathbb{R}^{n\times p}$ having all its entries equaling a largest possible common value δ_* such that we can satisfy our assumptions with $\nu = 0.25$.

Scenario (i). To satisfy the requirements from Section II-A with $f_0 = 0$, we first chose $V = [2, 0.2]^\top$, $c = 0.22$, $r = 0.06$, $p_* = 0.05$, $f_0 = 0$, $\overline{\tau} = 0.15$, $\Delta_A = 0$, $\Delta_B = 0$, and

$$\Gamma = \begin{bmatrix} 0.5 & 0.012 \\ 0.6 & 0.7 \end{bmatrix}. \tag{51}$$

Then we used a bisection method in Mathematica to find that the largest value of the lower bound ν for the inter-execution times that satisfies the assumptions of Theorem 1 (up to the second decimal place) was $\nu = 0.26$.

Scenario (ii). Next, we applied Theorem 1 with $\overline{\tau} = 0$, and with the growth rate $f_0 = 0.05$ and $h = 1.5$, which allowed $f = [f_1, f_2]^\top$ with $f_1(t, x_t) = 0.05\sin(x(t-1.5))$ and $f_2 = 0$. In this case, the assumptions of Theorem 1 were satisfied with the parameter values from Scenario (i) above, except with ν reduced from 0.26 to 0.15, r increased from 0.06 to 0.065, and p_* increased from 0.05 to 0.06. We increased r and p_* and reduced ν, because the assumptions of Theorem 1 were not satisfied with $f_0 = 0.05$ with the r, p_*, and ν values that we used in the $f_0 = 0$ case above. This illustrates a trade-off that we observed in numerical experiments, where allowing the uncertain nonlinearity f required smaller lower bounds ν on the inter-execution times, as compared to cases where the uncertainty was instead in the unknown input delay τ.

Scenario (iii). Consider the case where $f_0 = \overline{\tau} = 0$, $\overline{\Delta}_A$ is the zero matrix, and both entries of $\overline{\Delta}_B \in \mathbb{R}^{2\times 1}$ equaling some value $\delta_* > 0$. With the choice $\nu = 0.25$, $p_* = 0.0625$, and all other parameters chosen the same as in Scenario (ii), we then found that the largest possible δ_* that satisfied the requirements of Theorem 1 in this scenario was $\delta_* = 0.06$, which is larger than the first entry of B in (49). Hence, our work covers significant cases such as $\Delta_B(t) = [0.06\sin(t), 0]^\top$ where some components of Δ_B can have larger suprema than the corresponding components of the unperturbed coefficient B.

B. Robustness Example

To compare Theorem 1 from Section II-B above with the simpler static event-triggered interval observer control result from [19] which also allowed added uncertainty on A, we revisit the example from [19, Section 6.2], which had $A = [a_{ij}] \in \mathbb{R}^{2\times 2}$ having the entries $a_{11} = 1$, $a_{12} = 1/2$, $a_{21} = 3/2$, and $a_{22} = 0$, $f = 0$, $\overline{\tau} = 0$, $\Delta_B = 0$, $\Delta_A = [\delta_{ij}]$ having the form $\delta_{11} = \delta_1$, $\delta_{12} = \delta_2/2$, $\delta_{21} = 1.5\delta_3$, and $\delta_{22} = 0$, $K = [-4/3, -1/3]$, and $B = [1, 1]^\top$. In [19], the maximum allowable lower bound ν on the inter-execution times was $\nu = 0.02$ if the time-varying uncertainties δ_{ij} were each bounded by 0.01. This was based on choosing all entries of $\Gamma \in \mathbb{R}^{2\times 2}$ to be 0.045 in the requirement $|\Omega^{-1}(s) - I| \leq \Gamma$ for all $s \in [0, \nu]$. By contrast, here we used the Mathematica program to show that we can satisfy the requirements of Theorem 1 above with the preceding choices, except with each entry of Γ being 0.04455, $V = [1, 1]^\top$, $c = 1/6$, $r = 0.18$, $p_* = 0.017$, the same bound on the δ_{ij} as in [19, Section 6.2], and $\nu = 0.15$. This 7-fold increase in ν (from $\nu = 0.02$ in [19] to $\nu = 0.15$ here) is significant, and can be attributed to our new condition (8), our new dynamic event-triggered control, and our novel use of Halanay's inequality, and we found similar benefits under uncertain nonlinearities f or input delays τ.

IV. Conclusion

We advanced the state-of-the-art for dynamic event-triggered control under unknown input delays, uncertain coefficients, and unknown nonlinearities. Key novel features included (i) our relaxed requirement on ν and new trigger rules, using suprema of available state measurements over suitable intervals, (ii) our use of Halanay's inequality to handle unknown delays and uncertainties, and (iii) our ability to approximate the suprema that arise from our novel trigger rules using a variant of Halanay's inequality with gain terms. Our applications illustrated a tradeoff, where nonlinearites can reduce the lower bound on the inter-execution times, compared to significantly larger lower bounds on the inter-execution times occurring in other scenarios where the nonlinearity is not present, but where input delays or uncertain control gains occur. We aim to provide extensions for systems with outputs.

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