Almost Fixed Time Observers for Parameters and for State Variables of Nonlinear Systems

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***Abstract*— We provide new finite-time and fast-converging observers for continuous-time nonlinear systems that contain unknown constant parameters and measurement delays. When the dynamics are affine in the unmeasured state, we obtain fixed time observers that identify unknown states and unknown model parameters. When the dynamics contain a nonlinearity that depends on the unmeasured state, we instead obtain asymptotic convergence of our observers, whose rate of convergence converges to infinity as the growth rate of this nonlinearity converges to zero, so in this case we call the observers almost fixed time observers. Our examples illustrate how our assumptions can be easily checked.**

***Index Terms*— Observers, nonlinear, estimation**

I. Introduction

THIS letter continues our search (started in [1], [2], [3], [4], [5], [6], and [7]) for finite-time observers for nonlinear systems under suitable structural conditions. While our prior works provided arbitrarily fast convergence of observers for states of perturbed systems (which gave fixed time convergence when the uncertainties are zero and when the dynamics were affine in the unmeasured state, where by fixed time, we mean finite time convergence where the convergence time is independent of the initial state), here we consider a more difficult problem where in addition to state observers, one must identify unknown model parameters. This letter is motivated by the ubiquity of unmeasured states in applications, and the value of using observers to solve feedback control problems; see, e.g., [8]–[10], and [11].

An important setting where unknown constant model parameters occur is artificial neural network expansions, where an unknown time-varying function is a linear combination of known basis functions with unknown constant weights, where the goal is to compute the unknown weights to identify the uncertainty; see, e.g., Section IV-A below. One standard approach to estimating unknown model parameters is adaptive control. However, adaptive control usually does not yield fixed time parameter identification, and is not amenable to estimating states.

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This motivates this letter, which addresses the preceding challenges when there is also an affine output measurement with measurement delays. This makes it possible to prove fixed time convergence when the dynamics are affine in the unmeasured state, and almost finite convergence in the sense of [6] when there are nonlinearities depending on the unmeasured state. As in the global convergence results in [6], we assume that the nonlinearity that depends on the unmeasured state satisfies a uniform global Lipschitzness condition in its state argument that is uniform in its time argument, and then we place bounds on a corresponding global Lipschitz constant. However, since [6] did not identify unknown model parameters, this letter adds significant value as compared to [6]. Our work also differs from notable works such as [12] that did not address the challenge of parameter identification. This work improves on our conference version [13], which was confined to cases where the nonlinearity only depended on the measured state and time, and which can only identify one model parameter when measurement delays were present.

We use standard notation. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise, and $|\cdot|$ is the standard Euclidean norm and corresponding matrix norm. We set $g_t(s) = g(t+s)$ for functions g and all $s \leq 0$ and $t \geq 0$ such that $t+s$ is in the domain of g, $|f|_J$ is the usual supremum over any interval J in the domain of functions f, and $|f|_\infty$ is the supremum of a function f.

II. Main Observer Design

A. Studied System, Objectives, and Assumptions

We consider the system

$$\left\{\begin{array}{rcl} \dot{\xi}(t) & = & A\xi(t) + \varphi(C\xi(t), u(t)) + \Delta(\xi(t), t) \\ & & +\epsilon_1\gamma_1(t) + \ldots + \epsilon_q\gamma_q(t) \\ y_\xi(t) & = & C\xi(t-h) \end{array}\right. \tag{1}$$

where ξ is valued in $\mathbb{R}^n$, $q \geq 1$, y_ξ is valued in $\mathbb{R}$, the constants $\epsilon_j \in \mathbb{R}$ are unknown, u is a known input, $h \geq 0$ is a known constant delay, $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{1\times n}$ are known constant matrices, each $\gamma_j : \mathbb{R} \to \mathbb{R}^n$ is locally bounded and piecewise continuous, and φ and Δ are locally Lipschitz functions such that (1) satisfies the standard forward completeness condition. Our goal is to build observers for the combined variable $(\xi(t-h), \epsilon)$ whose estimates are computed from the measurements $y_\xi(\ell)$ for $\ell \leq t$ (i.e., recovery of time lagged values $\xi(t-h)$ from available output measurements at times t, plus parameter identification), and which have the fixed time property when $\Delta = 0$, and the almost finite-time property discussed above when Δ is not the zero function, where $\epsilon = [\epsilon_1, \ldots, \epsilon_q]$.

Our first assumption is this standard observability condition, which precludes the possibility of extending the dynamics using the additional equations $\dot{\epsilon}_j = 0$ for $j = 1, 2, \ldots q$ to reduce the problem to one of identifying unknown states:

Assumption 1: The pair (A, C) is observable. □

Hence, for any constant $\tau > 0$, the $n \times n$ matrix

$$E = \int_{-\tau}^{0} e^{A^\top s} C^\top C e^{A(s-h)} \mathrm{d}s \tag{2}$$

is invertible (by [14, Section 3.5]). We also use the functional

$$\kappa(\phi) = \int_{-\tau}^{0} e^{A^\top r} C^\top C \int_{r-h}^{0} e^{A(r-h-\ell)} \phi(\ell) \mathrm{d}\ell \mathrm{d}r \tag{3}$$

and the function $W : \mathbb{R} \to \mathbb{R}^q$ that is defined by

$$\begin{aligned} & W(t) = [W_1(t), \ldots, W_q(t)]^\top, \text{ where for } i = 1, \ldots, q, \\ & W_i(t) = \int_{t-T-h}^{t-h} C[AE^{-1}\kappa(\gamma_{i,m}) + \gamma_i(m)]\mathrm{d}m, \end{aligned} \tag{4}$$

where $T > 0$ is also a constant, and $\gamma_{i,m}$ is defined by $\gamma_{i,m}(s) = \gamma_i(m + s)$ for $m \geq 0$, $s \leq 0$, and $i = 1, \ldots, q$.

Assumption 2: There exist q nonnegative constants $\tau_1, \ldots, \tau_q$ such that $\tau_i < \tau_j$ for pairs (i, j) with $j > i$ such that the matrix valued function $M : \mathbb{R} \to \mathbb{R}^{q \times q}$ defined by

$$M(t) = [W(t - \tau_1) \quad \ldots \quad W(t - \tau_q)] \tag{5}$$

is nonsingular for each $t \geq \tau_q + T + \tau$. □

Our assumptions are easily checked. For instance, we can check Assumption 2 by searching for a positive lower bound on the absolute value of the determinant of $M(t)$; see also Remark 4. In terms of the above notation and

$$\begin{aligned} & \bar{\beta} = \int_{-\tau}^{0} \int_{\ell-h}^{0} |E^{-1} e^{A^\top \ell} C^\top C e^{A(\ell-h-m)}| \mathrm{d}m \mathrm{d}\ell \\ & \text{and } \kappa^\sharp(t) = \left[E^{-1}\kappa(\gamma_{1,t}) \quad \ldots \quad E^{-1}\kappa(\gamma_{q,t})\right] \end{aligned} \tag{6}$$

and the inverse values $M(t)^{-1}$ for (5), our last assumption is as follows, but this third assumption will only be required by our theorem when the Δ in (1) is not the zero function:

Assumption 3: The function Δ admits a constant $L_\Delta > 0$ such that $|\Delta(\xi_a, t) - \Delta(\xi_b, t)| \leq L_\Delta |\xi_a - \xi_b|$ for all $\xi_a \in \mathbb{R}^n$, $\xi_b \in \mathbb{R}^n$, and $t \in \mathbb{R}$. Also, the function $\kappa_*(t) = \kappa^\sharp(t)(M(t)^{-1})^\top$ is bounded on $J = [\tau_q + T + \tau, \infty)$, and

$$L_\Delta \left(\bar{\beta}\left\{T\sqrt{q}|\kappa_*|_J |CA| + 1\right\} + T|\kappa_*|_J |C|\sqrt{q}\right) < 1 \tag{7}$$

is satisfied. □

On the other hand, the $\Delta = 0$ case occurs in notable applications; see [1]–[4], and Section IV below. Our delay h is strongly motivated by the prevalence of measurement delays in information theory and aerospace systems, and (1) can also model input delays, e.g., by replacing $u(t)$ by $u(t - h_I)$, or $\Delta(\xi(t), t)$ by $\Delta(\xi(t), u(t - h_I))$, for an input delay h_I. The nonzero Δ case covers bilinear systems with bounded controls, i.e., having products of state and bounded control components, which are common in power electronics [15]. This gives terms $\Delta(\xi, t) = \sum_{j=1}^{m} \sigma_j(t) B_j \xi$ satisfying our uniform global Lipschitzness condition in ξ, where each B_j is a constant matrix and each σ_j is a real valued component of a bounded feedback $\sigma(t) = U(\hat{x}_0(t - h_I))$ using known values of our estimate $\hat{x}_0$ for x from (16) with input delays h_I.

B. Simplification

We use the dynamic extension and new variable

$$\begin{aligned} & \dot{\hat{\xi}}(t) = A\hat{\xi}(t) + \varphi(y_\xi(t), u(t-h)) + \Delta(\hat{\xi}(t), t-h) \\ & \text{and } \ x(t) = \xi(t) - \hat{\xi}(t+h). \end{aligned} \tag{8}$$

By combining (1) and (8), we obtain the system

$$\begin{cases} \dot{x}(t) = Ax(t) + \Delta_d(t) + \epsilon_1 \gamma_1(t) + \ldots + \epsilon_q \gamma_q(t) \\ y(t) = Cx(t-h), \end{cases} \tag{9}$$

where

$$\Delta_d(t) = \Delta(x(t) + \hat{\xi}(t+h), t) - \Delta(\hat{\xi}(t+h), t), \tag{10}$$

and where y is known since $y_\xi(t)$ and $\hat{\xi}$ are being measured. A benefit of this simplification is that it allows us to avoid adding global Lipschitzness or other assumptions on φ.

Hence, we focus on (9), to estimate x and ϵ. Then at each time t, (8) and our theorem below will provide an observer for $\xi(t-h) = x(t-h) + \hat{\xi}(t)$ and ϵ that is expressed in terms of the available measurements $\hat{\xi}(\ell)$ and $y(\ell)$ for values $\ell \leq t$. This will allow us to meet our goal of identification of values of the ϵ_j's and of the time lagged state $\xi(t-h)$ of (1) using the available measurements $y(\ell)$ for times $\ell \leq t$.

C. Almost Fixed Time Observer

With the above notation, we next introduce the functionals

$$\begin{aligned} & \Psi(y_t) = \int_{t-\tau}^{t} e^{A^\top(s-t)} C^\top y(s) \mathrm{d}s, \\ & \mu(t) = y(t) - y(t-T) - CAE^{-1} \int_{t-h-T}^{t-h} \Psi(y_m)\mathrm{d}m, \\ & \text{and } \mathcal{V}(y_t) = [\mu(t-\tau_1) \quad \ldots \quad \mu(t-\tau_q)] \end{aligned} \tag{11}$$

for values $\tau_1, \ldots, \tau_q$ that satisfy Assumption 2. We also use

$$\begin{aligned} & \hat{\mathcal{G}}(t) = E^{-1}\Psi(y_t) + \sum_{j=1}^{q} \epsilon_j E^{-1}\kappa(\gamma_{j,t}) \ \text{ and} \\ & \hat{x}(t) = \hat{\mathcal{G}}(t) + \hat{z}(t) - E^{-1}\big(\hat{q}(t) - e^{-A^\top \tau}\hat{q}(t-\tau)\big), \end{aligned} \tag{12}$$

where the dynamics for $\hat{z}$ and $\hat{q}$ are

$$\begin{cases} \dot{\hat{z}}(t) &= A\hat{z}(t) + \hat{\Delta}_d(t) \\ \dot{\hat{q}}(t) &= -A^\top \hat{q}(t) + C^\top C\hat{z}(t-h) \end{cases} \tag{13}$$

with the initial conditions $\hat{z}(\ell) = \hat{q}(\ell) = 0$ for all $\ell \leq 0$, and

$$\hat{\Delta}_d(t) = \Delta(\hat{x}(t) + \hat{\xi}(t+h), t) - \Delta(\hat{\xi}(t+h), t). \tag{14}$$

In Section II-D, we will prove that, in terms of the variable $\hat{x}$, the state of (9), and the estimation error $\tilde{x} = x - \hat{x}$, we can construct positive constants $\bar{c}_1$ and $\bar{c}_2$ such that

$$|\tilde{x}(t)| \leq \bar{c}_1 e^{-\bar{c}_2 t} |\tilde{x}|_{[0,\tau+h]} \tag{15}$$

holds for all $t \geq \tau + h$ and all initial states $x(0) \in \mathbb{R}^n$. The bound (15) will play a key role in our observer design.

However, since $\hat{x}$ in (12) contains $\hat{\mathcal{G}}$ and so also the entries of the unknown vector $\epsilon = [\epsilon_1, \ldots \epsilon_q]$, it is not available for measurement. Therefore, we will instead use the estimators

$$\begin{aligned} \hat{x}_0(t) &= E^{-1}\Psi(y_t) \\ & \quad + \kappa^\sharp(t)(M^{-1}(t))^\top [\mathcal{V}(y_t) - C\hat{W}_0^d(t)]^\top \\ & \quad + \hat{z}_0(t) + E^{-1}\left[\hat{q}_0(t) - \hat{q}_0(t-\tau)\right] \\ \hat{\epsilon}(t) &= (\mathcal{V}(y_t) - C\hat{W}_0^d(t))M(t)^{-1}, \end{aligned} \tag{16}$$

where

$$\hat{W}_0^d(t) = [\hat{W}_0(t-\tau_1-h) - \hat{W}_0(t-T-\tau_1-h),\\ \dots, \hat{W}_0(t-\tau_q-h) - \hat{W}_0(t-T-\tau_q-h)] \tag{17}$$

and where we use (6), (11), and the dynamic extensions

$$\begin{cases} \dot{\hat{z}}_0(t) &= A\hat{z}_0(t) + \hat{\Delta}_{d0}(t) \\ \dot{\hat{q}}_0(t) &= -A^\top \hat{q}_0(t) + C^\top C\hat{z}_0(t-h) \\ \dot{\hat{W}}_0(t) &= A[\hat{z}_0(t) - E^{-1}(\hat{q}_0(t) - e^{-A^\top\tau}\hat{q}_0(t-\tau))] \\ & \quad +\hat{\Delta}_{d0}(t), \text{ where} \end{cases} \tag{18}$$

$$\hat{\Delta}_{d0}(t) = \Delta(\hat{x}_0(t) + \hat{\xi}(t+h), t) - \Delta(\hat{\xi}(t+h), t), \tag{19}$$

with the initial conditions $\hat{z}_0(\ell) = \hat{q}_0(\ell) = \hat{W}_0(\ell) = 0$ for all $\ell \leq 0$. In the next subsection, we prove the following, which uses the sup notation $|\cdot|_J$ that we defined in Section I in the special case where the interval is $J = [0, T+\tau+\tau_q+h]$:

Theorem 1: Let Assumptions 1-2 hold. Then the following are true: (i) If Assumption 3 also holds, then we can find constants $c_1 > 0$ and $c_2 > 0$ such that

$$\left|\left(M(t)^\top(\hat{\epsilon}(t)-\epsilon)^\top, \hat{x}_0(t) - x(t)\right)\right| \\ \leq c_1 e^{-c_2 t}|(\tilde{x}, \hat{x}_0 - x)|_{[0, T+\tau+\tau_q+2h]} \tag{20}$$

is satisfied for all $t \geq 2(T+\tau+\tau_q+2h)$ and all initial states $x(0) \in \mathbb{R}^n$ for (9). (ii) If Δ is the zero function, then for all $t \geq \tau_q + T + \tau + h$, the equalities

$$x(t) = E^{-1}\Psi(y_t) + \kappa^\sharp(t)\left(\mathcal{V}(y_t)M(t)^{-1}\right)^\top \\ \text{and } [\epsilon_1 \quad \dots \quad \epsilon_q] = \mathcal{V}(y_t)M(t)^{-1} \tag{21}$$

hold for all initial states $x(0) \in \mathbb{R}^n$ for (9).

Remark 1: Although $\hat{\xi}(t+h)$ appears in (19) and so is needed to compute the estimated value $\hat{x}_0(t)$ of $x(t)$, we saw in Section II-B that only time lagged values $x(t-h)$ of $x(t)$ are needed to meet our goal of computing values of the time lagged state $\xi(t-h) = x(t-h) + \hat{\xi}(t)$ of (1). Hence, no future $\hat{\xi}$ or y values are needed to estimate $\xi(t-h)$ at times t, so our observer design is causal. In particular, (21) gives $\xi(t-h) = E^{-1}\Psi(y_{t-h}) + \kappa^\sharp(t-h)\left(\mathcal{V}(y_{t-h})M(t-h)^{-1}\right)^\top + \hat{\xi}(t)$ for all $t \geq \tau_q + T + \tau + 3h$ when $\Delta = 0$. Also, since $M(t)$ is invertible and known for all $t \geq \tau_q + T + \tau$, $(\hat{\epsilon}(t) - \epsilon)M(t)$ and $\hat{\epsilon}(t) - \epsilon$ contain the same information. We used $(\hat{\epsilon}(t) - \epsilon)M(t)$ in (20) instead of $\hat{\epsilon}(t) - \epsilon$ to ensure existence of c_1 without requiring the function $M_*(t) = M(t)^{-1}$ to be bounded. Our proof of Theorem 1 goes beyond establishing the existence of c_1 and c_2 (which are independent of the state variables), by showing how c_1 and c_2 can be constructed easily in a step by step way from standard small-gain methods and contractivity arguments from [16, Lemma 1]. Due to space constraints, we leave these easy constructions to the reader. □

D. Proof of Theorem 1

The proof has three parts. First, we prove the key error estimate (15) for the error $\tilde{x} = x - \hat{x}$ between values of the state and preliminary estimator, which will also show that

$$x(t) = E^{-1}\Psi(y_t) + \kappa^\sharp(t)\epsilon^\top \tag{22}$$

for all $t \geq \tau + h$ when $\Delta = 0$. In the second part, we use the first part to prove a decay estimate for the error $x - \hat{x}_0$ between the state x of (9) and the state observer $\hat{x}_0$ from (16). In the third part, we find an analogous estimate for the estimation error $\epsilon - \hat{\epsilon}$, which allows us to use small gain arguments to combine the error estimates to finish the proof.

First Part. Applying variation of parameters to (9) on $[s-h, t]$, and left multiplying the result by $e^{A(s-h-t)}$, gives

$$e^{A(s-h-t)}x(t) = \sum_{i=1}^{q} \epsilon_i \int_{s-h}^{t} e^{A(s-h-m)}\gamma_i(m)\mathrm{d}m \\ + x(s-h) + \int_{s-h}^{t} e^{A(s-h-m)}\Delta_d(m)\mathrm{d}m \tag{23}$$

for all $s \geq h$ and $t \geq s-h$. By left multiplying (23) by $e^{A^\top(s-t)}C^\top C$, and then integrating the resulting equality over all $s \in [t-\tau, t]$ when $t \geq \tau$, we obtain

$$Ex(t) = E\hat{\mathcal{G}}(t) \\ + \int_{t-\tau}^{t} e^{A^\top(\ell-t)}C^\top C\left[\int_{\ell-h}^{t} e^{A(\ell-h-m)}\Delta_d(m)\mathrm{d}m\right]\mathrm{d}\ell \tag{24}$$

for all $t \geq \tau + h$, with E defined in (2) and $\hat{\mathcal{G}}$ is from (12).

Recalling the invertibility of E, it follows that

$$x(t) = \hat{\mathcal{G}}(t) \\ + E^{-1}\int_{t-\tau}^{t} e^{A^\top(\ell-t)}C_a\left[\int_{\ell-h}^{t} e^{A(\ell-h-m)}\Delta_d(m)\mathrm{d}m\right]\mathrm{d}\ell \tag{25}$$

for all $t \geq \tau + h$, where $C_a = C^\top C$. When $\Delta = 0$, (25) and our formula (6) give (22) for all $t \geq \tau + h$, since $\Delta_d = 0$ in that case. When Δ is not the zero function, then using variation of parameters, it follows from (12)-(14) that

$$\hat{x}(t) = \hat{\mathcal{G}}(t) \\ + E^{-1}\int_{t-\tau}^{t} e^{A^\top(\ell-t)}C_a\left[\int_{\ell-h}^{t} e^{A(\ell-h-m)}\hat{\Delta}_d(m)\mathrm{d}m\right]\mathrm{d}\ell \tag{26}$$

for all $t \geq \tau + h$, where (26) follows by using variation of parameters and (13) to write the double integral in (26) as

$$\int_{t-\tau}^{t} e^{A^\top(\ell-t)}C_a e^{A(\ell-h-t)}\left[\int_{\ell-h}^{t} e^{A(t-m)}\hat{\Delta}_d(m)\mathrm{d}m\right]\mathrm{d}\ell \\ = \int_{t-\tau}^{t} C^\sharp(\ell-t)e^{-Ah}\left[\hat{z}(t) - e^{A(t-\ell+h)}\hat{z}(\ell-h)\right]\mathrm{d}\ell \\ = E\hat{z}(t) - \int_{t-\tau}^{t} e^{-A^\top(t-\ell)}C^\top C\hat{z}(\ell-h)\mathrm{d}\ell \\ = E\hat{z}(t) - (\hat{q}(t) - e^{-A^\top\tau}\hat{q}(t-\tau)), \tag{27}$$

where $C^\sharp(r) = e^{A^\top r}C^\top C e^{Ar}$. Recalling (10), (14), and our choices of $\bar{\beta}$ in (6) and $\tilde{x} = x - \hat{x}$, it follows from subtracting (26) from (25) that $|\tilde{x}(t)| \leq \rho|\tilde{x}|_{[t-\tau-h, t]}$ for all $t \geq \tau + h$, where $\rho = \bar{\beta}L_\Delta \in (0, 1)$ (because of (7) from Assumption 3). Formulas for constants $\bar{c}_1$ and $\bar{c}_2$ that satisfy (15) then follow from applying [16, Lemma 1] to the function $w(t) = |\tilde{x}(t+\tau+h)|$, which gives $\bar{c}_2 = -\ln(\rho)/(\tau+h)$.

Second Part. We deduce from our choice (12) of $\hat{x}$ and our formula $x(t) = \hat{x}(t) + \tilde{x}(t)$ that

$$x(t) = \tilde{x}(t) + \hat{\mathcal{G}}(t) + \hat{z}(t) - E^{-1}\left(\hat{q}(t) - e^{-A^\top\tau}\hat{q}(t-\tau)\right) \tag{28}$$

for all $t \geq \tau$, so (9) and our choice (12) of $\hat{\mathcal{G}}$ give

$$\dot{x}(t) = AE^{-1}\Psi(y_t) + \sum_{i=1}^{q}\epsilon_i AE^{-1}\kappa(\gamma_{i,t}) \\ + \sum_{i=1}^{q}\epsilon_i\gamma_i(t) + H(t) \tag{29}$$

(by left multiplying (28) by A), where

$$H(t) = A[\tilde{x}(t) + \hat{z}(t) - E^{-1}(\hat{q}(t) - e^{-A^\top \tau}\hat{q}(t-\tau))] + \Delta_d(t). \tag{30}$$

By reorganizing terms in (29), then integrating the result on $[t-T-h, t-h]$ where $T>0$ is the constant in (4), we get

$$\begin{aligned}&\epsilon_1 \int_{t-h-T}^{t-h}[AE^{-1}\kappa(\gamma_{1,m}) + \gamma_1(m)]\mathrm{d}m \\ &+ ... + \epsilon_q \int_{t-T-h}^{t-h}[AE^{-1}\kappa(\gamma_{q,m}) + \gamma_q(m)]\mathrm{d}m \\ &= x(t-h) - x(t-T-h) - AE^{-1}\int_{t-T-h}^{t-h}\Psi(y_m)\mathrm{d}m \\ &- \int_{t-T-h}^{t-h} H(\ell)\mathrm{d}\ell\end{aligned} \tag{31}$$

for all $t \geq T+h$. By left multiplying (31) by C, we obtain

$$\begin{aligned}&[\epsilon_1 \quad ... \quad \epsilon_q]W(t) = y(t) - y(t-T) \\ &-CAE^{-1}\int_{t-T-h}^{t-h}\Psi(y_m)\mathrm{d}m - C\int_{t-T-h}^{t-h}H(\ell)\mathrm{d}\ell,\end{aligned} \tag{32}$$

with W as defined in (4). By replacing t in (32) by the values $t-\tau_j$ for $j = 1, 2, \ldots, q$, and then collecting the resulting q equalities into one row vector, we deduce that

$$\begin{aligned}&[\epsilon_1 \quad ... \quad \epsilon_q]M(t) = \mathcal{V}(y_t) \\ &-C\left[\int_{t-T-\tau_1-h}^{t-\tau_1-h}H(\ell)\mathrm{d}\ell, \ldots, \int_{t-T-\tau_q-h}^{t-\tau_q-h}H(\ell)\mathrm{d}\ell\right]\end{aligned} \tag{33}$$

when $t \geq T+h+\tau_q$, where M and $\mathcal{V}$ are from (5) and (11). By Assumption 2, we right multiply (33) by $M(t)^{-1}$ to get

$$\begin{aligned}&[\epsilon_1 \quad ... \quad \epsilon_q] = \mathcal{V}(y_t)M(t)^{-1} \\ &-C\left[\int_{t-T-\tau_1-h}^{t-\tau_1-h}H(\ell)\mathrm{d}\ell, .., \int_{t-T-\tau_q-h}^{t-\tau_q-h}H(\ell)\mathrm{d}\ell\right]M(t)^{-1}\end{aligned} \tag{34}$$

which is the second equality of (21) (which we can combine with (22) to get (21)) when Δ is the zero function, because (25)-(26) give $\hat{q}(t) = \hat{q}(t-\tau) = \hat{z}(t) = \tilde{x}(t) = x(t) - \hat{x}(t) = 0$ and so also $H(t) = 0$ for all $t \geq \tau + h$ when Δ is the zero function. Hence, we can assume that Δ is not the zero function, and therefore, that Assumption 3 is satisfied in the remainder of the proof.

Then, our formulas from (16) and (30), and the argument that led to (27) (with $\hat{\Delta}_d$ replaced by $\hat{\Delta}_{d0}$), give

$$\begin{aligned}&H(t) = A\tilde{x}(t) + \Delta_d(t) + \\ &AE^{-1}\int_{t-\tau}^{t}C^\sharp(\ell-t)\left[\int_{\ell-h}^{t}e^{A(t-h-m)}\hat{\Delta}_d(m)\mathrm{d}m\right]\mathrm{d}\ell\end{aligned} \tag{35}$$

$$\begin{aligned}&\text{and } [\hat{\epsilon}_1(t) \quad ... \quad \hat{\epsilon}_q(t)] = \mathcal{V}(y_t)M(t)^{-1} \\ &-C\left[\int_{t-T-\tau_1-h}^{t-\tau_1-h}\hat{H}(\ell)\mathrm{d}\ell, .., \int_{t-T-\tau_q-h}^{t-\tau_q-h}\hat{H}(\ell)\mathrm{d}\ell\right]M(t)^{-1}\end{aligned} \tag{36}$$

and so also

$$\begin{aligned}\hat{x}_0(t) = &\Bigg\{E^{-1}\Psi(y_t) + \kappa^\sharp(t)(M^{-1}(t))^\top\Bigg[\mathcal{V}(y_t) \\ &- \left[\int_{t-T-\tau_1-h}^{t-\tau_1-h}C\hat{H}(\ell)\mathrm{d}\ell, .., \int_{t-T-\tau_q-h}^{t-\tau_q-h}C\hat{H}(\ell)\mathrm{d}\ell\right]\Bigg]^\top\Bigg\} \\ &+ E^{-1}\int_{t-\tau}^{t}C^\sharp(\ell-t)\left[\int_{\ell-h}^{t}e^{A(t-h-m)}\hat{\Delta}_{d0}(m)\mathrm{d}m\right]\mathrm{d}\ell\end{aligned} \tag{37}$$

for all $t \geq \tau + h + \tau_q$, where $\kappa^\sharp$ is defined in (6) and

$$\begin{aligned}&\hat{H}(t) = \hat{\Delta}_{d0}(t) + \\ &AE^{-1}\int_{t-\tau}^{t}C^\sharp(\ell-t)\left[\int_{\ell-h}^{t}e^{A(t-h-m)}\hat{\Delta}_{d0}(m)\mathrm{d}m\right]\mathrm{d}\ell\end{aligned} \tag{38}$$

and $\hat{\Delta}_{d0}$ is defined in (19). Also, formulas (12) and (34) give

$$\begin{aligned}\hat{\mathcal{G}}(t) = &E^{-1}\Psi(y_t) + \kappa^\sharp(t)(M(t)^{-1})^\top\Bigg[\mathcal{V}(y_t) \\ &- \left[\int_{t-T-\tau_1-h}^{t-\tau_1-h}CH(\ell)\mathrm{d}\ell, ..., \int_{t-T-\tau_q-h}^{t-\tau_q-h}CH(\ell)\mathrm{d}\ell\right]\Bigg]^\top,\end{aligned} \tag{39}$$

and (35) and (38) give

$$\begin{aligned}&|CH(\ell) - C\hat{H}(\ell)| \leq L_\Delta|C||x(\ell) - \hat{x}_0(\ell)| \\ &+ |CA||\tilde{x}(\ell)| + L_\Delta\bar{\beta}|CA||\hat{x} - \hat{x}_0|_{[\ell-\tau-h,\ell]}\end{aligned} \tag{40}$$

for all $\ell \in [t-T-\tau_q-h, t-\tau_1]$ and $t \geq T+\tau_q+h$, where the last term on the right side in (40) was used to upper bound the norm of the difference between the double integrals in (35) and (38), and was obtained by using the formulas for $\hat{\Delta}_d$ and $\hat{\Delta}_{\mathrm{do}}$ from (14) and (19), and the formula (6) for $\bar{\beta}$.

Hence, since our choice $\tilde{x} = x - \hat{x}$ and the triangle inequality give $|\hat{x}(\ell) - \hat{x}_0(\ell)| \leq |x(\ell) - \hat{x}_0(\ell)| + |\tilde{x}(\ell)|$ for all $\ell \in [t-T-\tau_q-h, t-\tau_1]$, our formulas (25), (35), and (37) and the choice $J = [\tau_q + T + \tau, \infty)$ give

$$\begin{aligned}&|x(t) - \hat{x}_0(t)| \leq \bar{\beta}L_\Delta|x - \hat{x}_0|_{[t-T-\tau_q-\tau-h,t]} \\ &+ \sqrt{q}|\kappa_*|_J|CA|\bar{\beta}TL_\Delta|x - \hat{x}_0|_{[t-T-\tau_q-\tau-2h,t]} \\ &+ \sqrt{q}T|\kappa_*|_J|CA|(1+\bar{\beta}L_\Delta)|\tilde{x}|_{[t-T-\tau_q-\tau-2h,t]} \\ &+ \sqrt{q}T|\kappa_*|_J|C|L_\Delta|x - \hat{x}_0|_{[t-T-\tau_q-\tau-h,t]}\end{aligned} \tag{41}$$

for all $t \geq T + \tau + \tau_q + 2h$, where the function κ_* is defined in Assumption 3, and where the last three terms in the sum on the right side of (41) came from upper bounding the difference between $\hat{\mathcal{G}}(t)$ in (25) and the terms in curly braces in (37) (by substituting (39) into (25) and using (40)), and where the first term in (41) came from upper bounding the difference between the double integrals in (25) and (37).

Therefore, with the choices

$$\bar{\beta}_1^\sharp = L_\Delta\left(\bar{\beta}[\sqrt{q}|\kappa_*|_J|CA|T+1] + |\kappa_*|_J|C|\sqrt{q}T\right) \tag{42}$$

and $\bar{\beta}_2^\sharp = \sqrt{q}T|\kappa_*|_J|CA|(1+\bar{\beta}L_\Delta)$, we get $|x(t) - \hat{x}_0(t)| \leq \bar{\beta}_1^\sharp|x - \hat{x}_0|_{[t-T-\tau_q-\tau-2h,t]} + \bar{\beta}_2^\sharp|\tilde{x}|_{[t-T-\tau_q-\tau-2h,t]}$ for all $t \geq T + \tau + \tau_q + 2h$. Since Assumption 3 gives $\bar{\beta}_1^\sharp \in (0,1)$, we can reapply [16, Lemma 1] (using $w(t) = |x(t+T+\tau+\tau_q+2h) - \hat{x}_0(t+T+\tau+\tau_q+2h)|$ in the statement of [16, Lemma 1]) to find positive constants $\bar{c}_3$, $\bar{c}_4$, and $\bar{c}_5$ such that

$$|x(t) - \hat{x}_0(t)| \leq \bar{c}_3e^{-\bar{c}_4t}|x - \hat{x}_0|_{[0,T+\tau+\tau_q+2h]} + \bar{c}_5|\tilde{x}|_{[0,t]} \tag{43}$$

for all $t \geq \tau + T + \tau_q + 2h$, with $\bar{c}_4 = -\ln(\bar{\beta}_1^\sharp)/(T + \tau + \tau_q + 2h)$.

Third Part. By (15) and (43) and using standard small-gain arguments, we obtain positive constants $\bar{c}_6$ and $\bar{c}_7$ such that

$$|(\tilde{x}(t), \hat{x}_0(t) - x(t))| \leq \bar{c}_6e^{-\bar{c}_7t}|(\tilde{x}, \hat{x}_0 - x)|_{[0,T+\tau+\tau_q+2h]} \tag{44}$$

for all $t \geq T + \tau + \tau_q + 2h$. Also, using (35) and (38) and (40), we can use our formulas for ϵ and $\hat{\epsilon}$ in (34) and (36) to find a constant $\bar{c}_8 > 0$ such that

$$|(\hat{\epsilon}(t) - \epsilon)M(t)| \leq \bar{c}_8|(\tilde{x}, \hat{x}_0 - x)|_{[t-T-\tau-\tau_q-2h,t]} \tag{45}$$

for all $t \geq T + \tau + \tau_q + 2h$. Hence, the formulas for c_1 and c_2 follow from taking the maximum on both sides of (44) to upper bound $|(\hat{\epsilon}(t) - \epsilon)M(t)|$ for $t \geq 2(T + \tau + \tau_q + 2h)$, and then adding the result to (44).

Remark 2: Our proof of Theorem 1 shows that $c_2 \to +\infty$ as $L_\Delta \to 0$ (since $-\ln(\bar{\beta}_1^\sharp) \to +\infty$ as $L_\Delta \to 0$), which is the almost-finite time property. By linear growth properties of the dynamics and Gronwall's inequality, we can bound the supremum on the right side of (20) by a multiple of $|x|_{[-h,0]} + |\epsilon|$. In practice, bounds on $x(0)$ and ϵ are known. Hence, we get upper bounds on the left side of (20) in terms of $|x|_{[-h,0]} + |\epsilon|$ for all $t \geq 2(T+\tau+\tau_q+2h)$. □

Remark 3: We can replace the requirement that κ_* is bounded and (7) by the two requirements that $\kappa^\sharp$ (as defined in (6)) is bounded and $L_\Delta \bar{\beta} < 1$, and then we instead obtain an input-to-state stability (or ISS) conclusion with respect to the approximation error $\hat{\epsilon}(t) - \epsilon$. In fact, with these replacements, the part of the proof of the theorem through (37) except with the terms in curly braces in (37) replaced by $E^{-1}\Psi(y_t)+\hat{\epsilon}_1(t)E^{-1}\kappa(\gamma_{1,t})+\ldots+\hat{\epsilon}_q(t)E^{-1}\kappa(\gamma_{q,t})$ provides positive constants c_1, c_2, and c_3 such that

$$\begin{aligned}&|\hat{x}_0(t) - x(t)| \leq \\ &c_1 e^{-c_2 t}|(\hat{x}_0 - x)|_{[0,T+\tau+\tau_q+2h]} + c_3|\epsilon - \hat{\epsilon}|_{[0,t]}\end{aligned} \tag{46}$$

for all $t \geq T + \tau + \tau_q + 2h$. Also, Theorem 1 remains true by a similar proof if we replace the estimator $\hat{\epsilon}$ from (16) by $\hat{\epsilon}(t) = \mathrm{sat}_{\bar{\epsilon},\underline{\epsilon}}\{(\mathcal{V}(y_t) - C\hat{W}_0^d(t))M(t)^{-1}\}$, where the saturation $\mathrm{sat}_{\bar{\epsilon},\underline{\epsilon}}\{v\}$ has the ith entry $\mathrm{sat}_{\bar{\epsilon}_i,\underline{\epsilon}_i}(v_i) = \max\{\underline{\epsilon}_i, \min\{v_i, \bar{\epsilon}_i\}\}$ for $v \in \mathbb{R}^{1\times q}$ and $i = 1,\ldots,q$ and using known bounds $\epsilon_i \in [-\underline{\epsilon}_i, \bar{\epsilon}_i]$ on the ϵ_i's. This means that smaller known intervals $[-\underline{\epsilon}_i, \bar{\epsilon}_i]$ containing the ϵ_i's produce smaller overshoots in the ISS estimate (46). □

III. Implementations

The integrals in (4) and (11) can be replaced by states of dynamical extensions consisting of ordinary differential equations with constant delays that are easily solved numerically, e.g., using `NDSolve` in Mathematica. This replacement can facilitate checking our assumptions, while also eliminating integrals from our expressions for the observer values to facilitate implementing our observer. To see how this can be done, we first show that for $i = 1,\ldots,q$, we have $W_i(t) = V_i(t) + W_i(0)$, where V_i is the first component of

$$\left\{\begin{aligned}\dot{V}_i(t) \;=\;& C[AE^{-1}(EL_{1i}(t-h) - L_{2i}(t-h) \\ &+ e^{-A^\top\tau}L_{2i}(t-\tau-h)) + \gamma_i(t-h) \\ &- AE^{-1}(EL_{1i}(t-T-h) - L_{2i}(t-T-h) \\ &+ e^{-A^\top\tau}L_{2i}(t-T-\tau-h)) \\ &-\gamma_i(t-T-h)] \\ \dot{L}_{1i}(t) \;=\;& AL_{1i}(t) + \gamma_i(t) \\ \dot{L}_{2i}(t) \;=\;& -A^\top L_{2i}(t) + C^\top CL_{1i}(t-h)\end{aligned}\right. \tag{47}$$

with the initial states $V_i(0) = 0$ and $L_{1i}(s) = L_{2i}(s) = 0$ for all $s \in [-T-\tau-h, 0]$. To this end, first note that

$$\begin{aligned}&\kappa_i(\gamma_{i,t}) \\ &= \int_{t-\tau}^t C^\sharp(r-t)e^{-Ah}\left[\int_{r-h}^t e^{A(t-m)}\gamma_i(m)\mathrm{d}m\right]\mathrm{d}r \\ &= \int_{t-\tau}^t C^\sharp(r-t)e^{-Ah}\left[L_{1i}(t) - e^{(t-r+h)A}L_{1i}(r-h)\right]\mathrm{d}r \\ &= EL_{1i}(t) - \int_{t-\tau}^t e^{-A^\top(t-r)}C^\top CL_{1i}(r-h)\mathrm{d}r \\ &= EL_{1i}(t) - (L_{2i}(t) - e^{-A^\top\tau}L_{2i}(t-\tau))\end{aligned} \tag{48}$$

for each i and $t \geq 0$, where $C^\sharp(r) = e^{A^\top r}C^\top Ce^{Ar}$, and the second and last equalities in (48) follow by applying variation of parameters to the L_{1i} and then the L_{2i} dynamics. Hence, (4) gives $\dot{W}_i = \dot{V}_i$ for all i, so the Fundamental Theorem of Calculus gives $W_i(t) = V_i(t) + W_i(0)$. This reduces the calculations of $W_i(t)$ to computing $W_i(0)$, which can be done easily, e.g., using `Simplify` in Mathematica to simplify the integrands in the $W_i(0)$ formulas before integrating. Then we can check that $M(t)$ is nonsingular, by checking that the absolute value of the determinant of $M(t)$ has a positive infimum over all $t \in [\tau_q + T + \tau, +\infty)$.

Similar arguments give $\Psi(y_t) = L_3(t) - e^{-A^\top\tau}L_3(t-\tau)$, and $\mu(t) = y(t) - y(t-T) - CAE^{-1}(L_4(t-h) - L_4(t-T-h))$, where L_3 and L_4 are solutions of $\dot{L}_3(t) = -A^\top L_3(t) + C^\top y(t)$ and $\dot{L}_4(t) = L_3(t) - e^{-A^\top\tau}L_3(t-\tau)$ with initial functions 0, by applying the method of variation of parameters to the L_3 dynamics to eliminate the integral from the Ψ formula in (11), and then using the result to eliminate the integral in the formula for μ in (11).

IV. Illustrations

We revisit the main example from [4], by showing how Theorem 1 makes it possible to identify the weighting coefficients ϵ_i in an unknown uncertainty that was not present in [4]. Then, we provide a generalization of the example from [13] that allows measurement delays and nonzero Δ's.

A. Motor Example

In [4], the dynamics of a single-link robotic manipulator coupled to a DC motor with a nonrigid joint after a change in coordinates produced (1) with

$$\begin{aligned}&A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{K}{J_1} & -\frac{F_1}{J_1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{K^2}{N^2 J_1 J_2} & 0 & -\frac{K}{N^2 J_2} & -\frac{F_2}{J_2} & 1 \\ 0 & 0 & 0 & -\frac{K_b K_t}{J_2 \mathcal{L}} & -\frac{R}{\mathcal{L}} \end{bmatrix}, \\ &\varphi(C\xi, u) = \begin{bmatrix} 0 \\ \frac{mg_* d}{J_1}[1 - \cos(\xi_1)] \\ 0 \\ 0 \\ \frac{K_t K}{J_1 J_2 N \mathcal{L}}u \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top\end{aligned} \tag{49}$$

and each γ_i and Δ being the zero function, where the positive constants J_1 and J_2 are the inertias, F_1 and F_2 are positive viscous friction constants, K is the positive spring constant, K_t is the positive torque constant, K_b is the positive back EMF constant, R and $\mathcal{L}$ are the armature resistance and inductance respectively and are positive constants, m is the constant positive link mass, the positive constant d is the position of the link's center of gravity, $N > 0$ is the positive gear ratio, and $g_* > 0$ is the constant gravity acceleration.

However, here we study a more general situation for the preceding dynamics where there is also an added uncertainty $\delta(t) = \epsilon_1\gamma_1(t) + \ldots + \epsilon_q\gamma_q(t)$ on its right side. Here, we take $q = 3$ and each entry of $\gamma_i : \mathbb{R} \to \mathbb{R}^5$ to be $e^{-(t-\lambda_i)^2}/(2w_i^2)$ for positive constants λ_i and w_i for $i = 1,2,3$ we will specify, which are commonly used in basis functions in neural network expansions, but similar reasoning applies for other functions γ_i. Since $\Delta = 0$, it suffices to check Assumptions 1-2, with $n = 5$ and $q = 3$. These assumptions hold for many choices of the parameters, except with the invertibility condition from Assumption 2 only holding for $t \in [\tau_q + T + \tau, T_*)$ for a large enough $T_* > 0$ that still allows us to conclude the fixed time convergence at time $t = \tau_q + T + \tau + h$. For instance, choosing all constants in A in (49) to be 1 as in [4], $\lambda_i = w_i = i$ for $i = 1, 2, 3$ in the γ_i's, and $\tau = T = 1$ and $\tau_1 = 0$, $\tau_2 = 0.2$, and $\tau_3 = 0.75$, we used Mathematica to check that the assumptions hold for all $h \in [0, 0.2]$. In our examples, we used Mathematica mainly to check that $M(t)$ is nonsingular when $t \geq \tau_q + T + \tau$, by checking that the absolute value of the determinant of $M(t)$ has a positive infimum on $[\tau_q + T + \tau, T_*)$. This extends [4] to more realistic cases with measurement delays and additive uncertainties whose weights ϵ_i and states we identify in fixed time.

B. Nonzero Nonlinearity Δ

Next we consider the system (1) with $n = 2$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } C = [1 \quad 0] \tag{50}$$

with $\varphi = 0$ and $q = 1$. Moreover, we assume that $\gamma_1(\ell) = [0, 1]^\top$ for all ℓ. Then Assumption 1 is satisfied. Let us check that Assumption 2 is satisfied as well, and then we find conditions on the uniform Lipschitz constant L_Δ on the additive state-dependent uncertainty Δ on the right side of the dynamics such that Assumption 3 is satisfied.

Recalling that

$$e^{A\ell} = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \tag{51}$$

for all $\ell \in \mathbb{R}$, simple calculations show that (51) gives

$$\kappa(\gamma_{1,t}) = \begin{bmatrix} -\frac{h\tau^2}{2} - \frac{h^2\tau}{2} - \frac{\tau^3}{6} \\ \frac{\tau^4}{8} + \frac{h\tau^3}{3} + \frac{h^2\tau^2}{4} \end{bmatrix}, \tag{52}$$

by our formula (3) for κ. It then readily follows that

$$\begin{aligned} &\textstyle\int_{t-T-h}^{t-h} C[AE^{-1}\kappa(\gamma_{1,m}) + \gamma_1(m)]\mathrm{d}m \\ &= T\left(\tfrac{1}{2}\tau + h\right) \neq 0. \end{aligned} \tag{53}$$

Thus Assumption 2 holds for any $T > 0$, $h > 0$, and $\tau > 0$ and $\tau_1 = 0$. Also, with the preceding parameters, $T = 2$, and $\tau = 0.5$, and with $h = 0$, we used the Mathematica program to check that (7) is equivalent to $L_\Delta < 0.44619$. If instead $h = 0.125$ and the other parameters are kept the same, then we found that the bound on L_Δ from (7) is $L_\Delta < 0.295463$. Finally, with $h = 0.25$ and all of the other parameters kept as before, we found that our condition (7) is $L_\Delta < 0.212663$. This illustrates a trade-off we observed in numerical experiments, where larger h's produced smaller bounds on the allowable growth rates L_Δ on Δ.

Remark 4: In the preceding examples, the function $M(t)$ from (5) was invertible for all $t \geq \tau_q + T + \tau$, so the conclusions of Theorem 1 hold for all $t \geq 2(T + \tau + \tau_q + 2h)$. However, as in [13], our approach is also valid when $M(t)$ is only invertible on $[\tau_q + T + \tau, \bar{t}]$ for a value $\bar{t}$, in which case the conclusions of our theorem only hold for $t \in [2(\tau_q + T + \tau + 2h), \bar{t}]$.

V. Conclusions

We provided observers for unknown parameters and unknown states. Our assumptions cover artificial neural network expansions. When the unmeasured state enters affinely, our observers are fixed time converging. This affineness holds for a broad class of dynamics (e.g., motor dynamics, and vibrating membranes) [1]–[4]. When the nonlinearity depends on the unmeasured state, we proved an analog of the almost finite-time work [6]. This identified model parameters, which was not possible using [6]. Our examples included a significant motor application, and illustrated a trade-off between the growth rate of the state dependent nonlinearity and the allowable measurement delays, which we aim to relax to compensate for arbitrarily long measurement delays. We also aim to find reduced order versions, to build on [3], which did not identify model parameters. Our other goals include multi-output versions, analogs with measurement noise, and applications to models of drones whose wind disturbances are weighted sums of known functions with unknown weights.

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