Finding a Fundamental Matrix: Perspective and Summary

Our purpose is to show you a pencil-and-paper procedure for solving

\[ x' = Ax. \]

This method always works and gives us a fundamental matrix. The matrix \( e^{At} \), the transition matrix based at \( t = 0 \), is in many ways the nicest fundamental matrix to have, because it is a transition matrix based at \( t = 0 \). But \( e^{At} \), being an infinite series, may be difficult to compute directly. So we will settle for finding, however we can, \( n \) independent solutions in the form of column vectors. Put them together and they form a fundamental matrix \( X(t) \). Then, if we still want \( e^{At} \), we can compute \( X(t)X(0)^{-1} \).

Understanding \( e^{At} \) is the key to our procedure for finding a fundamental matrix. Note that for every column vector \( k \), the matrix product \( e^{At}k \) is a combination of the columns of \( e^{At} \) and therefore is a solution. Next, observe that for every scalar \( \lambda \) and every vector \( k \),

\[ e^{At}k = e^{\lambda t} e^{(A - \lambda I)t} k. \]

For suitable choices of \( \lambda \) and \( k \), that solution will in fact be easy to calculate because there will be only a small number of nonzero summands in the infinite series

\[ e^{At}k = e^{\lambda t} \left( k + t(A - \lambda I)k + \frac{t^2}{2}(A - \lambda I)^2k + \cdots \right). \]  

(1)

Thus for example if \( \lambda \) is an eigenvalue of \( A \) and \( k \) is in the corresponding eigenspace, we have simply

\[ e^{At}k = e^{\lambda t}k. \]

If \( A \) is not deficient, we are especially lucky; we get \( n \) independent solutions of the form \( e^{\lambda t}k \). Otherwise, for each case of an eigenvalue \( \lambda \) whose multiplicity exceeds the dimension of its eigenspace, find any \( k \) such that \((A - \lambda I)k \neq 0 \) and \((A - \lambda I)^2k = 0\); then, for such a \( k \), only the first and second summands of the infinite series (1) can be nonzero, and we get a new independent solution-column of the form

\[ e^{\lambda t}(k + t(A - \lambda I)k). \]

Now suppose that after putting all of those on our list we find that there is an eigenvalue \( \lambda \) that still has not supplied us with a number of independent solutions equal to its (algebraic) multiplicity. Take any \( k \) such that \((A - \lambda I)^2k \neq 0 \) and \((A - \lambda I)^3k = 0\). Then, for such a \( k \), the infinite series stops after the third summand and we get a new independent solution-column of the form

\[ e^{\lambda t} \left( k + t(A - \lambda I)k + \frac{t^2}{2}(A - \lambda I)^2k \right). \]

And so forth! The process always lead us to \( n \) independent columns. Put them together and we’ve got a fundamental matrix \( X(t) \).