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Bounded-from-below solutions of the Hamilton-Jacobi equation for optimal control problems with exit times: vanishing lagrangians, eikonal equations, and shape-from-shading*†

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Abstract. We study the Hamilton-Jacobi equation for undiscounted exit time control problems with general nonnegative Lagrangians using the dynamic programming approach. We prove theorems characterizing the value function as the unique bounded-from-below viscosity solution of the Hamilton-Jacobi equation that is null on the target. The result applies to problems with the property that all trajectories satisfying a certain integral condition must stay in a bounded set. We allow problems for which the Lagrangian is not uniformly bounded below by positive constants, in which the hypotheses of the known uniqueness results for Hamilton-Jacobi equations are not satisfied. We apply our theorems to eikonal equations from geometric optics, shape-from-shading equations from image processing, and variants of the Fuller Problem.

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1 Introduction

Viscosity solutions form the basis for much current work in control theory and optimization (cf. [3, 4, 6, 11, 25, 29]). In a recent series of papers (cf. [17, 18, 19, 20, 22]), we presented results characterizing the value function in optimal control as the unique viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation (HJBE) that satisfies appropriate side conditions. These results apply to very general classes of exit time problems with unbounded dynamics and nonnegative Lagrangians, including H.J. Sussmann's Reflected Brachystochrone Problem (cf. [34, 35]) and other problems with non-Lipschitz dynamics (cf. [18, 20]). They also apply to the Fuller Problem and eikonal equations where the Lagrangians are not bounded below by positive constants and may even vanish outside the target for some values of the control (cf. [17, 19, 20, 22]). In this note, we extend some results of [17, 19] on proper viscosity solutions of the HJBE by characterizing the exit time value function as the unique *bounded-from-below* viscosity solution of the corresponding HJBE that is null on the target. (Recall that properness of a function $w : \mathbb{R}^N \to \mathbb{R}$ is the condition that $w(x) \to +\infty$ as $\|x\| \to \infty$, which is a more stringent requirement than boundedness from below.) This refinement applies to a large class of deterministic exit time problems for which the Lagrangian is not uniformly bounded below by a positive constant and for which an extra affordability condition (namely, (H_6) below) is also satisfied. We apply this result to several physical problems studied in [19, 29], including eikonal and shape-from-shading equations, as well as variants of the Fuller Problem which are not tractable using the well-known results or using our earlier results. (For example, see [29], which imposes the requirement, which is not needed below, that the light intensity I for shape-from-shading satisfies $I(x) \leq C < 1$ for all x and some constant C; [30], which considers solutions of eikonal and shape-from-shading equations on bounded sets; [16, 26] for uniqueness of bounded solutions of shape-from-shading equations; and [22, 29] which impose asymptotics, given in (11) below, which will not in general be satisfied for the problems we consider here.)

Value function characterizations of this kind have been studied by many authors for a variety of stochastic and deterministic optimal control problems and for dynamic games. The characterizations have been applied to the convergence of numerical schemes for approximating value functions and differential game values with error estimates, synthesis of optimal controls, singular perturbation problems, asymptotics problems, H^∞-control, and much more. See for example [3, 13] and the hundreds of references in these books. For surveys of numerical analysis applications of viscosity solutions, see [5, 31], and for uniqueness characterizations for the HJBE of *discounted* exit time problems, see [3]. For uniqueness characterizations for general Hamilton-Jacobi equations that do not necessarily arise as Bellman equations, see [1, 10, 14]. For an appropriate stronger solution concept for a subclass of problems, leading to a characterization of a maximal solution as a unique solution, see [8]. However, these earlier characterizations cannot in general be applied to exit time problems whose Lagrangians are not uniformly bounded below by positive constants. In fact, one easily finds exit time problems for which the Lagrangian is not bounded below by a positive constant and for which the corresponding HJBE has more than one bounded-from-below solution that vanishes on the target. Here is an example from [19] where this occurs:

Example 1.1 Choose the dynamics and Lagrangian

$$\dot{x}(t) = u(t) \in [-1, 1], \quad \ell(x, a) \equiv L(x) := (x+2)^2\ (x-2)^2 x^2 (x+1)^2 (x-1)^2, \quad (1)$$

respectively. Let v_1 and v_2 denote the value functions for the exit time problem of bringing points to the targets $\mathcal{T}_1 = \{0\}$ and $\mathcal{T}_2 = \{0, 2, -2\}$, respectively, using the data (1) (cf. (8) below). Therefore, if we let $\mathcal{M}$ denote the set of all measurable functions $u : [0, \infty) \to [-1, +1]$, then

$$v_j(x) = \inf_{u \in \mathcal{M}} \left\{ \int_0^{t_{\star j}} L(\phi(s))\, ds : \, t_{\star j} < \infty, \, \phi(0) = x, \dot{\phi} = u \text{ a.e.} \right\} \quad \text{for } j = 1, 2$$

where $t_{\star j} = \inf\{t \geq 0 : \phi(t) \in \mathcal{T}_j\}$ for $j = 1, 2$. One can easily check that v_1 and v_2 are both viscosity solutions of the associated HJBE

$$\|Dv(x)\| = (x+2)^2\ (x-2)^2 x^2 (x+1)^2 (x-1)^2 \quad (2)$$

on $\mathbb{R} \setminus \mathcal{T}$ with the target $\mathcal{T} := \mathcal{T}_1$ that vanish on $\mathcal{T}$. One checks that with the target $\mathcal{T} := \mathcal{T}_1$, the problem satisfies all hypotheses of the well-known theorems that characterize value functions of exit time control problems as unique viscosity solutions of the HJBE that are zero on $\mathcal{T}$ (cf. [3, 7, 27]) except that the positive lower bound requirement on ℓ is not satisfied.

Remark 1.2 One of the hypotheses we will make on the exit time problems in the rest of this paper is that the running costs of trajectories starting outside $\mathcal{T}$ and running for any positive time are always positive (cf. condition (H_5) below). This positivity hypothesis is not satisfied in the previous example, since the trajectory $x(t) \equiv -1 \notin \mathcal{T}$ gives $\int_0^t L(x(s))\, ds \equiv 0$ for all t. On the other hand, all other hypotheses we make in §2 below *do* hold for Example 1.1. Therefore, under the set of assumptions in our setting, condition (H_5) cannot be removed.

This note is organized as follows. In §2, we introduce the notation and hypotheses in force throughout most of the sequel, including the definitions of the exit time HJBE, relaxed controls and viscosity solutions. In §3, we state our main result, and we also explain how this result improves what was already known about viscosity solutions of the HJBE. Our results apply to exit time problems that violate the usual positivity condition on the Lagrangian (namely, (10) below) and that are also not tractable by means of [17, 18, 19, 20, 22]. This is followed in §4 by statements of the main lemmas. In §5, we prove our main result, and §6 gives physical applications, including cases which are not tractable using the known results or any of our earlier results. This is followed in §7 by variants of our main result for discontinuous viscosity solutions and local solutions. We conclude in §8 by showing how to use the methods of [19] to extend our results to cases where the control set is unbounded.

2 Definitions and Hypotheses

This note is concerned with problems of the form

$$\begin{aligned} &\text{For each } x \in \mathbb{R}^N, \text{ infimize } \int_0^{t_x(\beta)} \ell^r(y_x(s,\beta),\beta(s))\,ds \\ &\text{over all } \beta \in \mathcal{A} \text{ for which } t_x(\beta) < \infty, \end{aligned} \tag{3}$$

where $y_x(\cdot,\beta)$ is defined to be the solution of the initial value control problem

$$\dot{y}(t) = f^r(y(t),\beta(t)) \quad \text{a.e.}, \quad y(0) = x \tag{4}$$

for each $x \in \mathbb{R}^N$ and each $\beta \in \mathcal{A} := \{\text{measurable functions } [0,\infty) \to A^r\}$ for a given fixed compact metric space A and possibly unbounded nonlinear control system f, and $t_x(\beta) := \inf\{t \geq 0 : y_x(t,\beta) \in \mathcal{T}\}$ for a given fixed set $\mathcal{T} \subset \mathbb{R}^N$. (Depending on f, some choices of x could give $t_x(\beta) = +\infty$ for all β, in which case the infimum for (3) is $+\infty$.) Here, A^r denotes the set of all Radon probability measures on A viewed as a subset of the dual of the set $C(A)$ of all real-valued continuous functions on A, and $\mathcal{A}$ has the weak-$\star$ topology, so $\mathcal{A}$ is the set of relaxed controls from [2, 3, 36]. Notice that $\mathcal{A}$ includes all measurable $\alpha : [0,\infty) \to A$, which can be viewed as Dirac measure valued relaxed controls, and that A^r is compact. We also consider (3) for cases where $A \subset \mathbb{R}^M$ is closed but not bounded, in which case we set $A^r = A$ and

$$\begin{aligned} \mathcal{A} := &\{\text{measurable functions } [0,\infty) \to S^r : S \subseteq A \text{ compact}\} \\ &\bigcup \{\text{measurable functions } [0,\infty) \to A\} \end{aligned} \tag{5}$$

which of course reduces to the usual definition of $\mathcal{A}$ when A is compact. For compact $S \subseteq A$ and measurable $\alpha_n, \alpha, m : [0,\infty) \to S^r$, we set $h^r(x,m) := \int_S h(x,a)dm(a)$ for $x \in \mathbb{R}^N$ and $h = f, \ell$ for suitable f and ℓ specified below, and $\alpha_n \to \alpha$ weak-$\star$ means that for all $t \geq 0$ and for all Lebesgue integrable functions $B : [0,t] \to C(S)$, we have

$$\lim_{n\to\infty} \int_0^t \int_S (B(s))(a)\,\mathrm{d}(\alpha_n(s))(a)\,\mathrm{d}s = \int_0^t \int_S (B(s))(a)\,\mathrm{d}(\alpha(s))(a)\,\mathrm{d}s \tag{6}$$

Also, recall that $STC\mathcal{T}$ is the **small-time controllability condition** that

$$\mathcal{T} \subseteq \operatorname{int}(\mathcal{R}^\varepsilon) \quad \text{for all } \varepsilon > 0,$$

where

$$\mathcal{R}^{\varepsilon} := \{x \in \mathbb{R}^N : \exists t \in [0, \varepsilon) \ \& \ \beta \in \mathcal{A} \ \text{ s.t. } \ y_x(t, \beta) \in \mathcal{T}\}.$$

Roughly speaking, $STC\mathcal{T}$ means points near $\mathcal{T}$ can be brought to $\mathcal{T}$ in small time. We remark for later reference that $STC\mathcal{T}$ is a property of the restriction of the vector fields $f(\cdot, a)$ to neighborhoods of $\mathcal{T}$. In most of what follows, we assume the following standing hypotheses (but see §8 for analogs for cases where the control set A is not assumed to be compact):

- (H_1) A is a nonempty compact metric space.
- (H_2) $\mathcal{T} \subset \mathbb{R}^N$ is closed and nonempty, $STC\mathcal{T}$.
- (H_3) f is continuous, and $\exists L > 0$ such that $\|f(x,a) - f(y,a)\| \leq L\|x - y\|$ $\forall x, y \in \mathbb{R}^N$ & $a \in A$.
- (H_4) $\ell : \mathbb{R}^N \times A \to [0, \infty)$ is continuous.
- (H_5) If $t \in (0, \infty)$, $\beta \in \mathcal{A}$, and $x \in \mathbb{R}^N \setminus \mathcal{T}$, then $\int_0^t \ell^r(y_x(s, \beta), \beta(s))\, ds > 0$.
- (H_6) If $x \in \mathbb{R}^N$ and $\beta \in \mathcal{A}$ are such that $\limsup_{s\to\infty} \|y_x(s, \beta)\| = \infty$, then $\int_0^{\infty} \ell^r(y_x(s, \beta), \beta(s))\, ds = +\infty$.

Remark 2.1 Assumptions (H_5)–(H_6) are expressed in terms of the trajectories, rather than the HJBE data. From the PDE point of view, it is desirable to be able to check all of our assumptions directly from the data $f^r = (f_1, f_2, \ldots, f_N)$, ℓ^r, and $\mathcal{T}$ from the PDE, rather than assuming complete knowledge of the trajectories. One set of conditions on the data implying (H_5) is (i) there are constants $K > 0$ and $C > 0$ such that $\ell(x, a) \geq K|x_1|^C$ for all $a \in A$ and $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, and (ii) if $y \in \mathbb{R}^{N-1}$ and $(0, y) \in \mathbb{R}^N \setminus \mathcal{T}$, then $0 \notin \{f_1(0, y, m) : m \in A^r\}$. Conditions (i)–(ii) ensure that there is a positive cost assigned to staying outside $\mathcal{T}$ on each interval of positive length. These conditions will hold for example in the Fuller Problem discussed below (cf. §6.1). By using a generalized version of "Barbălat's lemma", (H_6) can also be checked from the HJBE data (cf. [21], §2).

Before discussing the motivation for these hypotheses, note that by the Filippov Selection Theorem (cf. [36]), all of our results remain true if $\mathcal{A}$ is replaced by {measurable functions $[0, \infty) \to A$} throughout the preceding definitions and hypotheses as long as the sets

$$\mathcal{D}(x) := \{(f(x, a), \ell(x, a)) : a \in A\}$$

are convex for all $x \in \mathbb{R}^N$. This follows from the fact that if all the sets $\mathcal{D}(x)$ are convex, then each relaxed control $\beta \in \mathcal{A}$ admits a measurable function $\alpha : [0, \infty) \to A$ for which

$$\int_0^t h^r(y_x(s, \beta), \beta(s))\, ds = \int_0^t h(y_x(s, \alpha), \alpha(s))\, ds \;\; \forall t \geq 0, \;\; h = f, \ell$$

We call $\mathcal{T}$, A, f, and ℓ the **target, control set, dynamics, and Lagrangian** for the problem (3), respectively. We let ∂S and $\bar{S}$ denote the boundary and closure for any set $S \subseteq \mathbb{R}^M$, respectively.

The interpretation of our standing hypotheses is as follows. Condition (H_5) has the economic interpretation that all movement outside the target states is costly. Notice that (H_5) is less stringent than requiring $\ell(x, a) > 0$ for all $x \in \mathbb{R}^N$ and $a \in A$, since it could be that points p for which $\min\{\ell(p, a) : a \in A\} = 0$ have the property that all inputs immediately bring p to points x where $\min\{\ell(x, a) : a \in A\} > 0$, which can give ($H_5$) (cf. §6 for problems with this property). The condition (H_6) has the interpretation that trajectories which go further and further from the starting point without bound are unaffordable. In other words, trajectories which give finite total costs over $[0, \infty)$ must stay in some bounded set. As we show in §6 below, (H_6) holds for a general class of shape-from-shading equations from image processing, as well as for problems with vanishing Lagrangians that are not tractable using the known results (cf. §6.1 below). However, (H_6) does not follow from (H_1)–(H_5) (cf. Remark 6.5 below). Finally, we recall (cf. [3], Chapter 3) that (H_3) guarantees that (4) admits a unique solution $y_x(\cdot, \beta)$ defined on $[0, \infty)$ which satisfies

$$\sup_{u \in \mathcal{A}} \|y_x(t, u) - x\| \leq M_x t \;\; \text{for all} \;\; t \in [0, 1/M_x], \tag{7}$$

where $M_x := \sup\{\|f(z, a)\| : a \in A, \|z - x\| \leq 1\}$ if this supremum is nonzero and $M_x = 1$ otherwise.

The **value function** v of (3) is defined by

$$v(x) = \inf \left\{ \int_0^{t_x(\beta)} \ell^r(y_x(s, \beta), \beta(s))\, ds : \beta \in \mathcal{A}, t_x(\beta) < \infty \right\} \in [0, \infty] \tag{8}$$

(but see Remark 3.1 for extensions to problems with exit costs). This note will study viscosity solutions w of the HJBE

$$\sup_{a \in A^r} \{-f^r(x, a) \cdot Dw(x) - \ell^r(x, a)\} = 0, \quad x \notin \mathcal{T} \tag{9}$$

associated with the exit problem (3) which satisfy the following side condition:

(SC_w) w is bounded-from-below, and $w \equiv 0$ on $\mathcal{T}$

We remark that the LHS in (9) equals $\sup\{-f(x, a) \cdot Dw(x) - \ell(x, a) : a \in A\}$ (cf. [3]). When we say that a function w is **bounded-from-below**, we mean that there is a finite constant b so that $w(x) \geq b$ for all x in the domain of w. In some of what follows, we use the notation

$$H_B(x,p) := \sup_{a \in B}\{-f(x,a) \cdot p - \ell(x,a)\}$$

for closed $B \subseteq A$. From (H_1)–(H_4), we know that H_B is continuous for all compact sets $B \subseteq A$. We sometimes write $H(x,p)$ to mean $H_A(x,p)$. We also set

$$B_q(p) := \{x \in \mathbb{R}^N : \|x - p\| < q\} \;\; \forall q > 0, \; p \in \mathbb{R}^N.$$

Letting $C^1(S)$ denote the set of all real-valued continuously differentiable functions on any open subset S of a Euclidean space, the definition of viscosity solutions can then be stated as follows:

Definition 2.2 Assume $\mathcal{G} \subseteq \mathbb{R}^N$ is open, $S \supseteq \mathcal{G}$, and $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $w : S \to \mathbb{R}$ are continuous. We will say that w is a **viscosity solution** of $F(x, Dw(x)) = 0$ on $\mathcal{G}$ provided the following conditions hold:

- (C_1) If $\gamma \in C^1(\mathcal{G})$ and $x_o \in \mathcal{G}$ are such that x_o is a local minimizer of $w - \gamma$, then $F(x_o, D\gamma(x_o)) \geq 0$.
- (C_2) If $\lambda \in C^1(\mathcal{G})$ and $x_1 \in \mathcal{G}$ are such that x_1 is a local maximizer of $w - \lambda$, then $F(x_1, D\lambda(x_1)) \leq 0$.

We also use the following equivalent definition of viscosity solutions based on the **superdifferentials** $D^+w(x)$ and **subdifferentials** $D^-w(x)$ of w. Let $\mathcal{G}$, S, F, and w be as in Definition 2.2, and define

$$D^+w(x) := \left\{ p \in \mathbb{R}^N : \limsup_{\mathcal{G} \ni y \to x} \frac{w(y) - w(x) - p \cdot (y - x)}{\|x - y\|} \leq 0 \right\}$$

$$D^-w(x) := \left\{ p \in \mathbb{R}^N : \liminf_{\mathcal{G} \ni y \to x} \frac{w(y) - w(x) - p \cdot (y - x)}{\|x - y\|} \geq 0 \right\}$$

One checks (cf. [3]) that conditions (C_1) and (C_2) are equivalent to

- (C_1') $F(x,p) \geq 0$ for all $x \in \mathcal{G}$ and $p \in D^-w(x)$
- (C_2') $F(x,p) \leq 0$ for all $x \in \mathcal{G}$ and $p \in D^+w(x)$

respectively. Therefore, we equivalently define viscosity solutions by saying that w is a viscosity solution of $F(x, Dw(x)) = 0$ on $\mathcal{G}$ provided conditions (C_1')–(C_2') hold. Our results can also be extended to the case of *discontinuous* viscosity solutions (cf. §7.1 below for the definitions and extensions).

3 Statement of Main Result and Remarks

Our main result will be the following:

Theorem 1 *Assume* (H_1)–(H_6). *If* $w : \mathbb{R}^N \to \mathbb{R}$ *is a continuous function which is a viscosity solution of the HJBE* (9) *on* $\mathbb{R}^N \setminus \mathcal{T}$, *and if* w *satisfies* (SC_w), *then* $w \equiv v$.

Remark 3.1 Under the standing hypotheses (H_1)–(H_6), if the value function v is finite and continuous on $\mathbb{R}^N$, then v itself is a viscosity solution of the HJBE (9) on $\mathbb{R}^N \setminus \mathcal{T}$ (cf. [3]). Since v satisfies (SC_v), Theorem 1 then characterizes v as the unique viscosity solution of the HJBE (9) on $\mathbb{R}^N \setminus \mathcal{T}$ in the class of continuous functions $w : \mathbb{R}^N \to \mathbb{R}$ which satisfy (SC_w). The assumption that the control set A is compact can be relaxed in various ways (cf. §8 below). Also, the statement of the theorem remains true, with minor changes in the proof, if we replace v with

$$v_g(x) = \inf \left\{ \int_0^{t_x(\beta)} \ell^r(y_x(s,\beta), \beta(s))\, ds + g(y_x(t_x(\beta), \beta)) : \beta \in \mathcal{A}, t_x(\beta) < \infty \right\}$$

for any continuous bounded-from-below final cost function $g : \mathbb{R}^N \to \mathbb{R}$, except that the boundary condition in (SC_w) that $w \equiv 0$ on $\mathcal{T}$ is replaced by $w \equiv g$ on $\mathcal{T}$. For extensions of Theorem 1 to *discontinuous* and local viscosity solutions with possibly unbounded control sets, see §§7-8.

Remark 3.2 Theorem 1 applies to problems which are not tractable by means of the standard results from [3] or using [17, 18, 19, 20, 22]. For example, the undiscounted exit time problem results of [3, 27] require

$$\forall \varepsilon > 0,\ \exists C_\varepsilon > 0 \ \text{ s.t. } \ \ell(x,a) \geq C_\varepsilon \ \forall a \in A \ \&\ \forall x \notin B(\mathcal{T}, \varepsilon), \tag{10}$$

where $\text{dist}(x, \mathcal{T}) := \inf\{\|x-b\| : b \in \mathcal{T}\}$ and $B(\mathcal{T}, \varepsilon) := \{p \in \mathbb{R}^N : \text{dist}(p, \mathcal{T}) < \varepsilon\}$, i.e., uniform positive lower bounds for ℓ, outside neighborhoods of $\mathcal{T}$. In particular, (10) does not allow $\inf_a \ell(\cdot, a)$ to vanish at any point outside $\mathcal{T}$, nor does it allow control values a for which $\ell(x,a) \to 0$ as $\|x\| \to \infty$ when $\mathcal{T}$ is compact. Moreover, as we saw in Example 1.1 above, this condition cannot be dropped. The examples we consider in this paper do not in general satisfy (10) (cf. §6 below). The results of [17, 19] apply to exit time problems violating (10) and give conditions guaranteeing that v is the unique viscosity solution of the HJBE in a certain class of functions which are either proper (where properness of a function w means that $w(x) \to +\infty$ as $\|x\| \to \infty$) or which satisfy a suitable generalized properness notion. The results of [17, 19] require the positivity condition (H_5), but they do not require (H_6). In [22], uniqueness results are given for problems which violate (10) but which do satisfy

$$\int_0^\infty \ell^r(y_x(s,\beta), \beta(s))\, ds < \infty \ \Rightarrow \ \lim_{s \to \infty} y_x(s,\beta) \in \mathcal{T}\ . \tag{11}$$

As we will show in §6 below, Theorem 1 applies to physical problems from optics and image processing and to problems violating *both* (10) and (11), including variants of the Fuller Problem (cf. [17, 19]). We remark that while the results of [17, 19] apply to cases where (10) and (11) both fail, the conclusions of those results are that if the value function is proper, then it is the unique *proper* solution of the HJBE satisfying appropriate side conditions. Since we do not need to assume properness in Theorem 1, our results can be viewed as an improvement of the results of [17] and [19] for cases where the extra affordability condition (H_6) is also satisfied. Notice too that (H_6) can be expressed as

$$\int_0^\infty \ell^r(y_x(s,\beta),\beta(s))\,ds < \infty \;\Rightarrow\; \sup_s \|y_x(s,\beta)\| < \infty, \tag{12}$$

which is of course less restrictive than (11) for problems with bounded targets (cf. §6.1 below).

4 Main Lemmas

Under our standing hypotheses (H_1)–(H_6), one proves (cf. [3]) that the value function v is a viscosity solution of the HJBE (9) on $\mathbb{R}^N \setminus \mathcal{T}$ when v is finite and continuous. The proof follows easily from the fact that v satisfies the Dynamic Programming Principle, which asserts that

$$v(x) \;=\; \inf_{\alpha\in\mathcal{A}} \left\{ \int_0^t \ell^r(y_x(s,\alpha),\alpha(s))\,ds \;+\; v(y_x(t,\alpha)) \right\} \quad \forall\, x \in \mathbb{R}^N \tag{13}$$

for all $t \in [0, \inf_\alpha t_x(\alpha)[$. Our uniqueness characterizations are based on the following representation lemmas which say that viscosity solutions of the HJBE (9) on $\mathbb{R}^N \setminus \mathcal{T}$ satisfy analogs of (13). The proofs of these lemmas are based on uniqueness characterizations for finite horizon control (cf. Chapter 3 of [3]).

Lemma 4.1 *Assume* (H_1)–(H_4) *are satisfied and* $u \in C(\bar{E})$ *is a viscosity solution of* $H(x, Du(x)) = 0$ *on* E*, where* $E \subset \mathbb{R}^N$ *is bounded and open. If we set* $\tau_q(\beta) = \inf\{t \geq 0 : y_q(t,\beta) \in \partial E\}$ *for each* $\beta \in \mathcal{A}$ *and* $q \in E$*, then, for all* $\beta \in \mathcal{A}$ *and* $q \in E$*, we have*

$$u(q) \;\leq\; \int_0^\delta \ell^r(y_q(s,\beta),\beta(s))\,ds \;+\; u(y_q(\delta,\beta)) \tag{14}$$

for $0 \leq \delta < \tau_q(\beta)$.

Lemma 4.2 *Assume that the standing hypotheses* (H_1)–(H_4) *hold and that* $w \in C(\bar{B})$ *is a viscosity solution of the* HJBE $H(x, Dw(x)) = 0$ *on* B*, where* B *is open and bounded. Set*

$$T_\delta(p) \;:=\; \inf\{\, t : \mathrm{dist}(y_p(t,\alpha),\, \partial B) \leq \delta,\ \alpha \in \mathcal{A}\}$$

for each $p \in B$ and $\delta > 0$. Then for any $p \in B$ and any $\delta \in]0, \mathrm{dist}(p, \partial B)/2]$, we have

$$w(p) \;\; \geq \;\; \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^t \ell^r(y_p(s,\alpha),\alpha(s))\, ds \; + \; w(y_p(t,\alpha)) \right\} \tag{15}$$

for all $t \in]0, T_\delta(p)[$.

Notice for future use that we can also put $\delta = \tau_q(\beta)$ in (14) when $\tau_q(\beta) < \infty$. We also need the following consequence of the Bellman-Gronwall Inequality and the sequential compactness of $\mathcal{A}$ (cf. [36]):

Lemma 4.3 *Let A be a compact metric space, let $\{\alpha_n\}$ be a sequence in $\mathcal{A}$, and let $c > 0$. Assume $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ satisfies* (H_3). *Then there exists a subsequence of $\{\alpha_n\}$ (which we do not relabel) and an $\alpha \in \mathcal{A}$ such that the following conditions hold:*

1. *$\alpha_n \to \alpha$ weak-$\star$ on $[0, c]$.*
2. *If $x_n \to x$ in $\mathbb{R}^N$, then $y_{x_n}(\cdot, \alpha_n) \to y_x(\cdot, \alpha)$ uniformly on $[0, c]$.*

Finally, we need the following variant of Barbălat's Lemma shown in [22]. Recall (cf. [22]) that a continuous function $g : \mathbb{R} \to [0, \infty)$ is said to be of class $\mathcal{MK}$ provided that $g(0) = 0$ and that g is even and strictly increasing on $[0, \infty)$. For example, $x \mapsto |x|^q$ is of class $\mathcal{MK}$ for all constants $q > 1$. Also, if G is any function of Sontag's Class $\mathcal{K}$ (cf. [12]), then $g(s) := G(|s|)$ is of class $\mathcal{MK}$. From [22], we recall the following:

Lemma 4.4 *Let g be of class $\mathcal{MK}$, $\phi : [0, \infty) \to \mathbb{R}$ be differentiable, ϕ' be Lipschitz, and $\int_0^\infty g(\phi(s))\, ds < \infty$. Then $\lim_{s \to \infty} \phi(s) = \lim_{s \to \infty} \phi'(s) = 0$.*

5 Proof of Main Result

The proof that $w \leq v$ pointwise is a repeated application of Lemma 4.1 which we leave to the reader (cf. [17] for details). It remains to show that $w \geq v$. We omit the superscripts r to simplify notation in some of what follows. The proof that $w \geq v$ is similar in spirit to an argument from [17, 19] but with a weak-$\star$ argument and a localization based on (H_6) replacing the 'strong controllability' and properness conditions used in [17]. Fix $x \in \mathbb{R}^N \setminus \mathcal{T}$, a constant $\kappa > w(x)$, and an integer J for which $x \in B_J(0)$. Set

$$S_\kappa = \{x \in \mathbb{R}^N : w(x) < \kappa\},$$

which is open by the hypothesis that w is continuous. Set $\mathcal{S} = S_\kappa \cap B_J(0)$, which is bounded and open. For each $p \in \mathbb{R}^N$ and $\beta \in \mathcal{A}$, set

$$\tau_p(\beta) := \inf\{t \geq 0 : y_p(t, \beta) \in \partial(\mathcal{S} \setminus \mathcal{T})\}.$$

Fix

$$\varepsilon \in]0, \kappa - w(x)[.$$

Set

$$I(x,t,\alpha) \ := \ \int_0^t \ell(y_x(s,\alpha),\alpha(s))ds \ + \ w(y_x(t,\alpha))$$

wherever the RHS is defined. We also set

$$T_\delta(p) \ = \ \inf\{t \geq 0 : \mathrm{dist}\,(y_p(t,\alpha), \partial(\mathcal{S} \setminus \mathcal{T})) < \delta, \alpha \in \mathcal{A}\}$$

for all $p \in \mathbb{R}^N$ and $\delta > 0$, and we define $x_1 := x$, $\tau_1 := T_1(x_1)$ when $T_1(x_1) < +\infty$, and τ_1:=10 when $T_1(x_1) = +\infty$. We can then use (15) of Lemma 4.2 to get an $\alpha_1 \in \mathcal{A}$ such that

$$w(x_1) \geq I(x_1, \tau_1, \alpha_1) - \varepsilon/4.$$

(We will always assume that δ of that lemma can be taken to be 1. Otherwise, replace $T_{1/k}(x_k)$ in what follows with $T_{\delta_k}(x_k)$ for an appropriate sequence $\delta_k \downarrow 0$.) Note that $y_{x_1}(\tau_1, \alpha_1) \in \mathcal{S} \setminus \mathcal{T}$. By induction, we define

$$x_k := \ y_{x_{k-1}}(\tau_{k-1}, \alpha_{k-1}) \ \in \ \mathcal{S} \setminus \mathcal{T} \quad \text{for} \quad k = 2, 3, \dots, \qquad \text{where}$$

$$\tau_k \ := \ \begin{cases} T_{1/k}(x_k) & \text{if } T_{1/k}(x_k) < +\infty \\ 10^k & \text{otherwise} \end{cases}. \tag{16}$$

Since $x_k \in \mathcal{S} \setminus \mathcal{T}$, we can use (15) to get an $\alpha_k \in \mathcal{A}$ such that

$$w(x_k) \ \geq \ I(x_k, \tau_k, \alpha_k) \ - \ 2^{-(k+1)}\varepsilon \quad \text{for all} \quad k \in \mathbb{N}. \tag{17}$$

We also set $\sigma_o = 0$, $\sigma_k := \tau_1 + \cdots + \tau_k$, $\bar{\sigma}_J = \limsup_k \sigma_k$, and, for an arbitrary $\bar{a} \in A$,

$$\bar{\alpha}_J(s) := \begin{cases} \alpha_1(s) & \text{if } 0 \leq s < \sigma_1, \\ \alpha_2(s - \sigma_1) & \text{if } \sigma_1 \leq s < \sigma_2, \\ \vdots & \\ \alpha_k(s - \sigma_{k-1}) & \text{if } \sigma_{k-1} \leq s < \sigma_k, \\ \vdots & \\ \bar{a} & \text{if } \bar{\sigma}_J \leq s, \end{cases}$$

with the last line used if $\bar{\sigma}_J < +\infty$. (We use the subscript J to indicate the choice of radius in $B_J(0)$.) From the definitions of x_k and $\bar{\alpha}_J$, we know that

$$y_x(s, \bar{\alpha}_J) \ = \ y_{x_k}(s - \sigma_{k-1}, \alpha_k) \ \in \ \mathcal{S} \setminus \mathcal{T} \quad \text{when} \quad s < \bar{\sigma}_J \tag{18}$$

and

$$\int_0^{\tau_k} \ell(y_{x_k}(s, \alpha_k), \alpha_k(s)) \, ds \ = \ \int_{\sigma_{k-1}}^{\sigma_k} \ell(y_x(s, \bar{\alpha}_J), \bar{\alpha}_J(s)) \, ds \geq 0 \text{ for all } k. \tag{19}$$

Reapplying (17), we therefore get

$$\begin{aligned} w(x) &\geq \int_0^{\tau_1} \ell(y_x(s,\bar\alpha_J),\bar\alpha_J(s))\,ds + w(x_2) - \varepsilon/4 \\ &\geq \int_0^{\sigma_2} \ell(y_x(s,\bar\alpha_J),\bar\alpha_J(s))\,ds + w(x_3) - \varepsilon\left(\frac{1}{4}+\frac{1}{8}\right) \\ &\geq \cdots \\ &\geq I(x,\sigma_k,\bar\alpha_J) - \frac{\varepsilon}{2}\left(1-\frac{1}{2^k}\right) \quad \forall k \in \mathbb{N}. \end{aligned} \tag{20}$$

By (16) and the boundedness of $\mathcal{S}$, we can find $\bar x_J \in \bar{\mathcal{S}}$ and a subsequence (which we will not relabel) for which $x_n \to \bar x_J$. (We later show that $\bar x_J \in \partial[B_J(0)] \cup \mathcal{T}$.) We claim that

$$\bar\tau_J := \inf\{\tau_{\bar x_J}(\alpha) : \alpha \in \mathcal{A}\} \leq \limsup_k \tau_k. \tag{21}$$

To see why (21) holds, first let $\delta \in (0,\infty)$ be given. Assume first that $\bar\tau_J < \infty$. Suppose that for k as large as desired we had $\tau_k < \bar\tau_J - \delta$. Passing to a subsequence, we can assume that $\tau_k \to z \in [0, \bar\tau_J - \delta]$. There would then exist a sequence $\tilde\tau_k \to z$ and a control $u \in \mathcal{A}$ such that

$$\mathrm{dist}(y_{\bar x_J}(z,u), \partial(\mathcal{S}\setminus\mathcal{T})) \leftarrow \mathrm{dist}(y_{x_k}(\tilde\tau_k,u_k), \partial(\mathcal{S}\setminus\mathcal{T})) \leq 1/k \to 0,$$

where we used the definition of the τ_k's and u is a weak-$\star$ limit of the u_k's on $[0,\bar\tau_J-\delta]$ (cf. Lemma 4.3). Since $z < \bar\tau_J$, this contradicts the definition of $\bar\tau_J$. If on the other hand we had $\bar\tau_J = \infty$, then we arrive at the same contradiction by replacing $\bar\tau_J - \delta$ with an arbitrary finite positive number in the previous argument. This establishes the claim (21).

Using (21) and passing to a further subsequence without relabeling, we can fix a constant $l \in [0,+\infty]$ so that

$$l \geq \bar\tau_J \quad \text{and} \quad \tau_k \uparrow l.$$

Moreover, the estimate (7) for Lipschitz dynamics easily gives $\bar\tau_J = 0$ iff $\bar x_J \in \partial(\mathcal{S}\setminus\mathcal{T})$ (cf. [19] for details).

We now use a variant of an argument from [17] to show that $\bar x_J \in \partial(\mathcal{S}\setminus\mathcal{T})$. This argument, which is a consequence of the assumption (H_5), is as follows. Suppose that $\bar x_J \notin \partial(\mathcal{S}\setminus\mathcal{T})$, so $l \geq \bar\tau_J > 0$. Let $M \in (0,l)$, and let $\tilde\alpha \in \mathcal{A}$ be a weak-$\star$ limit of a subsequence of the α_k's in $\mathcal{A}$ on $[0,M]$, which we assume to be the sequence itself for brevity (cf. Lemma 4.3). We conclude from (20) that

$$\begin{aligned} 0 &\leftarrow \int_{\sigma_{k-1}}^{\sigma_k \wedge \{\sigma_{k-1}+M\}} \ell(y_x(s,\bar\alpha_J),\bar\alpha_J(s))\,ds \\ &= \int_0^{\tau_k\wedge M} \ell(y_{x_k}(s,\alpha_k),\alpha_k(s))\,ds \quad \to \quad \int_0^M \ell(y_{\bar x_J}(s,\tilde\alpha),\tilde\alpha(s))\,ds. \end{aligned} \tag{22}$$

The left arrow is by the divergence test applied to the integrals in (20), since w is bounded below and ℓ is nonnegative. The right arrow is justified by the argument of [17, 19].

If we had $\int_0^{\bar{\tau}_J} \ell^r(y_{\bar{x}_J}(s,\tilde{\alpha}),\tilde{\alpha}(s))\ ds > 0$, then $\int_0^G \ell^r(y_{\bar{x}_J}(s,\tilde{\alpha}),\tilde{\alpha}(s))\ ds > 0$ for some $G \in (0,\bar{\tau}_J)$. Since $l \geq \bar{\tau}_J$, we would reach a contradiction by putting $M = G$ in (22). It follows that $\int_0^{\bar{\tau}_J} \ell^r(y_{\bar{x}_J}(s,\tilde{\alpha}),\tilde{\alpha}(s))\ ds = 0$. Since we were assuming that $\bar{x}_J \notin \partial(\mathcal{S}\setminus\mathcal{T})$, we have $\bar{\tau}_J > 0$ and $\bar{x}_J \notin \mathcal{T}$, so this contradicts (H_5). Therefore, it must have been the case that $\bar{x}_J \in \partial(\mathcal{S}\setminus\mathcal{T})$, as needed. Since

$$\partial(\mathcal{S}\setminus\mathcal{T}) \subseteq \partial(\mathcal{S}_\kappa) \cup \mathcal{T} \cup \partial(B_J(0)), \tag{23}$$

we have the following cases to consider:

Case 1 If $\bar{x}_J \in \partial(\mathcal{S}_\kappa)$, then the continuity of w gives $w(\bar{x}_J) = \kappa$. Using (20), the nonnegativity of ℓ, and the fact that $\varepsilon < \kappa - w(x)$, we conclude that

$$w(x)\ \geq\ w(x_k) - \varepsilon\ \rightarrow\ w(\bar{x}_J) - \varepsilon\ >\ \kappa - (\kappa - w(x))\ =\ w(x),$$

which is a contradiction. Therefore, $\bar{x}_J \notin \partial(\mathcal{S}_\kappa)$.

Case 2 If $\bar{x}_J \in \mathcal{T}$, then it follows from the controllability hypothesis STC$\mathcal{T}$, the continuity of w, (SC_w), and the estimate (7) that there exist $p \in \mathbb{N}$, $\tilde{t} > 0$, and $\tilde{\beta} \in \mathcal{A}$ which are such that

$$w(x_p) > -\varepsilon/4,\ \ \tilde{t} := t_{x_p}(\tilde{\beta}) < \infty,\ \ \text{and}\ \ \int_0^{\tilde{t}} \ell(y_{x_p}(s,\tilde{\beta}),\tilde{\beta}(s))\,ds < \varepsilon/4. \tag{24}$$

Combining (20) and (24) now gives

$$w(x) \geq \int_0^{t_\star} \ell(y_x(s,\bar{\bar{\alpha}}),\bar{\bar{\alpha}}(s))\,ds - \varepsilon \geq v(x) - \varepsilon,$$

where $\bar{\bar{\alpha}}$ is the concatenation of $\bar{\alpha}_J\lceil[0,\sigma_{p-1}]$ followed by $\tilde{\beta}$, and $t_\star := t_x(\bar{\bar{\alpha}}) < \infty$. This establishes that $w(x) \geq v(x)$, by the arbitrariness of ε.

Case 3 Since Case 1 cannot occur, and since Case 2 gives the desired conclusion $w(x) \geq v(x)$, it follows from (23) that we can assume that $\bar{x}_J \in \partial\,[B_J(0)]$ in what follows.

We may assume $\bar{\sigma}_J < \infty$. (Otherwise, in what follows, replace $\bar{x}_J$ with one of the x_k's for which $\|x_k\| \geq J - 2^{-J}$ and replace $\bar{\sigma}_J$ with the corresponding σ_{k-1}. This is possible since $x_k \rightarrow \bar{x}_J \in \partial[B_J(0)]$.) Notice that $w(\bar{x}_J) < \kappa$ and $\bar{x}_J = y_x(\bar{\sigma}_J,\bar{\alpha}_J)$. Now repeat this procedure but with the initial value x replaced by $\bar{x}_J$, $\mathcal{S}$ replaced by $\mathcal{S}_\kappa \cap B_{J+1}(0)$, and ε replaced by any positive number $\varepsilon_1 < \varepsilon/2 \wedge [\kappa - w(\bar{x}_J)]$ to get a trajectory for an input $\bar{\alpha}_{J+1}$ starting at $\bar{x}_J$ which wlog reaches $\partial(B_{J+1}(0))$ at time $\bar{\sigma}_{J+1} < \infty$. If we now concatenate this result with $y_x(\cdot,\bar{\alpha}_J)\lceil[0,\bar{\sigma}_J]$, then we get a trajectory which coincides with $y_x(\cdot,\bar{\alpha}_J)$ on $[0,\bar{\sigma}_J]$ and reaches $\partial[B_{J+1}(0)]$ in finite time $\bar{\sigma}_J + \bar{\sigma}_{J+1}$.

This process can be repeated, with ε replaced by any positive number $\varepsilon_q < \varepsilon/2^q \wedge [\kappa - w(\bar{x}_{J+q-1})]$ and the starting point x replaced by $\bar{x}_{J+q-1}$ in the qth iteration of this process. We can assume $\bar{\sigma}_{J+q} < \infty$ and that all the points $\bar{x}_{J+q} = y_{\bar{x}_{J+q-1}}(\bar{\sigma}_{J+q}, \bar{\alpha}_{J+q})$ obtained lie in $\partial\,[B_{J+q}(0)]$ for all q, by the preceding argument. Set

$$\bar{\bar{\sigma}}_q = \bar{\sigma}_J + \bar{\sigma}_{J+1} + \cdots + \bar{\sigma}_q \quad \text{and} \quad \bar{s} = \limsup_q \bar{\bar{\sigma}}_q$$

Fix $\bar{b} \in A$. We can then set

$$\hat{\alpha}(s) := \begin{cases} \bar{\alpha}_J(s) & \text{if } 0 \le s < \bar{\bar{\sigma}}_J, \\ \bar{\alpha}_{J+1}(s - \bar{\bar{\sigma}}_J) & \text{if } \bar{\bar{\sigma}}_J \le s < \bar{\bar{\sigma}}_{J+1}, \\ \vdots & \\ \bar{\alpha}_{J+q}(s - \bar{\bar{\sigma}}_{J+q-1}) & \text{if } \bar{\bar{\sigma}}_{J+q-1} \le s < \bar{\bar{\sigma}}_{J+q}, \\ \vdots & \\ \bar{b} & \text{if } \bar{s} \le s \end{cases}$$

to define an input $\hat{\alpha} \in \mathcal{A}$. A passage to the limit as $k \to \infty$ in (20) and a summation then gives

$$w(x) \ge \int_0^{\bar{\bar{\sigma}}_q} \ell(y_x(s, \hat{\alpha}), \hat{\alpha}(s))\, ds + w(\bar{x}_q) - 2\varepsilon \quad \text{for} \quad \mathbb{N} \ni q \ge J\,. \tag{25}$$

If $\bar{s}$ is finite, then we get

$$\partial\,[B_{J+q+1}(0)] \ni y_{\bar{x}_{J+q}}(\bar{\sigma}_{J+q+1}, \bar{\alpha}_{J+q+1}) = y_x(\bar{\bar{\sigma}}_{J+q+1}, \hat{\alpha}) \to y_x(\bar{s}, \hat{\alpha}) \ \text{ as } \ q \to \infty$$

which is impossible. Using the fact that w is bounded-from-below, a passage to the limit as $q \to \infty$ in (25) therefore gives

$$\int_0^\infty \ell(y_x(s, \hat{\alpha}), \hat{\alpha}(s))\, ds \ \le \ w(x) \ + \ \text{constant} \ < \ \infty \tag{26}$$

Since

$$y_x(\bar{\bar{\sigma}}_{J+q+1}, \hat{\alpha}) = y_{\bar{x}_{J+q}}(\bar{\sigma}_{J+q+1}, \bar{\alpha}_{J+q+1}) \in \partial[B_{J+q+1}(0)]$$

for $q = 1, 2, \ldots$, we also have

$$\limsup_{s \to \infty} \|y_x(s, \hat{\alpha})\| = \infty. \tag{27}$$

But (26)–(27) stand in contradiction with (H_6). Consequently, it must be the case that $\bar{x}_{J+q} \in \mathcal{T}$ for large enough q. By the argument above, this gives the desired inequality $w(x) \ge v(x)$ and completes the proof.

6 Three Applications

This section shows how Theorem 1 applies to exit time HJBE's which are not tractable by means of the well-known methods, including cases where the methods of [17, 18, 19, 20, 22] cannot be applied. We also show how Theorem 1 extends results from [19, 29] on degenerate eikonal and shape-from-shading equations from optics and image processing.

6.1 Vanishing Lagrangians

Theorem 1 can be used to give uniqueness characterizations for HJBE's which are not tractable using [17, 18, 19, 20, 22] or [3, 13]. For example, fix $k \geq 0$, take $N = 2$, and use the exit time data

$$\mathcal{T} = \{(k,k)\}, \quad A = [-1,+1],$$
$$f(x,y,a) = (y - k\Phi(x,y), a), \quad \ell(x,y,a) = x^2 + k(1-|a|)^2, \tag{28}$$

where $\Phi : \mathbb{R}^2 \to [0,1]$ is any C^1 function which is 1 on $B_{k/4}((k,k))$ and 0 on $\mathbb{R}^2 - B_{k/2}((k,k))$. The physical interpretation of this data is that Φ guarantees STC$\mathcal{T}$ (cf. below), and the structure of ℓ penalizes inputs which are not bang-bang. This is a generalization of the Fuller Problem exit time problem data (cf. [15, 17, 19, 22, 37]), which is the case where $k = 0$ in (28). Recall (cf. [24]) that the Fuller Problem admits a cost-minimizing control β_z for each initial state $z \in \mathbb{R}^2$, which is defined as follows. Set

$$\zeta := \{(x_1, x_2) : |x_1| = Cx_2^2,\ x_1x_2 \leq 0\} \subset \mathbb{R}^2,$$

set $\zeta^{\pm} = \{(x_1,x_2) \in \zeta : \pm x_1 > 0\}$, and let A^- and A^+ denote the regions lying above and below ζ respectively, where $C > 0$ is the constant root specified in [24]. Define the feedback $k : \mathbb{R}^2 \to [-1,+1]$ by $k(q) = -1$ if $q \in A^- \cup \zeta^-$, $k(q) = 1$ if $q \in A^+ \cup \zeta^+$, and $k(0,0) = 0$, and let γ_z be the closed-loop trajectory for the feedback k starting at z. We then take $\beta_z(t) = +1$ if $\gamma_z(t) \in A^+$, $\beta_z(t) = -1$ if $\gamma_z(t) \in A^-$, and $\beta_z(t) = 0$ if $\gamma_z(t) = (0,0)$. Let v_k denote the value function (8) for the exit time problem with data (28).

As shown in [22] (see also [29]), the value function $v = v_o$ for the Fuller Problem is the unique bounded-from-below viscosity solution of the corresponding HJBE on $\mathbb{R}^2 \setminus \{0\}$ in the class of all continuous functions $w : \mathbb{R}^2 \to \mathbb{R}$ which are null at $(0,0)$. This result uses the fact that the Fuller Problem satisfies (11). On the other hand, for $k > 0$, the exit time data (28) violate *both* (10) *and* (11). For example, (10) is violated since $\ell(0,p,1) \equiv 0$, even though $(0,0) \notin \mathcal{T}$. Therefore, the data (28) is not tractable using [3, 7, 27].

To see why (11) fails for $k > 0$, let $y_q^k(\cdot,\alpha)$ denote the trajectory for the data (28), the control α, and the initial position q. For $n \in \mathbb{N}$ and β_z as defined above, let $p(n) := (1/(2n^2), 1/n) = y^o_{(0,0)}(1/n, \alpha \equiv 1)$ and $t_n := \inf\{t \geq 0 :$

$y^o_{p(n)}(t, \beta_{p(n)}) = (0,0)\}$. Using [37], we have $M := \sup\{t_n : n \in \mathbb{N}\} < \infty$. Let β denote the concatenation of $\beta_{p(1)}\lceil[0,t_1]$ followed by $\alpha \equiv 1\lceil[0,1/2]$ followed by $\beta_{p(2)}\lceil[0,t_2]$ followed by $\alpha \equiv 1\lceil[0,1/3]$ followed by $\beta_{p(3)}\lceil[0,t_3]$ followed by $\alpha \equiv 1\lceil[0,1/4]$ and so on. Since the norm of the first coordinate of $y^o_{p(n)}(\cdot, \beta_{p(n)})$ is always below $1/n^2$ (cf. [24]), $v_o(p(n)) \leq M/n^4$ for all n. For $n \geq 2$, set

$$\tilde{t}_n = \sum_{j=1}^{n-1}\left[t_j + (j+1)^{-1}\right] \quad \text{and} \quad \gamma_n(s) = \beta(s + \tilde{t}_n),$$

so $p(n) = y^o_{p(1)}(\tilde{t}_n, \beta)$. Since (28) agrees with the Fuller Problem data for (x,y) in some neighborhood of 0 and $|a| = 1$, each $k > 0$ admits an $n(k) \in \mathbb{N}$ such that $y^o_{p(n(k))}(s, \gamma_{n(k)}) = y^k_{p(n(k))}(s, \gamma_{n(k)})$ for all $s \geq 0$, so

$$\begin{aligned}\int_0^\infty \ell(y^k_{p(n(k))}(s, \gamma_{n(k)}), \gamma_{n(k)}(s))\, ds &= \sum_{n=n(k)}^{\infty}\left[v_o(p(n)) + \int_0^{1/(n+1)} [s^2/2]^2\, ds\right] \\ &\leq \sum_{n=n(k)}^{\infty}[M/n^4 + 1/(20\,n^5)] < \infty,\end{aligned}$$

even though $y^k_{p(n(k))}(s, \gamma_{n(k)}) \to (0,0) \notin \mathcal{T}$ as $s \to +\infty$.

One checks that (H_1)–(H_6) hold for (28) for *all* $k \geq 0$. For example, (H_5) holds since the dynamics in (28) agrees with the Fuller dynamics in a neighborhood of the y-axis and the Lagrangian ℓ assigns a positive cost to staying at $(0,0)$ when $k > 0$ and the Fuller Problem satisfies (H_5). The fact that STC$\mathcal{T}$ holds for (28) follows since $f(x,y,a) = (y-k, a)$ near (k,k) and the Fuller Problem satisfies STC$\{(0,0)\}$ (cf. [19]), along with a change of coordinates. Finally, condition (H_6) holds by Lemma 4.4 with $g(x) := x^2$. This application of Lemma 4.4 is based on the fact that Φ' has compact support, which guarantees that the second derivative of the first component of $y^k_x(s, \beta)$ is globally bounded. We conclude as follows:

Corollary 6.1 *Let $k \geq 0$ be constant, and choose the exit time problem data* (28). *If $w : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function which is a bounded-from-below viscosity solution of the corresponding* HJBE

$$[-y + k\Phi(x,y)]\,(Dw(x,y))_1 + |(Dw(x,y))_2| - x^2 = 0$$

on $\mathbb{R}^2 \setminus \mathcal{T}$ that is null at $\mathcal{T}$, then $w \equiv v_k$.

Taking $k = 0$ in Corollary 6.1 gives the uniqueness characterization for the Fuller Problem HJBE asserted in [22]. The novelty of Corollary 6.1 is that it applies to problems violating both the usual positivity condition (10) and the asymptotics condition (11) from [22], and that it establishes uniqueness of solutions of the HJBE in a class of functions which includes functions which are not proper.

Remark 6.2 Using the fact that $x \mapsto x^2$ is convex, one shows that v_o is convex on $\mathbb{R}^2$ and therefore continuous. Moreover, using Soravia's Backward Dynamic Programming Principle (cf. [3, 27]), one can show that $(x, y) \mapsto w(x, y) := -v_o(-x, y)$ is also a viscosity solution of the Fuller Problem HJBE on $\mathbb{R}^2 \setminus \{0\}$ vanishing at the origin. The argument is based on the facts that v_o is a *bilateral* viscosity solution of the HJBE and that each $p \in \mathbb{R}^2$ is an optimal point (cf. [3] for the definitions) and the fact that $(p_1, p_2) \in D^+w(x, y) \Rightarrow (p_1, -p_2) \in D^-v_o(-x, y)$ and that $(p_1, p_2) \in D^-w(x, y) \Rightarrow (p_1, -p_2) \in D^+v_o(-x, y)$. It follows that v_o is the unique continuous bounded-from-below viscosity solution of the corresponding HJBE on $\mathbb{R}^2 \setminus \{0\}$ that vanishes at the origin and that the boundedness from below hypothesis of Corollary 6.1 cannot be removed.

Remark 6.3 Corollary 6.1 can be generalized. For example, the corollary remains true if the Lagrangian ℓ in (28) is replaced by $\ell(x, y, a) = g(x) + k(1 - |a|)^2$ for any g of class $\mathcal{MK}$, e.g., $g(x) = |x|^q$ for any $q > 0$. The proof goes through without changes if the data are modified in this way. Also, the target $\mathcal{T} = \{(k, k)\}$ can be replaced by $\{(k, m)\}$ for any $k \neq 0$ and any $m \in \mathbb{R}$ if Φ is chosen to be 1 near (k, m) and zero in some open set containing the y-axis. Moreover, using the methods of §7 below, the above corollary can be extended to cover local and *discontinuous* viscosity solutions.

6.2 Degenerate Eikonal Equations

This subsection shows how Theorem 1 applies to the HJBE's for a class of exit time problems from geometric optics. The problems have the dynamics $f(x, y, a, b) = (a, b) \in \overline{B_1(0)} \subseteq \mathbb{R}^2$ and the Lagrangians

$$\ell(x, y, a, b) \;=\; [1 + \sqrt{\|(x, y)\|}]^{-p}, \tag{29}$$

where $p \geq 0$ is a constant which will be further specify below. (The argument we are about to give also applies if we instead take the Lagrangian $(1 + \sqrt{|x|})^p$ or $(1 + \sqrt{|y|})^p$, or if the state space and compact control set are in $\mathbb{R}^M$ for M arbitrary.)

We choose any nonempty closed target $\mathcal{T} \subseteq \mathbb{R}^2$, and we let $v_{e,p}$ denote the value function for the exit time problem we have defined for each $p \geq 0$. The corresponding HJBE is

$$\|Dv(x, y)\| = \frac{1}{[1 + \sqrt{\|(x, y)\|}]^p}, \tag{30}$$

which is the eikonal equation of geometric optics for the propogation of light in a medium with speed

$$c(x, y) = [1 + \sqrt{\|(x, y)\|}]^p.$$

Viscosity solutions of eikonal equations have been studied extensively (cf.[3], which covers cases where the speed of the medium is bounded and also uniqueness questions for eikonal equation solutions on bounded sets, and [30]). However, (30) is not covered by these results since c is unbounded and $\mathcal{T}$ may be unbounded. It is easy to check that for $0 \le p \le 2$, the exit time problems for these data satisfy (H_1)–(H_6). Indeed, if $q \in \mathbb{R}^2$ and if ϕ is any trajectory for f starting at q, then we can find a $K > 0$ so that, for each $L > K$, we have

$$\int_0^L \frac{ds}{[1+\sqrt{\|\phi(s)\|}]^p} \ \ge \ \int_0^L \frac{ds}{[1+\sqrt{\|q\|+s}]^p} \ \ge \ \frac{1}{2}\int_K^L \frac{ds}{s^{p/2}} \ \rightarrow \ \infty$$

as $L \to \infty$, so (H_6) is satisfied vacuously. We conclude as follows:

Corollary 6.4 *Let $p \in [0,2]$ and $\mathcal{T} \subseteq \mathbb{R}^2$ be closed and nonempty. If $w : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function which is a bounded-from-below viscosity solution of* (30) *on $\mathbb{R}^2 \setminus \mathcal{T}$ which is null on $\mathcal{T}$, then $w \equiv v_{e,p}$.*

Remark 6.5 It was not necessary to assume that the target $\mathcal{T}$ is bounded. If $p > 2$ in (29), then Theorem 1 may not apply, since (H_6) could fail. For example, if $p = 4$, and $\mathcal{T} = \{(x,0) \in \mathbb{R}^2 : x \le -1\}$ and $\beta \equiv (1,0)$, then (29) gives $\int_0^\infty \ell(y_{(0,0)}(s,\beta),\beta(s))\, ds < \infty$, even though the trajectory does not remain bounded. Moreover, the standard uniqueness characterizations for exit time HJBE's (e.g., Corollary IV.4.3 of [3]) would not apply, since (10) is not satisfied. However, using [20], one can show that the statement of Corollary 6.4 remains true *even without* the restriction $p \in [0,2]$. This is done by rewriting the HJBE (30) as

$$[1+\sqrt{\|(x,y)\|}]^p \|Dv(x,y)\| - 1 = 0 \tag{31}$$

and then viewing (31) as the HJBE for the exit time problem with the non-Lipschitz dynamics

$$\tilde{f}(x,y,a,b) = [1+\sqrt{\|(x,y)\|}]^p (a,b)$$

(with $(a,b) \in \overline{B_1(0)}$ as before) and the Lagrangian $\tilde{\ell} \equiv 1$. The dynamics $\tilde{f}$ is then approximated by locally Lipschitz dynamics, and then Theorem IV.4.4 of [3] is applied. For details, see §6.1 of [20].

6.3 Shape-From-Shading Equations

Our results also apply to equations of the form

$$I(x)\Psi(Du(x)) - b(x)\cdot Du(x) - h^2(x) = 0$$

for I nonnegative and $\Psi : \mathbb{R}^N \to \mathbb{R}$ any convex function with $\Psi(0) = 0$. This equation is studied in [29]. Taking the Legendre transform $\Psi^\star$ of Ψ, which is nonnegative, we can rewrite this equation as

$$\max_{a\in\text{domain}(\Psi^\star)} \left\{-(b(x)-I(x)a)\cdot Du(x) - \left[h^2(x)+I(x)\Psi^\star(a)\right]\right\} = 0.$$

A particular case of this equation (cf. [29]) is

$$I(x)\left[1+\|Du(x)\|^2\right]^{1/2} - 1 = 0, \quad x\in\Omega\subseteq\mathbb{R}^2 \tag{32}$$

for open sets Ω, which in fact can be written as

$$\max_{\|a\|\le 1}\{I(x)a\cdot Du(x) - [1-I(x)\left(1-\|a\|^2\right)^{1/2}]\} = 0. \tag{33}$$

The equation (33) arises in shape-from-shading models in image processing, where $I(x)\in[0,1)$ is the intensity of light reflected by an object (cf. [30]). The objective in image processing is to reconstruct the unknown function u, representing the height of the surface on some subset Ω of the plane, from the brightness of a single two-dimensional image of the surface. For the case of a Lambertian surface which is not self-shadowing and which is illuminated by a single distant vertical light source, the height u is a viscosity solution of (33).

Now pick any closed nonempty target $\mathcal{T}\subseteq\mathbb{R}^2\setminus\{0\}$ and $\Omega := \mathbb{R}^2\setminus\mathcal{T}$, and choose the intensity function

$$I(x) := \frac{\|x\|}{1+\|x\|}. \tag{34}$$

Then (33) is an HJBE for an exit time problem with the dynamics

$$f(x,u) := -I(x)u, \tag{35}$$

the control set $A=\overline{B_1(0)}\subseteq\mathbb{R}^2$, and the Lagrangian

$$\ell(x,u) = 1 - \frac{\|x\|}{1+\|x\|}\left(1-\|u\|^2\right)^{1/2}. \tag{36}$$

As explained in Remark 3.2, for general $\mathcal{T}\subseteq\mathbb{R}^2\setminus\{0\}$, ℓ violates the positivity condition (10) (since $\ell(x,0)\to 0$ as $\|x\|\to\infty$), so the well-known results (e.g., those of [3]) cannot be used to get uniqueness characterizations for solutions of (32). On the other hand, using the fact that

$$\|y_q(s,\beta)\| \le \|q\| + s$$

for all $\beta\in\mathcal{A}$, $s\ge 0$, and $q\in\mathbb{R}^2$, one can easily check that (H_1)–(H_6) hold. The argument is similar to the validation of (H_6) in §6.2. Therefore, we conclude from Theorem 1 that if $w:\mathbb{R}^2\to\mathbb{R}$ is a continuous function which is a viscosity solution of (33) on $\mathbb{R}^2\setminus\mathcal{T}$ that satisfies (SC_w), then w coincides with the shape-from-shading value function. Local uniqueness characterizations and results for discontinuous viscosity solutions for the shape-from-shading equation can also be given using the results in §7 below.

Remark 6.6 As in the case of eikonal equations, it was not necessary to assume that the target $\mathcal{T}$ was bounded. It is worth remarking that if we replace the light intensity $I(x)$ with

$$\tilde{I}(x) := \frac{3e^{2\|x\|}}{1+3e^{2\|x\|}} \quad \in \quad [3/4, 1)$$

in the previous example and keep the example the same otherwise, then Theorem 1 would no longer apply, since condition (H_6) may not be satisfied. However, for such cases, we can still apply [19] to get uniqueness of *proper* solutions of the corresponding HJBE. For example, take $\mathcal{T} = \{(0,r) : r \leq -1\}$ and the control

$$\beta(t) \equiv (0, -1/(t+1)),$$

and let $t \mapsto x(t) = (x_1(t), x_2(t))$ denote the trajectory of $\tilde{f}(p,u) := -\tilde{I}(p)u$ for the initial position $p(0) = (0,1)$ and the control $u = \beta(t)$. For all $t > 0$, we then have $x_1(t) = 0$,

$$x_2(t) = 1 + \int_0^t \tilde{I}(x(s))\frac{1}{s+1}\,ds \;\geq\; 1 + \frac{3}{4}\int_0^t \frac{1}{s+1}\,ds \;=\; 1 + \frac{3}{4}\ln(t+1),$$

so $x(t)$ is not bounded. However,

$$\begin{aligned} e^{2\|x(t)\|}\|\beta(t)\|^2 &= \frac{1}{(t+1)^2} e^{\left[2+2\int_0^t \tilde{I}(x(s))\frac{1}{s+1}\,ds\right]} \\ &\leq \frac{1}{(t+1)^2} e^2 e^{2\ln(t+1)} \\ &\leq e^2. \end{aligned} \tag{37}$$

Therefore, if $\tilde{\ell}(p,u) = 1 - \tilde{I}(p)[1 - \|u\|^2]^{1/2}$ denotes the corresponding Lagrangian, then since we have

$$\tilde{\ell}(p,u) \leq 1 - \tilde{I}(p)[1 - \|u\|^2] \;\; \forall p \in \mathbb{R}^2,\; u \in \overline{B_1(0)},$$

(37) gives

$$\begin{aligned} \int_0^\infty \tilde{\ell}(x(s),\beta(s))\,ds &\leq \int_0^\infty \frac{1+3e^{2\|x(s)\|}\|\beta(s)\|^2}{1+3e^{2\|x(s)\|}}\,ds \\ &\leq \left[1+3e^2\right]\int_0^\infty \frac{dt}{1+3e^{2+3/2\ln(t+1)}} \\ &\leq \left(\left[1+3e^2\right]/\left[3e^2\right]\right)\int_0^\infty \frac{dt}{(t+1)^{3/2}} \;<\; \infty, \end{aligned}$$

even though $t \mapsto x(t)$ is not bounded, which shows (H_6) is not satisfied. Moreover, the standard uniqueness characterizations for exit time HJBE's (cf. [3, 7]) would again not apply, since the Lagrangian $\tilde{\ell}$ is not uniformly bounded below by positive constants. However, since (H_5) holds, one can use [17] to show that for any nonempty closed target $\mathcal{T} \subseteq \mathbb{R}^2 \setminus \{0\}$, any proper continuous viscosity solution of the corresponding HJBE

$$\sup_{\|a\|\leq 1} \{\tilde{I}(x)a \cdot Du(x) - [1 - \tilde{I}(x)\left(1 - \|a\|^2\right)^{1/2}]\} \;=\; 0$$

on $\mathbb{R}^2 \setminus \mathcal{T}$ which is null on $\mathcal{T}$ must in fact be identically equal to the shape-from-shading exit time value function v_{sfs} for the target $\mathcal{T}$, the dynamics $\tilde{f}$, and the Lagrangian $\tilde{\ell}$.

Remark 6.7 Notice that it was not necessary to assume that the domain set Ω for (32) was bounded. It is worth pointing out that one cannot in general expect uniqueness of solutions for the shape-from-shading HJBE for cases where I is allowed to take the value 1, since the surface u and $-u$ could both be viscosity solutions of (32). For example, take the light intensity $I(x) = (1 + 4\|x\|^2)^{-1/2}$, $\mathcal{T} = \mathbb{R}^2 \setminus B_1(0)$, and the surface $u(x) = 1 - \|x\|^2$ on $B_1(0)$ and zero elsewhere. Clearly, u and $-u$ are both solutions of (32). However, (H_5)–(H_6) are not satisfied, since the trajectory $\phi(t) \equiv 0$ gives zero integrated costs on $[0, \infty)$ without ever reaching the target, so this case is not covered by Theorem 1. For the analysis of cases where $\#\{x : I(x) = 1\} = 1$, see [16], and for bounded viscosity solutions of (33), see [26].

7 Discontinuous and Local HJBE Solutions

This section gives variants of Theorem 1 for discontinuous and local HJBE solutions. We study discontinuous solutions using the envelopes approach from [3].

7.1 A Remark on Discontinuous Viscosity Solutions

Under (H_1)–(H_6), the value function v_g could be discontinuous (cf. [3], pp. 248–249). This suggests the question of how one can characterize v as the unique *discontinuous* solution of the HJBE on $\mathbb{R}^N \setminus \mathcal{T}$ that satisfies (SC_v). By a discontinuous solution, we mean the following. For each locally bounded function $w : S \to \mathbb{R}$ on a set $S \subseteq \mathbb{R}^N$, we define the following semicontinuous envelopes:

$$w_\star(x) := \liminf_{S \ni y \to x} w(y) \quad \text{and} \quad w^\star(x) := \limsup_{S \ni y \to x} w(y).$$

We call $w_\star$ the **lower envelope** of w, and we call $w^\star$ the **upper envelope** of w. For $\mathcal{G}$, S, and F satisfying the requirements of Definition 2.2, we then say that a locally bounded function $w : S \to \mathbb{R}$ is a **discontinuous subsolution** (resp., **supersolution**) of $F(x, Dw(x)) = 0$ on $\mathcal{G}$ provided $F(x_o, D\gamma(x_o)) \leq 0$ (resp., ≥ 0) for each $\gamma \in C^1(\mathcal{G})$ at each local maximizer (resp., minimizer) of $w^\star - \gamma$

(resp., $w_\star - \gamma$) on $\mathcal{G}$.[1] A **(discontinuous viscosity) solution** of $F(x, Dw(x)) = 0$ on $\mathcal{G}$ is then a function which is simultaneously a discontinuous subsolution and a discontinuous supersolution of $F(x, Dw(x)) = 0$ on $\mathcal{G}$. Lemma 4.1 remains true if $u \in C(\bar{E})$ is replaced by any bounded discontinuous subsolution of the HJBE on E and u in (14) is replaced by $u^\star$. Also, Lemma 4.2 remains true if $w \in C(\bar{B})$ is replaced by any bounded discontinuous supersolution of the HJBE on B and w in (15) is replaced by $w_\star$. Using these facts, one can prove the following generalization of Theorem 1: If (H_1)–(H_6) hold, if $w : \mathbb{R}^N \to \mathbb{R}$ is a discontinuous viscosity solution of the HJBE on $\mathbb{R}^N \setminus \mathcal{T}$ that satisfies (SC_w), and if $w_\star$ is continuous on $\mathbb{R}^N$, then $w \equiv v$ on $\mathbb{R}^N$.[2] The proof is almost identical to the proof of Theorem 1 but with w replaced by w_* in the proof of the inequality $w \geq v$. For cases where v is continuous, this establishes that all solutions w of the HJBE (9) on $\mathbb{R}^N \setminus \mathcal{T}$ that satisfy (SC_w) and continuity of w_* agree with v, and therefore are continuous.

7.2 Local Solutions of the HJBE

This subsection shows how to extend Theorem 1 to get uniqueness of solutions of the HJBE on sets of the form $\Omega \setminus \mathcal{T}$ for open sets Ω. We set

$$\mathcal{R} = \left\{ x \in \mathbb{R}^N : \inf\{t_x(\beta) : \beta \in \mathcal{A}\} < \infty \right\},$$

so $\mathcal{R}$ is the set of points that can be brought to $\mathcal{T}$ in finite time using the dynamics f. Using (H_2)–(H_3), one shows that $\mathcal{R}$ is open (cf. [3]). In many classical cases where ℓ is bounded below by a positive constant, one has

$$v \text{ is continuous on } \mathcal{R}, \quad \text{and} \quad \lim_{x \to x_o} v(x) = +\infty \ \forall x_o \in \partial\mathcal{R}. \tag{38}$$

On the other hand, one easily finds examples where ℓ is not bounded below by a positive constant and the limit condition in (38) fails. Here is an elementary example where this occurs:

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[1] In this context, 'discontinuous' means "not necessarily continuous".

[2] The continuity of $w_\star$ is used to ensure that the sets S_κ in the proof of Theorem 1 are open. The condition that $w_\star$ is continuous of course holds automatically if w is continuous. However, (SC_w) and continuity of $w_\star$ can even be satisfied by functions which are nowhere continuous. For example, if we take the indicator function $w \equiv \mathbf{1}_{\mathbb{Q}} : \mathbb{R} \to \{0, 1\}$, then $w_\star \equiv 0$. This generalized version of Theorem 1 remains true if the pointwise condition that $w \equiv 0$ on $\mathcal{T}$ is replaced by the less restrictive requirement that there be a locally bounded function $g : \mathbb{R}^N \to \mathbb{R}$ for which

$$\forall x \in \mathcal{T}, \ \ w_\star(x) \geq g_\star(x) \ \text{ and } \ w^\star(x) \leq g^\star(x)$$

except that the conclusion that $w \equiv v$ is replaced by the following inequalities on $\mathbb{R}^N$ (cf. Remark 3.1): $w_* \geq v_{g_*}$ and $w^\star \leq v_{g^\star}$. In case $g \in C(\mathcal{T})$, this implies $w \equiv v_g$ on $\mathbb{R}^N$.

Example 7.1 Take $N = 1$, $\mathcal{T} = [1, +\infty)$, $A = \{+1\}$, $f(x, a) = |x|a$, and $\ell(x, a) = |x|$. In this case,

$$v(\bar{x}) = \int_0^{\ln(1/\bar{x})} \bar{x}e^t \, dt = 1 - \bar{x} \to 1 \quad \text{as} \quad \bar{x} \downarrow 0,$$

even though $0 \in \partial\mathcal{R}$.

This motivates the question of how one can characterize v as a unique viscosity solution of the HJBE on $\mathcal{R} \setminus \mathcal{T}$ for cases where $\mathcal{R} \neq \mathbb{R}^N$ and the extra condition (38) holds. To address this question, we assume the following relaxed version

(H_5') If $x \in \mathcal{R}\setminus\mathcal{T}$ and $\beta \in \mathcal{A}$, then $\int_0^t \ell^r(y_x(s, \beta), \beta(s)) \, ds > 0$ for all $t \in (0, \infty)$.

of (H_5). We also fix an open set $\Omega \subseteq \mathcal{R}$ containing $\mathcal{T}$, and we consider viscosity solutions of the HJBE (9) on $\Omega \setminus \mathcal{T}$ that satisfy the localization

($OSC_{w,\Omega}$) w is bounded-from-below on Ω, $w \equiv 0$ on $\mathcal{T}$, and $\lim_{x \to x_o} w(x) = +\infty \ \forall x_o \in \partial\Omega$.

Noting that v satisfies ($OSC_{v,\mathcal{R}}$) if (38) holds, we then have the following local version of Theorem 1:

Theorem 2 *Let* (H_1)–(H_4), (H_5'), *and* (H_6) *hold. Let $\Omega \subset \mathcal{R}$ be an open set containing $\mathcal{T}$. Let $w : \Omega \to \mathbb{R}$ be a continuous function which is viscosity solution of the* HJBE (9) *on $\Omega\setminus\mathcal{T}$ that satisfies ($OSC_{w,\Omega}$). Then, $w \equiv v$ on Ω. In particular, if v satisfies* (38)*, then v is the unique viscosity solution w of the* HJBE *on $\mathcal{R} \setminus \mathcal{T}$ in the class of all continuous functions $w : \mathcal{R} \to \mathbb{R}$ that satisfy ($OSC_{w,\mathcal{R}}$).*

Remark 7.2 The proof of the inequality $w \leq v$ for Theorem 2 is exactly the proof of that inequality in [19]. The proof is slightly more complicated than the proof that $w \leq v$ for Theorem 1, since one must consider trajectories that reach $\mathcal{T}$ in finite time but which exit Ω before the first time they ever reach $\mathcal{T}$. The proof of the reverse inequality closely follows the proof of Theorem 1 except that instead of setting $\mathcal{S} = \mathcal{S}_\kappa \cap B_J(0)$, we set $\mathcal{S} = \mathcal{S}_\kappa \cap B_J(0) \cap \Omega$. We rule out cases where $\bar{x}_J \in \partial\Omega$ using the limit condition in ($OSC_{w,\Omega}$). Theorem 2 can also be generalized to the case of discontinuous viscosity solutions using the method of §7.1.

8 Problems with Unbounded Control Sets

We close by giving two variants of Theorem 1 which can be applied for cases where the control set $A \subseteq \mathbb{R}^M$ is closed but possibly unbounded. In the first variant, we impose regularity conditions on the data which penalize the use of control set values of large norm. In the second variant, we replace the possibly unbounded control set A with a suitable compact set of vector field valued controls. Recall the definition (5) of $\mathcal{A}$ which applies to possibly noncompact control sets.

8.1 Penalization Method

For simplicity, let us assume that all the sets

$$\mathcal{D}(x) := \{(f(x,a), \ell(x,a)) : a \in A\}$$

are convex. As explained in §2, the set of inputs $\alpha \in \mathcal{A}$ can then be taken to be the measurable functions valued in A (by the Filippov Selection Theorem). We assume that (H_2)–(H_6) are satisfied, where $0 \in A \subseteq \mathbb{R}^M$ for $M \in \mathbb{N}$ and A is closed but not necessarily compact. Following [4, 11, 19], we then add the following conditions on f and ℓ:

(H_7) f is bounded on $B_R(0) \times A$ for each $R > 0$.

(H_8) There is a modulus ω such that $|\ell(x,u) - \ell(y,u)| \leq \omega(\|x-y\|)$ for all $x, y \in \mathbb{R}^N$ and $u \in A$.

(H_9) There exist constants $\ell_o > 0$, $C_o \geq 0$, $\beta \in (0,1]$, $\delta_2 \geq 0$, $\bar{\ell} \geq 0$, and $\delta_1 > 1$ such that the following conditions hold for all $x, y \in \mathbb{R}^N$ and $a \in A$:

(a) $\ell(x,a) \geq \ell_o\|a\|^{\delta_1} - C_o$

(b) $|\ell(x,a) - \ell(y,a)| \leq \bar{\ell}\|x-y\|^{\beta}(1 + \|a\|^{\delta_1} + \|x\|^{\delta_2} + \|y\|^{\delta_2})$

(Recall that a **modulus** is a nondecreasing continuous function $\omega : [0,\infty) \to [0,\infty)$ for which $\omega(0) = 0$.) As shown in [3], Lemmas 4.1 and 4.2 remain true if (H_2)–(H_8) are assumed instead of the assumptions (H_1)–(H_6). These assumptions penalize the use of control set values of large norm. We then consider only viscosity solutions w of the HJBE on $\mathbb{R}^N \setminus \mathcal{T}$ for which the subdifferential sets $D^-w(x)$ are locally bounded, i.e., such that $\sup\{\|p\| : p \in D^-w(x), x \in K\} < \infty$ for each compact set $K \subseteq \mathbb{R}^N$. As shown in Theorem I.7.3 of [9], this is equivalent to considering only locally Lipschitz solutions of the HJBE on $\mathbb{R}^N \setminus \mathcal{T}$. In this case, the infimizations in the restriction of the HJBE to any $B_J(0)$ can be taken over a corresponding compact set $C_J \subset A$, i.e., in the notation we introduced in §2, $H_A\lceil[B_J(0) \times D_J] = H_{C_J}\lceil[B_J(0) \times D_J]$, where D_J is a bounded set large enough to contain $\{p \in D^-w(x) : x \in B_J(0)\}$ (cf. [4, 11] for the proof). Then the arguments in §5 on $B_J(0)$ apply with the compact control set C_J replacing A, and then we iterate on J to get an input $\hat{\alpha} : [0,\infty) \to A$ as before. We then invoke (H_6) to conclude as follows:

Theorem 3 *Assume hypotheses* (H_2)–(H_9)*, with A a closed set containing $0 \in \mathbb{R}^M$. Let $w : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz function which is a viscosity solution of* (9) *on $\mathbb{R}^N \setminus \mathcal{T}$ that satisfies (SC_w). Then $w \equiv v$.*

8.2 Vector Field Valued Controls Method

Another way to extend Theorem 1 to the case of noncompact control sets is as follows. As in the previous subsection, we assume the sets $\mathcal{D}(x)$ are all convex. We give $C(\mathbb{R}^N, \mathbb{R}^N \times \mathbb{R})$ the topology of compact convergence (cf. [23]). We continue to assume (H_2)–(H_8) and that $A \subseteq \mathbb{R}^M$ is closed and nonempty but possibly unbounded. We also add the following assumptions:

(NC_1) $\sup\{\ell(0,u) : u \in A\} < \infty$.

(NC_2) $\{(f(\cdot,u), \ell(\cdot,u)) : u \in A\} \subseteq C(\mathbb{R}^N, \mathbb{R}^N \times \mathbb{R})$ is closed.

These guarantee that the supremum in the definition of the HJBE is always finite. It follows from the Ascoli-Arzelá Theorem that $K := \{k_u(\cdot) := (f(\cdot,u), \ell(\cdot,u)) : u \in A\}$ is a compact subset of the metric space $C(\mathbb{R}^N, \mathbb{R}^N \times \mathbb{R})$ (cf. [19, 23]). Define the projection mappings π_j on K by

$$\pi_j(k_u(\cdot)) = \begin{cases} f(\cdot,u), & j = 1 \\ \ell(\cdot,u), & j = 2 \end{cases} \qquad \forall u \in A.$$

We now apply the method of our proofs to the new exit time problem whose dynamics F, Lagrangian Λ, and set $\tilde{\mathcal{A}}$ of admissible controls are

$$F(x,k) = (\pi_1 \circ k)(x), \quad \Lambda(x,k) = (\pi_2 \circ k)(x) \ \& \ \tilde{\mathcal{A}} := \{[0,\infty) \ni t \mapsto k_{\beta(t)} : \beta \in \mathcal{A}\}$$

with the same target $\mathcal{T}$. Notice that $\{(F(x,k), \Lambda(x,k)) : k \in K\}$ is convex for each $x \in \mathbb{R}^N$. Let $\tilde{v}$ denote the value function of this new problem. Since the trajectories of F with the controls $\tilde{\mathcal{A}}$ are exactly the trajectories of f with controls in $\mathcal{A}$, it follows that $\tilde{v} \equiv v$. Moreover, the new problem satisfies (H_1)–(H_6) (with K replacing A, F replacing f, and Λ replacing ℓ). Our proof of Theorem 1 then gives the following:

Theorem 4 *Let $\emptyset \neq A \subseteq \mathbb{R}^M$ be closed. Assume* (H_2)-(H_8) *and* (NC_1)-(NC_2). *Let $w : \mathbb{R}^N \to \mathbb{R}$ be a continuous function which is a viscosity solution of* (9) *on $\mathbb{R}^N \setminus \mathcal{T}$ that satisfies (SC_w). Then $w \equiv v$.*

We remark that if (H_2)–(H_8) and (NC_1)–(NC_2) all hold with $A \neq \emptyset$ a closed subset of $\mathbb{R}^N$, and if $\mathcal{R} = \mathbb{R}^N$, then the value function v is a discontinuous viscosity solution of the HJBE on $\mathbb{R}^N \setminus \mathcal{T}$ (cf. [3]). If we also assume $v_\star$ is continuous, then a generalization of Theorem 4 characterizes v as the unique discontinuous viscosity solution w of the HJBE in the class of functions $w : \mathbb{R}^N \to \mathbb{R}$ that satisfy (SC_w) and continuity of $w_\star$. The generalization of Theorem 4 to discontinuous solutions follows from the argument of §7.1. Also, the theorem extends to local HJBE solutions using the arguments of the previous section.

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