The Distance Between Sets - Due Tuesday, April 9

Let $P$ and $Q$ be two subsets of a metric space $(S, d)$. We define the distance between $P$ and $Q$ as follows:

$$
\text{dist}(P, Q) := \inf \{d(p, q) \mid p \in P \text{ and } q \in Q\}.
$$

Let’s make a few easy observation. If the two sets intersect—that is, if there is a point $p \in P \cap Q$—then of course $d(p, p) = 0$ and so obviously $\text{dist}(P, Q) = 0$. However, the distance between the two sets may be zero even if they don’t intersect. For an example, let $S$ be the plane with the usual metric, let $P$ be the $y$-axis, and let $Q$ be the graph of the function $y = 1/x$ for $x > 0$. For each $x > 0$, the point $p := (0, 1/x)$ is in $P$ and the point $q := (x, 1/x)$ is in $Q$, and $d(p, q) = x$. Thus $\text{dist}(P, Q) = 0$ even though the two sets have no points in common.

Prove the following Theorem.

**Theorem.** Let $P$ and $Q$ be two subsets of a metric space $(S, d)$. Suppose that $P \cap Q = \emptyset$. Suppose that $P$ is compact and that $Q$ is closed. Then $\text{dist}(P, Q) > 0$.

Here are a hint as to one way to go about the proof. If it were true that $\text{dist}(P, Q) = 0$, then there would be a sequence of pairs $p_n \in P$, $q_n \in Q$ such that

$$
d(p_n, q_n) < \frac{1}{n}.
$$

(why?) The sequence $p_n$ would have a subsequence $p_{n_k}$ converging to some point $p_0 \in P$ (why?), and then the sequence $q_{n_k}$ would also converge to $p_0$. But then $p_0$ would belong to $Q$ (why?), bringing us to a contradiction (why?).