Remarks on the Littlewood Conjecture
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(Text for transparencies, smaller-type version)

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Let E be a set of N integers, and let

$$f(x) = \sum_{n \in E} \hat{f}(n) e^{inx},$$

where $|\hat{f}(n)| \geq 1$ for $n \in E$. The Conjecture is that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| dx \;\; \geq \;\; C \, log \, N, \tag{1}$$

where C is a constant independent of N and E. Let $A = A(Z) = \widehat{L^1(T)}$; with $\|\hat{f}\|_A$ defined to mean $\|f\|_{L^1}$, what (1) says in the case when $\hat{f} = 1$ on E is that

$$\|\chi_E\|_A \; > \; C \, log(\#E). \tag{2}$$

J. E. Littlewood is said to have proposed this result over 60 years ago; a formulation appeared in a 1948 paper by Hardy and Littlewood.

Does (1) hold at least for *some* function of N on the right-hand side that tends to infinity with N?

Paul Cohen's 1960 paper told us that the answer was Yes. That result is both an instance and the source of a set of principles which are important for understanding the behavior of Fourier transforms.

We will come back later to the story of "$C \, log \, N$."

Some Principles of Transform Behavior

Let G be a locally compact abelian group, Γ the dual group of G. Let $M(G)$ denote the convolution algebra of bounded complex-valued Borel measures on G with the total-mass norm, and let $B = B(\Gamma) = \widehat{M(G)}$. Define $\|\hat{\mu}\|_B$ to mean $\|\mu\|_{M(G)}$.

The *indicator function* of a set E is denoted by χ_E and defined as follows: $\chi_E(n) = 1$ if $n \in E$, $\chi_E(n) = 0$ otherwise.

What Henry Helson had shown for Z in 1953, Cohen proved for an arbitrary Γ:

For $E \subset \Gamma$, $\chi_E \in B(\Gamma)$ if and only if E is in the coset ring of Γ.

That statement is not the end of the structure theory. It has a quantitative aspect.

$\|\chi_E\|_{B(\Gamma)}$ *equals* 1 *only when E is itself a coset.*

The more complicated E is as a combination of cosets, the larger $\|\chi_E\|_B$ must be.

One instance is (1) itself. A finite set of N integers is a combination of no fewer than N cosets. Accordingly, the norm of its indicator function is bounded below by a function that tends to infinity with N.

Another instance (Saeki [1,2]):

Let E belong to the coset ring. If E is not itself a coset, then

$$\|\chi_E\|_B \geq \frac{1+\sqrt{2}}{2} \simeq 1.2071.$$

If

$$\frac{1+\sqrt{2}}{2} \leq \|\chi_E\|_B \leq \frac{1+\sqrt{17}}{4} \simeq 1.2808,$$

then E is the union of two cosets of one subgroup.

Beyond those results of Helson and Cohen, we now know that:

If χ_E is merely uniformly close to something in $B(\Gamma)$ that is not too big, then E is in the coset ring.

Jean-François Méla, perfecting work begun by L. T. Ramsey, proved this elegant theorem. Its quantitative aspects are very nearly best possible.

Let $E \subset Z$. If h is a Fourier-Stieltjes transform and

$$|h(n) - \chi_E(n)| < \epsilon \quad \forall n \in Z,$$

and if

$$\|h\|_B < 1.135\,|log\,\epsilon| - 2,$$

then E is in the coset ring.

Such a result is not just about idempotent functions. The principle involved is something like this:

Let g be a function on a discrete Γ. If the range of g has a gap in it, and g takes on the values on each side of the gap on a substantial set, and if the corresponding break in the domain does not respect the group structure of Γ, then either g is not a transform or, if it is, then $\|g\|_B$ must be large.

A transform on a non-discrete group must observe that principle with respect to discrete subgroups.

So the principle is not limited to discrete groups. It is important quite generally in understanding what transforms can and cannot do.

N. N. Lusin asked whether, for every continuous F on T, there is a homeomorphism $\phi : T \to T$ such that the rearrangement of F by ϕ, $F \circ \phi$, is in $A(T)$.

Adapting the procedure in Cohen's proof of (1), A. M. Olevskiǐ constructed a continuous real-valued F of which every rearrangement violates the principle and thus fails to be in $A(T)$!

Here is a simple and familiar instance of the principle: If $h \in A(T)$ and h is monotone near x_0, then

$$|h(x) - h(x_0)| = o\left(\frac{1}{log|x - x_0|}\right) \quad \text{as } x \to 0. \tag{3}$$

Using just that, Kahane and Katznelson showed that the answer to Lusin's question is No with a complex-valued example, $F = f + ig$, as follows. Let $f(x) = x$ near 0 and construct g such that $g \circ \phi$ is not in A, not even locally at zero, if ϕ is any monotone function satisfying (3) at $x_0 = 0$. Suppose now that for some homeomorphism ϕ, $F \circ \phi$ and hence both $f \circ \phi$ and $g \circ \phi$ were in A. We may suppose that $\phi(0) = 0$. If $f \circ \phi \in A$, it must equal ϕ near 0, so ϕ satisfies (3) at $x_0 = 0$, so that by the choice of g, $g \circ \phi$ cannot be in A, a contradiction.

Return we now to the Conjecture. Why might one expect (1) or (2) to be true, with $C\,log\,N$ on the right? Look at examples. Consider first the set E consisting of all the integers from $-N$ to N. Its indicator function is the transform of the N^{th} entry in the Dirichlet kernel. As we all know, as $N \to \infty$,

$$\|\chi_{[-N,N]}\|_A \;\sim\; \left(\frac{4}{\pi^2}\right) log\,N. \tag{4}$$

Secondly, consider a lacunary set E. It is easy to show that

$$\|\chi_{\{0,3,9,\ldots,3^N\}}\|_A \;\geq\; \left(\frac{N}{3e}\right)^{1/2}. \tag{5}$$

Let us try to explain what is going on with these examples. We will assume $0 \in E$ to allow simpler statements.

Let $E \subset Z$. Consider *in steps* the process as E generates a group. Here is one way to do so. First take the sums and differences of pairs of elements of E; then take such combinations of triples, getting the elements $\pm x \pm y \pm z$; and so forth.

If the very first step gives elements that all lie outside E, as with the powers-of-three set in (5), then there is very little of Z's group structure within E; the break between E and its complement does not respect the group structure in the least.

But an arithmetic progression, as in (4), is somewhat better in that respect. The first step produces many elements that are within E. A progression already contains many of the points that arrive early in the stepwise process. That is why the norm in (5) is larger than in (4).

Thus the two examples represent two extremes of how a set of N elements can "lie" in the group Z. That makes it reasonable to expect (2) to be true for all $E \subset Z$. The examples suggest more:

If we want a Fourier transform to equal one on a big set E, and to equal zero on most of its complement, and also to have a relatively small norm, then we must *allow it to take on some values which are intermediate between zero and one at certain points. Just allow the function to move gently down from* 1 *to* 0 *along points that are arithmetically close to E.*

It is well-known how this works with (4) and (5): Replace the box with a trapezoid, or extend the indicator function of the powers-of-three set by the transform of a Riesz product, and you get an object whose norm is under control—no greater than 2 or 3, regardless of N.

More generally: We can find a Fourier transform with prescribed large values on E, and with relatively small values on most of the complement of E, and still keep some control on the norm of f, provided we let f do what it wants to do on the points outside E that are arithmetically close to E.

That principle finds very sophisticated expression in the separation theorems developed by S. W. Drury, N. Th. Varopoulos, and C. S. Herz to solve the union problem for Helson sets. See Graham & McGehee, Chapter 3, for an exposition. Here is a sample of their results:

Theorem. *Let $E \subset Z$. For each integer $K > 0$, let E_K be the set of sums $\sum_{n\in E} u_n n$, considering all the sequences $\{u_n\}_{n\in E}$ such that*

$$\sum |u_n|^2 < 2K \quad \text{and} \quad \sum_{n\in E} u_n = 1.$$

Let F be a set that does not intersect E_K. Let $\epsilon > 0$. Let w be a function defined on E such that $|w(n)| \leq 1$ for each $n \in E$. Then there exists $f \in A$ such that

$$(i)\ f = w \text{ on } E,$$
$$(ii)\ |f| < \epsilon \text{ on } F, \text{ and}$$
$$(iii)\ \|f\|_A < \epsilon^{-1/2K}(\#E)^{1+1/2K}.$$

A Brief History of $C \log N$

Paul Cohen's procedure (1960) gave (1) with

$$C\left(\frac{\log N}{\log\log N}\right)^{1/8}.$$

A modification due to H. Davenport gave

$$C\left(\frac{\log N}{\log\log N}\right)^{1/4}.$$

An improvement by S. K. Pichorides (1974) gave

$$C\left(\frac{\log N}{\log\log N}\right)^{1/2}.$$

J. J. F. Fournier [1] (1978) discovered an advanced version of Cohen's proof which yields

$$C(logN)^{1/2}.$$

Pichorides [4] (1977) had obtained the same result in the case when $|\hat{f}| = 1$ on E, and his 1978 paper [5] removed that condition.

Pichorides's approach was completely different, and turned out to be most productive. In 1979, he [6] got

$$C\,\frac{logN}{(log\,logN)^2}.$$

That result appeared first in the Orsay publications of that year.

Pichorides's ideas underlie the proof of (1) given by S. V. Konjagin (1981).

A Proof of (1)

We present here the proof due to Brent Smith and others (1981), but with the refinement due to J. D. Stegeman (and also to Kôzô Yabuta) giving

$$C > 0.1293, \text{ which exceeds } \frac{4}{\pi^3} \simeq 0.1290.$$

The constant can of course be no greater than

$$\frac{4}{\pi^2} \simeq 0.4053.$$

The procedure is similar to Cohen's in that, given the function f, whose transform vanishes off E, we undertake to construct a function G such that $\sum \hat{f}\hat{G}$ is large (like $log\,N$) while $\|G\|_\infty$ is bounded. In other words, we hope to fill in the blanks satisfactorily in the following line:

$$\cdots \leq \int fG = \sum \hat{f}\hat{G} \leq \tag{6}$$

$$\leq \|f\|_1\|G\|_\infty \leq \|f\|_1 \cdot \cdots.$$

The procedure is similar to Cohen's also in that we break up E into subsets $E_j \subset E$, for $j = 0, 1, \ldots, m$. Choose integers $d \geq 2$ and m such that

$$m \leq \frac{log\,N}{log(1+d)} < m+1. \tag{7}$$

Then

$$\sum_{j=0}^{m} d^j \leq (1+d)^m \leq N.$$

Let E_0 contain only the first element of E. Let E_1 contain the next d elements, E_2 the next d^2 elements, and so forth up to E_m.

The exponential growth rate in the size of the sets E_j allows the advantageous use of the Cauchy-Schwarz inequality, which replaces the counting arguments in Cohen's procedure.

Here is a preliminary definition of G; it will not quite work, and we will need to modify it. Let

$$P_j(x) = \sum_{n\in E_j} d^{-j} e^{-i\theta_n} e^{inx},$$

where θ_n is chosen so that

$$\hat{f}(n) = |\hat{f}(n)| e^{i\theta_n}.$$

Then

$$\int P_j f \geq 1, \quad \|P_j\|_2 = d^{-j/2}, \quad \text{and} \quad \|P_j\|_\infty \leq 1.$$

Let

$$G = \sum_{j=0}^{m} P_j$$

The integral of Gf is now at least $m+1$. The difficulty is that we do not have satisfactory control on the sup-norm of G. So we will revise our choice of G.

For each j, write $|P_j|$ in its Fourier series expansion:

$$|P_j(x)| = \sum c_n e^{inx}.$$

Then $c_n = \bar{c}_{-n}$, c_0 is real, and $\sum |c_n|^2 = d^{-j}$. Let $a > 0$. Let

$$h_j(x) = a\left(c_0 + 2\sum_{n<0} c_n e^{inx}\right).$$

Then

$$Re\, h_j(x) = a|P_j(x)| \quad \text{and} \quad \|h_j\|_2 \leq a\sqrt{2}\|P_j\|_2.$$

With those definitions and observations, we are prepared to define the function G that will make (6) effective; G will be G_m, where the sequence $\{G_j\}$ is defined inductively: $G_0 = P_0$ and

$$G_j = P_j + G_{j-1} e^{-h_j}.$$

It is a calculus exercise to show that

$$t + \frac{e^{-at}}{1-e^{-a}} \leq \frac{1}{1-e^{-a}} \quad \text{for} \quad 0 \leq t \leq 1. \tag{8}$$

We claim that for each j,

$$|G_j(x)| \leq \frac{1}{1-e^{-a}} \quad \forall x.$$

It is trivial for $j = 0$, and if it is true for $j-1$ then it is also true for j by (8), since

$$|G_j(x)| \leq |P_j(x)| + \frac{e^{-a|P_j(x)|}}{1-e^{-a}}.$$

With G defined to be G_m, we have a usable bound on its sup-norm:

$$\|G\|_\infty \leq \frac{1}{1-e^{-a}}. \tag{9}$$

With the old G, $G = \sum_0^m P_j$, we had

$$\int Gf = \sum_0^m \int P_j f \geq m+1.$$

We must now find a lower bound for $\int Gf$ using the new G:

$$\begin{aligned} G = &P_0\ exp(-h_1 - h_2 - \ldots - h_m) \\ &+P_1\ exp(-h_2 - \ldots - h_m) + \ldots + \\ &+P_j\ exp(-h_{j+1} - \ldots - h_m) + \ldots + \\ &+P_{m-1}\ exp(-h_m)\ +\ P_m. \end{aligned} \tag{10}$$

The question is whether the exponential factors in (10) change the values of $\hat{G}$ on E by too much.

It is helpful that each h_j, and hence each exponential in (10), has a Fourier transform that vanishes on the positive integers.

Thus the factors by which $P_j, P_{j+1}, \ldots$ are multiplied in (10) will affect the Fourier coefficients of G on E_j but the factors by which $P_{j-1}, P_{j-2}, \ldots$ are multiplied will not.

Thus on E_j, $\hat{G} - \hat{P}_j$ agrees with the transform of

$$\begin{aligned} q_j = &P_j\ [exp(-h_{j+1} - \ldots - h_m) - 1] + \\ &+P_{j+1}\ [exp(-h_{j+2} - \ldots - h_m) - 1] + . \\ &+ \ldots + P_{m-1}\ [exp(-h_m) - 1] \end{aligned}$$

Using the fact that $|exp(-z) - 1| \leq |z|$ for $Re\ z \geq 0$, estimate q_j pointwise:

$$\begin{aligned} |q_j| \leq |P_j| \cdot |\sum_{j+1}^m h_k| + |P_{j+1}| \cdot |\sum_{j+2}^m h_k| + \\ + \ldots + |P_{m-1}| \cdot |h_m|; \end{aligned}$$

then obtain a bound on $\hat{q}_j$:

$$\begin{aligned} |\hat{q}_j| \leq \|q_j\|_1 &\leq \sum_{k=j}^{m-1} \|P_k\|_2 \sum_{l=k+1}^{m} \|h_l\|_2 \leq \\ &\leq \sum_{k=j}^{\infty} d^{-k/2} \sum_{l=k+1}^{\infty} a\sqrt{2}\, d^{-l/2} = \\ &= \frac{a\sqrt{2}\, d}{(\sqrt{d}-1)(d-1)} d^{-j}. \end{aligned}$$

The last number is a bound for the difference between $\hat{G}(n)$ and $d^{-j}e^{-i\theta_n}$ for each $n \in E_j$. Therefore, with $G = G_m$,

$$Re \int fG = Re \sum_{j=0}^{m} \sum_{n \in E_j} \hat{f}(n)\hat{G}(n) \geq$$
$$\geq (m+1)\left(1 - \frac{a\sqrt{2}d}{(\sqrt{d}-1)(d-1)}\right). \tag{11}$$

By (7), (9), and (11), then, for every choice of $d \geq 2$ and every $a > 0$,

$$\|f\|_1 \geq (\log N) \times \left(\frac{1-e^{-a}}{\log(1+d)}\right) \times$$
$$\times \left(1 - \frac{a\sqrt{2}d}{(\sqrt{d}-1)(d-1)}\right).$$

Stegeman's computations show that with $d = 95$, and the optimal choice of a, one obtains

$$\|f\|_1 \geq 0.1293 \ \log N.$$

The proof is complete.

Hardy's Inequality

Hardy proved that

$$\|f\|_1 \geq \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{|\hat{f}(k)|}{k+1} \quad \text{for } f \in H^1(T).$$

The proof above shows that if the support of $\hat{f}$ is $\{n_1 < n_2 < \ldots\}$, then

$$\|f\|_1 \geq c \sum_{k=1}^{\infty} \frac{|\hat{f}(n_k)|}{k},$$

where the constant is at least .032.

Ivo Klemes [1] used that generalization of Hardy's inequality to prove the following result, which was a conjecture of A. Pełczyński:

Theorem. *Let E be a set of nonnegative integers. Then χ_E is the transform of an idempotent multiplier of H^1 if and only if E is a finite Boolean combination of lacunary sets, finite sets, and intersections with Z^+ of arithmetic sequences.*

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