Lyapunov Functions under LaSalle Conditions with an Application to Lotka-Volterra Systems

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Converse Lyapunov theory guarantees the *existence* of strict Lyapunov functions in many cases.

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Using LaSalle Invariance, we can often use nonstrict Lyapunov functions to prove stability.

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However, explicit strict Lyapunov function *constructions* are often needed in applications.

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First Construction

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Let $V \in C^\infty$ be a nonstrict Lyapunov function for $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$, with f and V having period T in t.

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$a_1 = -\dot{V}$ and $a_{i+1} = -\dot{a}_i$ for $i \geq 1$.
$A_j(t,x) = \sum_{m=1}^{j} a_{m+1}(t,x) a_m(t,x)$.

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Theorem 1

Assume $\exists$ constants $\tau \in (0, T]$ and $\ell \in \mathbb{N}$ and a positive definite continuous function ρ such that for all $x \in \mathbb{R}^n$ and all $t \in [0, \tau]$,

$$a_1(t,x) + \sum_{m=2}^{\ell} a_m^2(t,x) \geq \rho(V(t,x)) \ . \quad (1)$$

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Then we can explicitly determine functions $\mathcal{F}_j$ and $\mathcal{G}$ such that

$$V^\sharp(t,x) = \sum_{j=1}^{\ell-1} \mathcal{F}_j\big(V(t,x)\big) A_j(t,x) + \mathcal{G}\big(t, V(t,x)\big) \tag{2}$$

is a strict Lyapunov function, giving UGAS of the dynamics.

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Significance: The construction can be done locally near the origin, when the assumptions only hold near the origin.

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Significance: The function (2) is a simple weighted sum involving the easily calculated iterated Lie derivatives a_i.

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is a strict Lyapunov function, giving UGAS of the dynamics.

Significance: Simpler than Mazenc-Nesic-TAC'04, which is limited to time invariant systems under a stronger NDC.

Second Construction for $\dot{x} = f(x)$, $x \in \mathcal{X}$

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Assumptions 1

There exist a storage function $V_1 : \mathcal{X} \to [0, \infty)$*; functions* $h_1, \ldots, h_m$ *such that* $h_j(0) = 0$ *for all* j*; everywhere positive functions* $r_1, \ldots, r_m$ *and* ρ*; and an integer* $N > 0$ *for which*

$$\nabla V_1(x) f(x) \ \leq \ -r_1(x) h_1^2(x) - \ldots - r_m(x) h_m^2(x) \ \ \forall x \in \mathcal{X} \tag{3}$$

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$$\text{and} \quad \sum_{k=0}^{N-1} \sum_{j=1}^{m} \left[L_f^k h_j(x) \right]^2 \ \geq \ \rho(V_1(x)) V_1(x) \ \ \forall x \in \mathcal{X}. \tag{4}$$

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Also, $f \in C^\infty(\mathbb{R}^n)$*, and* V_1 *has a positive definite quadratic lower bound in some neighborhood of* $0 \in \mathbb{R}^n$*.*

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$$V_i(x) = -\sum_{\ell=1}^{m} L_f^{i-2} h_\ell(x) L_f^{i-1} h_\ell(x) \ , \quad i = 2, \ldots, N \ . \tag{5}$$

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One can determine explicit functions $k_\ell, \Omega_\ell \in \mathcal{K}_\infty \cap C^1$ such that

$$S(x) = \sum_{\ell=1}^{N} \Omega_\ell \Big(k_\ell(V_1(x)) + V_\ell(x) \Big) \qquad (6)$$

is a strict Lyapunov function on $\mathcal{X}$ satisfying $S(x) \geq V_1(x)$ on $\mathcal{X}$.

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Significance: New theorem says which functions V_i to pick.

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Significance: Allows any open state space $\mathcal{X}$ containing $0 \in \mathbb{R}^n$.

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Significance: Readily extends to t-v systems.

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Assume $\alpha > d$. Want a global strict Lyapunov function for (8).

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Nonstrict Lyapunov decay condition: $\dot{V}_1(\tilde{x},\tilde{y}) \le -|\tilde{x}|^2$.

Auxiliary function from theorem: $V_2(\tilde{x},\tilde{y}) = \tilde{x}[\tilde{x}+\alpha\tilde{y}](\tilde{x}+x_*)$.

Strict Lyapunov Function Construction

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$$
\begin{aligned}
S(\tilde{x}, \tilde{y}) \quad = \quad & V_2(\tilde{x}, \tilde{y}) + \int_0^{V_1(\tilde{x}, \tilde{y})} \phi_1(r)\,\mathrm{d}r \\
& + \left[p_1\left(V_1(\tilde{x}, \tilde{y})\right) + 1\right] V_1(\tilde{x}, \tilde{y}),
\end{aligned}
\tag{11}
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where

$$\phi_1(r) = 2\left[(289x_* + 144\alpha y_*)^2 + 144\alpha^2 x_* y_*\right] e^{2\left(\frac{1}{x_*}+\frac{1}{y_*}\right)r}$$

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Our strict Lyapunov function for the Lotka-Volterra model leads to an ISS robustness analysis under uncertain death rates.

Summary and Extensions (cont'd)

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Further Reading

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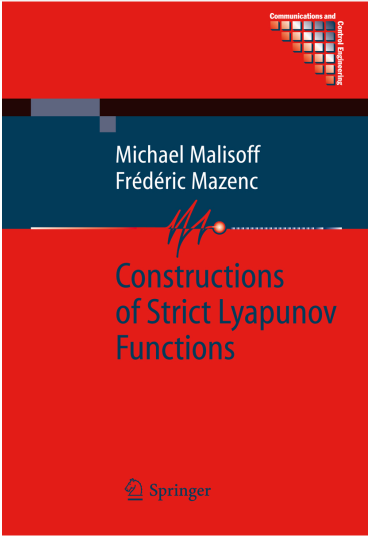

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Hence, (1) holds with $\tau = \frac{\pi}{4}$ and $\rho(r) = r^2/\{200(r+1)\}$.

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$\Omega_N(r) = r$, and $\{\Omega_i\}_{i=1}^{N-1}$ satisfy

$$\Omega_i'(U_i) \geq (N-1)^2 \frac{8\phi_1^2(V_1)}{\tilde{\rho}(V_1)} \sum_{r=1+i}^{N} \Omega_r'(U_r)^2, \tag{16}$$

with $\Omega_i' : [0, \infty) \to [1, \infty)$ continuous and increasing for each i.