Further Remarks on Strict Input-to-State Stable Lyapunov Functions for Time-Varying Systems

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Abstract

We study the stability properties of a class of time-varying nonlinear systems. We assume that nonstrict input-to-state stable (ISS) Lyapunov functions for our systems are given. We posit a mild nondegeneracy condition on our given Lyapunov functions under which strict ISS Lyapunov functions for our systems are also known to exist. However, the known existence result is based on value function representations and is therefore nonconstructive. We provide simple direct constructions of explicit strict ISS Lyapunov functions for our systems by applying an integral smoothing method. We illustrate our constructions using a tracking problem for a rotating rigid body.

Key Words: Lyapunov functions, input-to-state stabilization, nonautonomous systems.

1 Introduction

The theory of input-to-state stable (ISS) systems plays a central role in modern nonlinear control analysis and controller design (see [7, 8, 14, 16, 17]). The ISS property was introduced by Sontag in [13] and an ISS Lyapunov characterization was obtained by Sontag and Wang in [17]. The ISS Lyapunov characterization provides necessary and sufficient conditions for time-invariant systems to be ISS, in terms of the existence of so-called strict ISS Lyapunov functions; see Section 2 below for the relevant definitions and [3] for an extension to time-varying systems. Strict Lyapunov functions have been used to design stabilizing feedback laws that render asymptotically controllable systems ISS to actuator errors and small observation noise; see [8, 16]. Such control laws are expressed in terms of gradients of Lyapunov functions and therefore require explicit strict Lyapunov functions in order to be implemented. This has motivated a great deal of research devoted to constructing explicit strict Lyapunov functions.

One obstacle to these constructions is that the known strict Lyapunov functions from the existence theory are optimal control value functions, involving a supremum of a cost criterion over infinitely many possible solution paths (see [1, 3, 17, 18]), and therefore are not explicit. Although value functions can often be expressed as unique solutions of Hamilton-Jacobi (HJ) equations subject to appropriate side conditions, the usual techniques for computing value functions in terms of HJ equation solutions can be difficult to implement. For certain special kinds of systems, strict ISS Lyapunov functions can be explicitly constructed by ad hoc means. On the other hand, there are numerous important cases where it is relatively straightforward to use backstepping or other known methods to construct explicit nonstrict

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ISS Lyapunov functions (see our definitions of ISS and nonstrict ISS Lyapunov functions in Section 2 and Section 5 for an explicit example). For instance, applying the methods of [4] to tracking problems for nonholonomic systems in chained form results in nonstrict Lyapunov functions. The constructions in [10] also frequently give rise to nonstrict Lyapunov functions.

This motivates the search for techniques for constructing strict ISS Lyapunov functions for time-varying systems, in terms of known nonstrict ISS Lyapunov functions. This search is the focus of this note. For time-varying systems with no controls, the paper [9] constructed strict globally asymptotically stable (GAS) Lyapunov functions in terms of given nonstrict GAS Lyapunov functions. Here we further develop the approach in [9]. We provide the necessary background on ISS systems and Lyapunov functions in Section 2. We then introduce a nonstrict generalization of ISS in which the dissipation rate depends on a nonnegative time-dependent decay parameter. The parameter can be zero along intervals of positive length. However, when the parameter is identically one, our nonstrict ISS property agrees with the usual ISS condition. Under a mild nondegeneracy assumption on this parameter, we show that our nonstrict ISS property is equivalent to the existence of a strict ISS Lyapunov function and is therefore also equivalent to the standard ISS condition. We prove these equivalences in Section 3.

The proof of our equivalences is based on our new methods for explicitly constructing strict ISS Lyapunov functions. These methods are the main contributions of our work, and are the subject of Section 4. In Section 5, we illustrate our constructions using a tracking example. For instance, applying the methods of [4] to tracking problems also give rise to nonstrict Lyapunov functions. The constructions in [10] also frequently give rise to nonstrict Lyapunov functions.

2 Preliminaries

Let $\mathcal{K}_{\infty}$ denote the set of all continuous functions $\rho : [0, \infty) \rightarrow [0, \infty)$ for which (i) $\rho(0) = 0$ and (ii) $\rho$ is strictly increasing and unbounded. Note that $\mathcal{K}_{\infty}$ is closed under inverse and composition; i.e., if $\rho_1, \rho_2 \in \mathcal{K}_{\infty}$, then we have $\rho_1^{-1}, \rho_1 \circ \rho_2 \in \mathcal{K}_{\infty}$. We let $\mathcal{KL}$ denote the set of all continuous functions $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ for which (1) $\beta(\cdot, t) \in \mathcal{K}_{\infty}$ for each $t \geq 0$, (2) $\beta(s, \cdot)$ is nonincreasing for each $s \geq 0$, and (3) $\beta(s, t) \rightarrow 0$ as $t \rightarrow +\infty$ for each $s \geq 0$. When we say that a function $\rho$ is smooth (a.k.a. $C^1$), we mean it is continuously differentiable, in which case we write $\rho \in C^1$.

We study the stability properties of the fully nonlinear nonautonomous system

\begin{equation}
\dot{x} = f(t, x, u), \quad t \geq 0, \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m
\end{equation}

where we always assume $f$ is locally Lipschitz in $(t, x, u)$. Following [9], we also assume $f$ is periodic in $t$, which means there exists a constant $T > 0$ such that $f(t + T, x, u) = f(t, x, u)$ for all $t \geq 0$, $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$. However, most of our arguments remain valid if this periodicity assumption is weakened to requiring $f$ to be uniformly locally bounded in $t$, meaning,

\begin{equation}
\sup \{|f(t, x, u)| : (t, x, u) \in K, t \geq 0\} < +\infty \ \forall \text{ compact } K \subseteq \mathbb{R}^n \times \mathbb{R}^m,
\end{equation}

where $|\cdot|$ is the usual Euclidean norm. The control functions (a.k.a. inputs) for our system (1) comprise the set of all measurable locally essentially bounded functions $\alpha : [0, \infty) \rightarrow \mathbb{R}^m$; we denote this set by $\mathcal{U}$. We let $|\alpha|$ denote the essential supremum of any control $\alpha \in \mathcal{U}$ restricted to any interval $I \subseteq [0, \infty)$. For each $t_o \geq 0$, $x_o \in \mathbb{R}^n$, and $\alpha \in \mathcal{U}$, we let $I \supset t \mapsto \phi(t; x_o, t_o, \alpha)$ denote the unique trajectory of (1) for the input $\alpha$ satisfying $x(t_o) = x_o$ and defined on its maximal interval $I \subseteq [t_o, \infty)$. We let $T_{t_o, x_o, \alpha}$ denote the supremum of $I$. This trajectory will be denoted by $\phi$ when this would not lead to confusion. We say that $f$ is forward complete provided each such trajectory $\phi$ is defined on all of $[t_o, \infty)$; i.e., $T_{t_o, x_o, \alpha} = +\infty$.

A $C^1$ function $V : [0, \infty) \times \mathbb{R}^n \rightarrow [0, \infty)$ is said to be of class UPPD (written $V \in \text{UPPD}$) provided it is uniformly proper and positive definite, which means there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_{\infty}$ such that

\begin{equation}
\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \quad \text{and} \quad |\nabla V(t, x)| \leq \alpha_3(|x|) \quad \forall t \geq 0, \ x \in \mathbb{R}^n.
\end{equation}
We say that $V$ has period $\tau$ in $t$ provided there exists a constant $\tau > 0$ such that $V(t + \tau, x) = V(t, x)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$; in this case, the bound on $\nabla V$ in (3) is redundant. We assume $\alpha_1$ and $\alpha_2$ in (3) are $C^1$, e.g., by taking $\alpha_2(s) = \int_0^s \alpha_1(r)dr$ and minorizing $\alpha_1$ by a $C^1$ function of class $K_\infty$. Given $V \in \text{UPPD}$, we set
\[
\dot{V}(t, x, u) := \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) \cdot f(t, x, u).
\]
Notice that for each $\chi \in K_\infty$, the mapping $s \mapsto \sup\{|\dot{V}(t, x, u)| : t \geq 0, |x| \leq \chi(s), |u| \leq s\}$ is of class $K_\infty$ (by (2)-(3)). We let $\mathcal{P}$ denote the set of all continuous functions $p : \mathbb{R} \to [0, \infty)$ that admit constants $\tau, \varepsilon, \bar{p} > 0$ for which
\[
\int_{t-\tau}^t p(s)ds \geq \varepsilon \quad \text{and} \quad p(t) \leq \bar{p} \quad \forall t \geq 0. \tag{4}
\]
We write $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$ to indicate that (i) $p \in \mathcal{P}$ and (ii) $\tau, \varepsilon, \bar{p} > 0$ are constants such that (4) holds. In particular, any continuous periodic function $p : \mathbb{R} \to [0, \infty)$ that is not identically zero admits constants $\tau, \varepsilon, \bar{p} > 0$ satisfying (4). On the other hand, (4) also allows nonperiodic $p$ with arbitrarily large null sets, e.g., for fixed $r \in \mathbb{N}$, set
\[
p_r(t) = \begin{cases} (1 + e^{-|k|}) \sin^2(t) & \text{if } t \in [rk\pi, (rk + 1)\pi] \text{ and } k \in \mathbb{Z} \\
0 & \text{otherwise} \end{cases}
\]
The elements of $\mathcal{P}$ serve as the decay rates for our nonstrict Lyapunov functions as follows (but see Section 4 for an extension to decay rates that also depend on the state):

**Definition 1.** Let $p \in \mathcal{P}$. A function $V \in \text{UPPD}$ is called a nonstrict ISS Lyapunov function for (1) and $p$, a.k.a. an ISS($p$) Lyapunov function, provided there exist $\chi \in K_\infty$ and $\mu \in K_\infty \cap C^1$ such that $|x| \geq \chi(|u|) \Rightarrow \dot{V}(t, x, u) \leq -p(t)\mu(|x|) \quad \forall t \geq 0$. An ISS($p$) Lyapunov function for (1) and $p(t) \equiv 1$ is also called a strict ISS Lyapunov function.

Notice that (5) allows $\dot{V}(t, x, u) = 0$ for those $t$ where $p(t) = 0$. This corresponds to allowing $V$ to nonstrictly decrease along the trajectories $\phi$ of $f$ that we defined above.

**Definition 2.** Let $p \in \mathcal{P}$. We say that (1) is input-to-state stable (ISS) with decay rate $p$, a.k.a. ISS($p$), provided there exist $\beta \in K\mathcal{L}$ and $\gamma \in K_\infty$ such that for all $t_o \geq 0$, $x_o \in \mathbb{R}^n$, $u_o \in \mathcal{U}$ and $h \geq 0$,
\[
|\phi(t_o + h; x_o, t_o, u_o)| \leq \beta \left( |x_o|, \int_{t_o}^{t_o+h} p(s)ds \right) + \gamma \left( |u_o|_{[t_o, t_o+h]} \right). \tag{6}
\]
If (1) is ISS($p$) with $p \equiv 1$, then we say that (1) is ISS.

Notice that ISS($p$) systems are automatically forward complete. We also study dissipation-type decay conditions as follows:

**Definition 3.** Let $p \in \mathcal{P}$. A function $V \in \text{UPPD}$ is called a nonstrict dissipative Lyapunov function for (1) and $p$, a.k.a. a DIS($p$) Lyapunov function, provided there exist $\Omega \in K_\infty$ and $\mu \in K_\infty \cap C^1$ such that $\dot{V}(t, x, u) \leq -p(t)\mu(|x|) + \Omega(|u|) \quad \forall t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m$. A DIS($p$) Lyapunov function for (1) and $p(t) \equiv 1$ is also called a strict DIS Lyapunov function.

Under our boundedness assumption (2) on $f$, one can check (see Steps 2-3 in Section 3) that a function $V \in \text{UPPD}$ is a strict DIS Lyapunov function for (1) if and only if it is a strict ISS Lyapunov function for the system. One can also check (see below) that ISS($p$) and ISS are equivalent conditions for any $p \in \mathcal{P}$. The main contribution of our work is a simple direct formula for a strict ISS Lyapunov function for (1) in terms of a given ISS($p$) or DIS($p$) Lyapunov function for $f$, using the $K_\infty$ functions $\alpha_2$ and $\mu$ from (3) and (7) (see (11) below). We use the following elementary observations:
Lemma 4. Let $\tau, \varepsilon, \bar{\rho} > 0$ be constants and $p \in \mathcal{P}(\tau, \varepsilon, \bar{\rho})$ be given. Then:

(i) $\int_{t}^{t+\tau} \left( \int_{t}^{t+h} p(r) dr \right) ds = \int_{t}^{t+\tau} (r - t + \tau) p(r) dr \leq \frac{\tau^2 \rho}{2}$ for all $t \geq 0$ and

(ii) $[0, \infty) \ni h \mapsto p(h) = \inf \left\{ \int_{t}^{t+h} p(r) dr : t \geq 0 \right\}$ is continuous, nondecreasing, and unbounded.

The proof of (i) is an elementary integration exercise based on Fubini’s Theorem, and the continuity of $p$ is easily checked. To check the monotonicity of $p$, notice that if $t \geq 0$ and $h_1 \leq h_2 \in [0, \infty)$, then

$$\int_{t}^{t+h_2} p(r) dr \geq \int_{t}^{t+h_1} p(r) dr,$$

so $\int_{t}^{t+h_2} p(r) dr \geq p(h_1)$.

Applying the infimum gives $\bar{p}(h_2) \geq p(h_1)$. Finally, for all $k \geq 1$,

$$\int_{t}^{t+k\tau} p(r) dr = \sum_{i=0}^{k-1} \int_{t+(i+1)\tau}^{t+(i+1)\tau+\tau} p(r) dr \geq k\varepsilon \rightarrow +\infty \text{ as } k \rightarrow +\infty$$

for all $t \geq 0$, so $\bar{p}$ is unbounded.

### 3 Equivalent Characterizations of Nonstrict ISS

We next relate the Lyapunov function and stability notions we introduced in the last section. We show that ISS(p) is equivalent to the existence of an ISS(p) Lyapunov function and the existence of a strict ISS Lyapunov function. Our proof explicitly constructs a strict ISS Lyapunov function for (1) in terms of a given DIS(p) Lyapunov function. Moreover, if $p \in \mathcal{P}(\tau, \varepsilon, \bar{\rho})$ and our given DIS(p) Lyapunov function both have period $\tau$, then the strict ISS Lyapunov function we construct also has period $\tau$. This is desirable for applications because nonstrict Lyapunov functions are often readily available. We further discuss our strict ISS Lyapunov function constructions in the next section. We next prove:

**Theorem 5.** Let $p \in \mathcal{P}$ and $f$ be as above. The following are equivalent:

(C1) $f$ admits an ISS(p) Lyapunov function.

(C2) $f$ admits a strict ISS Lyapunov function.

(C3) $f$ admits a DIS(p) Lyapunov function.

(C4) $f$ admits a strict DIS Lyapunov function.

(C5) $f$ is ISS(p).

(C6) $f$ is ISS.

We prove the following implications: $\text{(C1)} \Rightarrow \text{(C2)} \Rightarrow (\text{C4}) \Rightarrow (\text{C1})$, $(\text{C3}) \Rightarrow (\text{C4})$, $(\text{C2}) \Rightarrow (\text{C6})$, and $(\text{C5}) \Rightarrow (\text{C6})$. We assume that $\tau, \varepsilon, \bar{\rho} > 0$ are constants such that $p \in \mathcal{P}(\tau, \varepsilon, \bar{\rho})$.

**Step 1:** $(\text{C1}) \Rightarrow (\text{C2}).$

If $(\text{C1})$ holds, then we can find an ISS(p) Lyapunov function $V$ for $f$, and therefore $\alpha_1, \alpha_2 \in \mathcal{K}_\infty \cap C^1$ satisfying (3) and $\chi \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}_\infty \cap C^1$ satisfying (5). Set

$$\tilde{\alpha}_2(s) := \max \left\{ \frac{\tau \bar{\rho}}{2}, 1 \right\} (\alpha_2(s) + \mu(s) + s), \quad w(s) := \frac{1}{47} \mu(\tilde{\alpha}_2^{-1}(s)). \tag{8}$$

Then $\tilde{\alpha}_2, \tilde{\alpha}_2^{-1} \in \mathcal{K}_\infty \cap C^1$. Since $V(t, x) \leq \tilde{\alpha}_2(|x|)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$,

$$|x| \geq \chi(|u|) \Rightarrow \tilde{V}(t, x, u) \leq -p(t)\mu(\tilde{\alpha}_2^{-1}(V(t, x))) \quad \forall t \geq 0. \tag{9}$$
Note too that \( w \in \mathcal{K}_\infty \cap C^1 \). In particular, we later use the fact that
\[
0 \leq w'(s) = \frac{\mu'((\hat{\alpha}_2^{-1}(s)))}{4\tau \hat{\alpha}_2^{-1}(s)} \leq \frac{\mu'(\hat{\alpha}_2^{-1}(s))}{4\tau \max\{\frac{p}{2\tau^2}, 1\} \{\mu'(\hat{\alpha}_2^{-1}(s)) + 1\}} \leq \frac{1}{2\tau^2 p}
\]
for all \( s \geq 0 \). Consider the UPPD function
\[
V^\sharp(t, x) = V(t, x) + \left[ \int_{t-\tau}^{t} \left( \int_{s}^{t} p(r) \, dr \right) \, ds \right] w(V(t, x)).
\]
The term in brackets in (11) can also be written \( \int_{t-\tau}^{t} (r - t + \tau)p(r) \, dr \), so
\[
\dot{V}^\sharp(t, x, u) = \left[ 1 + \left[ \int_{t-\tau}^{t} \left( \int_{s}^{t} p(r) \, dr \right) \, ds \right] w'(V(t, x)) \right] \dot{V}(t, x, u) + \left[ \tau p(t) - \int_{t-\tau}^{t} p(r) \, dr \right] w(V(t, x)).
\]
When \( |x| \geq \chi(|u|) \), condition (9) gives \( \dot{V}(t, x, u) \leq 0 \) and therefore also
\[
\dot{V}^\sharp(t, x, u) \leq -p(t) \mu(\hat{\alpha}_2^{-1}(V(t, x))) + \left[ \tau p(t) - \int_{t-\tau}^{t} p(r) \, dr \right] \frac{1}{2\tau^2} \mu(\hat{\alpha}_2^{-1}(V(t, x)))
\leq -\frac{3}{4} p(t) \mu(\hat{\alpha}_2^{-1}(V(t, x))) - \left( \int_{t-\tau}^{t} p(r) \, dr \right) \frac{1}{2\tau^2} \mu(\hat{\alpha}_2^{-1}(V(t, x)))
\leq -\frac{3}{4} \mu(\hat{\alpha}_2^{-1}(\alpha_1(|x|))) \forall t \geq 0.
\]
Since \( \mu \circ \hat{\alpha}_2^{-1} \circ \alpha_1 \in C^1 \cap \mathcal{K}_\infty \), it follows that \( V^\sharp \) is a strict ISS Lyapunov function for (1).

\textbf{Step 2:} (C2) \( \Rightarrow \) (C4).

Assume (C2), so \( f \) admits a strict ISS Lyapunov function \( V \). Let \( \mu \) and \( \chi \) satisfy condition (5) with \( p = 1 \). Then the strict dissipative condition (7) with \( p = 1 \) follows by choosing any \( \Omega \in \mathcal{K}_\infty \) satisfying
\[
\Omega(s) = \max\{\dot{V}(t, x, u) + \mu(|x|) : t \geq 0, |x| \leq \chi(s), |u| \leq s\} \quad \forall s \geq 0.
\]
Such an \( \Omega \) exists by our assumptions (2)-(3) on \( f \) and \( \nabla V \). Therefore, \( V \) is itself a strict DIS Lyapunov function for \( f \).

\textbf{Step 3:} (C4) \( \Rightarrow \) (C1).

Assume (C4), so \( f \) admits a strict DIS Lyapunov function \( V \). Let \( \mu, \Omega \in \mathcal{K}_\infty \) satisfy (7) with \( p = 1 \). It follows from (7) that if \( |x| \geq \chi(|u|) := \mu^{-1}(2\Omega(|u|)) \), then \( \dot{V}(t, x, u) \leq -\frac{1}{2} \mu(|x|) \) and therefore also
\[
\dot{V}(t, x, u) \leq -\frac{p(t)}{2\mu} \mu(|x|) \quad \forall t \geq 0.
\]
Therefore, \( V \) is also an ISS(p) Lyapunov function for \( f \), so (C1) is satisfied.

\textbf{Step 4:} (C3) \( \Leftrightarrow \) (C4).

Since \( p \in \mathcal{P} \) is bounded, we easily conclude that (C4) implies (C3). Conversely, assume \( V \in \text{UPPD} \) is a DIS(p) Lyapunov function for \( f \) and \( \alpha_1, \alpha_2, \mu, \Omega \in \mathcal{K}_\infty \) satisfy (3) and the DIS(p) requirements. Define \( \hat{\alpha}_2, w \in \mathcal{K}_\infty \cap C^1 \) and \( V^\sharp \) by (8) and (11). As before, when \( \hat{\mu} = \mu \circ \hat{\alpha}_2^{-1} \), we have
\[
\dot{V}(t, x, u) \leq -p(t) \hat{\mu}(V(t, x)) + \Omega(|u|) \quad \forall t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m.
\]
(12)
It follows from Lemma 4(i) and (10) that
\[
1 + \left[ \int_{t-\tau}^{t} \left( \int_{s}^{t} p(r) \, dr \right) \, ds \right] w'(V(t, x)) \in \left[ 1, \frac{5}{4} \right] \quad \forall t \geq 0, x \in \mathbb{R}^n.
\]
(13)
Since \( w = \frac{1}{\tau} \tilde{\mu} \), it follows that
\[
\dot{V}(t, x, u) = \left[ 1 + \int_{t-\tau}^{t} \left( \int_{s}^{t} p(r)dr \right) ds \right] w'(V(t, x)) \dot{V}(t, x, u) + \left[ \tau p(t) - \int_{t-\tau}^{t} p(r)dr \right] w(V(t, x)) \\
\leq -p(t)\tilde{\mu}(V(t, x)) + \frac{5}{4} \Omega(|u|) + \tau p(t)w(V(t, x)) - \left( \int_{t-\tau}^{t} p(r)dr \right) w(V(t, x)) \\
\leq -\frac{3}{4} p(t)\tilde{\mu}(V(t, x)) + \frac{5}{4} \Omega(|u|) - \left( \int_{t-\tau}^{t} p(r)dr \right) w(V(t, x)) \\
\leq -\varepsilon w(\alpha_1(|x|)) + \frac{5}{4} \Omega(|u|).
\]
Since \( w \circ \alpha_1 \in C^1 \cap K_\infty \), it follows that \( V \) is the desired strict DIS Lyapunov function.

**Step 5:** \((C_2) \iff (C_6)\)

The implication \((C_2) \Rightarrow (C_6)\) follows from [5, Theorem 4.19, p.176]. In [5], the controls are the bounded piecewise continuous functions \( \alpha : [0, \infty) \to \mathbb{R}^m \), but the result from [5] can be extended to our general control set \( \mathcal{U} \) using a standard denseness argument (see e.g. Remark C.1.2 and the proof of Theorem 1 in [15]). The converse was announced in [3, Theorem 1] and can be deduced from [1] as follows. If \( F \) is ISS, then \( \tilde{\mu} \in K_\infty \) such that for each \( t_0 \geq 0 \) and \( x_0 \in \mathbb{R}^n \) and each trajectory \( \gamma \) of \( F \) satisfying \( \gamma(t_0) = x_0 \), we have \( |\gamma(t_0 + h)| \leq \beta(|x_0|, h) \) for all \( h \geq 0 \). By minorizing \( \chi^{-1} \), we can assume it is \( C^1 \). This means the locally Lipschitz set-valued dynamics \( F(t, x, u) = \{ f(t, x, u) : \chi(|\tilde{u}|) \leq |x| \} \) is UGAS, as is its convexification \( \overline{\text{co}}(F) \), namely \( (t, x) \mapsto \overline{\text{co}} \) \( F(t, x) \) where \( \overline{\text{co}}(F) \) denotes the closed convex hull (cf. [1, Proposition 4.2]). Since \( \overline{\text{co}}(F) \) is continuous and compact and convex valued, [1, Theorem 4.5] provides a time-periodic \( V \in UPPD \) such that
\[
\frac{\partial V}{\partial t}(t, x, u) + \frac{\partial V}{\partial x}(t, x)w \leq -V(t, x) \quad \forall x \in \mathbb{R}^n, \ t \geq 0, \ w \in F(t, x).
\]
Recalling the definition of \( F \) and assuming without loss of generality that \( V \) satisfies (3) for some \( \alpha_1 \in K_\infty \cap C^1 \) gives \( |x| \geq \chi(|\tilde{u}|) \Rightarrow f(t, x, u) \in F(t, x) \Rightarrow \dot{V}(t, x, u) \leq -V(t, x) \leq -\alpha_1(|x|) \) for all \( t \geq 0 \), so \( V \) is the desired strict ISS Lyapunov function for \( f \). This establishes \((C_6) \Rightarrow (C_2)\).

**Step 6:** \((C_5) \Rightarrow (C_6)\)

If \((C_6)\) holds, then we can find \( \beta \in K_L \) such that for all \( t_0 \geq 0, x_0 \in \mathbb{R}^n, u_0 \in \mathcal{U} \), and \( h \geq 0 \),
\[
|\phi(t_0 + h; x_0, t_0, u_0) - \beta(|x_0|, \tilde{p}h) + \gamma(|u_0|_{[t_0, t_0+h]})| \leq \beta(|x_0|, \int_{t_0}^{t_0+h} p(s)ds) + \gamma(|u_0|_{[t_0, t_0+h]}),
\]
where \( \phi \) is the trajectory of \( f \) we defined in Section 2. Therefore, if \( F \) is ISS(p), then we can find \( \beta \in K_L \) such that for all \( t_0 \geq 0, x_0 \in \mathbb{R}^n, u_0 \in \mathcal{U} \), and \( h \geq 0 \),
\[
|\phi(t_0 + h; x_0, t_0, u_0) - \beta(|x_0|, \int_{t_0}^{t_0+h} p(s)ds) + \gamma(|u_0|_{[t_0, t_0+h]})| \leq \beta(|x_0|, p(h)) + \gamma(|u_0|_{[t_0, t_0+h]}).
\]
By Lemma 4(ii), \( \tilde{\beta}(s, t) := \beta(s, p(t)) \in K_L \), so \((C_5) \Rightarrow (C_6)\), as desired. This proves Theorem 5.

**Remark 6.** Our proof of Theorem 5 shows that if \( V \) is a strict ISS Lyapunov function for \( f \), then \( V \) is also a strict DIS Lyapunov function for \( f \). The preceding implication is no longer true if our boundedness requirement (2) on \( f \) is dropped, as illustrated by the following example from [3]: Take the one-dimensional single input system
\[
\dot{x} = f(t, x, u) := -x + (1 + t)q(u - |x|),
\]
where \( q : \mathbb{R} \to \mathbb{R} \) is any smooth function for which \( q(r) \equiv 0 \) for \( r \leq 0 \) and \( q(r) > 0 \) otherwise. Then \( V(x) = x^2 \) is a strict ISS Lyapunov function for the system (since \( |x| \geq |u| \Rightarrow \dot{V} \leq -x^2 \)) but \( V \) does not satisfy the strict DIS condition (7) for any choices of \( \mu \) and \( \Omega \). This does not contradict our results because (2) is not satisfied. This contrasts with the time-invariant case where strict ISS Lyapunov functions are automatically strict DIS Lyapunov functions.
4 Strict Lyapunov Function Constructions

4.1 First Construction

Our proof of Theorem 5 provides a mechanism for constructing an explicit strict ISS Lyapunov function in terms of a given nonstrict ISS Lyapunov function. Since the existence of a nonstrict ISS(p) Lyapunov function implies ISS (Theorem 5), strict ISS Lyapunov functions were already known to exist under our ISS(p) hypothesis. However, the known existence theory was nonconstructive. The following theorem summarizes our main strict Lyapunov function construction and follows from the proof of Theorem 5:

**Theorem 7.** Let \( \tau, \varepsilon, p > 0 \) be constants, let \( p \in \mathcal{P}(\tau, \varepsilon, p) \), let \( V \) be a DIS(p) Lyapunov function for \( f \), and let \( \alpha_2, \mu \in \mathcal{K}_\infty \cap C^1 \) satisfy the UPPD and DIS(p) requirements from (3) and (7). Define \( V^\sharp \) by

\[
V^\sharp(t, x) = V(t, x) + \left[ \int_{t-\tau}^{t} \left( \int_{s}^{t} p(r) \, dr \right) \, ds \right] w(V(t, x)),
\]

where

\[
w(s) = \frac{1}{4\tau} (\tilde{\alpha}_2^{-1}(s)) \quad \text{and} \quad \tilde{\alpha}_2(s) = \max \left\{ \frac{\tau p}{2}, 1 \right\} (\alpha_2(s) + \mu(s) + s).
\]

Then \( V^\sharp \) is a strict ISS Lyapunov function for (1). If \( V \) and \( p \) have period \( \tau \) in \( t \), then so does \( V^\sharp \).

4.2 Second Construction

The preceding construction can be generalized to cases where our decay rates \( p \) also depend on the state, as follows. We are given \( V \in \text{UPPD} \), a \( C^1 \) function \( W : \mathbb{R} \times [0, \infty) \to [0, \infty) \) \((t, q) \mapsto W(t, q)\), a constant \( \tau > 0 \), a continuous function \( \delta : [0, \infty) \to [0, \infty), \Omega \in \mathcal{K}_\infty, \) and \( \varepsilon \in \mathcal{K}_\infty \cap C^1 \) satisfying:

\[
\begin{align*}
\dot{V}(t, x) &\leq -W(t, |x|) + \Omega(|u|) \quad \forall t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m \\
0 &\leq \frac{\partial W}{\partial q}(t, q) \leq \delta(q) \quad \forall q \geq 0, t \in \mathbb{R}.
\end{align*}
\]

This includes the case where \( V \) is a DIS(p) Lyapunov function by taking \( W(t, q) = p(t)\mu(q) \) where \( \mu \) is as in the DIS(p) definition. Let \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \cap C^1 \) satisfy the UPPD requirement (3) for \( V \). Set \( \tilde{\alpha}_2(s) := \alpha_2(s) + \int_{0}^{s} \delta(r) \, dr + s \) and \( \tilde{W}(t, q) = W(t, \tilde{\alpha}_2^{-1}(q)) \). Reasoning as in Section 3 shows

\[
0 \leq \frac{\partial \tilde{W}}{\partial q}(t, q) \leq 1 \quad \forall t \in \mathbb{R}, q \geq 0, \quad \text{and} \quad \tilde{V}(t, x, u) = -\tilde{W}(t, V(t, x)) + \Omega(|u|) \quad \forall t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m
\]

and \( \tilde{\alpha}_2^{-1} \in C^1 \cap \mathcal{K}_\infty \). Set

\[
V^\sharp(t, x) = V(t, x) + \frac{1}{\tau} \left[ \int_{1-\tau}^{t} \left( \int_{s}^{t} \frac{\partial \tilde{W}}{\partial q}(r, V(t, x)) \, dr \right) \, ds \right].
\]

By (15), we know that \( r \mapsto \sup\{W(t, q) : t \geq 0, q \leq r\} \) can be majorized by a class \( \mathcal{K}_\infty \) function. It follows that \( V^\sharp \in \text{UPPD} \). Moreover, along the trajectories of \( f \),

\[
\begin{align*}
\dot{V}^\sharp(t, x, u) &\leq -\tilde{W}(t, V(t, x)) + \Omega(|u|) + \tilde{W}(t, V(t, x)) - \frac{1}{\tau} \int_{t-\tau}^{t} \tilde{W}(r, V(t, x)) \, dr \\
&\quad + \frac{1}{\tau} \left[ \int_{1-\tau}^{t} \left( \int_{s}^{t} \frac{\partial \tilde{W}}{\partial q}(r, V(t, x)) \, dr \right) \, ds \right] \tilde{V}(t, x, u) \\
&\leq -\frac{1}{\tau} \varepsilon(\tilde{\alpha}_2^{-1}(V(t, x))) + \Omega(|u|) \\
&\quad + \frac{1}{\tau} \left[ \int_{1-\tau}^{t} \left( \int_{s}^{t} \frac{\partial \tilde{W}}{\partial q}(r, V(t, x)) \, dr \right) \, ds \right] \Omega(|u|) \\
&\leq -\frac{1}{\tau} \varepsilon(\tilde{\alpha}_2^{-1}(V(t, x))) + \left[ 1 + \frac{\tau}{2} \right] \Omega(|u|) \leq -\frac{1}{\tau} \varepsilon(\tilde{\alpha}_2^{-1}(\alpha_1(|x|))) + \left[ 1 + \frac{\tau}{2} \right] \Omega(|u|),
\end{align*}
\]

so \( V^\sharp \) is a strict DIS Lyapunov function for \( f \), and therefore also a strict ISS Lyapunov function for \( f \). Note that if \( W \) and \( V \) have period \( \tau > 0 \) in \( t \), then so does \( V^\sharp \).
4.3 Third Construction

We next show how the preceding strict ISS Lyapunov function constructions can be simplified if our dynamics take the control affine form

\[ \dot{x} = f(t, x, u) := h(t, x) + g(t, x)u. \]  

We fix constants \( \tau, \varepsilon, \tilde{p} > 0 \) and \( p \in \mathcal{P}(\tau, \varepsilon, \tilde{p}) \), and we assume there exists a time-independent function \( W \in \text{UPPD} \) and a constant \( \bar{g} > 0 \) such that

\[ \left| \frac{\partial W}{\partial x}(x)h(t, x) \right| \leq \frac{\varepsilon}{\tau^2 \tilde{p}} W(x) \quad \text{and} \quad \left| \frac{\partial W}{\partial x}(x)g(t, x) \right| \leq \bar{g} \quad \forall t \geq 0, x \in \mathbb{R}^n. \]

**Theorem 8.** Let \( p, g, h, f, \) and \( W \) be as above. Let \( V \in \text{UPPD} \) and \( \chi \in \mathcal{K}_\infty \) be such that

\[ |x| \geq \chi(|u|) \quad \Rightarrow \quad \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) [h(t, x) + g(t, x)u] \leq -p(t)W(x) \quad \forall t \geq 0. \]  

Then

\[ U(t, x) := V(t, x) + \frac{1}{\tau} \left[ \int_{t-\tau}^t \left( \int_s^t p(r) \, dr \right) \, ds \right] W(x) \]

is a strict ISS Lyapunov function for (16). If \( p \) and \( V \) have period \( \tau \) in \( t \), then so does \( U \).

The proof is as follows. Choose \( w \in \mathcal{K}_\infty \cap C^1 \) such that \( W(x) \geq w(|x|) \) for all \( x \in \mathbb{R}^n \). Set

\[ \delta(s) := \min \left\{ \chi^{-1}(s), \frac{\varepsilon w(s)}{2\tau^2 \tilde{p}} \right\}, \quad \tilde{\chi} := \delta^{-1} \in \mathcal{K}_\infty. \]

If \( |x| \geq \tilde{\chi}(|u|) \), then Lemma 4(i) and (17) give

\[ \dot{U}(t, x, u) \leq -\frac{1}{\tau} \left[ \int_{t-\tau}^t p(r) \, dr \right] W(x) + \frac{1}{\tau} \left[ \int_{t-\tau}^t \left( \int_s^t p(r) \, dr \right) \, ds \right] \frac{\partial W}{\partial x}(x) [h(t, x) + g(t, x)u] \]

\[ \leq -\frac{\varepsilon}{\tau} W(x) + \frac{\tau \tilde{p}}{2} \left( \frac{\tilde{p}}{\tau^2 \tilde{p}} W(x) + \bar{g} |u| \right) \leq -\frac{\varepsilon}{2\tau} W(x) + \frac{\tau \bar{p} \tilde{p}}{2} |u| \leq -\frac{\varepsilon}{4\tau} w(|x|) \quad \forall t \geq 0 \]

so \( U \) is a strict ISS Lyapunov function for \( f \). The periodicity assertion is also easily checked.

**Remark 9.** Our results can be extended to integral-input-to-state stable (iISS) Lyapunov functions (see [8, 14]). A nonstrict iISS Lyapunov function for (1) and \( p \in \mathcal{P} \) (a.k.a. an iISS(p) Lyapunov function) is defined to be a function \( V \in \text{UPPD} \) satisfying (7) for some positive definite function \( \mu \in C^1 \) and some \( \Omega \in \mathcal{K}_\infty \). This is a less restrictive condition than \( \text{DIS}(p) \) since \( \mu \) need not be in \( \mathcal{K}_\infty \). An iISS(p) Lyapunov function for (1) and \( p \equiv 1 \) is also called a strict iISS Lyapunov function for (1). Given \( p \in \mathcal{P} \), our results provide a method for constructing a strict iISS Lyapunov function for (1) in terms of a given iISS(p) Lyapunov function for the system. This is done by minorizing the positive definite function \( \mu \) in (7) by a strictly increasing positive definite smooth function that we again call \( \mu \) (see [5, Lemma 4.3, p. 145]) and then using the \( V^2 \) from the proof of Theorem 5.

5 Illustration

We next use our results to construct a strict ISS Lyapunov function for a tracking problem for a rotating rigid body (see [2, 11, 12] for the background and motivation for this problem). Following Leféver [6, p.31], we only consider the dynamics of the velocities, namely,

\[ \dot{\omega}_1 = \frac{I_2 - I_3}{I_1} \omega_2 \omega_3 + d_1 + u_1, \quad \dot{\omega}_2 = \frac{I_3 - I_1}{I_2} \omega_3 \omega_1 + d_2 + u_2, \quad \dot{\omega}_3 = \frac{I_1 - I_2}{I_3} \omega_1 \omega_2 \]  

(18)
where the \( \omega \)'s are the angular velocities, and \( I_1 > I_2 \) and \( I_3 > 0 \) are the principal moments of inertia. The change of feedback and change of coordinate

\[
\delta_1 = \frac{I_2 - I_3}{I_1} \omega_2 \omega_3 + d_1, \quad \delta_2 = \frac{I_3 - I_1}{I_2} \omega_3 \omega_1 + d_2, \quad Z_3 = \frac{I_3}{I_1 - I_2} \omega_3
\]

transform (18) as follows:

\[
\dot{\omega}_1 = \delta_1 + u_1, \quad \dot{\omega}_2 = \delta_2 + u_2, \quad \dot{Z}_3 = \omega_1 \omega_2.
\]

We consider the reference state trajectory

\[
\omega_{1r}(t) = \cos^2(t), \quad \omega_{2r}(t) = Z_{3r}(t) = 0
\]

but our method applies to more general reference trajectories as well; see Remark 10 below. The substitutions

\[
\dot{\omega}_1 = \omega_1(t) - \omega_{1r}(t), \quad \dot{\omega}_2 = \omega_2(t) - \omega_{2r}(t), \quad \dot{Z}_3 = Z_3(t) - Z_{3r}(t)
\]

\[
k_1(t) = \delta_1(t) + 2 \cos(t) \sin(t), \quad k_2(t) = \delta_2(t).
\]

transform (20) into the error equations

\[
\dot{\omega}_1 = k_1 + u_1, \quad \dot{\omega}_2 = k_2 + u_2, \quad \dot{Z}_3 = (\omega_1 + \cos^2(t)) \omega_2.
\]

Consider the control laws

\[
k_1 = -\dot{\omega}_1 - \frac{1}{2} [\ddot{\omega}_2 + 2 \ddot{Z}_3 \ddot{\omega}_2], \quad k_2 = -\cos^2(t) (\ddot{\omega}_2 + \ddot{Z}_3)
\]

and the \( C^1 \) positive definite functions

\[
Q(\dot{\omega}_2, \dot{Z}_3) = \ddot{\omega}_2^2 + \dot{Z}_3^2 + \ddot{Z}_3 \omega_2, \quad R(\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3) = Q(\dot{\omega}_2, \dot{Z}_3) + \omega_1^2.
\]

Along the trajectories of (22), our control laws (23) give

\[
\dot{Q} = 2\ddot{\omega}_2 [- \cos^2(t) (\ddot{\omega}_2 + \ddot{Z}_3) + u_2] + 2 \ddot{Z}_3 (\dot{\omega}_1 + \cos^2(t)) \ddot{\omega}_2
\]

\[
+ [- \cos^2(t) (\ddot{\omega}_2 + \dot{Z}_3) + u_2] \ddot{Z}_3 + \omega_2 (\dot{\omega}_1 + \cos^2(t)) \ddot{\omega}_2
\]

\[
= -\cos^2(t) \dddot{\omega}_2 - 2 \cos^2(t) \ddot{\omega}_2 \dddot{Z}_3 + 2 \ddot{Z}_3 \dddot{\omega}_2 + 2 \dddot{Z}_3 \omega_2 \dddot{\omega}_2 + 2 \cos^2(t) \dddot{Z}_3 \dddot{\omega}_2
\]

\[
- \cos^2(t) \dddot{\omega}_2 \dddot{Z}_3 - \cos^2(t) \dddot{Z}_3 + u_2 \dddot{Z}_3 + \omega_2 \dddot{\omega}_1
\]

\[
\dot{R} = -\cos^2(t) Q(\dot{\omega}_2, \dot{Z}_3) + [\dddot{\omega}_2^2 + 2 \dddot{Z}_3 \dddot{\omega}_2] \dddot{\omega}_1 + [\dddot{Z}_3 + 2 \dddot{\omega}_2] u_2
\]

\[
= -\cos^2(t) Q(\dot{\omega}_2, \dot{Z}_3) + [\dddot{\omega}_2^2 + 2 \dddot{Z}_3 \dddot{\omega}_2] \dddot{\omega}_1 + [\dddot{Z}_3 + 2 \dddot{\omega}_2] u_2 + 2 \dddot{\omega}_1 [k_1 + u_1]
\]

\[
\leq -\cos^2(t) R(\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3) + [\dddot{Z}_3 + 2 \dddot{\omega}_2] u_2 + 2 \dddot{\omega}_1 u_1
\]

Setting

\[
V(\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3) := \sqrt{R(\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3)} + 1 - 1, \quad \tilde{\mu}(s) := \frac{s}{2}, \quad \Omega(s) := 2 s, \quad p(t) := \cos^2(t)
\]

and noting that \( Q(\dot{\omega}_2, \dot{Z}_3) = \frac{1}{4} (\dddot{\omega}_2^2 + \dddot{Z}_3^2) \) everywhere gives

\[
\dot{V} \leq -\cos^2(t) \frac{R(\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3)}{2 \sqrt{R(\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3) + 1}} \dddot{\omega}_2 + \frac{2 \dddot{\omega}_1}{2 \sqrt{R(\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3) + 1}} u_2 + \frac{2 \dddot{\omega}_1}{2 \sqrt{R(\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3) + 1}} u_1
\]

\[
\leq -\frac{1}{2} \cos^2(t) V(\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3) + |u_1| + |u_2|
\]

\[
\leq -p(t) \tilde{\mu}(V(\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3)) + \Omega(|u|) \quad \forall u \in \mathbb{R}^2, \quad t \geq 0, \quad (\dot{\omega}_1, \dot{\omega}_2, \dot{Z}_3) \in \mathbb{R}^3
which is the DIS(p) condition (12) for the error equations (22) in closed-loop with the control laws (23). In this case, \( p \in \mathcal{P}(\pi, \frac{\pi}{2}, 1) \). Setting \( \tau = \pi \) and \( w(s) = \frac{1}{4\tau} \mu(s) = \frac{1}{4\tau^2} \), it follows that (13) also holds. Therefore, Steps 3-4 from our proof of Theorem 5 show that

\[
V^\sharp(t, \omega_1, \omega_2, \tilde{Z}_3) = V(\omega_1, \omega_2, \tilde{Z}_3) + \left[ \int_{t-\tau}^t \left( \int_{s}^{t} p(r)dr \right) ds \right] w(V(\omega_1, \omega_2, \tilde{Z}_3)) = \left( \sqrt{\omega_1^2 + \omega_2^2 + \tilde{Z}_3^2 + \omega_2 \tilde{Z}_3 + 1} - 1 \right) \left( 1 + \frac{\tau}{32} + \frac{1}{32} \sin(2t) \right)
\]

is a strict DIS Lyapunov function and also a strict ISS Lyapunov function for the closed-loop system.

**Remark 10.** We chose to work with the reference trajectory (21) because it leads to the simple error equations (22). However, one can easily check that a strict ISS Lyapunov function can be constructed for any reference state trajectory \( (\omega_1r(t), \omega_2r(t), Z_3r(t)) \) such that

\[
\sup_{t} \left| \int_{0}^{t} \omega_1r(s)\omega_2r(s)ds \right| < \infty \quad \text{and} \quad \int_{t-\tau}^{t} [\omega_1r(s)^2 + \omega_2r(s)^2]ds \geq \varepsilon \quad \forall t \geq \tau
\]

for some constants \( \tau, \varepsilon > 0 \).

**References**


