Constructions of Strict Lyapunov Functions: Stability, Robustness, Delays, and State Constraints

Matrosov's Approach

Michael Malisoff

Outline

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- Strict and nonstrict Lyapunov functions

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- Input-to-state stability and point stabilization

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M. Malisoff and F. Mazenc. Constructions of Strict Lyapunov Functions. Communications and Control Engineering Series, Springer-Verlag London Ltd., London, UK, 2009.

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A Lyapunov function for a system $\dot{x} = \mathcal{F}(t, x)$ with state space $\mathcal{X}$ is a positive definite proper function $V : [0, \infty) \times \mathcal{X} \to [0, \infty)$ such that $\dot{V}(t, x) := V_t(t, x) + V_x(t, x)\mathcal{F}(t, x) \leq 0$ on $[0, \infty) \times \mathcal{X}$.

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For example, $V(x) = \ln(1 + x^2)$ is a Lyapunov function for $\dot{x} = -x/(1 + x^2)$ because $\dot{V} \leq -x^2/(1 + x^2)^2$, which gives global asymptotic stability, i.e., attractivity and local stability.

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However, for each constant $\bar{\delta} > 0$, we can find an x_0 such that the trajectory for $\dot{x} = -x/(1 + x^2) + \bar{\delta}$ starting at $x(0) = x_0$ is unbounded, which means we lack input-to-state stability.

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Using LaSalle Invariance, we can often use nonstrict ones to prove GAS, e.g., for $\dot{x} = f(x)$ where $\dot{V}(x) := \nabla V(x) f(x)$.

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If V is a nonstrict Lyapunov function such that the only solution that remains in $\{x : \dot{V}(x) = 0\}$ is $x = 0$, then conclude GAS to 0.

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For example, take $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_2^3$. Use $V(x) = 0.5|x|^2$.
Then $\dot{V} = -x_2^4$.

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For example, take $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_2^3$. Use $V(x) = 0.5|x|^2$.
Then $\dot{V} = -x_2^4$. The largest invariant set in $\{x : x_2 = 0\}$ is $\{0\}$.

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However, explicit strict Lyapunov function *constructions* are often needed in applications to certify robustness.

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This has led to significant research on explicitly constructing strict Lyapunov functions.

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We assume standard assumptions on the dynamics which hold under smooth forward completeness and time-periodicity.

ISS Motivation-Part 1 of 3

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The disturbances $d : [0, \infty) \rightarrow D$ are measurable essentially bounded functions valued in some subset D of a Euclidean space. See our CCE book for standing assumptions on $\mathcal{F}$.

ISS Motivation-Part 2 of 3

We say that $\dot{x} = \mathcal{F}(t, x, d)$ is ISS provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ and $\bar{\alpha} \in \mathcal{K}_\infty$ s.t. for all initial conditions $x(t_0) = x_0 \in \mathcal{X}$ and all disturbances d, the corresponding trajectories $t \mapsto \zeta(t; t_0, x_0, d)$ satisfy

$$|\zeta(t; t_0, x_0, d)| \leq \beta\Big(\bar{\alpha}(|x_0|), t - t_0\Big) + \gamma(|d|_\infty) \ \ \forall t \geq t_0 \ . \qquad (2)$$

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UGAS: Special case where $d = 0$.

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Example:

ISS Motivation-Part 3 of 3

Example: Assume that

$$\dot{x} = \mathcal{F}_{\mathrm{cl}}(t, x) := f(t, x) + g(t, x)K(t, x) \tag{3}$$

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Need $K(t,x)$ and $D_xV(t,x)\cdot g(t,x)$.

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Assume $\dot{x} = f(x)$ has a nonstrict Lyapunov function V so that:

$$\exists N_* > 0 \text{ s.t. } \forall q \in \mathbb{R}^n \setminus \{0\},\ \exists i \in [1, N_*] \text{ s.t. } L_f^i V(q) \neq 0. \quad \text{(NDC)}$$

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A: Yes, and we can allow time varying systems and relax NDC.

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Then we can explicitly determine functions $\mathcal{F}_j$ and $\mathcal{G}$ such that

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Assumption A *There are functions h_j such that $h_j(0) = 0$ for all j; everywhere positive functions $r_1, \ldots, r_m$ and ρ; a proper positive definite function $V_1 : \mathcal{X} \to [0, \infty)$; and an integer $N > 0$ for which*

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Also, $f \in C^\infty(\mathbb{R}^n)$, and V_1 has a positive definite quadratic lower bound in some neighborhood of $0 \in \mathbb{R}^n$.

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Significance: New theorem says which functions V_i to pick.

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Significance: Allows any open state space $\mathcal{X}$ containing $0 \in \mathbb{R}^n$.

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Assume $\alpha > d$. Want a global strict Lyapunov function for (12).

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Along the trajectories of the L-V error dynamics,

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Thank you for your attention and interest!