Stabilization in a Chemostat with Sampled and Delayed Measurements

Michael Malisoff

Joint with Jerome Harmand and Frederic Mazenc
Background and Motivation

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Rate of change of organism = growth - washout.
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\begin{align*}
    s' &= (s_{in} - s)D - \frac{ms}{a+s} \frac{x}{\gamma} \\
    x' &= x \left( \frac{ms}{a+s} - D \right)
\end{align*}
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\[ X'(t) = G(t, X(t), X(t - \tau(t))), \quad X(t) \in \mathcal{X} \]  \hspace{1cm} (\Sigma)
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\[ X'(t) = \mathcal{G}(t, X(t), X(t - \tau(t))), \quad X(t) \in \mathcal{X} \]  \hspace{1cm} (\Sigma)

\[ |X(t)| \leq \gamma_1 \left( e^{t_0 - t} \gamma_2(|X|_{[t_0 - \bar{\tau}, t_0]}) \right) \]  \hspace{1cm} (UGAS)

\( \gamma_i \in \mathcal{K}_\infty: 0 \text{ at } 0, \text{ strictly increasing, unbounded.} \sup_{t \geq 0} \tau(t) \leq \bar{\tau}. \)
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Find \( \gamma_i \)'s by building Lyapunov-Krasovskii functionals (LKFs).
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Find \( \gamma_i \)'s by building Lyapunov-Krasovskii functionals (LKFs).

Equivalent to \( \mathcal{KL} \) formulation; see Sontag 1998 SCL paper.
Uncertain Controlled Chemostat with Sampling

\[
\begin{aligned}
\dot{s}(t) &= D(s(t - \tau(t)))[s_{\text{in}} - s(t)] - (1 + \delta(t))\mu(s(t))x(t) \\
\dot{x}(t) &= [(1 + \delta(t))\mu(s(t)) - D(s(t - \tau(t)))]x(t)
\end{aligned}
\]  
\tag{C}

\[\tau(t) = \begin{cases} 
\tau_f, & t \in [0, \tau_f) \\
\tau_f + t - t_j, & t \in [t_j + \tau_f, t_{j+1} + \tau_f) \text{ and } j \geq 0
\end{cases}\]

0 < \epsilon_1 \leq t_{i+1} - t_i \leq \epsilon_2. \quad \delta : [0, \infty) \rightarrow [d, \infty), \text{ with } d \in (-1, 0].
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\[\mu(s) = \frac{\mu_1(s)}{1+\gamma(s)}, \text{ with a unique maximizer } s_M \in (0, s_{in}]\]
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Assumption 1: The function \(\mu\) is \(C^1\) and \(\mu(0) = 0\). Also, there is a constant \(s_M \in (0, s_{\text{in}}]\) such that \(\mu'(s) > 0\) for all \(s \in [0, s_M]\) and \(\mu'(s) \leq 0\) for all \(s \in [s_M, \infty)\). Finally, \(\mu(s) > 0\) for all \(s > 0\).
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\[\mu(s) \overset{(\ast)}{=} \frac{\mu_1(s)}{1 + \gamma(s)}, \text{ with a unique maximizer } s_M \in (0, s_{in}]\]

**Lemma:** Under Assumption 1, there are \(\mu_1 \in C^1 \cap \mathcal{K}_\infty\) and a nondecreasing \(C^1\) function \(\gamma : \mathbb{R} \rightarrow [0, \infty)\) such that \((\ast)\) holds for all \(s \geq 0\), \(\mu'_1(s) > 0\) on \([0, \infty)\), and \(\gamma'(s) > 0\) on \([s_M, \infty)\).
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\((1 + \delta)\mu(s)\) for Different Constant \(\delta\) Choices, \(s_M = 1/\sqrt{2}\) and \(s_{\text{in}} = 1\)
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\[\mu(s) \overset{(*)}{=} \frac{\mu_1(s)}{1 + \gamma(s)}, \text{ with a unique maximizer } s_M \in (0, s_{in}]\]

**Lemma:** Under Assumption 1, there are \(\mu_1 \in C^1 \cap K_\infty\) and a nondecreasing \(C^1\) function \(\gamma : \mathbb{R} \rightarrow [0, \infty)\) such that \((*)\) holds for all \(s \geq 0, \mu_1'(s) > 0\) on \([0, \infty)\), and \(\gamma'(s) > 0\) on \([s_M, \infty)\).
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Goal: Under suitable conditions on an upper bound \(\tau_M\) for the delay \(\tau(t)\), and for constants \(s_* \in (0, s_{\text{in}})\), design the control \(D\) to render the dynamics for \(X(t) = (s(t), x(t)) - (s_*, s_{\text{in}} - s_*)\) ISS.
Main Result for Unperturbed Case

\[ \omega_s = \inf_{s \in [0,s_{in}]} \mu'_1(s), \quad \omega_l = \sup_{s \in [0,s_{in}]} \mu'_1(s), \quad \rho_l = \sup_{s \in [0,s_{in}]} \gamma'(s), \]

\[ \rho_m = \frac{\rho_l^2}{2\omega_s} \max_{l \in [0,s_{in}]} \frac{\mu_1^2(l+1.1\mu_1(s_*)s_{in}\tau_M)}{1+\gamma(l)}, \quad \text{where } \mu(s) = \frac{\mu_1(s)}{1+\gamma(s)} \]
Main Result for Unperturbed Case

\[ \omega_s = \inf_{s \in [0,s_{in}]} \mu_1'(s), \quad \omega_I = \sup_{s \in [0,s_{in}]} \mu_1'(s), \quad \rho_I = \sup_{s \in [0,s_{in}]} \gamma'(s), \]

\[ \rho_m = \frac{\rho_I^2}{2\omega_s} \max_{l \in [0,s_{in}]} \frac{\mu_1^2(l+1.1\mu_1(s_*)s_{in}\tau_M)}{1+\gamma(l)}, \text{ where } \mu(s) = \frac{\mu_1(s)}{1+\gamma(s)} \]

Assume that \( \frac{\mu_1(s_{in})}{1+\gamma(s_{in})} - \frac{\mu_1(s_*)}{1+\gamma(s_{in}-\mu_1(s_*)s_{in}\tau_M)} > 0 \) \( (a) \)

and \( \tau_M < \max \left\{ \frac{1}{2s_{in}\sqrt{2\rho_m\omega_I}}, \frac{1}{2\rho_I s_{in}\mu_1(s_{in})} \right\} \), with \( s_* < s_{in} \).
Main Result for Unperturbed Case

$$\omega_s = \inf_{s \in [0, s_{\text{in}}]} \mu'_1(s), \quad \omega_l = \sup_{s \in [0, s_{\text{in}}]} \mu'_1(s), \quad \rho_l = \sup_{s \in [0, s_{\text{in}}]} \gamma'(s),$$

$$\rho_m = \frac{\rho_l^2}{2\omega_s} \max_{l \in [0, s_{\text{in}}]} \frac{\mu_1^2(l+1.1\mu_1(s_*))s_{\text{in}}\tau_M}{1+\gamma(l)}, \quad \text{where} \quad \mu(s) = \frac{\mu_1(s)}{1+\gamma(s)}$$

Assume that

$$\frac{\mu_1(s_{\text{in}})}{1+\gamma(s_{\text{in}})} - \frac{\mu_1(s_*)}{1+\gamma(s_{\text{in}}-\mu_1(s_*)s_{\text{in}}\tau_M)} > 0$$

and

$$\tau_M < \max \left\{ \frac{1}{2s_{\text{in}}\sqrt{2\rho_m\omega_l}}, \frac{1}{2\rho_l s_{\text{in}} \mu_1(s_{\text{in}})} \right\}, \quad \text{with} \ s_* < s_{\text{in}}.$$

**Theorem 1:** For all componentwise positive initial conditions, all solutions of the chemostat system (C) with $\delta(t) = 0$ and

$$D(s(t - \tau(t))) = \frac{\mu_1(s_*)}{1+\gamma(s(t-\tau(t)))}$$

remain in $(0, \infty)^2$ and converge to $(s_*, s_{\text{in}} - s_*)$. \[\square\]
Proof Outline for $\delta = 0$ Unperturbed Case

Step 1: For any fixed $\bar{s} \geq s_{in}$, show that $z = s_{in} - s - x$ satisfies

$$|z(t)| \leq |z(0)| e^{-t\mu_1(s_*)}$$

for all $t \geq 0$. (ES)
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**Step 1:** For any fixed $\bar{s} \geq s_{\text{in}}$, show that $z = s_{\text{in}} - s - x$ satisfies

\[
|z(t)| \leq |z(0)| e^{-t\mu_1(s_*)} \quad \text{for all } t \geq 0. \tag{ES}
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$z = -(x - s_{\text{in}} + s_*) - (s - s_*) = -X_2 - X_1$. $X$= error variable.
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Step 1: For any fixed $\bar{s} \geq s_{in}$, show that $z = s_{in} - s - x$ satisfies

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for all $t \geq 0$. (ES)

$z = -(x - s_{in} + s_*) - (s - s_*) = -X_2 - X_1$. $X =$ error variable.

Step 2: Build $T \in K_\infty$ and a constant $\bar{c} > 0$ such that

$$U_1(s) = \int_0^{s-s_*} \frac{m}{s_{in}-s_*-m} dm, \quad (2)$$

satisfies

$$\dot{U}_1(t) \leq \frac{(s(t)-s_*)(\mu_1(s_*)-\mu_1(s(t)))}{2[1+\gamma(s(t-\tau(t)))]}$$

$$+ \rho m \tau M \int_{t-\tau(t)}^t (\dot{s}(m))^2 dm + \bar{c} |s(t) - s_*| |z(t)|$$

for all $t \geq T(||X(0)||)$ where $X(t) = (s(t), x(t)) - (s_*, s_{in} - s_*)$. 
Proof Outline for $\delta = 0$ Unperturbed Case

Step 3: Find constants $c_i > 0$ such that

$$U_2(s_t) = \int_0^{s(t)-s_*} \frac{m}{s_{in}-s_*-m} dm + 2\rho m \tau M \int_{t-M}^{t-M} \int_{\ell}^{t} (\dot{s}(m))^2 dm d\ell. \quad (4)$$

satisfies

$$\dot{U}_2(t) \leq -c_1 U_2(s_t) + c_2 z^2(t) + \bar{c} |s(t) - s_*| |z(t)| \quad (5)$$

for all $t \geq T(|X(0)||)$. 
Proof Outline for $\delta = 0$ Unperturbed Case

**Step 3:** Find constants $c_i > 0$ such that

$$U_2(s_t) = \int_0^{s(t) - s_*} \frac{m}{s_{in} - s_* - m} \, dm + 2\rho m\tau M \int_{t - \tau}^t \int_\ell \dot{s}(m)^2 \, dm \, d\ell. \quad (4)$$

satisfies

$$\dot{U}_2(t) \leq -c_1 U_2(s_t) + c_2 z^2(t) + \bar{c} \lVert s(t) - s_* \rVert \|z(t)\| \quad (5)$$

for all $t \geq T(\|X(0)\|)$.

**Step 4:** The sum $U_3$ of a quadratic Lyapunov function for the $z$ variable and $U_2$ admits a constant $c_3 > 0$ such that

$$\dot{U}_3(t) \leq -c_3 U_3(s_t, z(t)) \quad (6)$$

for all $t \geq T(\|X(0)\|)$. 
Extensions and Applications

\[
\begin{align*}
\text{ISS with respect to } \delta(t) \text{ without upper bounds on } |\delta|_{\infty} & \\
= & \left(1 + \delta(t)\right) \mu(s(t)) \\
> & 0 \\
\end{align*}
\]

Functions \(\gamma_i\) from ISS condition measure distance from equilibria at all times, providing both transient and asymptotic information.

Extensions and Applications

ISS with respect to $\delta(t)$ without upper bounds on $|\delta|_\infty$...
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\frac{(1+d)\mu_1(s_{in})}{1+\gamma(s_{in})} - \frac{\mu_1(s_*)}{1+\gamma(s_{in}-\mu_1(s_*)s_{in}\tau_M)} > 0 \quad (a)
\]

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\[ \mathcal{U}_2(s_t) = \int_{s(t) - s_*} s(t) - s_* \frac{m}{s_{in} - s_* - m} dm + 2\rho_m \tau_M \int_{t-\tau_M}^{t} \int_{\ell}^{t} (\dot{s}(m))^2 dm d\ell. \]
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Mathematica Simulations of (C)

\[ s_{in} = 1, \quad \mu(s) = \frac{0.5s}{1+0.25s+2s^2}, \quad t_j = 0.24j, \quad \delta(t) = 0. \]

\( s(t) \) in Red, \( x(t) \) in Blue, \( D(t) \) in Green.
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\[ s_{in} = 1, \quad \mu(s) = \frac{0.5s}{1+0.25s+2s^2}, \quad t_j = 0.24j, \quad \delta(t) = 0.15(1 + \sin(t)). \]

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Thank you for your attention!
References with Hyperlinked Paper Titles


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