Strict Lyapunov Function Constructions under LaSalle Conditions with an Application to Lotka-Volterra Systems

Michael Malisoff
LSU Department of Mathematics
malisoff@lsu.edu

Joint with Frédéric Mazenc

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Converse Lyapunov theory guarantees the *existence* of strict Lyapunov functions in many cases.

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Using LaSalle Invariance, we can often use nonstrict Lyapunov functions to prove stability.

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However, explicit strict Lyapunov function *constructions* are often needed in applications.

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ISS is defined using comparison functions.

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A modulus with respect to $\mathcal{X}$ is any continuous positive definite function $\alpha : \mathcal{X} \to [0, \infty)$ such that $\alpha(\zeta) \to +\infty$ as ζ approaches the boundary of $\mathcal{X}$, or as $|\zeta| \to \infty$ with ζ remaining in $\mathcal{X}$ (the latter possibility being ruled out if $\mathcal{X}$ is bounded).

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A continuous function $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is of class $\mathcal{KL}$ provided $\beta(\cdot, t) \in \mathcal{K}_\infty$ for all $t \geq 0$ and $\beta(s, \cdot)$ is decreasing and asymptotically approaches zero for each $s \geq 0$.

ISS Motivation-Part 2/3

We say that (1) is ISS provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ and a modulus $\bar{\alpha}$ with respect to $\mathcal{X}$ s.t. for all initial conditions $x(t_0) = x_0 \in \mathcal{X}$ and all disturbances d, the corresponding trajectories $t \mapsto \zeta(t; t_0, x_0, d)$ satisfy

$$|\zeta(t; t_0, x_0, d)| \leq \beta\Big(\bar{\alpha}(x_0), t - t_0\Big) + \gamma(|d|_\infty) \ \ \forall t \geq t_0 \ . \qquad (2)$$

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The special case where γ and d are not present is UGAS.

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$$\dot{x} = f(t,x) + g(t,x)\Big[K(t,x) - \partial_x V(t,x)g(t,x) + d\Big] \quad (4)$$

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Need $\partial_x V(t,x)g(t,x)$.

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$$\exists N_* > 0 \text{ s.t. } \forall q \in \mathbb{R}^n \setminus \{0\},\ \exists i \in [1, N_*] \text{ s.t. } L_f^i V(q) \neq 0. \quad \text{(NDC)}$$

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Answer: Yes. (Mazenc-Nesic, IEEE T-AC, 2004).

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Objective: Find a simpler construction that also applies to t-v systems, and that has a much less restrictive NDC on V.

First Construction

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Let $V \in C^{\infty}$ be a nonstrict Lyapunov function for $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$, with f and V having period T in t.

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$a_1 = -\dot{V}$ and $a_{i+1} = -\dot{a}_i$ for $i \geq 1$.
$A_j(t,x) = \sum_{m=1}^{j} a_{m+1}(t,x)a_m(t,x)$.

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Theorem 1

Assume $\exists$ constants $\tau \in (0,T]$ and $\ell \in \mathbb{N}$ and a positive definite continuous function ρ such that for all $x \in \mathbb{R}^n$ and all $t \in [0,\tau]$,

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Then we can explicitly determine functions $\mathcal{F}_j$ and $\mathcal{G}$ such that

$$V^\sharp(t,x) = \sum_{j=1}^{\ell-1} \mathcal{F}_j\big(V(t,x)\big)A_j(t,x) + \mathcal{G}\big(t, V(t,x)\big) \tag{6}$$

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Significance: The construction can be done locally near the origin, when the assumptions only hold near the origin.

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Significance: The function (6) is a simple weighted sum involving the easily calculated iterated Lie derivatives a_i.

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Significance: Simpler than Mazenc-Nesic-TAC'04, which is limited to time invariant systems under a stronger NDC.

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$$a_1(t,x) + a_2^2(t,x) + a_3^2(t,x) \;\; \geq \;\; \frac{4\cos^4(t)}{200(V(x)+1)} V^2(x) \, .$$

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Hence, (5) holds with $\tau = \frac{\pi}{4}$ and $\rho(r) = r^2/\{200(r+1)\}$.

Idea of Proof of Thm 1, Part 1/3

Let $\Gamma \in C^1$ be any everywhere positive increasing function s.t.

$$\Gamma(V(t,x)) \geq (\ell + 2)|a_m(t,x)| + 1$$

for all $m \in \{1, ..., \ell + 1\}$ and all $(t,x) \in [0,\infty) \times \mathbb{R}^n$.

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Pick $\omega \in \mathcal{K}_\infty \cap C^1$ and the strictly increasing everywhere positive function $K \in C^1$ such that

$$\rho(r) \geq \frac{\omega(r)}{K(r)} \quad \forall r \geq 0 \ . \tag{8}$$

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$$k_{\ell-1}(v) = \omega^{2^{\ell-1}}(v) \text{ and } k_p(v) = k_{\ell-1}(v)\Omega^{1-2^{\ell-p-1}}(v)$$

for $1 \leq p \leq \ell - 2$

Idea of Proof of Thm 1, Part 2/3

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$$k_{\ell-1}(v) = \omega^{2^{\ell-1}}(v) \text{ and } k_p(v) = k_{\ell-1}(v)\Omega^{1-2^{\ell-p-1}}(v)$$
$$\text{for } 1 \le p \le \ell-2, \text{ where } \Omega(v) = \frac{2\tau\omega(v)}{3T(\ell-2)\Gamma^2(v)K(v)} \tag{9}$$

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and

$$M_p(t,x) = \sum_{m=1}^{p} a_{m+1}(t,x)a_m(t,x) + \int_0^{V(t,x)} \Gamma(r)\mathrm{d}r. \tag{10}$$

Idea of Proof of Thm 1, Part 2/3

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Let k_0 be any C^1 increasing function such that

$$k_0\big(V(t,x)\big) + k_0'\big(V(t,x)\big)\,V(t,x) \;\geq\; \sum_{p=1}^{\ell-1} \big|k_p'\big(V(t,x)\big)\big|\,|M_p(t,x)| + 1 \tag{11}$$

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and $q : \mathbb{R} \to [0,1]$ be any continuous function with period T s.t. $q(t) = 0$ for all $t \in [\tau, T]$ and $q(t) = 1$ for all $t \in [\frac{\tau}{3}, \frac{2\tau}{3}]$.

Idea of Proof of Thm 1, Part 2/3

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Let G be any C^1 function such that

$$G'(v) \;\geq\; T\left|k_{\ell-1}(v)\frac{\omega'(v)K(v)-\omega(v)K'(v)}{K^2(v)}+k'_{\ell-1}(v)\frac{\omega(v)}{K(v)}\right|$$

for all $v \geq 0$.

Idea of Proof of Thm 1, Part 3/3

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and $\kappa \in C^1$ is any increasing function such that $\kappa(V(t,x)) \geq |S_3(t,x)| + 1$ everywhere.

Second Construction for $\dot{x} = f(x)$, $x \in \mathcal{X}$

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Assumptions 1

There exist a storage function $V_1 : \mathcal{X} \to [0, \infty)$*; functions* $h_1, \ldots, h_m$ *such that* $h_j(0) = 0$ *for all* j*; everywhere positive functions* $r_1, \ldots, r_m$ *and* ρ*; and an integer* $N > 0$ *for which*

$$\nabla V_1(x) f(x) \;\leq\; -r_1(x) h_1^2(x) - \ldots - r_m(x) h_m^2(x) \;\; \forall x \in \mathcal{X} \quad (12)$$

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Also, $f \in C^\infty(\mathbb{R}^n)$*, and* V_1 *has a positive definite quadratic lower bound in some neighborhood of* $0 \in \mathbb{R}^n$*.*

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One can determine explicit functions $k_\ell, \Omega_\ell \in \mathcal{K}_\infty \cap C^1$ such that

$$S(x) = \sum_{\ell=1}^{N} \Omega_\ell \Big(k_\ell(V_1(x)) + V_\ell(x) \Big) \tag{15}$$

is a strict Lyapunov function on $\mathcal{X}$ satisfying $S(x) \geq V_1(x)$ on $\mathcal{X}$.

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Significance: New theorem says which functions V_i to pick.

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Significance: Allows any open state space $\mathcal{X}$ containing $0 \in \mathbb{R}^n$.

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Significance: Readily extends to time periodic t-v systems.

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Find everywhere positive C^1 increasing ϕ_1 and p_1 s.t.

$$\nabla V_i(x) f(x) \ \le \ -\mathcal{N}_i(x) + \phi_1\big(V_1(x)\big)\sqrt{\mathcal{N}_{i-1}(x)}\sqrt{V_1(x)} \quad (16)$$

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Finally, find a continuous everywhere positive $\tilde{\rho}$ so that

$$\sum_{i=1}^{N} \mathcal{N}_i(x) \ \geq \ \tilde{\rho}(V_1(x))V_1(x) \tag{18}$$

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$\Omega_N(r) = r$, and $\{\Omega_i\}_{i=1}^{N-1}$ satisfy

$$\Omega_i'(U_i) \geq (N-1)^2 \frac{8\phi_1^2(V_1)}{\tilde{\rho}(V_1)} \sum_{r=1+i}^{N} \Omega_r'(U_r)^2, \tag{21}$$

with $\Omega_i' : [0, \infty) \to [1, \infty)$ continuous and increasing for each i.

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Assume $\alpha > d$. Want a global strict Lyapunov function for (23).

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Nonstrict Lyapunov decay condition:

Use of Theorem 2

There are many Lyapunov constructions for Lotka-Volterra models available based on computing the LaSalle invariant set.

Our result is original and significant because we provide a *global strict Lyapunov function*.

$$V_1(\tilde{x},\tilde{y}) = \tilde{x} - x_* \ln\left(1+\frac{\tilde{x}}{x_*}\right) + \tilde{y} - y_* \ln\left(1+\frac{\tilde{y}}{y_*}\right) \qquad (25)$$

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Along the trajectories of the L-V error dynamics,

$$\dot{S} \le -\frac{1}{4}\left[\tilde{x}^2 + \left\{(\tilde{x}+\alpha\tilde{y})(\tilde{x}+x_*)\right\}^2\right]. \tag{27}$$

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Our strict Lyapunov function constructions are valid on general state spaces and have desirable local properties.

For example, their time derivatives are frequently bounded above by negative definite quadratic functions of the state.

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Further Reading on Strictification

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