Stabilization in a Chemostat with Sampled and Delayed Measurements

Michael Malisoff

Joint with Jerome Harmand and Frederic Mazenc

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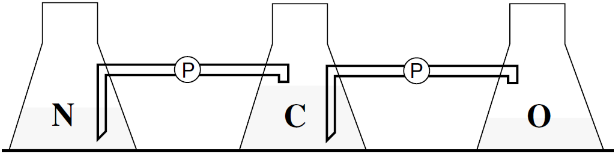

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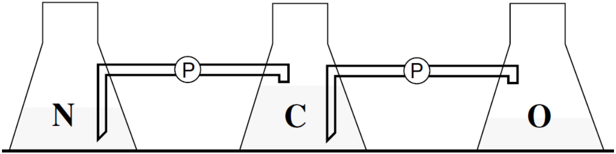

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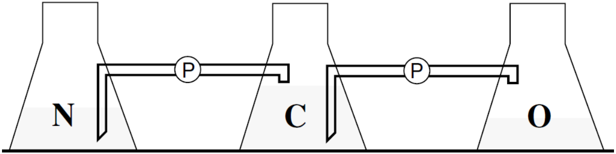

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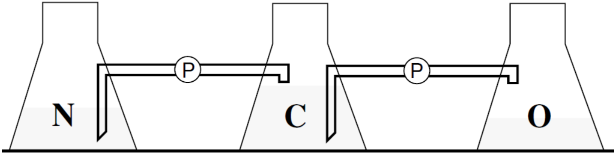

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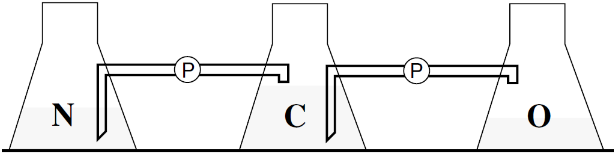

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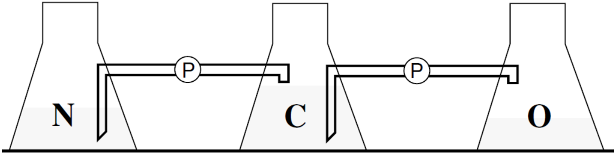

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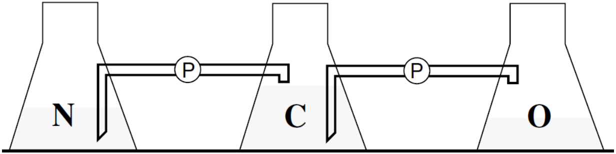

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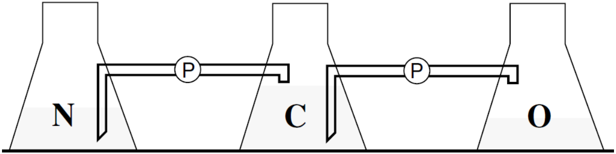

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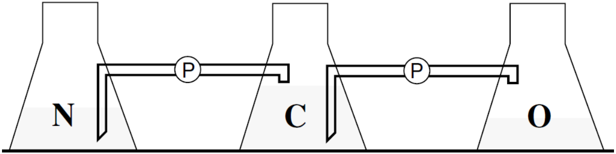

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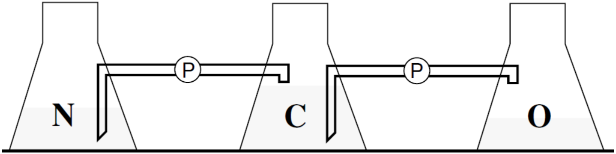

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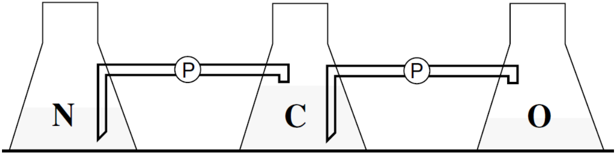

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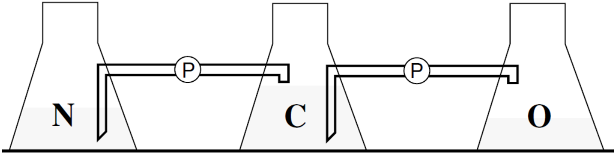

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Equivalent to $\mathcal{KL}$ formulation; see Sontag 1998 SCL paper.

Uncertain Controlled Chemostat with Sampling

$$\begin{cases} \dot{s}(t) = D(s(t-\tau(t))[s_{\text{in}} - s(t)] - (1+\delta(t))\mu(s(t))x(t) \\ \dot{x}(t) = [(1+\delta(t))\mu(s(t)) - D(s(t-\tau(t))]x(t) \end{cases} \quad \text{(C)}$$

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Assumption 1: The function μ is C^1 and $\mu(0) = 0$. Also, there is a constant $s_M \in (0, s_{\text{in}}]$ such that $\mu'(s) > 0$ for all $s \in [0, s_M)$ and $\mu'(s) \leq 0$ for all $s \in [s_M, \infty)$. Finally, $\mu(s) > 0$ for all $s > 0$.

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Lemma: Under Assumption 1, there are $\mu_1 \in C^1 \cap \mathcal{K}_\infty$ and a nondecreasing C^1 function $\gamma : \mathbb{R} \to [0, \infty)$ such that $(\star)$ holds for all $s \geq 0$, $\mu_1'(s) > 0$ on $[0, \infty)$, and $\gamma'(s) > 0$ on $[s_M, \infty)$.

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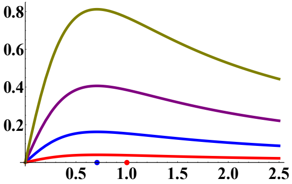

$(1+\delta)\mu(s)$ for Different Constant δ Choices, $s_M = 1/\sqrt{2}$ and $s_{\text{in}} = 1$

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Goal: Under suitable conditions on an upper bound τ_M for the delay $\tau(t)$, and for constants $s_* \in (0, s_{\text{in}})$, design the control D to render the dynamics for $X(t) = (s(t), x(t)) - (s_*, s_{\text{in}} - s_*)$ ISS.

Main Result for Unperturbed Case

$$\omega_s = \inf_{s\in[0,s_{\mathrm{in}}]} \mu_1'(s)\ , \ \ \omega_l = \sup_{s\in[0,s_{\mathrm{in}}]} \mu_1'(s)\ , \ \ \rho_l = \sup_{s\in[0,s_{\mathrm{in}}]} \gamma'(s),$$

$$\rho_m = \frac{\rho_l^2}{2\omega_s} \max_{l\in[0,s_{\mathrm{in}}]} \frac{\mu_1^2(l+1.1\mu_1(s_*)s_{\mathrm{in}}\tau_M)}{1+\gamma(l)}, \ \text{where} \ \mu(s) = \frac{\mu_1(s)}{1+\gamma(s)}$$

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Assume that $\frac{\mu_1(s_{\text{in}})}{1+\gamma(s_{\text{in}})} - \frac{\mu_1(s_*)}{1+\gamma(s_{\text{in}}-\mu_1(s_*)s_{\text{in}}\tau_M)} \overset{(a)}{>} 0$

and $\tau_M \overset{(b)}{<} \max\left\{\frac{1}{2s_{\text{in}}\sqrt{2\rho_m\omega_l}}, \frac{1}{2\rho_l s_{\text{in}}\mu_1(s_{\text{in}})}\right\}$, with $s_* < s_{\text{in}}$.

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Theorem 1: For all componentwise positive initial conditions, all solutions of the chemostat system (C) with $\delta(t) = 0$ and

$$D(s(t-\tau(t))) = \frac{\mu_1(s_*)}{1+\gamma(s(t-\tau(t)))} \qquad (1)$$

remain in $(0,\infty)^2$ and converge to $(s_*, s_{\text{in}} - s_*)$. □

Proof Outline for $\delta = 0$ Unperturbed Case

Step 1: For any fixed $\bar{s} \geq s_{\text{in}}$, show that $z = s_{\text{in}} - s - x$ satisfies

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Step 2: Build $\mathcal{T} \in \mathcal{K}_\infty$ and a constant $\bar{c} > 0$ such that

$$\mathcal{U}_1(s) = \int_0^{s-s_*} \frac{m}{s_{\rm in}-s_*-m} \mathrm{d}m, \tag{2}$$

satisfies

$$\begin{aligned} \dot{\mathcal{U}}_1(t) \quad \leq \quad & \frac{(s(t)-s_*)(\mu_1(s_*)-\mu_1(s(t)))}{2[1+\gamma(s(t-\tau(t)))]} \\ & + \rho_m \tau_M \int_{t-\tau(t)}^{t} (\dot{s}(m))^2 \mathrm{d}m + \bar{c}|s(t) - s_*||z(t)| \end{aligned} \tag{3}$$

for all $t \geq \mathcal{T}(|X(0)|)$ where $X(t) = (s(t), x(t)) - (s_*, s_{\rm in} - s_*)$.

Proof Outline for $\delta = 0$ Unperturbed Case

Step 3: Find constants $c_i > 0$ such that

$$\mathcal{U}_2(s_t) = \int_0^{s(t)-s_*} \frac{m}{s_{\rm in} - s_* - m} \mathrm{d}m + 2\rho_m \tau_M \int_{t-\tau_M}^{t} \int_{\ell}^{t} (\dot{s}(m))^2 \mathrm{d}m \, \mathrm{d}\ell. \quad (4)$$

satisfies

$$\dot{\mathcal{U}}_2(t) \leq -c_1 \mathcal{U}_2(s_t) + c_2 z^2(t) + \bar{c}|s(t) - s_*||z(t)| \quad (5)$$

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Step 4: The sum $\mathcal{U}_3$ of a quadratic Lyapunov function for the z variable and $\mathcal{U}_2$ admits a constant $c_3 > 0$ such that

$$\dot{\mathcal{U}}_3(t) \;\leq\; -c_3 \mathcal{U}_3(s_t, z(t)) \tag{6}$$

for all $t \geq \mathcal{T}(|X(0)|)$.

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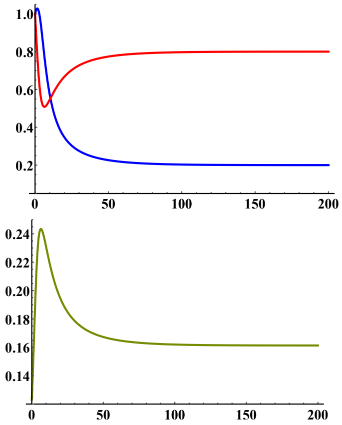
Mazenc, F., J. Harmand, and M. Malisoff, "Stabilization in a chemostat with sampled and delayed measurements and uncertain growth functions," *Automatica*, 2017.

Mathematica Simulations of (C)



$$s_{\text{in}} = 1,\ \mu(s) = \frac{0.5s}{1+0.25s+2s^2},\ t_j = 0.24j,\ \delta(t) = 0.$$

$s(t)$ in Red, $x(t)$ in Blue, $D(t)$ in Green.

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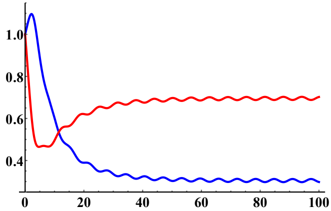

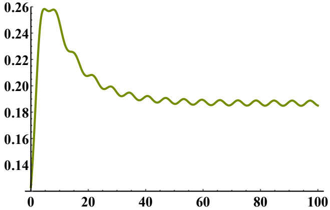

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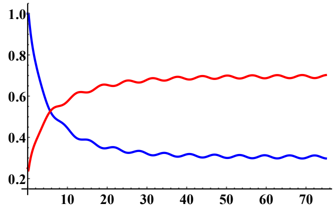

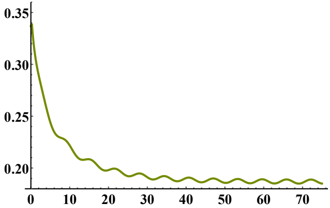

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Thank you for your attention!

References with Hyperlinked Paper Titles

Mazenc, F., G. Robledo, and M. Malisoff, “Stability and robustness analysis for a multispecies chemostat model with delays in the growth rates and uncertainties," *Discrete and Continuous Dynamical Systems Series B*, to appear.

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