Constructions of Strict Lyapunov Functions: Stability, Robustness, Delays, and State Constraints

Matrosov’s Approach

Michael Malisoff
Outline

- Strict and nonstrict Lyapunov functions
- Input-to-state stability and point stabilization
- Strictification to certify good performance
- LaSalle strictification
- Matrosov approaches

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A Lyapunov function for a system \( \dot{x} = F(t, x) \) with state space \( X \) is a positive definite proper function \( V : [0, \infty) \times X \rightarrow [0, \infty) \) such that

\[
\dot{V}(t, x) := V_t(t, x) + V_x(t, x) F(t, x) \leq 0 \quad \text{on} \quad [0, \infty) \times X.
\]

By positive definite, we mean inf \( t V(t, x) \) is zero when \( x = 0 \) and positive for all \( x \in X \setminus \{0\} \).

Proper means that inf \( t V(t, x) \rightarrow \infty \) as \( x \) approaches boundary (\( X \)) or \( |x| \rightarrow \infty \).

For example, \( V(x) = \ln(1 + x^2) \) is a Lyapunov function for

\[
\dot{x} = -x / (1 + x^2)
\]

because \( \dot{V} \leq -x^2 / (1 + x^2)^2 \), which gives global asymptotic stability, i.e., attractivity and local stability.
A Lyapunov function for a system $\dot{x} = \mathcal{F}(t, x)$ with state space $\mathcal{X}$ is a positive definite proper function $V : [0, \infty) \times \mathcal{X} \to [0, \infty)$ such that $\dot{V}(t, x) := V_t(t, x) + V_x(t, x)\mathcal{F}(t, x) \leq 0$ on $[0, \infty) \times \mathcal{X}$. For example, $V(x) = \ln(1 + x^2)$ is a Lyapunov function for $\dot{x} = -x/(1 + x^2)$ because $\dot{V}(t, x) \leq -x^2/(1 + x^2)^2$, which gives global asymptotic stability, i.e., attractivity and local stability.
Basic Vocabulary and Simple Example

A **Lyapunov function** for a system $\dot{x} = \mathcal{F}(t, x)$ with state space $\mathcal{X}$ is a positive definite proper function $V : [0, \infty) \times \mathcal{X} \to [0, \infty)$ such that $\dot{V}(t, x) := V_t(t, x) + V_x(t, x)\mathcal{F}(t, x) \leq 0$ on $[0, \infty) \times \mathcal{X}$. By **positive definite**, we mean $\inf_t V(t, x)$ is zero when $x = 0$ and positive for all $x \in \mathcal{X} \setminus \{0\}$.
A Lyapunov function for a system $\dot{x} = \mathcal{F}(t, x)$ with state space $\mathcal{X}$ is a positive definite proper function $V : [0, \infty) \times \mathcal{X} \to [0, \infty)$ such that $\dot{V}(t, x) := V_t(t, x) + V_x(t, x)\mathcal{F}(t, x) \leq 0$ on $[0, \infty) \times \mathcal{X}$.

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A Lyapunov function for a system \( \dot{x} = \mathcal{F}(t, x) \) with state space \( \mathcal{X} \) is a positive definite proper function \( V : [0, \infty) \times \mathcal{X} \to [0, \infty) \) such that \( \dot{V}(t, x) := V_t(t, x) + V_x(t, x)\mathcal{F}(t, x) \leq 0 \) on \( [0, \infty) \times \mathcal{X} \).

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For example, \( V(x) = \ln(1 + x^2) \) is a Lyapunov function for \( \dot{x} = -x/(1 + x^2) \) because \( \dot{V} \leq -x^2/(1 + x^2)^2 \), which gives global asymptotic stability, i.e., attractivity and local stability.
A Lyapunov function for a system $\dot{x} = \mathcal{F}(t, x)$ with state space $\mathcal{X}$ is a positive definite proper function $V : [0, \infty) \times \mathcal{X} \to [0, \infty)$ such that $\dot{V}(t, x) := V_t(t, x) + V_x(t, x)\mathcal{F}(t, x) \leq 0$ on $[0, \infty) \times \mathcal{X}$.

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However, for each constant $\bar{\delta} > 0$, we can find an $x_0$ such that the trajectory for $\dot{x} = -x/(1 + x^2) + \bar{\delta}$ starting at $x(0) = x_0$ is unbounded, which means we lack input-to-state stability.
Background

Strict Lyapunov function decay:

\[ \dot{V}(t, x) \leq -W(x), \]

with \( W(x) \) positive definite.

Nonstrict Lyapunov function decay:

\[ \dot{V}(t, x) \leq -W(x), \]

with \( W(x) \) nonnegative definite.

Either way, \( \inf t V(t, x) \) is assumed proper and positive definite.

Converse Lyapunov theory often guarantees the existence of strict Lyapunov functions.

See Bacciotti-Rosier CCE Book.
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Using LaSalle Invariance, we can often use nonstrict ones to prove GAS, e.g., for \( \dot{x} = f(x) \) where \( \dot{V}(x) := \nabla V(x)f(x) \).
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If \( V \) is a nonstrict Lyapunov function such that the only solution that remains in \( \{ x : \dot{V}(x) = 0 \} \) is \( x = 0 \), then conclude GAS to 0.
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Either way, \( \inf_t V(t, x) \) is assumed **proper** and **positive definite**.

For example, take \( \dot{x}_1 = x_2, \dot{x}_2 = -x_1 - x_2^3. \)
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For example, take \( \dot{x}_1 = x_2, \dot{x}_2 = -x_1 - x_2^3. \) Use \( V(x) = 0.5|x|^2. \)
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Either way, \( \inf_t V(t, x) \) is assumed **proper** and **positive definite**.

For example, take \( \dot{x}_1 = x_2, \dot{x}_2 = -x_1 - x_2^3 \). Use \( V(x) = 0.5|x|^2 \). Then \( \dot{V} = -x_2^4 \).
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For example, take \( \dot{x}_1 = x_2, \dot{x}_2 = -x_1 - x_2^3 \). Use \( V(x) = 0.5|x|^2 \). Then \( \dot{V} = -x_2^4 \). The largest invariant set in \( \{x : x_2 = 0\} \) is \( \{0\} \).
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Either way, \( \inf_t V(t, x) \) is assumed proper and positive definite.

However, explicit strict Lyapunov function constructions are often needed in applications to certify robustness.
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This has led to significant research on explicitly constructing strict Lyapunov functions.
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We assume standard assumptions on the dynamics which hold under smooth forward completeness and time-periodicity.
Input-to-state stability is a robustness property for systems
\[ \dot{x} = F(t, x, d) \] (1).

Invented by E. Sontag; see CDC'88, T -AC'89.

The state space \( X \) is a general open subset of Euclidean space containing 0.

Assume \( F(t, 0, 0) = 0 \) for all \( t \).

E.g., \[ \dot{x} = f(t, x) + g(t, x) \] \( d \) if \( f(t, 0) = 0 \) for all \( t \).

That's the control-affine case.

The disturbances \( d: [0, \infty) \rightarrow D \) are measurable essentially bounded functions valued in some subset \( D \) of a Euclidean space.

See our CCE book for standing assumptions on \( F \).
Input-to-state stability is a robustness property for systems

\[ \dot{x} = \mathcal{F}(t, x, d) . \]
ISS Motivation-Part 1 of 3

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ISS Motivation-Part 1 of 3

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The disturbances \( d : [0, \infty) \rightarrow D \) are measurable essentially bounded functions valued in some subset \( D \) of a Euclidean space.
ISS Motivation-Part 1 of 3

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The disturbances \( d : [0, \infty) \to D \) are measurable essentially bounded functions valued in some subset \( D \) of a Euclidean space. See our CCE book for standing assumptions on \( \mathcal{F} \).
ISS Motivation-Part 2 of 3

We say that \( \dot{x} = \mathcal{F}(t, x, d) \) is ISS provided there exist functions \( \beta \in \mathcal{KL} \) and \( \gamma \in \mathcal{K}_\infty \) and \( \bar{\alpha} \in \mathcal{K}_\infty \) s.t. for all initial conditions \( x(t_0) = x_0 \in \mathcal{X} \) and all disturbances \( d \), the corresponding trajectories \( t \mapsto \zeta(t; t_0, x_0, d) \) satisfy

\[
|\zeta(t; t_0, x_0, d)| \leq \beta\left(\bar{\alpha}(|x_0|), t - t_0\right) + \gamma(|d|_\infty) \quad \forall t \geq t_0.
\] (2)
ISS Motivation-Part 2 of 3

We say that \( \dot{x} = F(t, x, d) \) is ISS provided there exist functions \( \beta \in \mathcal{KL} \) and \( \gamma \in \mathcal{K}_\infty \) and \( \bar{\alpha} \in \mathcal{K}_\infty \) such that for all initial conditions \( x(t_0) = x_0 \in \mathcal{X} \) and all disturbances \( d \), the corresponding trajectories \( t \mapsto \zeta(t; t_0, x_0, d) \) satisfy

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\( \mathcal{K}_\infty \): continuous, strictly increasing, unbounded, 0 at 0.
ISS Motivation-Part 2 of 3

We say that $\dot{x} = F(t, x, d)$ is ISS provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ and $\bar{\alpha} \in \mathcal{K}_\infty$ s.t. for all initial conditions $x(t_0) = x_0 \in \mathcal{X}$ and all disturbances $d$, the corresponding trajectories $t \mapsto \zeta(t; t_0, x_0, d)$ satisfy

$$|\zeta(t; t_0, x_0, d)| \leq \beta\left(\bar{\alpha}(|x_0|), t - t_0\right) + \gamma(|d|_\infty) \ \forall t \geq t_0.$$  \hspace{1cm} (2)

$\mathcal{K}_\infty$: continuous, strictly increasing, unbounded, 0 at 0.

$\mathcal{KL}$: continuous, $\beta(\cdot, t) \in \mathcal{K}_\infty$ for all $t$, $\lim_{t \to \infty} \beta(s, t) = 0$ for all $s$. 
We say that $\dot{x} = \mathcal{F}(t, x, d)$ is ISS provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ and $\bar{\alpha} \in \mathcal{K}_\infty$ s.t. for all initial conditions $x(t_0) = x_0 \in \mathcal{X}$ and all disturbances $d$, the corresponding trajectories $t \mapsto \zeta(t; t_0, x_0, d)$ satisfy

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ISS Lyapunov function decay:
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ISS Lyapunov function decay: $\dot{V} \leq -\alpha_1(V) + \alpha_2(|d|), \alpha_i \in \mathcal{K}_{\infty}$. 
ISS Motivation-Part 2 of 3

We say that \( \dot{x} = \mathcal{F}(t, x, d) \) is ISS provided there exist functions \( \beta \in \mathcal{K}\mathcal{L} \) and \( \gamma \in \mathcal{K}_\infty \) and \( \bar{\alpha} \in \mathcal{K}_\infty \) s.t. for all initial conditions \( x(t_0) = x_0 \in \mathcal{X} \) and all disturbances \( d \), the corresponding trajectories \( t \mapsto \zeta(t; t_0, x_0, d) \) satisfy

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\( \mathcal{K}\mathcal{L} \): continuous, \( \beta(\cdot, t) \in \mathcal{K}_\infty \) for all \( t \), \( \lim_{t \to \infty} \beta(s, t) = 0 \) for all \( s \).

ISS Lyapunov function decay: \( \dot{V} \leq -\alpha_1(V) + \alpha_2(|d|), \) \( \alpha_i \in \mathcal{K}_\infty \).

UGAS: Special case where \( d = 0 \).
Example: Assume that
\[ \dot{x} = F_{\text{cl}}(t, x) := f(t, x) + g(t, x)K(t, x) \]
is UGAS to the origin. Assume that we have a strict Lyapunov function \( V \) so that
\[ W(x) = \inf_t \{-V(t, x) + V(t, x)F_{\text{cl}}(t, x)\} \]
is proper. Then
\[ \dot{x} = f(t, x) + g(t, x)\left[K(t, x) - D_x V(t, x) \cdot g(t, x) + d\right] \]
is ISS with respect to actuator errors \( d \).

Need \( K(t, x) \) and \( D_x V(t, x) \cdot g(t, x) \).
Example:

\[
\dot{x} = F_{\text{cl}}(t, x) := f(t, x) + g(t, x)K(t, x) \tag{3}
\]

is UGAS to the origin. Assume that we have a strict Lyapunov function \(V\) so that

\[
W(x) = \inf_{t} \{-V_t(t, x) + V_x(t, x)F_{\text{cl}}(t, x)\}
\]

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\[
\dot{x} = f(t, x) + g(t, x)\left[K(t, x) - D_xV(t, x) \cdot g(t, x) + d\right] \tag{4}
\]

is ISS with respect to actuator errors \(d\). Need \(K(t, x)\) and \(D_xV(t, x) \cdot g(t, x)\).
ISS Motivation-Part 3 of 3

Example: Assume that

\[ \dot{x} = \mathcal{F}_{\text{cl}}(t, x) := f(t, x) + g(t, x)K(t, x) \]  \hspace{1cm} (3)

is UGAS to the origin.
ISS Motivation-Part 3 of 3

Example: Assume that

\[ \dot{x} = \mathcal{F}_{\text{cl}}(t, x) := f(t, x) + g(t, x)K(t, x) \]  

(3)

is UGAS to the origin.

Assume that we have a strict Lyapunov function \( V \) so that

\[ W(x) = \inf_t \{-[V_t(t, x) + V_x(t, x)\mathcal{F}_{\text{cl}}(t, x)]\} \]

is proper.
ISS Motivation-Part 3 of 3

Example: Assume that

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\[ W(x) = \inf_t \{-[V_t(t, x) + V_x(t, x)\mathcal{F}_{\text{cl}}(t, x)]\} \] is proper.

Then

\[ \dot{x} = f(t, x) + g(t, x) \left[ K(t, x) - D_x V(t, x) \cdot g(t, x) + d \right] \]  \hspace{1cm} (4)

is ISS with respect to actuator errors \( d \).
Example: Assume that

\[ \dot{x} = \mathcal{F}_{cl}(t, x) := f(t, x) + g(t, x)K(t, x) \]  

(3)

is UGAS to the origin.

Assume that we have a strict Lyapunov function \( V \) so that

\[ W(x) = \inf_t \{ -[V_t(t, x) + V_x(t, x)\mathcal{F}_{cl}(t, x)] \} \]

is proper.

Then

\[ \dot{x} = f(t, x) + g(t, x) \left[ K(t, x) - D_x V(t, x) \cdot g(t, x) + d \right] \]  

(4)

is ISS with respect to actuator errors \( d \).

Need \( K(t, x) \) and \( D_x V(t, x) \cdot g(t, x) \).
Strictification under LaSalle Assumptions

Assume \( \dot{x} = f(x) \) has a nonstrict Lyapunov function \( V \) so that:

\[ \exists N^* > 0 \text{ s.t. } \forall q \in \mathbb{R}^n \setminus \{0\}, \exists i \in [1, N^*] \text{ s.t. } L f V(q) \neq 0. \] (NDC)

Then GAS holds.

In fact, if \( L f V(x(t, x_0)) \equiv 0 \) along some trajectory, then \( L^k f V(x(t, x_0)) \equiv 0 \) for all \( t \geq 0 \) and \( k \in \mathbb{N} \), so \( L^k f V(x_0) \equiv 0 \).

Q: Can we transform \( V \) into a strict Lyapunov function?

A: Yes, and we can allow time varying systems and relax NDC.

Let \( V \in C^\infty \) be a nonstrict Lyapunov function for \( \dot{x} = f(t, x) \), \( x \in \mathbb{R}^n \), with \( f \) and \( V \) having period \( T \) in \( t \).

Goal: Strictify it.
Strictification under LaSalle Assumptions

Assume \( \dot{x} = f(x) \) has a nonstrict Lyapunov function \( V \) so that:

\[
\exists N_* > 0 \text{ s.t. } \forall q \in \mathbb{R}^n \setminus \{0\}, \exists i \in [1, N_*] \text{ s.t. } L^i_f V(q) \neq 0. \quad (\text{NDC})
\]
Strictification under LaSalle Assumptions

Assume $\dot{x} = f(x)$ has a nonstrict Lyapunov function $V$ so that:

$\exists N_* > 0 \text{ s.t. } \forall q \in \mathbb{R}^n \setminus \{0\}, \exists i \in [1, N_*] \text{ s.t. } L_f^i V(q) \neq 0. \text{ (NDC)}$

$L_f V = (\nabla V)f$, $L_f^i V = L_f(L_f^{i-1} V)$.
Strictification under LaSalle Assumptions

Assume $\dot{x} = f(x)$ has a nonstrict Lyapunov function $V$ so that:

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Strictification under LaSalle Assumptions

Assume \( \dot{x} = f(x) \) has a nonstrict Lyapunov function \( V \) so that:

\[
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\]

\[
L_f V = (\nabla V)f, \quad L^i_f V = L_f(L^{i-1}_f V). \text{ Then GAS holds.}
\]

In fact, if \( L_f V(x(t, x_0)) \equiv 0 \) along some trajectory, then

\[
L^k_f V(x(t, x_0)) \equiv 0 \text{ for all } t \geq 0 \text{ and } k \in \mathbb{N}, \text{ so } L^k_f V(x_0) \equiv 0.
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Q:
Strictification under LaSalle Assumptions

Assume $\dot{x} = f(x)$ has a nonstrict Lyapunov function $V$ so that:

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Q: Can we transform $V$ into a strict Lyapunov function?
Strictification under LaSalle Assumptions

Assume $\dot{x} = f(x)$ has a nonstrict Lyapunov function $V$ so that:

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$L_f V = (\nabla V)f$, $L_f^i V = L_f (L_f^{i-1} V)$. Then GAS holds.

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Q: Can we transform $V$ into a strict Lyapunov function?

A:
Strictification under LaSalle Assumptions

Assume $\dot{x} = f(x)$ has a nonstrict Lyapunov function $V$ so that:

$$\exists N_* > 0 \text{ s.t. } \forall q \in \mathbb{R}^n \setminus \{0\}, \exists i \in [1, N_*] \text{ s.t. } L_f^i V(q) \neq 0.$$ (NDC)

$L_f V = (\nabla V)f$, $L_f^i V = L_f(L_f^{i-1} V)$. Then GAS holds.

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Strictification under LaSalle Assumptions

Assume $\dot{x} = f(x)$ has a nonstrict Lyapunov function $V$ so that:

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A: Yes, and we can allow time varying systems and relax NDC.

Let $V \in C^\infty$ be a nonstrict Lyapunov function for $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$, with $f$ and $V$ having period $T$ in $t$. 
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Strictification under LaSalle Assumptions

Assume \( \dot{x} = f(x) \) has a nonstrict Lyapunov function \( V \) so that:
\[
\exists N_\star > 0 \text{ s.t. } \forall q \in \mathbb{R}^n \setminus \{0\}, \exists i \in [1, N_\star] \text{ s.t. } L^i_f V(q) \neq 0. \quad \text{(NDC)}
\]
\[
L_f V = (\nabla V)f, \quad L^i_f V = L_f(L_f^{i-1} V).
\]
Then GAS holds.

In fact, if \( L_f V(x(t, x_0)) \equiv 0 \) along some trajectory, then \( L^k_f V(x(t, x_0)) \equiv 0 \) for all \( t \geq 0 \) and \( k \in \mathbb{N} \), so \( L^k_f V(x_0) \equiv 0 \).

Q: Can we transform \( V \) into a strict Lyapunov function?

A: Yes, and we can allow time varying systems and relax NDC.

Let \( V \in C^\infty \) be a nonstrict Lyapunov function for \( \dot{x} = f(t, x) \), \( x \in \mathbb{R}^n \), with \( f \) and \( V \) having period \( T \) in \( t \). Goal: Strictify it.
Strictification under LaSalle Assumptions

\[ a_1 = -\dot{V}. \]
Strictification under LaSalle Assumptions

\[ a_1 = -\dot{V}. \quad a_{i+1} = -\dot{a}_i. \]
Strictification under LaSalle Assumptions

\[ a_1 = -\dot{V}. \quad a_{i+1} = -\dot{a}_i. \quad A_j(t, x) = \sum_{m=1}^{j} a_{m+1}(t, x)a_m(t, x). \]
Strictification under LaSalle Assumptions

\[ a_1 = -\dot{V}. \quad a_{i+1} = -\dot{a}_i. \quad A_j(t, x) = \sum_{m=1}^{j} a_{m+1}(t, x)a_m(t, x). \]

Theorem 1 (MM-FM, TAC’10)
Strictification under LaSalle Assumptions

\[ a_1 = -\dot{V}. \quad a_{i+1} = -\dot{a}_i. \quad A_j(t, x) = \sum_{m=1}^j a_{m+1}(t, x)a_m(t, x). \]

Theorem 1 (MM-FM, TAC’10) Assume \( \exists \) constants \( \tau \in (0, T] \) and \( \ell \in \mathbb{N} \) and a positive definite continuous function \( \rho \) such that for all \( x \in \mathbb{R}^n \) and all \( t \in [0, \tau] \), we have the NDC condition

\[ a_1(t, x) + \sum_{m=2}^\ell a_m^2(t, x) \geq \rho(V(t, x)). \]  

(5)
Strictification under LaSalle Assumptions

\[ a_1 = -\dot{V}. \quad a_{i+1} = -\dot{a}_i. \quad A_j(t, x) = \sum_{m=1}^{j} a_{m+1}(t, x)a_m(t, x). \]

**Theorem 1 (MM-FM, TAC’10)** Assume \( \exists \) constants \( \tau \in (0, T] \) and \( \ell \in \mathbb{N} \) and a positive definite continuous function \( \rho \) such that for all \( x \in \mathbb{R}^n \) and all \( t \in [0, \tau] \), we have the NDC condition

\[
a_1(t, x) + \sum_{m=2}^{\ell} a_m^2(t, x) \geq \rho(V(t, x)). \tag{5}
\]

Then we can explicitly determine functions \( \mathcal{F}_j \) and \( \mathcal{G} \) such that

\[
V^\#(t, x) = \sum_{j=1}^{\ell-1} \mathcal{F}_j(V(t, x))A_j(t, x) + \mathcal{G}(t, V(t, x)) \tag{6}
\]

is a strict Lyapunov function, giving UGAS of the dynamics.
Second Construction for \( \dot{x} = f(x) \), \( x \in X \)
Second Construction for \( \dot{x} = f(x), x \in \mathcal{X} \)

This Matrosov approach constructs auxiliary functions.
Second Construction for $\dot{x} = f(x)$, $x \in \mathcal{X}$

This Matrosov approach constructs auxiliary functions.

**Assumption A**  There are functions $h_j$ such that $h_j(0) = 0$ for all $j$; everywhere positive functions $r_1, \ldots, r_m$ and $\rho$; a proper positive definite function $V_1 : \mathcal{X} \to [0, \infty)$; and an integer $N > 0$ for which

$$\nabla V_1(x)f(x) \leq -r_1(x)h_1^2(x) - \cdots - r_m(x)h_m^2(x) \quad \forall x \in \mathcal{X} \quad (7)$$

Also, $f \in C^\infty(\mathbb{R}^n)$, and $V_1$ has a positive definite quadratic lower bound in some neighborhood of $0 \in \mathbb{R}^n$. 
Second Construction for $\dot{x} = f(x)$, $x \in \mathcal{X}$

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and

$$\sum_{k=0}^{N-1} \sum_{j=1}^{m} \left[ L_f^k h_j(x) \right]^2 \geq \rho(V_1(x))V_1(x) \quad \forall x \in \mathcal{X}. \quad (8)$$

Also, $f \in C^\infty(\mathbb{R}^n)$, and $V_1$ has a positive definite quadratic lower bound in some neighborhood of $0 \in \mathbb{R}^n$. 

Second Construction for $\dot{x} = f(x)$, $x \in \mathcal{X}$

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Second Construction for $\dot{x} = f(x), x \in \mathcal{X}$
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**Theorem 2 (MM-FM, TAC’10)** Assume that $\dot{x} = f(x)$ satisfies Assumption A.
Second Construction for $\dot{x} = f(x), \ x \in \mathcal{X}$

**Theorem 2 (MM-FM, TAC’10)** Assume that $\dot{x} = f(x)$ satisfies Assumption A. Set

$$V_i(x) = - \sum_{\ell=1}^{m} \left( \frac{L_f^{i-2} h_{\ell}(x)}{L_f^{i-1} h_{\ell}(x)} \right) \left( \frac{L_f^{i-1} h_{\ell}(x)}{L_f^{i} h_{\ell}(x)} \right), \ i = 2, \ldots, N.$$  \hspace{1cm} (9)
Second Construction for $\dot{x} = f(x), \ x \in \mathcal{X}$

Theorem 2 (MM-FM, TAC’10) Assume that $\dot{x} = f(x)$ satisfies Assumption A. Set

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One can determine explicit functions $k_\ell, \Omega_\ell \in \mathcal{K}_\infty \cap \mathcal{C}^1$ such that

$$S(x) = \sum_{\ell=1}^{N} \Omega_\ell \left( k_\ell(V_1(x)) + V_\ell(x) \right)$$ \hspace{1cm} (10)

is a strict Lyapunov function on $\mathcal{X}$ satisfying $S(x) \geq V_1(x)$ on $\mathcal{X}$. 

Significance:
Second Construction for $\dot{x} = f(x)$, $x \in \mathcal{X}$

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One can determine explicit functions $k_\ell, \Omega_\ell \in \mathcal{K}_\infty \cap C^1$ such that

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**Significance:** New theorem says which functions $V_i$ to pick.
Second Construction for $\dot{x} = f(x), \ x \in \mathcal{X}$

**Theorem 2 (MM-FM, TAC’10)** Assume that $\dot{x} = f(x)$ satisfies Assumption A. Set

$$V_i(x) = -\sum_{\ell=1}^{m} \left(L_{f}^{i-2} h_{\ell}(x)\right) \left(L_{f}^{i-1} h_{\ell}(x)\right), \quad i = 2, \ldots, N.$$  \hfill (9)

One can determine explicit functions $k_{\ell}, \Omega_{\ell} \in \mathcal{K}_{\infty} \cap C^1$ such that

$$S(x) = \sum_{\ell=1}^{N} \Omega_{\ell} \left(k_{\ell}(V_1(x)) + V_{\ell}(x)\right)$$  \hfill (10)

is a strict Lyapunov function on $\mathcal{X}$ satisfying $S(x) \geq V_1(x)$ on $\mathcal{X}$.

**Significance:** Allows any open state space $\mathcal{X}$ containing $0 \in \mathbb{R}^n$. 

Second Construction for $\dot{x} = f(x), \ x \in \mathcal{X}$

**Theorem 2 (MM-FM, TAC’10)** Assume that $\dot{x} = f(x)$ satisfies Assumption A. Set

$$V_i(x) = - \sum_{\ell=1}^{m} \left( L_i^{i-2} h_\ell(x) \right) \left( L_i^{i-1} h_\ell(x) \right), \ i = 2, \ldots, N. \tag{9}$$

One can determine explicit functions $k_\ell, \Omega_\ell \in \mathcal{K}_\infty \cap \mathcal{C}^1$ such that

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**Significance:** Readily extends to time periodic t-v systems.
Biological Application: Lotka-Volterra Dynamics

\[ \dot{x} = \gamma x \left(1 - \frac{x}{L}\right) - a x \zeta \]
\[ \dot{\zeta} = \beta x \zeta - \Delta \zeta \]

\[ x = \text{prey}. \]
\[ \zeta = \text{predator}. \]
\[ a, \beta, \gamma, \Delta, L = \text{positive constants}. \]

Change coordinates and rescale to get the error dynamics

\[ \dot{\tilde{x}} = -[\tilde{x} + \alpha \tilde{y}](\tilde{x} + x^*) \]
\[ \dot{\tilde{y}} = \alpha \tilde{x}(\tilde{y} + y^*) \]

with state space
\[ X = (-\infty, +\infty) \times (-\infty, +\infty) \]
\[ \alpha = \frac{\beta L}{\gamma}, \quad \Delta = \frac{\Delta}{\gamma}, \quad x^* = \frac{\alpha}{\alpha + \Delta} \]
\[ y^* = \frac{1}{\alpha} - \frac{\alpha + \Delta}{\alpha} \]

Assume \( \alpha > \Delta \).

Want a global strict Lyapunov function for (12).
Biological Application: Lotka-Volterra Dynamics

\[
\begin{align*}
\dot{\chi} &= \gamma \chi \left(1 - \frac{\chi}{L}\right) - a\chi \zeta \\
\dot{\zeta} &= \beta \chi \zeta - \Delta \zeta
\end{align*}
\] (11)
Biological Application: Lotka-Volterra Dynamics

\[
\begin{aligned}
\dot{\chi} &= \gamma \chi (1 - \frac{\chi}{L}) - a \chi \zeta \\
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\end{aligned}
\]

\(\zeta = \text{predator}. \ \chi = \text{prey.}\)
Biological Application: Lotka-Volterra Dynamics

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\end{align*}
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(11)

\[\zeta = \text{predator. } \chi = \text{prey. } a, \beta, \gamma, \Delta, L = \text{positive constants.}\]
Biological Application: Lotka-Volterra Dynamics

\[
\begin{align*}
\dot{\chi} &= \gamma \chi (1 - \frac{\chi}{L}) - a \chi \xi \\
\dot{\xi} &= \beta \chi \xi - \Delta \xi
\end{align*}
\]  \hspace{1cm} (11)

\(\xi = \text{predator. } \chi = \text{prey. } a, \beta, \gamma, \Delta, L = \text{positive constants.}\)

Change coordinates and rescale to get the error dynamics

\[
\begin{align*}
\dot{x} &= -[\ddot{x} + \alpha \ddot{y}](\ddot{x} + x_*) \\
\dot{y} &= \alpha \ddot{x}(\ddot{y} + y_*)
\end{align*}
\]  \hspace{1cm} (12)

with state space \(\mathcal{X} = (-x_*, +\infty) \times (-y_*, +\infty),\)
Biological Application: Lotka-Volterra Dynamics

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\dot{\chi} &= \gamma \chi \left(1 - \frac{\chi}{L}\right) - a\chi\zeta \\
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Change coordinates and rescale to get the error dynamics

\[
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\dot{\tilde{x}} &= -[\tilde{x} + \alpha \tilde{y}](\tilde{x} + x^*) \\
\dot{\tilde{y}} &= \alpha \tilde{x}(\tilde{y} + y^*),
\end{align*}
\]  (12)

with state space \(\mathcal{X} = (-x^*, +\infty) \times (-y^*, +\infty),\)

\[
\alpha = \frac{\beta L}{\gamma}, \quad d = \frac{\Delta}{\gamma}, \quad x^* = \frac{d}{\alpha} \quad \text{and} \quad y^* = \frac{1}{\alpha} - \frac{d}{\alpha^2}.\]  (13)
Biological Application: Lotka-Volterra Dynamics

\[
\begin{align*}
\dot{\chi} &= \gamma \chi (1 - \frac{\chi}{L}) - a \chi \zeta \\
\dot{\zeta} &= \beta \chi \zeta - \Delta \zeta
\end{align*}
\]  
(11)

\(\zeta = \) predator. \(\chi = \) prey. \(a, \beta, \gamma, \Delta, L = \) positive constants.

Change coordinates and rescale to get the error dynamics

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\begin{align*}
\dot{\tilde{x}} &= -[\tilde{x} + \alpha \tilde{y}](\tilde{x} + x^*) \\
\dot{\tilde{y}} &= \alpha \tilde{x}(\tilde{y} + y^*)
\end{align*}
\]  
(12)

with state space \(\mathcal{X} = (-x^*, +\infty) \times (-y^*, +\infty)\),

\[
\alpha = \frac{\beta L}{\gamma}, \quad d = \frac{\Delta}{\gamma}, \quad x^* = \frac{d}{\alpha} \text{ and } y^* = \frac{1}{\alpha} - \frac{d}{\alpha^2}.
\]  
(13)

Assume \(\alpha > d\).
Biological Application: Lotka-Volterra Dynamics

\[
\begin{align*}
\dot{\chi} &= \gamma \chi \left(1 - \frac{\chi}{L}\right) - a \chi \zeta \\
\dot{\zeta} &= \beta \chi \zeta - \Delta \zeta 
\end{align*}
\]

(11)

\(\zeta = \text{predator}. \ \chi = \text{prey}. \ a, \beta, \gamma, \Delta, L = \text{positive constants}.

Change coordinates and rescale to get the error dynamics

\[
\begin{align*}
\dot{\tilde{x}} &= -[\tilde{x} + \alpha \tilde{y}] (\tilde{x} + x_*) \\
\dot{\tilde{y}} &= \alpha \tilde{x} (\tilde{y} + y_*) 
\end{align*}
\]

(12)

with state space \(X = (-x_*, +\infty) \times (-y_*, +\infty)\),

\[
\alpha = \frac{\beta L}{\gamma}, \quad d = \frac{\Delta}{\gamma}, \quad x_* = \frac{d}{\alpha} \quad \text{and} \quad y_* = \frac{1}{\alpha} - \frac{d}{\alpha^2}.
\]

(13)

Assume \(\alpha > d\). Want a global strict Lyapunov function for (12).
Use of Theorem 2

There are many Lyapunov constructions for Lotka-Volterra models available based on computing the LaSalle invariant set. By contrast, our result provides a global strict Lyapunov function.

\[ V_1(\tilde{x}, \tilde{y}) = \tilde{x} - x^* \ln \left( 1 + \frac{\tilde{x}}{x^*} \right) + \tilde{y} - y^* \ln \left( 1 + \frac{\tilde{y}}{y^*} \right) \]  

(14)

Nonstrict Lyapunov decay condition:

\[ \dot{V}_1(\tilde{x}, \tilde{y}) \leq -|\tilde{x}|^2. \]

Auxiliary function from theorem:

\[ V_2(\tilde{x}, \tilde{y}) = \tilde{x}(\tilde{x} + \alpha \tilde{y})(\tilde{x} + x^*). \]
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There are many Lyapunov constructions for Lotka-Volterra models available based on computing the LaSalle invariant set. By contrast, our result provides a *global strict Lyapunov function*.

\[
V_1(\tilde{x}, \tilde{y}) = \tilde{x} - x_\ast \ln \left( 1 + \frac{\tilde{x}}{x_\ast} \right) + \tilde{y} - y_\ast \ln \left( 1 + \frac{\tilde{y}}{y_\ast} \right) \tag{14}
\]

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Auxiliary function from theorem:
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Nonstrict Lyapunov decay condition: \( \dot{V}_1(\tilde{x}, \tilde{y}) \leq -|\tilde{x}|^2. \)

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Strict Lyapunov Function Construction (MM-FM)
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\[
S(\tilde{x}, \tilde{y}) = V_2(\tilde{x}, \tilde{y}) + \int_0^{V_1(\tilde{x}, \tilde{y})} \phi_1(r) \, dr \\
+ [\rho_1(V_1(\tilde{x}, \tilde{y})) + 1] V_1(\tilde{x}, \tilde{y}),
\]  
(15)
Strict Lyapunov Function Construction (MM-FM)

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\]

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where

\[
\phi_1(r) = 2 \left[ \left( 289 x_* + 144 \alpha y_* \right)^2 + 144 \alpha^2 x_* y_* \right] e^{2 \left( \frac{1}{x_*} + \frac{1}{y_*} \right) r}
\]
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and

\[ p_1(r) = 1536(x_* + 1)(\alpha + 1)(1 + x_* + y_*)^4(1 + r)^3. \]
Strict Lyapunov Function Construction (MM-FM)

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Along the trajectories of the L-V error dynamics,

\[ \dot{S}(t, x) \leq -\frac{1}{4} \left[ \tilde{x}^2 + \left\{ (\tilde{x} + \alpha \tilde{y})(\tilde{x} + x_*) \right\}^2 \right]. \]
Conclusions
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- While UGAS can sometimes be proven using nonstrict Lyapunov functions, strict Lyapunov functions can give ISS.
- The LaSalle and Matrosov approaches transform nonstrict Lyapunov functions into strict ones.

Thank you for your attention and interest!
Conclusions

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