Further Remarks on Strict Input-to-State Stable Lyapunov Functions for Time-Varying Systems ⋆

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Abstract

We study the stability properties of a class of time-varying nonlinear systems. We assume that non-strict input-to-state stable (ISS) Lyapunov functions for our systems are given and posit a mild persistency of excitation condition on our given Lyapunov functions which guarantee the existence of strict ISS Lyapunov functions for our systems. Next, we provide simple direct constructions of explicit strict ISS Lyapunov functions for our systems by applying an integral smoothing method. We illustrate our constructions using a tracking problem for a rotating rigid body.

Key words: Lyapunov functions, input-to-state stabilization, nonautonomous systems.

1 Introduction

The theory of input-to-state stable (ISS) systems plays a central role in modern non-linear control analysis and controller design (see (Sontag, 1998, 2001; Sontag & Wang, 1995)). The ISS property was introduced in (Sontag, 1989) and an ISS Lyapunov characterization was obtained in (Sontag & Wang, 1995). The ISS Lyapunov characterization provides necessary and sufficient conditions for time-invariant systems to be ISS, in terms of the existence of so-called strict ISS Lyapunov functions; see Section 2 below for the relevant definitions and (Edwards *et al.*, 2000) for an extension to time-varying systems. Strict Lyapunov functions have been used to design stabilizing feedback laws that render asymptotically controllable systems ISS to actuator errors and small observation noise; see (Sontag, 2001). Such control laws are expressed in terms of gradients of Lyapunov functions and therefore require explicit strict Lyapunov functions in order to be implemented. This has motivated a great deal of research devoted to constructing explicit strict Lyapunov functions.

One obstacle to these constructions is that the known strict Lyapunov functions from the existence theory are optimal control value functions (see (Bacciotti & Rosier, 2001; Edwards *et al.*, 2000; Sontag & Wang, 1995; Teel & Praly, 2000)), and therefore are not explicit. Although value functions can often be expressed as unique solutions of Hamilton-Jacobi (HJ) equations subject to appropriate side conditions, the usual techniques for computing value functions in terms of HJ equation solutions can be difficult to implement. For special kinds of systems, strict ISS Lyapunov functions can be explicitly constructed by ad hoc means. On the other hand, there are numerous important cases where it is straightforward to use backstepping or other known methods to construct explicit *non*-strict ISS Lyapunov functions (see our definitions of ISS and non-strict ISS Lyapunov functions in Section 2 and Section 4 for an explicit ex-

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ample). For instance, applying the methods of (Jiang & Nijmeijer, 1997) to tracking problems for nonholonomic systems in chained form gives non-strict Lyapunov functions. The constructions in (Mazenc & Praly, 2000) also frequently give rise to non-strict Lyapunov functions.

This motivates the search for techniques for constructing strict ISS Lyapunov functions for time-varying systems, in terms of known non-strict ISS Lyapunov functions. This search is the focus of this note. For time-varying systems with no controls, the paper (Mazenc, 2003) constructed strict globally asymptotically stable (GAS) Lyapunov functions in terms of given non-strict GAS Lyapunov functions. See (Angeli, 1999; Mazenc & Nesic, 2004; Sontag & Teel, 1995) for constructions of strict Lyapunov functions from nonstrict ones, which apply to *autonomous* systems, and which use totally different techniques from what we consider below. Here we further develop the approach in (Mazenc, 2003). We provide the necessary background on ISS systems and Lyapunov functions in Section 2. We then introduce a non-strict generalization of ISS in which the dissipation rate depends on a non-negative time-dependent decay parameter. The parameter can be zero along intervals of positive length. However, when the parameter is identically one, our non-strict ISS property agrees with the usual ISS condition. Under a mild non-degeneracy assumption on this parameter, which is of persistency of excitation type (see for instance (Loria *et al.*, 2002) and (Loria & Panteley, 2002) for definitions and discussions of the concept of persistency of excitation), we show that our non-strict ISS property is equivalent to the existence of a strict ISS Lyapunov function and is therefore also equivalent to the standard ISS condition. We prove these equivalences in Section 3. They are proved by explicitly constructing strict ISS Lyapunov functions. In Section 4, we illustrate our constructions using a tracking example. Concluding remarks in Section 5 end the paper.

2 Preliminaries

Let $\mathcal{K}_\infty$ denote the set of all continuous functions $\rho : [0,\infty) \to [0,\infty)$ for which (i) $\rho(0) = 0$ and (ii) ρ is increasing and unbounded. Let $\mathcal{KL}$ denote the set of all continuous functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ for which (1) for each $t \geq 0$, $\beta(\cdot, t)$ is strictly increasing and $\beta(0,t) = 0$ (2) $\beta(s,\cdot)$ is non-increasing for each $s \geq 0$, and (3) $\beta(s,t) \to 0$ as $t \to +\infty$ for each $s \geq 0$. We study the stability properties of the system

$$\dot{x} = f(t,x,u), \quad t \geq 0,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \qquad (1)$$

where we always assume f is locally Lipschitz in (t,x,u). Following Mazenc (2003), we also assume f is periodic in t, which means there exists a constant $T > 0$ such that $f(t+T,x,u) = f(t,x,u)$ for all $t \geq 0$, $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$. However, most of our arguments remain valid if this periodicity assumption is weakened to requiring

$$\sup\{|f(t,x,u)| : (x,u) \in K, t \geq 0\} < +\infty \qquad (2)$$

where $|\cdot|$ is the usual Euclidean norm. The control functions for our system (1) comprise the set of all measurable locally essentially bounded functions $\alpha : [0,\infty) \to \mathbb{R}^m$; we denote this set by $\mathcal{U}$. We let $|\alpha|_I$ denote the essential supremum of any control $\alpha \in \mathcal{U}$ restricted to any interval $I \subseteq [0,\infty)$. For each $t_o \geq 0$, $x_o \in \mathbb{R}^n$, and $\alpha \in \mathcal{U}$, we let $I \ni t \mapsto \phi(t; x_o, t_o, \alpha)$ denote the unique trajectory of (1) for the input α satisfying $x(t_o) = x_o$ and defined on its maximal interval $I \subseteq [t_o, \infty)$. This trajectory will be denoted by ϕ when this would not lead to confusion. We say that f is *forward complete* provided each such trajectory ϕ is defined on all of $[t_o,\infty)$.

A C^1 function $V : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ is said to be of class UPPD (written $V \in$ UPPD) provided it is uniformly proper and positive definite, which means there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that, for all $t \geq 0, x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|), \quad |\nabla V(t,x)| \leq \alpha_3(|x|). \qquad (3)$$

We say that V has period τ in t provided there exists a constant $\tau > 0$ such that $V(t+\tau,x) = V(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$; in this case, the bound on ∇V in (3) is redundant. We assume α_1 and α_2 in (3) are C^1, e.g., by taking $\alpha_2(s) = \int_o^s \alpha_3(r)dr$ and minorizing α_1 by a C^1 function of class $\mathcal{K}_\infty$. Given $V \in$ UPPD, we set

$$\dot{V}(t,x,u) := \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x,u).$$

Notice that for each $\chi \in \mathcal{K}_\infty$, the mapping $s \mapsto \sup\{|\dot{V}(t,x,u)| : t \geq 0, |x| \leq \chi(s), |u| \leq s\} + s$ is of class $\mathcal{K}_\infty$ (by (2)-(3)). We let $\mathcal{P}$ denote the set of all continuous functions $p : \mathbb{R} \to [0,\infty)$ that admit constants $\tau, \varepsilon, \bar{p} > 0$ for which

$$\int_{t-\tau}^{t} p(s)ds \ \geq\ \varepsilon \quad \text{and} \quad p(t) \leq \bar{p}\,, \quad \forall t \geq 0. \qquad (4)$$

We write $p \in \mathcal{P}(\tau,\varepsilon,\bar{p})$ to indicate that (i) $p \in \mathcal{P}$ and (ii) $\tau, \varepsilon, \bar{p} > 0$ are constants such that (4) holds. In particular, any continuous periodic function $p : \mathbb{R} \to [0,\infty)$ that is not identically zero admits constants $\tau, \varepsilon, \bar{p} > 0$ satisfying (4), but (4) also allows non-periodic p with arbitrarily large null sets, e.g., for fixed $r > 0$, $p_r(t) = (1+e^{-t})\max\{0, \sin^3(\frac{t}{r})\}$. The elements of $\mathcal{P}$ serve as decay rates for non-strict Lyapunov functions as follows:

Definition 1 *Let $p \in \mathcal{P}$. A function $V \in$ UPPD is called an* ISS(p) Lyapunov function *for (1), provided there exist $\chi \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}_\infty \cap C^1$ such that*

$$|x| \geq \chi(|u|) \Rightarrow \dot{V}(t,x,u) \leq -p(t)\mu(|x|) \quad \forall t \geq 0. \qquad (5)$$

An ISS(p) Lyapunov function for (1) and $p(t) \equiv 1$ is also called a strict ISS Lyapunov function.

Notice that (5) allows $\dot{V}(t,x,u)=0$ for some t's so V can non-strictly decrease along solutions of (1).

Definition 2 *Let $p \in \mathcal{P}$. We say that (1) is* ISS(p)*, or that it is* input-to-state stable (ISS) with decay rate p, *provided there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $t_o \geq 0$, $x_o \in \mathbb{R}^n$, $u_o \in \mathcal{U}$ and $h \geq 0$,*

$$|\phi(t_o+h; x_o, t_o, u_o)| \leq \beta\left(|x_o|, \int_{t_o}^{t_o+h} p(s)ds\right) + \gamma\left(|u_o|_{[t_o,t_o+h]}\right). \tag{6}$$

If (1) is ISS(p) with $p \equiv 1$, then we say that (1) is ISS.

Notice that ISS(p) systems are always forward complete.

Definition 3 *Let $p \in \mathcal{P}$. A function $V \in$* UPPD *is called a* non-strict dissipative Lyapunov function *for (1) and p, or a* DIS(p) Lyapunov function, *provided there exist $\Omega \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}_\infty \cap C^1$ such that, for all $t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m$*

$$\dot{V}(t,x,u) \leq -p(t)\mu(|x|) + \Omega(|u|). \tag{7}$$

A DIS(p) Lyapunov function for (1) and $p(t) \equiv 1$ is also called a strict DIS Lyapunov function.

Remark 4 *For systems without inputs, it follows from (Loria & Panteley, 2002; Loria* et al.*, 2005) that our hypotheses imply uniform GAS since $p(t)\mu(|x|)$ is δ-PE. In contrast with (Loria* et al.*, 2005), which deals mainly with establishing stability, our work leads to simple, direct, explicit expressions for* strict ISS Lyapunov functions *for* general nonlinear systems*, which are not provided by the auxiliary function approach in (Loria* et al.*, 2005).*

We use the following elementary lemma whose proof we leave as a simple exercise:

Lemma 5 *Let $\tau, \varepsilon, \bar{p} > 0$ be constants and $p \in \mathcal{P}(\tau,\varepsilon,\bar{p})$ be given. Then:*
(i) $0 \leq \int_{t-\tau}^{t}\left(\int_s^t p(r)dr\right)ds \leq \frac{\tau^2\bar{p}}{2}$ for all $t \geq 0$ and
(ii) $[0,\infty) \ni h \mapsto \underline{p}(h) = \inf\{\int_t^{t+h} p(r)dr : t \geq 0\}$ is continuous, non-decreasing, and unbounded.

3 Characterizations of Non-Strict ISS

We next relate the Lyapunov functions and stability notions we introduced in the last section. We show that ISS(p) is equivalent to the existence of an ISS(p) Lyapunov function and the existence of a strict ISS Lyapunov function. Our proof explicitly constructs a strict ISS Lyapunov function for (1) in terms of a given DIS(p) Lyapunov function. Moreover, if $p \in \mathcal{P}(\tau,\varepsilon,\bar{p})$ and our given DIS(p) Lyapunov function both have period τ, then our strict ISS Lyapunov function also has period τ.

Theorem 6 *Let $p \in \mathcal{P}$ and f be as above. The following are equivalent:*
(C_1) f admits an ISS(p) Lyapunov function.
(C_2) f admits a strict ISS Lyapunov function.
(C_3) f admits a DIS(p) Lyapunov function.
(C_4) f admits a strict DIS Lyapunov function.
(C_5) f is ISS(p).
(C_6) f is ISS.

We prove: $(C_1) \Rightarrow (C_2) \Rightarrow (C_4) \Rightarrow (C_1)$, $(C_3) \Leftrightarrow (C_4)$, $(C_2) \Leftrightarrow (C_6)$, and $(C_5) \Leftrightarrow (C_6)$. We assume $\tau, \varepsilon, \bar{p} > 0$ are constants such that $p \in \mathcal{P}(\tau,\varepsilon,\bar{p})$.
Step 1: $(C_1) \Rightarrow (C_2)$. If (C_1) holds, then we can find an ISS(p) Lyapunov function V for f, and therefore $\alpha_1, \alpha_2 \in \mathcal{K}_\infty \cap C^1$ satisfying (3) and $\chi \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}_\infty \cap C^1$ satisfying (5). Set

$$\begin{aligned}\tilde{\alpha}_2(s) &:= \max\left\{\tfrac{\tau\bar{p}}{2}, 1\right\}(\alpha_2(s)+\mu(s)+s),\\ w(s) &:= \tfrac{1}{4\tau}\mu(\tilde{\alpha}_2^{-1}(s)).\end{aligned} \tag{8}$$

Then $\tilde{\alpha}_2, \tilde{\alpha}_2^{-1} \in \mathcal{K}_\infty \cap C^1$. Since $V(t,x) \leq \tilde{\alpha}_2(|x|)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$, the following holds for all $t \geq 0$:

$$|x| \geq \chi(|u|) \Rightarrow \dot{V}(t,x,u) \leq -p(t)\mu(\tilde{\alpha}_2^{-1}(V(t,x))). \tag{9}$$

Note too that $w \in \mathcal{K}_\infty \cap C^1$. We later use the fact that

$$\begin{aligned}0 \leq w'(s) &\leq \frac{\mu'(\tilde{\alpha}_2^{-1}(s))}{4\tau\max\{\frac{\tau\bar{p}}{2},1\}(\mu'(\tilde{\alpha}_2^{-1}(s))+1)}\\ &\leq \frac{1}{2\tau^2\bar{p}}\end{aligned} \tag{10}$$

for all $s \geq 0$. Consider the UPPD function

$$V^\sharp(t,x) = V(t,x) + \xi(t)w(V(t,x)) \tag{11}$$

with $\xi(t) = \int_{t-\tau}^{t}\left(\int_s^t p(r)\,dr\right)ds$. Then

$$\begin{aligned}\dot{V}^\sharp(t,x,u) &= [1+\xi(t)w'(V(t,x))]\dot{V}(t,x,u)\\ &\quad + \left[\tau p(t) - \int_{t-\tau}^{t} p(r)\,dr\right] w(V(t,x))\end{aligned}$$

follows from a simple calculation. When $|x| \geq \chi(|u|)$, condition (9) gives $\dot{V}(t,x,u) \leq 0$ and therefore also

$$\begin{aligned}\dot{V}^\sharp(t,x,u) &\leq -p(t)\mu(\tilde{\alpha}_2^{-1}(V(t,x)))\\ &\quad + \left[\tau p(t) - \int_{t-\tau}^{t} p(r)\,dr\right]\tfrac{1}{4\tau}\mu(\tilde{\alpha}_2^{-1}(V(t,x)))\\ &\leq -\tfrac{3}{4}p(t)\mu(\tilde{\alpha}_2^{-1}(V(t,x)))\\ &\quad - \left(\int_{t-\tau}^{t} p(r)\,dr\right)\tfrac{1}{4\tau}\mu(\tilde{\alpha}_2^{-1}(V(t,x)))\\ &\leq -\tfrac{\varepsilon}{4\tau}\mu(\tilde{\alpha}_2^{-1}(\alpha_1(|x|)))\ \ \forall t \geq 0.\end{aligned}$$

Since $\mu \circ \tilde{\alpha}_2^{-1} \circ \alpha_1 \in C^1 \cap \mathcal{K}_\infty$, it follows that $V^\sharp$ is a strict ISS Lyapunov function for (1).

Step 2: $(C_2) \Rightarrow (C_4)$. Assume (C_2), so f admits a strict ISS Lyapunov function V. Let μ and χ satisfy condition (5) with $p \equiv 1$. Then the strict dissipative condition (7) with $p \equiv 1$ follows by choosing any $\Omega \in \mathcal{K}_\infty$ satisfying

$$\Omega(s) \geq \max_{\{t \geq 0, |x| \leq \chi(s), |u| \leq s\}} \{\dot{V}(t,x,u) + \mu(|x|)\} \ \ \forall s \geq 0.$$

Such an Ω exists by our assumptions (2)-(3) on f and ∇V. Thus, V is a strict DIS Lyapunov function for f.
Step 3: $(C_4) \Rightarrow (C_1)$. Assume (C_4), so f admits a strict DIS Lyapunov function V. Let $\mu, \Omega \in \mathcal{K}_\infty$ satisfy (7) with $p \equiv 1$; then if $|x| \geq \chi(|u|) := \mu^{-1}(2\Omega(|u|))$, then

$$\dot{V}(t,x,u) \leq -\frac{1}{2}\mu(|x|), \ \text{ so } \ \dot{V}(t,x,u) \leq -\frac{p(t)}{2\bar{p}}\mu(|x|)$$

for all $t \geq 0$. Therefore, V is also an ISS(p) Lyapunov function for f, so (C_1) is satisfied.
Step 4: $(C_3) \Leftrightarrow (C_4)$. Since $p \in \mathcal{P}$ is bounded, we easily conclude that (C_4) implies (C_3). Conversely, assume $V \in$ UPPD is a DIS(p) Lyapunov function for f and $\alpha_1, \alpha_2, \mu, \Omega \in \mathcal{K}_\infty$ satisfy (3) and the DIS(p) requirements. Define $\tilde{\alpha}_2, w \in \mathcal{K}_\infty \cap C^1$ and $V^\sharp$ by (8) and (11). As before, when $\tilde{\mu} = \mu \circ \tilde{\alpha}_2^{-1}$, we have, for all $t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m$,

$$\dot{V}(t,x,u) \leq -p(t)\tilde{\mu}(V(t,x)) + \Omega(|u|). \tag{12}$$

It follows from Lemma 5(i) and (10) that

$$1 + \xi(t)w'(V(t,x)) \in \left[1, \frac{5}{4}\right] \ , \ \ \forall t \geq 0, x \in \mathbb{R}^n. \tag{13}$$

Since $w = \frac{1}{4\tau}\tilde{\mu}$, we deduce that

$$\begin{aligned}\dot{V}^\sharp(t,x,u) \ \ &\leq \ \ -p(t)\tilde{\mu}(V(t,x)) + \tfrac{5}{4}\Omega(|u|) \\ &+ \tau p(t) w(V(t,x)) - \left(\textstyle\int_{t-\tau}^{t} p(r)dr\right) w(V(t,x)) \\ &\leq -\varepsilon w(\alpha_1(|x|)) + \tfrac{5}{4}\Omega(|u|).\end{aligned}$$

Since $w \circ \alpha_1 \in C^1 \cap \mathcal{K}_\infty$, it follows that $V^\sharp$ is the desired strict DIS Lyapunov function.
Step 5: $(C_2) \Leftrightarrow (C_6)$. The implication $(C_2) \Rightarrow (C_6)$ follows from (Khalil, 2002, Theorem 4.19, p.176). (In (Khalil, 2002), the controls are bounded piecewise continuous functions $\alpha : [0,\infty) \to \mathbb{R}^m$, but the result from (Khalil, 2002) can be extended to our general control set $\mathcal{U}$ using a standard denseness argument (see e.g. Remark C.1.2 and the proof of Theorem 1 in (Sontag, 1998)).) The converse was announced in (Edwards *et al.*, 2000, Theorem 1) and can be deduced from (Bacciotti & Rosier, 2001) as follows. If f is ISS, then (Sontag & Wang, 1995) provides $\chi \in \mathcal{K}_\infty$ such that the constrained input system $\dot{x} = f_\chi(t,x,d) := f(t,x,d\chi^{-1}(|x|))$, $|d| \leq 1$ is uniformly globally asymptotically stable (UGAS); i.e., there exists $\beta \in \mathcal{KL}$ such that for each $t_o \geq 0$ and $x_o \in \mathbb{R}^n$ and each trajectory y of f_χ satisfying $y(t_o) = x_o$, we have $|y(t_o + h)| \leq \beta(|x_o|, h)$ for all $h \geq 0$. By minorizing χ^{-1}, we can assume it is C^1. This means the locally Lipschitz set-valued dynamics $F(t,x) = \{f(t,x,u) : \chi(|u|) \leq |x|\}$ is UGAS, as is its convexification $\overline{\text{co}}(F)$, namely $(t,x) \mapsto \overline{\text{co}}\{F(t,x)\}$ where $\overline{\text{co}}$ denotes the closed convex hull (cf. (Bacciotti & Rosier, 2001, Proposition 4.2)). Since $\overline{\text{co}}(F)$ is continuous and compact and convex valued, and since we are assuming f is periodic in t, (Bacciotti & Rosier, 2001, Theorem 4.5) provides a time-periodic $V \in$ UPPD such that, for all $x \in \mathbb{R}^n, \ t \geq 0, \ w \in F(t,x)$,

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)w \leq -V(t,x).$$

Recalling the definition of F and assuming (without loss of generality) that V satisfies (3) with $\alpha_1 \in \mathcal{K}_\infty \cap C^1$,

$$\begin{aligned}&|x| \geq \chi(|u|) \ \Rightarrow \ f(t,x,u) \in F(t,x) \\ &\Rightarrow \ \dot{V}(t,x,u) \leq -V(t,x) \leq -\alpha_1(|x|)\end{aligned}$$

for all $t \geq 0$, so V is the desired strict ISS Lyapunov function for f. This establishes $(C_6) \Rightarrow (C_2)$.
Step 6: $(C_5) \Leftrightarrow (C_6)$. Assuming (C_6), there are $\beta \in \mathcal{KL}$ such that for all $t_o \geq 0$, $x_o \in \mathbb{R}^n$, $u_o \in \mathcal{U}$, and $h \geq 0$,

$$\begin{aligned}&|\phi(t_o + h; x_o, t_o, u_o)| \ \leq \ \beta(|x_o|, \bar{p}h) + \gamma(|u_o|_{[t_o, t_o+h]}) \\ &\leq \ \beta(|x_o|, \textstyle\int_{t_o}^{t_o+h} p(s)ds) \ + \ \gamma(|u_o|_{[t_o, t_o+h]}),\end{aligned}$$

where ϕ is the trajectory of f we defined in Section 2. Therefore, f is ISS(p) so $(C_6) \Rightarrow (C_5)$. Conversely, if f is ISS(p), then we can find $\beta \in \mathcal{KL}$ such that for all $t_o \geq 0$, $x_o \in \mathbb{R}^n$, $u_o \in \mathcal{U}$, and $h \geq 0$,

$$\begin{aligned}&|\phi(t_o + h; x_o, t_o, u_o)| \ \leq \ \beta\left(|x_o|, \textstyle\int_{t_o}^{t_o+h} p(s)ds\right) \\ &+ \gamma(|u_o|_{[t_o, t_o+h]}) \ \leq \ \beta\left(|x_o|, \underline{p}(h)\right) + \gamma(|u_o|_{[t_o, t_o+h]}).\end{aligned}$$

By Lemma 5(ii) , $\hat{\beta}(s,t) := \beta(s, \underline{p}(t)) \in \mathcal{KL}$, so $(C_5) \Rightarrow (C_6)$, as desired. This proves Theorem 6.

Remark 7 *Observe that if the functions V, α_2, μ, p are of class C^k, where k is a positive integer or ∞, then the particular function $\tilde{\alpha}_2$ in (8) we have chosen implies that the function $V^\sharp(t,x)$ is of class C^k.*

Remark 8 *Our proof of Theorem 6 shows that if V is a strict ISS Lyapunov function for f, then V is also a strict DIS Lyapunov function for f. The preceding implication is no longer true if our boundedness requirement (2) on f is dropped, as illustrated by the following example from (Edwards* et al.*, 2000): Take the one-dimensional single input system*

$$\dot{x} = f(t,x,u) := -x + (1+t)q(u - |x|),$$

where $q : \mathbb{R} \to \mathbb{R}$ is any C^1 function for which $q(r) \equiv 0$ for $r \le 0$ and $q(r) > 0$ otherwise. Then $V(x) = x^2$ is a strict ISS Lyapunov function for the system since $|x| \ge |u| \Rightarrow \dot{V} \le -x^2$ but does not satisfy (7) for any choices of μ and Ω. This does not contradict our results because (2) is not satisfied. This contrasts the time-invariant case where strict ISS Lyapunov functions are automatically strict DIS Lyapunov functions.

4 Illustration

We next use our results to construct a strict ISS Lyapunov function for a tracking problem for the angular velocity subsystem of the model of a rotating rigid body (see (Crouch, 1984; Morin *et al.*, 1995; Morin & Samson, 1997) for the background and motivation for this problem). Following (Astolfi & Rapaport, 1998), where disturbance attenuation results are obtained through time-invariant control laws and (Lefeber, 2000, p.31), we only consider the dynamics of the velocities, which, after a change of feedback, are

$$\dot{\omega}_1 = \delta_1 + u_1 \ , \ \ \dot{\omega}_2 = \delta_2 + u_2 \ , \ \ \dot{\omega}_3 = \omega_1\omega_2. \tag{14}$$

where δ_1 and δ_2 are the inputs and u_1 and u_2 are the disturbances. We consider the reference state trajectory

$$\omega_{1r}(t) = \sin(t), \quad \omega_{2r}(t) = \omega_{3r}(t) = 0 \tag{15}$$

but our method applies to more general reference trajectories as well; see Remark 9 below. The substitution $\tilde{\omega}_i(t) = \omega_i(t) - \omega_{ir}(t)$ turns (14) into the error equations

$$\begin{array}{l} \dot{\tilde{\omega}}_1 \ = \ \delta_1 + u_1 - \cos(t), \qquad \dot{\tilde{\omega}}_2 \ = \ \delta_2 + u_2 \ , \\ \dot{\tilde{\omega}}_3 \ = \ (\tilde{\omega}_1 + \sin(t))\tilde{\omega}_2 \ . \end{array} \tag{16}$$

By applying the backstepping approach as it is applied in (Jiang & Nijmeijer, 1997), or through direct calculations, one shows that the derivative of the class UPPD function

$$V(t, \tilde{\omega}) = \frac{1}{2}\left[\tilde{\omega}_1^2 + (\tilde{\omega}_2 + \sin(t)\tilde{\omega}_3)^2 + \tilde{\omega}_3^2\right] \tag{17}$$

with $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)^\top$ along the trajectories of (16) in closed-loop with the control laws

$$\begin{array}{rcl} \delta_1(t, \tilde{\omega}) & = & -\tilde{\omega}_1 - \tilde{\omega}_2\tilde{\omega}_3 + \cos(t) \\ \delta_2(t, \tilde{\omega}) & = & -[1 + \sin(t)\tilde{\omega}_1 + \sin^2(t)]\tilde{\omega}_2 \\ & & -(2\sin(t) + \cos(t))\tilde{\omega}_3 \end{array} \tag{18}$$

satisfies

$$\begin{array}{rl} \dot{V} & = -\tilde{\omega}_1^2 - (\tilde{\omega}_2 + \sin(t)\tilde{\omega}_3)^2 - \sin^2(t)\tilde{\omega}_3^2 \\ & \quad +\tilde{\omega}_1 u_1 + (\tilde{\omega}_2 + \sin(t)\tilde{\omega}_3)u_2 \\ & \le -\frac{1}{2}\tilde{\omega}_1^2 - \frac{1}{2}(\tilde{\omega}_2 + \sin(t)\tilde{\omega}_3)^2 - \sin^2(t)\tilde{\omega}_3^2 \\ & \quad +\frac{1}{2}(u_1^2 + u_2^2) \ \ \le \ \ -p(t)\tilde{\mu}(V(\tilde{\omega})) + \Omega(|u|) \end{array} \tag{19}$$

with $u = (u_1, u_2)^\top \in \mathbb{R}^2$, $p(t) = \sin^2(t)$, $\tilde{\mu}(s) = s$ and $\Omega(s) = \frac{1}{2}s^2$. Therefore V is a DIS(p) Lyapunov function for (16) in closed-loop with the control laws (18). Observe that, in this case, $p \in \mathcal{P}(\pi, \pi/2, 1)$. Setting $\tau = \pi$ and $w(s) = \frac{1}{8\tau}\tilde{\mu}(s) = \frac{s}{8\pi}$, it follows that (13) also holds. Therefore, Steps 3-4 from our proof of Theorem 6 show

$$\begin{array}{rl} V^\sharp(t, \tilde{\omega}) = & V(t, \tilde{\omega}) + \left[\int_{t-\tau}^t \left(\int_s^t p(r)dr\right) ds\right] w(V(t, \tilde{\omega})) \\ & = \left[1 + \frac{\pi}{32} - \frac{1}{32}\sin(2t)\right] V(t, \tilde{\omega}) \end{array}$$

is a strict DIS Lyapunov function and also a strict ISS Lyapunov function for the system (16) in closed-loop with the control laws (18).

Remark 9 *We chose to work with the reference trajectory (15) because it leads to the simple error equations (16). However, one can easily check that a strict ISS Lyapunov function can be constructed for any reference state trajectory $(\omega_{1r}(t), \omega_{2r}(t), \omega_{3r}(t))$ such that, for some constants $\tau, \varepsilon > 0$, $\sup_t |\int_o^t \omega_{1r}(s)\omega_{2r}(s)ds| < \infty$ and $\int_{t-\tau}^t [\omega_{1r}^2(s) + \omega_{2r}^2(s)]ds \ge \varepsilon$, $\forall t \ge \tau$.*

5 Conclusion

For ISS time-varying systems, we provided explicit strict Lyapunov function *constructions* that can easily be performed in practice. The knowledge of these Lyapunov functions allows us to extend the well-known and useful theory of ISS systems to a broad class of time-varying nonlinear dynamics. We conjecture that a discrete-time version of our main result can be proved.

References

Angeli, D. (1999). Input-to-state stability of PD controlled robotic systems. *Automatica*, 35(7), 1285-1290.

Astolfi, A. & Rapaport, A. (1998). Robust stabilization of the angular velocity of a rigid body. *Systems & Control Letters*, 34(5), 257-264.

Bacciotti, A. & Rosier, L. (2001). *Liapunov Functions and Stability in Control Theory.* Springer, London.

Crouch, P. (1984). Spacecraft attitude control and stabilization: applications of geometric control theory to rigid body models. *IEEE Trans. Auto. Control*, 29(4), 321-331.

Edwards, H., Lin, Y. & Wang, Y. (2000). On input-to-state stability for time-varying nonlinear systems. *Proc. 39th IEEE Conf. Decis. Contr., Sydney.*

Jiang, Z.-P. & Nijmeijer, H. (1997). Tracking control of mobile robots: a case study in backstepping. *Automatica*, 33(7), 1393-1399.

Khalil, H. (2002). *Nonlinear Systems, Third Edition.* Prentice Hall, 2002.

Lefeber, E., (2000). *Tracking Control of Nonlinear Mechanical Systems.* PhD Thesis, University of Twente, Enschede, The Netherlands, April 2000.

Loria, A. & Panteley, E. (2002). Uniform exponential stability of linear time-varying systems: revisited. *Systems & Control Letters*, 47(1), 13-24.

Loria, A., Panteley, E., Popovic, D. & Teel, A. (2002). δ-persistency of excitation: a necessary and sufficient condition for uniform attractivity. *Proc. 41st IEEE Conf. Decis. Contr., Las Vegas, NV.*

Loria, A., Panteley, E., Popovic, D. & Teel, A. (2005). A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems. *IEEE Trans. Auto. Control*, 50(2), 183-198.

Mazenc, F. (2003). Strict Lyapunov functions for time-varying systems. *Automatica*, 39(2), 349-353.

Mazenc, F., & Nesic, D. (2004). Strong Lyapunov functions for systems satisfying the conditions of La salle. *IEEE Trans. Auto. Control*, 49(6), 1026-1030.

Mazenc, F. & Praly, L. (2000). Asymptotic tracking of a state reference for systems with a feedforward structure. *Automatica*, 36(2), 179-187.

Morin, P. & Samson, C. (1997). Time-varying exponential stabilization of a rigid spacecraft with two controls. *IEEE Trans. Auto. Control*, 42(4), 528-534.

Morin, P., Samson, C., Pomet, J.-B. & Jiang, Z.-P. (1995). Time-varying feedback stabilization of the attitude of a rigid spacecraft with two controls. *Systems & Control Letters*, 25(5), 375-385.

Sontag, E. (1989). Smooth stabilization implies coprime factorization. *IEEE Trans. Auto. Control*, 34(4), 435-443.

Sontag, E. (1998). Comments on integral variants of ISS. *Systems & Control Letters*, 34(1-2), 93-100.

Sontag, E. (1998). *Mathematical Control Theory. Deterministic Finite-Dimensional Systems. Second Edition.* Springer-Verlag, New York, 1998.

Sontag, E. (2001). *The ISS philosophy as a unifying framework for stability-like behavior.* In: Nonlinear Control in the Year 2000, Vol. 2. Springer, London. 443–467.

Sontag, E. & Teel, A. (1995). Changing supply functions in input/state stable systems. *IEEE Trans. Auto. Control*, 40(8), 1476-1478.

Sontag, E. & Wang, Y. (1995). On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5), 351-359.

Teel, A. & Praly, L. (2000). A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions. *ESAIM Control Optim. Calc. Var.*, 5, 313-367.