Tracking Control for Neuromuscular Electrical Stimulation

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Problem and Our Solution

NMES artificially stimulates skeletal muscles to restore function in human limbs (Crago, Jezernik, Koo-Leonessa, Levy-Mizrahi...). It entails voltage excitation of skin or implanted electrodes to produce muscle contraction, joint torque, and motion. Delays in muscle response come from finite propagation of chemical ions, synaptic transmission delays, and other causes. Delay compensating controllers have realized some tracking objectives including use on humans (Dixon, Sharma, 2011...). Our new control only needs sampled observations, allows any delay, and tracks position and velocity under a state constraint.
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Our new control only needs sampled observations, allows any delay, and tracks position and velocity under a state constraint.
What are Delayed Control Systems?

These are doubly parameterized families of ODEs of the form

\[ Y'(t) = F(t, Y(t), u(t, Y(t-\tau)), \delta(t)) \]

where \( Y \subseteq \mathbb{R}^n \).

We have freedom to choose the control function \( u \).

The functions \( \delta: [0, \infty) \rightarrow D \) represent uncertainty.

Typically we construct \( u \) such that all trajectories of (2) for all possible choices of \( \delta \) satisfy some control objective.

Karafyllis (NTUA), Krstic (UCSD), Malisoff (LSU), et al.

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\[ Y_t(\theta) = Y(t + \theta). \] Specify \( u \) to get a singly parameterized family

\[ Y'(t) = \mathcal{G}(t, Y_t, \delta(t)), \quad Y(t) \in \mathcal{Y}, \quad (2) \]

where \( \mathcal{G}(t, Y_t, \delta) = \mathcal{F}(t, Y(t), u(t, Y(t - \tau)), \delta). \)
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where \( G(t, Y_t, d) = F(t, Y(t), u(t, Y(t - \tau)), d) \).

Typically we construct \( u \) such that all trajectories of (2) for all possible choices of \( \delta \) satisfy some control objective.
What is One Possible Control Objective?

Input-to-state stability generalizes global asymptotic stability:

\[ \dot{Y}(t) = G(t, Y(t)), \quad Y(t) \in Y. \]

\[ \left| Y(t) \right| \leq \gamma_1 \left( e^{t_0 - t} \gamma_2 \left( \left| Y(t_0) \right| [\tau, 0] \right) \right) \] (UGAS)

Our \( \gamma_i \)'s are 0 at 0, strictly increasing, and unbounded. \( \gamma_i \in \mathbb{K}_\infty \).

Tracking Error:

\[ Y(t) = s(t) - s_r(t). \]

\[ s(t) = \text{state.} \]

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Find \( \gamma_i \)'s by building certain LKFs for \( \dot{Y}(t) = G(t, Y(t), 0) \).
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Find \( \gamma_i \)'s by building certain LKFs for \( Y'(t) = G(t, Y_t, 0) \).
Leg extension machine at Warren Dixon’s NCR Lab at U of FL
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Knee Joint Dynamics (Sharma et al, 2009)

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\[ M_I (\ddot{q}) = J\ddot{q}: \text{inertial effects of shank-foot complex about the knee-joint.} \quad J = \text{inertia of the combined shank and foot.} \]
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\( M_I (\ddot{q}) = J\ddot{q} \): **inertial effects** of shank-foot complex about the knee-joint. \( J \) = inertia of the combined shank and foot.

\( M_v (\dot{q}) = b_1 \dot{q} + b_2 \tanh (b_3 \dot{q}) \): **viscous effects** due to damping in the musculotendon complex, with constants \( b_i > 0 \).
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\[ M_e (q) = k_1 q e^{-k_2 q} + k_3 \tan (q): \text{elastic effects due to joint stiffness with constants } k_i > 0. \text{ We introduce the tan term to accommodate our state constraint } q \in (-\pi/2, \pi/2). \]
\[ M_I (\ddot{q}) + M_V (\dot{q}) + M_e (q) + M_g (q) = \mu_c \] (KD)

\[ M_g (q) = \mathcal{M}gl \sin(q): \text{gravitational component.} \quad \mathcal{M} = \text{mass of shank and foot,} \quad g = \text{gravitational acceleration,} \quad l = \text{distance between knee-joint and lumped center of mass of shank-foot.} \]
\[ M_I (\ddot{q}) + M_V (\dot{q}) + M_e (q) + M_g (q) = \mu_c \quad \text{(KD)} \]

\[ M_g (q) = M g l \sin(q): \text{gravitational component. } M = \text{mass of shank and foot, } g = \text{gravitational acceleration, } l = \text{distance between knee-joint and lumped center of mass of shank-foot.} \]

\[ \mu_c = \zeta(q) F: \text{knee torque. } F = \text{total muscle force at tendon.} \]

\[ \zeta(q) = \text{positive valued moment arm.} \]
\[ M_I (\ddot{q}) + M_V (\dot{q}) + M_e (q) + M_g (q) = \mu_c \]  

\( M_g (q) = Mgl \sin(q) \): gravitational component. \( M \) = mass of shank and foot, \( g \) = gravitational acceleration, \( l \) = distance between knee-joint and lumped center of mass of shank-foot.

\( \mu_c = \zeta(q)F \): knee torque. \( F \) = total muscle force at tendon. \( \zeta(q) \) = positive valued moment arm.

\( F = \xi(q, \dot{q})v(t - \tau) \): \( v \) = voltage across quadriceps. \( \tau \) = latency between applying voltage and force production.
\[
\begin{align*}
J\ddot{q} + b_1 \dot{q} + b_2 \tanh(b_3 \dot{q}) + k_1 q e^{-k_2 q} + k_3 \tan(q) + M gl \sin(q) = A(q, \dot{q}) v(t - \tau), & \quad q \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\
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\end{align*}
\]
Knee Joint Dynamics (Sharma et al, 2009)

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\begin{align*}
J\ddot{q} + b_1 \dot{q} + b_2 \tanh(b_3 \dot{q}) + k_1 q e^{-k_2 q} + k_3 \tan(q) + M g l \sin(q) &= A(q, \dot{q}) v(t - \tau), \quad q \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\
A &= \text{scaled moment arm, } v = \text{voltage control}
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J\ddot{q} + b_1 \dot{q} + b_2 \tanh(b_3 \dot{q}) + k_1 q e^{-k_2 q} + k_3 \tan(q) + M g l \sin(q) &= A(q, \dot{q}) \nu(t - \tau), \quad q \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\
A &= \text{scaled moment arm}, \quad \nu = \text{voltage control}
\end{align*}
\]

(3)

\[
\ddot{q}(t) = -\frac{dF}{dq}(q(t)) - H(\dot{q}(t)) + G(q(t), \dot{q}(t))\nu(t - \tau)
\]

(4)
\[ M_I(\dot{q}) J \ddot{q} + b_1 \dot{q} + b_2 \tanh(b_3 \dot{q}) + k_1 q e^{-k_2 q} + k_3 \tan(q) + M_g l \sin(q) = A(q, \dot{q}) v(t - \tau), \quad q \in (-\frac{\pi}{2}, \frac{\pi}{2}) \]

\( A = \text{scaled moment arm}, \ v = \text{voltage control} \)

\[ \ddot{q}(t) = -\frac{dF}{dq}(q(t)) - H(\dot{q}(t)) + G(q(t), \dot{q}(t)) v(t - \tau) \]

Our Requirements:

- \( F : (-\pi/2, \pi/2) \rightarrow [0, \infty) \) is \( C^2 \) and \( \lim_{q \rightarrow \pm \pi/2} F(q) = \infty. \)
- \( G : (-\pi/2, \pi/2) \times \mathbb{R} \rightarrow (0, \infty) \) is \( C^1 \) and bounded.
- \( H : \mathbb{R} \rightarrow \mathbb{R} \) is \( C^1 \) and \( \inf_{x \in \mathbb{R}} x H(x) \geq 0. \)
Tracking Problem

\[ \ddot{q}(t) = -\frac{dF}{dq}(q(t)) - H(\dot{q}(t)) + G(q(t), \dot{q}(t))v(t - \tau) \quad (4) \]

\[ F(q) = \frac{k_1 \exp(-k_2 q)}{J k_2^2} \left( \exp(k_2 q) - 1 - k_2 q \right) + \frac{m g l}{J} (1 - \cos(q)) + \frac{k_3}{J} \ln \left( \frac{1}{\cos(q)} \right), \quad (5) \]

\[ G(q, \dot{q}) = \frac{1}{J} A(q, \dot{q}), \text{ and} \]

\[ H(\dot{q}) = \frac{b_2}{J} \tanh(b_3 \dot{q}) + \frac{b_1}{J} \dot{q}. \]
Tracking Problem

\[ \ddot{q}(t) = -\frac{dF}{dq}(q(t)) - H(\dot{q}(t)) + G(q(t), \dot{q}(t))v(t - \tau) \quad (4) \]

\[ F(q) = \frac{k_1 \exp(-k_2 q)}{jk_2} \left( \exp(k_2 q) - 1 - k_2 q \right) + \frac{mg}{J} (1 - \cos(q)) + \frac{k_3}{J} \ln \left( \frac{1}{\cos(q)} \right), \]

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\[ \ddot{q}_d(t) = -\frac{dF}{dq}(q_d(t)) - H(\dot{q}_d(t)) + G(q_d(t), \dot{q}_d(t))v_d(t - \tau) \quad (6) \]
Tracking Problem

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\ddot{q}(t) = -\frac{dF}{dq}(q(t)) - H(\dot{q}(t)) + G(q(t), \dot{q}(t))v(t - \tau) \tag{4}
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\[
F(q) = \frac{k_1 \exp(-k_2 q)}{jk_2^2} \left( \exp(k_2 q) - 1 - k_2 q \right) + \frac{mgl}{J} (1 - \cos(q)) + \frac{k_3}{J} \ln \left( \frac{1}{\cos(q)} \right), \tag{5}
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\ddot{q}_d(t) = -\frac{dF}{dq}(q_d(t)) - H(\dot{q}_d(t)) + G(q_d(t), \dot{q}_d(t))v_d(t - \tau) \tag{6}
\]

\[
\max\{||\dot{q}_d||_{\infty}, ||v_d||_{\infty}, ||\dot{v}_d||_{\infty}\} < \infty \text{ and } ||q_d||_{\infty} < \frac{\pi}{2} \tag{7}
\]
Tracking Problem

\[ \ddot{q}(t) = -\frac{dF}{dq}(q(t)) - H(\dot{q}(t)) + G(q(t), \dot{q}(t))v(t - \tau) \]  \hspace{1cm} (4)

\[
F(q) = \frac{k_1 \exp(-k_2 q)}{\sqrt[4]{2}} \left( \exp(k_2 q) - 1 - k_2 q \right) \\
+ \frac{mgl}{J}(1 - \cos(q)) + \frac{k_3}{J} \ln \left( \frac{1}{\cos(q)} \right),
\]

\[
G(q, \dot{q}) = \frac{1}{J} \mathcal{A}(q, \dot{q}), \text{ and}
\]

\[
H(\dot{q}) = \frac{b_2}{J} \tanh(b_3 \dot{q}) + \frac{b_1}{J} \dot{q}.
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\[ \ddot{q}_d(t) = -\frac{dF}{dq}(q_d(t)) - H(\dot{q}_d(t)) + G(q_d(t), \dot{q}_d(t))v_d(t - \tau) \]  \hspace{1cm} (6)

\[
\max\{\|\dot{q}_d\|_\infty, \|v_d\|_\infty, \|\dot{v}_d\|_\infty\} < \infty \text{ and } \|q_d\|_\infty < \frac{\pi}{2}
\]  \hspace{1cm} (7)

We want \((q - q_d, \dot{q} - \dot{q}_d) \to 0\) in a UGAS exponential way.
Voltage Controller

Error variables:

\[ x_1 = \tan(q) - \tan(q_d) \]

\[ x_2 = \dot{q} \cos^2(q) - \dot{q}_d \cos^2(q_d) \]

Three parts of the control scheme, assuming \( t_0 = 0 \):

A numerical prediction \( \xi(T_i) = z_N \) of the error variables at time \( T_i + \tau \) using \( (q(T_i), \dot{q}(T_i)) \in (-\pi/2, \pi/2) \times \mathbb{R} \).

An intersample prediction \( \xi = (\xi_1, \xi_2) \) of the error variables for the time interval between two consecutive measurements.

Applying the predictor feedback \( v(t) \), i.e., the nominal control with the state variables replaced by their predicted values.
Error variables:
\[ x_1 = \tan(q) - \tan(q_d) \]
\[ x_2 = \frac{\dot{q}}{\cos^2(q)} - \frac{\dot{q}_d}{\cos^2(q_d)} \]
Voltage Controller

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Three parts of the control scheme, assuming \( t_0 = 0 \):
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Applying the predictor feedback \( v(t) \), i.e., the nominal control with the state variables replaced by their predicted values.
Voltage Controller

\[ v(t) = \frac{g_2(\zeta_d(t+\tau))v_d(t) - g_1(\zeta_d(t+\tau)+\xi(t)) + g_1(\zeta_d(t+\tau)) - (1+\mu^2)\xi_1(t) - 2\mu\xi_2(t)}{g_2(\zeta_d(t+\tau)+\xi(t))} \]

for all \( t \in [T_i, T_{i+1}) \) and each \( i \).
\[
V(t) = \frac{g_2(\zeta_d(t+\tau))v_d(t) - g_1(\zeta_d(t+\tau) + \xi(t)) + g_1(\zeta_d(t)) - (1+\mu^2)\xi_1(t) - 2\mu\xi_2(t)}{g_2(\zeta_d(t+\tau) + \xi(t))}
\]

for all \( t \in [T_i, T_{i+1}) \) and each \( i \), where

\[
g_1(x) = -(1 + x_1^2) \frac{dF}{dq}(\tan^{-1}(x_1)) + \frac{2x_1x_2^2}{1+x_1^2} - (1 + x_1^2)\ H \left( \frac{x_2}{1+x_1^2} \right),
\]

\[
g_2(x) = (1 + x_1^2) \ G \left( \tan^{-1}(x_1), \frac{x_2}{1+x_1^2} \right),
\]

\[
\zeta_d(t) = (\zeta_{1,d}(t), \zeta_{2,d}(t)) = \left( \tan(q_d(t)), \frac{\dot{q}_d(t)}{\cos^2(q_d(t))} \right),
\]

\[
\xi_1(t) = e^{-\mu(t-T_i)} \left\{ \left( \xi_2(T_i) + \mu\xi_1(T_i) \right) \sin(t - T_i) \right\} + \xi_1(T_i) \cos(t - T_i),
\]

\[
\xi_2(t) = e^{-\mu(t-T_i)} \left\{ - \left( \mu\xi_2(T_i) + (1 + \mu^2)\xi_1(T_i) \right) \sin(t - T_i) \right\} + \xi_2(T_i) \cos(t - T_i),
\]

and \( \xi(T_i) = z_{N_i} \).
Voltage Controller

\[ v(t) = \frac{g_2(\zeta_d(t+\tau))v_d(t) - g_1(\zeta_d(t+\tau) + \xi(t)) + g_1(\zeta_d(t+\tau)) - (1 + \mu^2)\xi_1(t) - 2\mu\xi_2(t)}{g_2(\zeta_d(t+\tau) + \xi(t))} \]

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\[ \xi_1(t) = e^{-\mu(t-T_i)} \left\{ (\xi_2(T_i) + \mu\xi_1(T_i)) \sin(t - T_i) + \xi_1(T_i) \cos(t - T_i) \right\}, \]

\[ \xi_2(t) = e^{-\mu(t-T_i)} \left\{ - (\mu\xi_2(T_i) + (1 + \mu^2)\xi_1(T_i)) \sin(t - T_i) + \xi_2(T_i) \cos(t - T_i) \right\}, \]

and \( \xi(T_i) = z_{N_i} \). The time-varying Euler iterations \( \{z_k\} \) at each time \( T_i \) use measurements \( (q(T_i), \dot{q}(T_i)) \).
Euler iterations used for control:

\[ z_{k+1} = \Omega(T_i + kh_i, h_i, z_k; \nu) \quad \text{for} \quad k = 0, ..., N_i - 1 \quad \text{, where} \]

\[ z_0 = \begin{pmatrix} \tan(q(T_i)) - \tan(q_{d}(T_i)) \\ \frac{\dot{q}(T_i)}{\cos^2(q(T_i))} - \frac{\dot{q}_{d}(T_i)}{\cos^2(q_{d}(T_i))} \end{pmatrix}, \quad h_i = \frac{\tau}{N_i}, \]

and \( \Omega : [0, +\infty)^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \) is defined by

\[ \Omega(T, h, x; \nu) = \begin{bmatrix} \Omega_1(T, h, x; \nu) \\ \Omega_2(T, h, x; \nu) \end{bmatrix} \quad (8) \]

and the formulas

\[ \Omega_1(T, h, x; \nu) = x_1 + hx_2 \quad \text{and} \]

\[ \Omega_2(T, h, x; \nu) = x_2 + \zeta_2,d(T) + \int_T^{T+h} g_1(\zeta_d(s) + x) ds \]

\[ + \int_T^{T+h} g_2(\zeta_d(s) + x) \nu(s - \tau) ds - \zeta_2,d(T + h). \]
For all positive constants $\tau$ and $r$, there exist a locally bounded function $N$, a constant $\omega \in (0, \mu/2)$ and a locally Lipschitz function $C$ satisfying $C(0) = 0$ such that: For all sample times $\{T_i\}$ in $[0, \infty)$ such that $\sup_{i \geq 0} (T_{i+1} - T_i) \leq r$ and each initial condition, the solution $(q(t), \dot{q}(t), v(t))$ with

$$N_i = N \left( \left| \left( \tan(q(T_i)), \frac{\dot{q}(T_i)}{\cos^2(q(T_i))} \right) - \zeta_d(T_i) \right| 
+ \left| \left| v - v_d \right| \right|[T_i - \tau, T_i] \right)$$

satisfies

$$|q(t) - q_d(t)| + |\dot{q}(t) - \dot{q}_d(t)| + \left| \left| v - v_d \right| \right|[t - \tau, t] \leq e^{-\omega t} C \left( \frac{|q(0) - q_d(0)| + |\dot{q}(0) - \dot{q}_d(0)|}{\cos^2(q(0))} + \left| \left| v_0 - v_d \right| \right|[-\tau, 0] \right)$$

for all $t \geq 0$. 
First Simulation

\[ J\ddot{q} + b_1 \dot{q} + b_2 \tanh(b_3 \dot{q}) + k_1 q e^{-k_2 q} + k_3 \tan(q) + Mgl \sin(q) = A(q, \dot{q}) v(t - \tau), \quad q \in (-\frac{\pi}{2}, \frac{\pi}{2}) \] (10)
\[ J \ddot{q} + b_1 \dot{q} + b_2 \tanh (b_3 \dot{q}) + k_1 q e^{-k_2 q} + k_3 \tan (q) + Mgl \sin (q) = A(q, \dot{q}) \nu (t - \tau), \quad q \in (-\frac{\pi}{2}, \frac{\pi}{2}) \]

\( \tau = 0.07s, \quad A(q, \dot{q}) = \bar{a}e^{-2q^2} \sin(q) + \bar{b} \)

\[ J = 0.39 \text{ kg}-\text{m}^2/\text{rad}, \quad b_1 = 0.6 \text{ kg}-\text{m}^2/(\text{rad}-\text{s}), \quad \bar{a} = 0.058, \]

\[ b_2 = 0.1 \text{ kg}-\text{m}^2/(\text{rad}-\text{s}), \quad b_3 = 50 \text{ s}/\text{rad}, \quad \bar{b} = 0.0284, \]

\[ k_1 = 7.9 \text{ kg}-\text{m}^2/(\text{rad}-\text{s}^2), \quad k_2 = 1.681/\text{rad}, \]

\[ k_3 = 1.17 \text{ kg}-\text{m}^2/(\text{rad}-\text{s}^2), \quad M = 4.38 \text{ kg}, \quad l = 0.248 \text{ m}. \]
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\( q(0) = 0.5 \text{ rad}, \quad \dot{q}(0) = 0 \text{ rad/s}, \quad v(t) = 0 \text{ on } [-0.07, 0), \]
\( N_i = N = 10, \quad \text{and } T_{i+1} - T_i = 0.014s, \quad \text{and } \mu = 2. \)
First Simulation

Tracking Control for Neuromuscular Electrical Stimulation

Karafyllis (NTUA), Krstic (UCSD), Malisoff (LSU), et al.
Simulated Robustness Test

We took $\tau = 0.07s$ and the same model parameters $J = 0.39 \text{kg-m}^2/\text{rad}$, $b_1 = 0.6 \text{kg-m}^2/(\text{rad-s})$, $\bar{a} = 0.058$, $b_2 = 0.1 \text{kg-m}^2/(\text{rad-s})$, $b_3 = 50 \text{s/\text{rad}}$, $\bar{b} = 0.0284$, $k_1 = 7.9 \text{kg-m}^2/(\text{rad-s}^2)$, $k_2 = 1.681/\text{rad}$, $k_3 = 1.17 \text{kg-m}^2/(\text{rad-s}^2)$, $M = 4.38 \text{kg}$, $l = 0.248 \text{m}$.

$q_d(t) = \pi/3 (1 - \exp(-3t)) \text{rad}$, $q(0) = \pi/18$, $\dot{q}(0) = v_0(t) = 0$, $N_i = N = 10$, $T_i + 1 - T_i = 0.014$.

We used these mismatched parameters in the control: $J' = 1.25 J$, $b_1' = 1.2 b_1$, $b_2' = 0.9 b_2$, $\bar{a}' = 1.185 \bar{a}$, $b_3' = 0.85 b_3$, $k_1' = 1.1 k_1$, $k_2' = 0.912 k_2$, $\bar{b}' = 0.98 \bar{b}$, $k_3' = 0.9 k_3$, $M' = 0.97 M$, and $l' = 1.013 l$. 

Karafyllis (NTUA), Krstic (UCSD), Malisoff (LSU), et al. Tracking Control for Neuromuscular Electrical Stimulation
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\(13\)
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\begin{align*}
J &= 0.39 \text{ kg-m}^2/\text{rad}, \quad b_1 = 0.6 \text{ kg-m}^2/(\text{rad-s}), \quad \ddot{a} = 0.058, \\
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\end{equation}

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(15)
Simulated Robustness Test

Tracking Control for Neuromuscular Electrical Stimulation
Summary of NMES Research

NMES is an important emerging technology that can help rehabilitate patients with motor neuron disorders. It produces difficult tracking control problems that contain delays, state constraints, and uncertainties. Our new sampled predictive control design overcame these challenges and can track a large class of reference trajectories. By incorporating the state constraint on the knee position, our control can help ensure patient safety for any input delay value. Our control used a new numerical solution approximation method that covers many other time-varying models. In future work, we hope to apply input-to-state stability to better understand the effects of uncertainties under state constraints.
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