The Properties of the Exponential Function,
Derived from its Differential Equation

Suppose that there exists a function $f : \mathbf{R} \to \mathbf{R}$ such that

$$f' = f \qquad \text{and} \qquad f(0) = 1. \tag{1}$$

Let's see what can be proved about f. First of all, since f is differentiable everywhere, it is continuous everywhere. By its continuity at zero, and since $f(0) = 1$, we know there exists $\delta > 0$ such that $f(x) > 0$ at least for $-\delta < x < \delta$.

I. For all $x \in \mathbf{R}$, $f(x)f(-x) = 1$. Proof: Let $\phi(x) = f(x)f(-x)$. Notice that $\phi(0) = 1$. Look at the derivative: $\phi'(x) = f'(x)f(-x) - f(x)f'(-x)$, which equals zero everywhere because $f = f'$. Therefore ϕ is constant and equals one everywhere.

II. $f(x) > 0$ for all $x \in \mathbf{R}$. Proof: By I, $f(x)$ can never be zero. We know $f(0)$ is positive. If $f(b)$ were negative for some b, by the Intermediate Value Theorem $f(c)$ would be zero for some c between 0 and b, which cannot be.

III. f is unique. Proof: Suppose there is another function g that satisfies (1). Then

$$\left(\frac{g}{f}\right)' = \frac{fg' - f'g}{f^2} = \frac{fg - fg}{f^2} = 0,$$

so g/f is constant; it equals 1 at 0, so it must equal 1 everywhere, so $f = g$.

IV. For all $x, y \in \mathbf{R}$, $f(x+y) = f(x)f(y)$. Proof: Let y be an arbitrary fixed real number, and consider the function

$$h(x) = \frac{f(x+y)}{f(x)}.$$

Notice that $h(0) = f(y)$. Look at the derivative:

$$h'(x) = \frac{f(x)f'(x+y) - f(x+y)f'(x)}{f(x)^2},$$

which equals zero because $f' = f$. Therefore h is constant, so it equals $f(y)$ everywhere.

V. Let k and c be nonzero constants. Notice that the function

$$p(x) = kf(cx)$$

satisfies

$$p' = cp \qquad \text{and} \qquad p(0) = k. \tag{2}$$

In fact, p is the only function satisfying (2). Proof: Suppose q also satisfies (2). Then

$$\left(\frac{q}{p}\right)' = \frac{pq' - qp'}{p^2} = \frac{pcq - qcp}{p^2} = 0,$$

so q/p is constant; it equals $\frac{k}{k} = 1$ at 0, so it must equal 1 everywhere; so $q = p$ everywhere.