Feedback Control under Input Delays

Michael Malisoff
Background on Research Area
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Feedback Control: Automatically adjust the system to respond to information about the system’s state and surroundings.

Delays and Sampling: Time-lagged state observations and/or observations at discrete instants instead of continuous ones.
Control Systems with Input Delays

System of ODEs with delays $\tau$, controls $u$, and perturbations $\delta$:

$$Y'(t) = F(t, Y(t), u(t, Y(t - \tau(t))), \delta(t)), \quad Y(t) \in \mathcal{Y}. \quad (1)$$

$\mathcal{Y} \subseteq \mathbb{R}^n$. $\delta : [0, \infty) \to \mathcal{D}$ is (nonstochastic) uncertainty. $\mathcal{D} \subseteq \mathbb{R}^m$.

Choose $u$ to achieve desired behavior for the solutions $Y(t)$. 
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$\tau$: time lags in computing or communicating control or state...
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Closed loop system:

$$Y'(t) = G(t, Y(t), Y(t - \tau(t)), \delta(t)), \quad Y(t) \in \mathcal{Y}, \quad (2)$$

where $G(t, Y(t), Y(t - \tau), \delta) = F(t, Y(t), u(t, Y(t - \tau)), \delta)$. 
Input-to-State Stable (ISS)
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ISS (Sontag, ’89) generalizes uniform global asymptotic stability.
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\[ Y'(t) = G(t, Y(t), Y(t - \tau(t))), \quad Y(t) \in \mathcal{Y} \quad (\Sigma) \]
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\[ Y'(t) = G(t, Y(t), Y(t - \tau(t))), \quad Y(t) \in \mathcal{Y} \quad (\Sigma) \]

\[ |Y(t)| \leq \gamma_1 \left( e^{t_0 - t} \gamma_2(|Y|_{[t_0 - \bar{\tau}, t_0]}) \right) \quad (\text{UGAS}) \]

\( \gamma_i \)'s are 0 at 0, strictly increasing, continuous, and unbounded.
\[ \sup_{t \geq 0} \tau(t) \leq \bar{\tau}. \]
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\[ Y'(t) = G(t, Y(t), Y(t - \tau(t)), \delta(t)), \quad Y(t) \in \mathcal{Y} \quad (\Sigma_{\text{pert}}) \]
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\[ |Y(t)| \leq \gamma_1 \left( e^{t_0 - t} \gamma_2(|Y|_{[t_0 - \bar{\tau}, t_0]}) \right) + \gamma_3(|\delta|_{[t_0, t]}) \quad (ISS) \]
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Without explicit flow maps, prove UGAS and ISS indirectly.
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Current or past AE for Automatica, IEEE Transactions on Automatic Control, SIAM Journal on Control and Optimization,..

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Our goals: Input-to-state stabilization of equilibria with uncertain uptake functions using only delayed sampled substrate values.
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O. Bernard, D. Dochain, J. Gouze, J. Monod, H. Smith, ...
Background on Chemostats

Constant volume. Substrate pumped in and substrate/biomass mixture pumped out at same rate
Uncertain Controlled Chemostat with Sampling

\[
\begin{align*}
\dot{s}(t) &= D(s(t - \tau(t))[s_{in} - s(t)] - (1 + \delta(t))\mu(s(t))x(t) \\
\dot{x}(t) &= [(1 + \delta(t))\mu(s(t)) - D(s(t - \tau(t))]x(t)
\end{align*}
\]

\[\tau(t) = \begin{cases} 
\tau_f, & t \in [0, \tau_f) \\
\tau_f + t - t_j, & t \in [t_j + \tau_f, t_{j+1} + \tau_f) \text{ and } j \geq 0
\end{cases}\]

\(0 < \epsilon_1 \leq t_{j+1} - t_j \leq \epsilon_2. \quad \delta : [0, \infty) \rightarrow [d, \infty), \text{ with } d \in (-1, 0].\)
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\mu(s) = \frac{\mu_1(s)}{1 + \gamma(s)}, \text{ with a unique maximizer } s_M \in (0, s_{in}]
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\mu(s) = \frac{\mu_1(s)}{1 + \gamma(s)}, \text{ with a unique maximizer } s_M \in (0, s_{\text{in}}]
\]

Assumption 1: The function \( \mu \) is \( C^1 \) and \( \mu(0) = 0 \). Also, there is a constant \( s_M \in (0, s_{\text{in}}] \) such that \( \mu'(s) > 0 \) for all \( s \in [0, s_M) \) and \( \mu'(s) \leq 0 \) for all \( s \in [s_M, \infty) \). Finally, \( \mu(s) > 0 \) for all \( s > 0 \).
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\dot{s}(t) &= D(s(t - \tau(t))[s_{in} - s(t)] - (1 + \delta(t))\mu(s(t))x(t) \\
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\[\mu(s) \overset{(*)}{=} \frac{\mu_1(s)}{1 + \gamma(s)}, \text{ with a unique maximizer } s_M \in (0, s_{in}]\]

**Lemma:** Under Assumption 1, there are \(\mu_1 \in C^1 \cap \mathcal{K}_\infty\) and a nondecreasing \(C^1\) function \(\gamma : \mathbb{R} \to [0, \infty)\) such that \((*)\) holds for all \(s \geq 0, \mu'_1(s) > 0\) on \([0, \infty)\), and \(\gamma'(s) > 0\) on \([s_M, \infty)\).
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\begin{aligned}
\dot{s}(t) &= D(s(t - \tau(t))[s_{\text{in}} - s(t)] - (1 + \delta(t))\mu(s(t))x(t) \\
\dot{x}(t) &= ((1 + \delta(t))\mu(s(t)) - D(s(t - \tau(t)))]x(t)
\end{aligned}
\]

\(\tau(t) = \begin{cases} 
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\end{cases}\)

\((1 + \delta)\mu(s)\) for Different Constant \(\delta\) Choices, \(s_M = 1/\sqrt{2}\) and \(s_{\text{in}} = 1\)
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0 < \epsilon_1 \leq t_{j+1} - t_j \leq \epsilon_2. \quad \delta : [0, \infty) \to [d, \infty), \text{ with } d \in (-1, 0].

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\mu(s) = \frac{\mu_1(s)}{1 + \gamma(s)}, \text{ with a unique maximizer } s_M \in (0, s_{in}]
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Lemma: Under Assumption 1, there are \( \mu_1 \in C^1 \cap K_\infty \) and a nondecreasing \( C^1 \) function \( \gamma : \mathbb{R} \to [0, \infty) \) such that (\( \ast \)) holds for all \( s \geq 0, \mu'_1(s) > 0 \) on \([0, \infty), \) and \( \gamma'(s) > 0 \) on \([s_M, \infty).\)
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\[\tau(t) = \begin{cases}
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\[\mu(s) = \frac{\mu_1(s)}{1 + \gamma(s)}, \text{ with a unique maximizer } s_M \in (0, s_{\text{in}}]\]

Goal: Under suitable conditions on an upper bound \(\bar{\tau}\) for the delay \(\tau(t)\), and for constants \(s_* \in (0, s_{\text{in}})\), design the control \(D\) to render the dynamics for \(Y(t) = (s(t), x(t)) - (s_*, s_{\text{in}} - s_*)\) ISS.
One of Our Results for Unperturbed Case

\[\omega_s = \inf_{s \in [0, s_{in}]} \mu'_1(s), \quad \omega_I = \sup_{s \in [0, s_{in}]} \mu'_1(s), \quad \rho_I = \sup_{s \in [0, s_{in}]} \gamma'(s),\]

\[\rho_m = \frac{\rho_I^2}{2\omega_s} \max_{l \in [0, s_{in}]} \frac{\mu_1^2(l+1.1\mu_1(s_*)s_{in}T)}{1+\gamma(l)} , \quad \text{where} \quad \mu(s) = \frac{\mu_1(s)}{1+\gamma(s)}\]
One of Our Results for Unperturbed Case

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where $$\mu(s) = \frac{\mu_1(s)}{1+\gamma(s)}$$

Assume that

$$\frac{\mu_1(s_{\text{in}})}{1+\gamma(s_{\text{in}})} - \frac{\mu_1(s_*)}{1+\gamma(s_{\text{in}}-\mu_1(s_*)s_{\text{in}}\bar{\tau})} \overset{(a)}{>} 0$$

and $$\bar{\tau} \overset{(b)}{<} \max \left\{ \frac{1}{2s_{\text{in}}\sqrt{2\rho_m\omega_I}}, \frac{1}{2\rho_I s_{\text{in}}\mu_1(s_{\text{in}})} \right\},$$

with $$s_* < s_{\text{in}}.$$
One of Our Results for Unperturbed Case

$$\omega_s = \inf_{s \in [0, s_{in}]} \mu'_1(s), \quad \omega_l = \sup_{s \in [0, s_{in}]} \mu'_1(s), \quad \rho_l = \sup_{s \in [0, s_{in}]} \gamma'(s),$$

$$\rho_m = \frac{\rho^2_l}{2\omega_s} \max_{l \in [0, s_{in}]} \frac{\mu^2_1(l+1.1\mu_1(s_*) s_{in} \tau)}{1+\gamma(l)},$$

where $$\mu(s) = \frac{\mu_1(s)}{1+\gamma(s)}$$

Assume that

$$\frac{\mu_1(s_{in})}{1+\gamma(s_{in})} - \frac{\mu_1(s_*)}{1+\gamma(s_{in} - \mu_1(s_*) s_{in} \tau)} > 0 \quad (a)$$

and $$\bar{\tau} < \max \left\{ \frac{1}{2s_{in} \sqrt{2\rho_m \omega_l}}, \frac{1}{2\rho_l s_{in} \mu_1(s_{in})} \right\} \quad (b),$$

with $$s_* < s_{in}.$$ 

**Theorem 1:** For all componentwise positive initial conditions, all solutions $$(s, x)(t)$$ of the chemostat system (C) with $$\delta(t) = 0$$ and

$$D(s(t - \tau(t))) = \frac{\mu_1(s_*)}{1+\gamma(s(t-\tau(t)))} \quad (3)$$

remain in $$(0, \infty)^2$$ and converge to $$(s_*, s_{in} - s_*)$$ as $$t \to +\infty.$$  □
Extensions and Ideas of Proof
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ISS with respect to $\delta(t)$ without upper bounds on $|\delta|_\infty$...

$$\frac{(1+d)\mu_1(s_{in})}{1+\gamma(s_{in})} - \frac{\mu_1(s_*)}{1+\gamma(s_{in}-\mu_1(s_*)s_{in}\bar{\tau})} > 0$$

$$\cdots (1 + \delta(t))\mu(s(t)) \cdots$$
Extensions and Ideas of Proof

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$$\frac{(1+d)\mu_1(s_{in})}{1+\gamma(s_{in})} - \frac{\mu_1(s_*)}{1+\gamma(s_{in}-\mu_1(s_*)s_{in}\bar{\tau})} > 0 \quad \ldots (1 + \delta(t))\mu(s(t)) \ldots$$

$$U_2(s_t) =$$

$$\int_{0}^{s(t)-s_*} \frac{m}{s_{in}-s_*-m} dm + 2\rho m \bar{\tau} \int_{t-\bar{\tau}}^{t} \int_{\ell}^{t} (\dot{s}(m))^2 dm d\ell.$$

Use $z = s_{in} - s - x = (s_{in} - s_* - x) + (s_* - s) \to 0$ exponentially.
Extensions and Ideas of Proof

ISS with respect to $\delta(t)$ without upper bounds on $|\delta|_\infty$...

$$\frac{(1+\alpha)d}{1+\gamma(e)} \mu_1(s_{in}) - \frac{\mu_1(s_*)}{1+\gamma(s_{in}-\mu_1(s_*)s_{in} \tau)} > 0 \quad (1 + \delta(t))\mu(s(t))...$$

$$\mathcal{U}_2(s_t) = \int_{s(t)-s_*}^{-s_\star} \frac{m}{s_{in}-s_\star-m} dm + 2\rho_{m} \int_{t-\bar{\tau}}^{t} \int_{\ell}^{t} (\dot{s}(m))^2 dm d\ell.$$  

Use $z = s_{in} - s - x = (s_{in} - s_\star - x) + (s_\star - s) \to 0$ exponentially.

\[ s_{in} = 1, \quad \mu(s) = \frac{0.5s}{1 + 0.25s + 2s^2}, \quad t_j = 0.24j, \quad \delta(t) = 0. \]

\[ s(t) \text{ in Red, } x(t) \text{ in Blue, } D(t) \text{ in Green}. \]
$s_{in} = 1$, $\mu(s) = \frac{0.5s}{1+0.25s+2s^2}$, $t_j = 0.24j$, $\delta(t) = 0.15(1 + \sin(t))$.

$s(t)$ in Red, $x(t)$ in Blue, $D(t)$ in Green.
\( s_{in} = 1, \ \mu(s) = \frac{0.5s}{1+0.25s+2s^2}, \ t_j = 0.24j, \ \delta(t) = 0.15(1 + \sin(t)). \)

\( s(t) \) in Red, \( x(t) \) in Blue, \( D(t) \) in Green.
Conclusions
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Sampling and delays are common in feedback control problems.
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Sampling and delays are common in feedback control problems. Bioreactors involve uncertainties, delays, and sampling. Discretization of continuous time controls can produce errors. Our control only needs discrete delayed substrate values.
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Bioreactors involve uncertainties, delays, and sampling.
Discretization of continuous time controls can produce errors.
Our control only needs discrete delayed substrate values.
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Thank you for your attention!