Finding a Fundamental Matrix: Perspective and Summary

Our purpose is to show you a pencil-and-paper procedure for solving

$$\mathbf{x}' = A\mathbf{x}.$$

This method always works and gives us a fundamental matrix. The matrix e^{At}, the transition matrix based at $t = 0$, is in many ways the nicest fundamental matrix to have, because it is a transition matrix based at $t = 0$. But e^{At}, being an infinite series, may be difficult to compute directly. So we will settle for finding, however we can, n independent solutions in the form of column vectors. Put them together and they form a fundamental matrix $X(t)$. Then, if we still want e^{At}, we can compute $X(t)X(0)^{-1}$.

Understanding e^{At} is the key to our procedure for finding a fundamental matrix. Note that for every column vector $\mathbf{k}$, the matrix product $e^{At}\mathbf{k}$ is a combination of the columns of e^{At} and therefore is a solution. Next, observe that for every scalar λ and every vector $\mathbf{k}$,

$$e^{At}\mathbf{k} = e^{\lambda t}e^{(A-\lambda I)t}\mathbf{k}.$$

For suitable choices of λ and $\mathbf{k}$, that solution will in fact be easy to calculate because there will be only a small number of nonzero summands in the infinite seties

$$e^{At}\mathbf{k} = e^{\lambda t}\left(\mathbf{k} + t(A-\lambda I)\mathbf{k} + \frac{t^2}{2}(A-\lambda I)^2\mathbf{k} + \cdots\right). \tag{1}$$

Thus for example if λ is an eigenvalue of A and $\mathbf{k}$ is in the corresponding eigenspace, we have simply

$$e^{At}\mathbf{k} = e^{\lambda t}\mathbf{k}.$$

If A is not deficient, we are especially lucky; we get n independent solutions of the form $e^{\lambda t}\mathbf{k}$. Otherwise, for each case of an eigenvalue λ whose multiplicity exceeds the dimension of its eigenspace, find any $\mathbf{k}$ such that $(A-\lambda I)\mathbf{k} \neq 0$ and $(A-\lambda I)^2\mathbf{k} = 0$; then, for such a $\mathbf{k}$, only the first and second summands of the infinite series (1) can be nonzero, and we get a new independent solution-column of the form

$$e^{\lambda t}(\mathbf{k} + t(A-\lambda I)\mathbf{k}).$$

Now suppose that after putting all of those on our list we find that there is an eigenvalue λ that still has not supplied us with a number of independent solutions equal to its (algebraic) multiplicity. Take any $\mathbf{k}$ such that $(A-\lambda I)^2\mathbf{k} \neq 0$ and $(A-\lambda I)^3\mathbf{k} = 0$. Then, for such a $\mathbf{k}$, the infinite series stops after the third summand and we get a new independent solution-column of the form

$$e^{\lambda t}\left(\mathbf{k} + t(A-\lambda I)\mathbf{k} + \frac{t^2}{2}(A-\lambda I)^2\mathbf{k}\right).$$

And so forth! The process always lead us to n independent columns. Put them together and we've got a fundamental matrix $X(t)$.

See Sections 8.9 and 8.10 of *Differential Equations and Linear Algebra*, second edition, by Stephen W. Goode.