

REVIEWS

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Niels Henrik Abel and His Times: Called Too Soon by Flames Afar. By Arild Stubhaug. Springer-Verlag, 2000, x + 580 pp., \$44.95. ISBN 3-540-66834-9.

Reviewed by **O. A. Laudal**

Niels Henrik Abel died on April 5th, 1829, at the age of 26, at the home of a rich industrialist not far from his father's and grandfather's parish Gjerstad. On his deathbed, wracked by tuberculosis, he seems to have cursed his fate. He would never be a *Herr Professor mit seiner Gehmahlin* as he had promised his fiancée. He would never reach the harbor of peace and financial tranquility that he had dreamt about. But he had already fulfilled the vision of the Danish professor Ferdinand Degen, who had suggested in 1821 that the study of elliptic integrals might enable him to "discover, like Magellan, passages into great regions of a single gigantic analytic ocean".

That year, barely 19 years old and still a student at the Christiania Cathedral School, Abel had proposed an algebraic solution to the quintic equation. Neither his teacher Bernt Michael Holmboe nor Degen was able to find the error, but both were skeptical. Two years later, Abel managed not only to see his error but also to prove the impossibility of solving the quintic equation by radicals, ending a centuries-old quest for a continuation of the work of Tartaglia, Cardano, and Ferrari. This was the start of a brief career that in many ways changed mathematics as a science. In five years of hectic travels and phenomenal mathematical creativity, the young Niels Henrik Abel opened up group theory to what we now call Galois theory, reformed part of analysis, and set the stage for modern algebraic and analytic geometry.

Tragically, he never saw his crowning work in print. Due to a not uncommon indifference on the part of the French Academy of Sciences, his masterpiece *Mémoire sur une propriété générale d'une classe très étendue des fonctions transcendentes*, which included the addition theorem for elliptic functions, collected dust in Cauchy's office until some months after Abel's death. It was then read and appreciated by members of the Academy, resulting in the posthumous award to Abel of the Academy's grand prize (jointly with Jacobi) in July 1830, but it disappeared for a time and was not published until 1841. Afterward, the manuscript disappeared from Paris again and ended up in the Guglielmo Libri nachlass in Florence, where it was found by my teacher Viggo Brun in 1952.

This is the skeleton of the book *Niels Henrik Abel and His Times: Called Too Soon by Flames Afar* by Arild Stubhaug. But Stubhaug takes us on a much longer and quite circuitous voyage, covering the history of the Abel family from 1640, when the two brothers Abild arrived at Trondheim as refugees from war-torn Europe. The book takes us on a fascinating historical journey through the dawn of enlightenment in Scandinavia, including the shaping of the new educational system from which the mathematician Abel could emerge.

Stubhaug debunks some of the popular legends of the poor peasant boy who was misunderstood and maltreated by the barbaric Norwegian society. Rather, Abel was the

second son in a family that combined top intellectual and political status in Denmark-Norway with a substantial fortune on his mother's side. The early death of his father and his grandfather's financial collapse, which may have worsened his mother's alcoholism, led to his situation as a *pauvre honteux* in his social class, needing the assistance and moral support of his many well-to-do friends and admirers.

The psychological effects of this family background on the young Niels Henrik are wisely kept at a distance throughout the book. Stubhaug has resisted the obvious temptation to speculate on the psychological effects of this family background on the young Abel or to "explain" Abel's mathematical creativity as some sort of sublimation. But he tells the whole story, down to the minute details. Were it not for his research, we would probably never know that the cousin of Abel's mother's grandfather, Jens Kraft, became professor at the Sorø Academy and was the foremost Danish mathematician of his time. The significance, if any, of these details for Abel's development is most graciously left to our imagination.

The effect of this background story is, however, that the reader becomes thoroughly acquainted with the social structure and the intellectual environment that formed Abel. The person Niels Henrik Abel, about whom one actually knows very little, thereby stands out like the shadow on a magic lantern. His personal traits are perhaps best transmitted to us via a few very touching letters to his "second mother", the wife of the geophysicist Hansteen. They reveal a quite childlike but pleasant character, struggling to cope with the harsh realities of the world and withdrawing as often as he could to theater and mathematics, his favorite occupations. In the shadow of the theater rotunda or concentrating at his work table, he could rid himself of the melancholy thoughts that haunted him elsewhere.

This book is meant for the general public rather than specialists in the history of mathematics, who will find little new beyond the biography written by Øystein Ore in 1954 [2]. Stubhaug treats the controversial Abel-Ruffini story without chauvinistic overtones, and he is quite neutral in the conflict between Abel and Jacobi. He writes about it but refrains from judgment, referring as best he can to the writings of the contemporary mathematicians. As far as I can see, he presents a quite complete overview of the available material pertaining to these and several other interesting scientific conflicts. In the process he also paints, most often with a delicate if rather broad brush, the pictures of the main characters of the mathematical play in which Abel took part. I think most mathematicians today will find his vivid picture of the life of a student in Berlin and Paris during the years 1825–27 very rewarding.

In the fall of 1825, together with a small but quite remarkable group of young students from the newly founded Royal Frederik University in Christiania (Oslo), Abel obtained a two-year travel grant from the Norwegian Government. Urged by the Danish mathematician von Schmidten, he went first to Berlin with a letter of introduction to August Leopold Crelle. Stubhaug presents a fine portrait of Crelle, the founder of the *Journal Für die Reine und Angewandte Mathematik*, where most of Abel's mathematical work was published. Stubhaug has been able to unearth Crelle's amazing web of connections among the Berlin nobility as well as in the scientific world. Abel was to Crelle what Poincaré later became to Mittag-Leffler, the mathematical genius promoting and giving fame to a new and ambitious mathematical journal. But Crelle was much more to Abel: he was the savior, the influential friend opening the doors of Berlin society to Abel, and his single best mathematical contact.

Stubhaug shows how Abel's self-confidence got a boost in Berlin. He decided to neglect the conditions of his travel grant, striking Gauss and Göttingen from his itinerary, and join his young Norwegian friends on a lengthy tour of Europe including Dresden, Prague, Vienna, Trieste, Venice, Verona, Bolzano, Innsbruck, Zürich, Basel, and Paris.

During this strenuous and obviously very exciting two months of travel, he must have had time to rethink ideas that had come to him in Berlin, and he became bold. In a letter to Hansteen he drew up a mathematical program that was quite audacious for a 23-year-old self-taught boy from Ultima Thule. He wrote:

I will engage all my powers to bring more light into the enormous darkness that is now unquestionably to be found in analysis. It lacks almost all forms of plan and system, so it is really a great surprise that it can be studied by so many, and now the situation exists wherein it is handled without the least degree of stringency. At the outset there are few propositions in higher analysis that are proven with convincing rigor. Above all one finds the unfortunate method of deducing the general from the special, and it is strange to the extreme that by following this sort of approach, so few of the so called paradoxes are to be found. It is really extremely interesting to look into the foundations of this phenomenon.

The outcome of this program is implemented in most textbooks on analysis today. The first paper in this direction seems to be the paper in the first volume of Crelle's Journal from 1826, on the binomial series, "Recherches sur la série $1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \dots$ ". Here Abel gives modern definitions of, and criteria for, convergence. He corrects false statements of Cauchy, introduces absolute convergence, and ends up with a complete analysis of the convergence of the binomial series. However, the remarkable clarity of Abel's work on this program and his amazing ability to find the proper setting and the proper generalization of the problem is best seen in the short paper "Note sur un mémoire de M.L. Olivier, ayant pour titre 'Remarques sur les séries infinies et leur convergence'" from 1828, where he first gives a counterexample to the contention that a series $\sum a_n$ converges if and only if $na_n \rightarrow 0$. Then he proves by a startlingly simple method that there can be no function $\phi(n)$ such that $\phi(n)a_n$ tends to zero if and only if the series $\sum a_n$ converges.

Stubhaug takes us with the group of young Norwegians through their fabulous journey, which for Abel ended in Paris on July 10, 1826. Here in Paris Abel did not have his friend Crelle to open doors for him. He found the French way of life different and difficult, and he was not able to form personal contacts to the extent he had hoped. He brought a letter of introduction to Professor Bouvard from Professor Joseph Johann von Littrow in Vienna, and Bouvard introduced him to the members of the Academy of Sciences. He saw Cauchy, Legendre, Laplace, Lacroix, and Baron de Ferrusac, and he got to know Dirichlet, Hachette, Saigey, and Raspail, but it does not seem that he ever attended any of Cauchy's classes at the École Polytechnique. However, he had high hopes for his stay in Paris and for his contact with the members of the Academy of Science. For this purpose he had prepared his best work, the one representing the high point of the "Magellanic voyage" charted by Degen for him in 1821. Stubhaug takes us through the half-year of hopes and disappointments and through an amusing crash course on the local history of Paris. Apparently it was in Paris that Abel first experienced the symptoms of his smoldering tuberculosis, which sent him to bed for days from time to time.

The lack of response from the Academy of Sciences, in particular from Cauchy and Legendre, to his Paris *Mémoire* on elliptic functions must have been a horrible blow for Abel, for in letters to Holmboe he had been quite sure about the quality of this paper. The history of mathematics has proved him right. This paper served as inspiration to Riemann in his work on geometry, and it still serves as inspiration for original research in algebraic as well as analytic geometry; see Griffiths's beautiful paper [1]. But nobody in Paris seemed to care.

Stubhaug accompanies Abel on his journey home via Berlin, where Crelle was eager to employ him as editor of his journal or as a professor at the new university he

planned. Here is the origin of Abel's hope of seeing himself as *Herr Professor mit seiner Gemahlin* of which he spoke to his fiancée after his homecoming. In Norway, however, Holmboe had been promoted to the only available permanent professorship in mathematics. Abel had no position to go to, and this made him melancholic, the more so because he was the only one in his family who could be expected to care for his alcoholic mother and his sister and brothers, all of whom were in dire need.

Stubhaug has done penetrating research on the situation of the final two years of Abel's life. Despite Abel's phenomenal pace in research, including papers on elliptic functions, differential equations, and his first love, the theory of algebraic equations and the problem of characterizing those equations that admit solutions by radicals, it soon became clear that he was sick, and the lack of economic security was wearing him down. He subsisted on a small grant from the Royal Frederik University and had to take out loans from friends and from the Bank of Norway. Finally, in the spring of 1828, he got a reasonable temporary position at the university, vacant in the absence of his mentor Hansteen. Moreover, Abel must have known that Crelle was working hard to find a professorship for him in Berlin. But tuberculosis was not curable at the time. The travel by horse-drawn sled through snow and bitter cold on his way to celebrate Christmas 1828 with his fiancée at Froland Mansion did not make his situation better. Abel's time had come; he fell seriously ill and never recovered. Stubhaug takes us close to Abel's deathbed and through the burial, and describes the local and European reactions to this terrible loss of a mathematical genius and a young nation's pride and glory.

Crelle wrote; "Let us honor the memory of this man who distinguished himself equally with his uncommon talents and the integrity of his character. He was one of the singular beings that nature rarely brings forth in the course of a century."

The English version of this book, translated from Norwegian by Richard Daly, is generally well written and a faithful rendition of the prize-winning volume published by Aschehoug, Oslo, in 1996. Springer-Verlag has given the book a beautiful layout with very nice illustrations. However, the translator has made several annoying errors in translating mathematical terms and expressions. One has to endure the notions of elliptical integrals and functions, primary numbers, square roots instead of radicals, etc. But this is a minor drawback.

In *Niels Henrik Abel and His Times*, Arild Stubhaug has given us a work of lasting value. It will serve as an indispensable source book for any one wanting to learn about Abel, particularly as we approach the Abel bicentennial conference at the University of Oslo in June 2002. Moreover, following up on the work done in connection with the 150th anniversary of Sophus Lie's birth in 1992, Stubhaug has just published another outstanding biography of this second great Norwegian mathematician. I understand that an English translation will soon be published by Springer-Verlag; it should be worth waiting for!

REFERENCES

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Reviewed by **O. Carruth McGehee**

Consider the following problem. We wish to offer a one-semester course in Fourier analysis, mostly for juniors and seniors. We want to attract and to serve the interests of students in engineering and the sciences, as well as our own majors. There must be a satisfactory development of whatever theory is employed. We must make the course accessible to students whose common background includes only the freshman and sophomore courses in calculus and differential equations. Finally, we must bring within plain view, and easy reach, the scientific applications about which the intended audience is curious.

Many of us see difficulties in teaching such a course and would expect to find serious limitations on what we could achieve. We think of Fourier analysis as a subject most naturally taught to students who have already studied Lebesgue integration and functional analysis. But the purpose at hand requires inventing a rather different course, building a spare theoretical structure, ruthlessly omitting the unnecessary, but still aiming ambitiously for an understanding of the subject's amazing array of uses.

The Mathematics Department at Louisiana State University began offering a course under the name "Fourier Transforms" in 1993. I have now taught the course four times. The students who enroll have an above-average mastery of the calculus. Most have previously acquired, from courses in science and engineering, an awareness of the importance and power of Fourier transforms. They arrive with a healthy curiosity, eyes open for information they can use. The population has been 44% physicists; 32% engineers, mostly mechanical; 16% mathematicians; and some from other fields such as economics and biology. Two-thirds have been undergraduates. During the course, the material holds their interest. Years afterward, graduates tell me of its usefulness.

The secret of the success enjoyed by the LSU course is that from the very beginning, David W. Kammler has allowed us to use preliminary versions of his excellent book, which has now been published. It is error-free, well organized, flexible, historically informed, equipped with a treasure of over 500 exercises, and written with care and enthusiasm from start to finish.

There are several good ways to choose a one-semester course from this 800-page book. I spend about two-thirds of the lectures discussing selected material from the first part, called "The Mathematical Core", which comprises seven chapters. I spend the rest of the time on topics chosen from the second part, called "Selected Applications", in which the chapters are "Sampling", "Partial Differential Equations", "Wavelets", "Musical Tones", and "Probability". Typically I find time to discuss signal recognition, Fourier methods for partial differential equations up to Schrödinger's equation, and probability. The students, depending on field of interest, tend to find most of those topics quite interesting. They often keep the book for future use.

I make the book's exercises a primary part of the course requirements. I allow the students almost complete freedom to choose which ones to do, so that each can find an appropriate level of difficulty, and select the applications and other topics that are of greatest interest. The students sign up for exercises in advance, so that no two students do the same problem. I distribute copies of the best solutions to the class.

Although the book is not at all tied to a computer program, specialized software is being developed to support the emphasis on visualization. I understand that in the near

future, the website <http://www.math.siu.edu/Kammler> will describe the software and provide other helpful material for teachers.

Chapter 1 introduces Fourier analysis on the real line \mathbb{R} , the circle group $\mathbb{T}_p \equiv \mathbb{R} \bmod p$, the integer group \mathbb{Z} , or the finite cyclic group $\mathbb{P}_N \equiv \mathbb{Z} \bmod N$. The author discusses the Fourier transform and its inverse for suitable classes of functions on each of the groups. Of course, one must do without the Lebesgue integral and forego the pursuit of various best-possible results in favor of something quick and clean. In Section 1.5, for example, the author establishes sufficient conditions on a function $f : \mathbb{R} \rightarrow \mathbb{C}$ for its Fourier transform F to be defined and for f to be recoverable from F :

$$F(s) = \int_{x=-\infty}^{\infty} f(x)e^{-2\pi isx} dx \quad \text{and} \quad f(x) = \int_{s=-\infty}^{\infty} F(s)e^{2\pi isx} ds.$$

The conditions are understandable in terms of the Riemann integral, they make the treatment logically complete, and they cover the functions that need to be dealt with.

The author introduces conventions that aid geometric insight. Numerous functions and their transforms are graphed side-by-side. Subgroups and quotient groups also appear (without the use of those words), and an 8-vertex commutative diagram summarizes the relations of the pertinent operators on functions. One of those, of course, is restriction to a subgroup, motivated as sampling.

Chapter 1 also develops the Parseval and Plancherel identities, treats Gibbs's phenomenon carefully, proves the Weierstrass approximation theorem using a kernel that converges weakly to the identity, derives the Tartaglia-Cardan formulas for the roots of a cubic, and previews several applications that are treated later. Among the forty-six exercises at the end of this chapter are several of historical interest. One (1.18) introduces the Hipparchus-Ptolemy model for planetary motion. Another (1.19) takes the student through Gauss's analysis of the orbit of the asteroid Pallas and his discovery therein of a Fast Fourier Transform algorithm. A third (1.45) introduces the Michelson-Stratton mechanical harmonic analyzer of the 1890s.

Chapters 2, 3, and 4 develop a computational calculus for finding convolutions and for obtaining Fourier transforms F of functions f on the aforementioned groups. Operating in a given fashion on f entails doing what to F ? A given property of F implies what property of f ? There is plenty of material here to develop a thorough geometric conceptualization of the transform, as well as the sense that f and F are two incarnations of the same object.

Incidentally, I've known students to enjoy, and to solve beautifully, Kasner's problem (Exercise 2.32). It is a matter of solving $Av = w$ for v , given the vector w and the $N \times N$ circulant matrix A . Equivalently, the problem is to solve the convolution equation $a * v = w$; the crux is whether the Fourier transform of a , as a function on \mathbb{P}_N , is everywhere nonzero.

Kammler's exercises in Chapter 3 bring out the fact that the Fourier transform calculus is competitive with the residue calculus for finding exact values for a wide variety of integrals.

I'm struck by how well-prepared the students are for Chapter 5, "Operator Identities Associated with Fourier Analysis", once they have had the experience of Chapters 1–4. They find the algebraic abstractions well-motivated, and appreciate the efficiency of the language and machinery. Chapter 5 introduces the Cosine, Sine, Hartley, and Hilbert transforms, among others, as well as numerous projections, identities, and commutative diagrams, and the idea of the Fourier transform of an operator. This chapter moves from the computational-rule-based calculus of the earlier material to an

operator-based calculus, and thus prepares the way for Chapter 6, “The Fast Fourier Transform”, where a quite satisfactory introduction is achieved in just the first two sections, and several FFT algorithms are treated in detail.

Chapter 7, “Generalized Functions on \mathbb{R} ”, is the gateway to the applications chapters. It presents distributions as functionals on the Schwartz class. They are realized as derivatives of continuous slowly-growing functions, and a serviceable computational calculus is developed. One studies the Fourier transforms, convolution products, and weak limits of distributions. Everything is done without benefit of functional analysis, avoiding the structural questions that would customarily be addressed in more advanced courses.

At this point in the review, I leave off commenting on individual chapters to address a few general issues. In his prefatory remarks to the instructor, the author writes, in an imprecise dichotomy: “As you teach this material you will face the constant temptation to prove too much. . . . For every hour that you spend presenting 19th-century advanced calculus arguments, you will have one less hour for explaining the 20th-century mathematics of generalized functions, sampling theory, wavelets, You must decide which of these alternatives will best serve your students.” I agree with the drift of this pointed remark. However, one may make some choices different from the author’s without losing the point.

I do think, for example, that it’s worthwhile to survey the subtleties of recovering a periodic function from its Fourier series. As for pointwise convergence, Kammler’s treatment of Gibbs’s phenomenon is fine. As for norm convergence, Kammler offers only a proof of the Weierstrass approximation theorem (trigonometric polynomials are dense in $C(\mathbb{T}_p)$), using a particular summability kernel. I prefer to take a bit more time, work with an arbitrary summability kernel on \mathbb{T}_p , and establish a more general array of results, obtaining, for example, the convergence of the Cesàro sums in L^1 norm. I find time as well to state Wiener’s classical result, which underlies modern Banach algebra theory: The reciprocal of a function f with absolutely convergent Fourier series also has absolutely convergent Fourier series if and only if f does not vanish.

The needs of the engineers and physicists are the occasion for creating this course. Nevertheless, it is a good thing for mathematics majors to have it as an option. Having taken it, a student in a modern functional analysis course will have a better feeling for the meaning and usefulness of the concept of weak convergence and will have more examples to feed the intuition. The student who undertakes to follow an advanced treatment of commutative harmonic analysis will be happy to have learned it first in the limited, concrete settings of this text. After all, many interesting phenomena that are important and proper to the subject get abstracted away in the setting of an arbitrary group. Harmonic analysis is more properly understood to be the union, not the intersection, of studies on the various groups. The choice of settings represented by Kammler’s book has ever been a crossroads of number theory, algebra, set theory, probability, and models fitted to physical problems.

The author has made a valuable contribution, obviously the product of many years’ experience at teaching the material to the intended audience. The book invents, and makes practical, a substantial upper-undergraduate-level course in Fourier analysis that is appealing for many curricula. It is a joy to use as a text. Its pages hold a good many surprises, more than a review has room to reveal. Give it a look.

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