Math 2040 (McGehee) Restoring Rubik's Cube December 4, 1997

These notes present an exposition of an algorithm more or less as found in *Handbook of Cubik Math* by Alexander H. Frey, Jr. and David Singmaster (Enslow Publishers, 1982), which I believe is now out of print. The book studies in detail the group of transformations of the cube.

To establish a coordinate system, hold the cube as you wish. The faces of the cube are then the Front, Back, Right, Left, Up, and Down. A *cubie* is one of the 26 parts of the cube that you can see. The six face cubies cannot be moved. There are 12 edge cubies, with two facelets each, and 8 corner cubies, with three facelets each. A *cubicle* is one of the 20 places among which the movable cubies can move. A cubie's *home cubicle* is where it belongs, which means where each of its facelets agrees in color with the face cubie it shares a face with.

The elementary transformations of the cube use the initials of the face names. Thus F is the clockwise quarter-turn of the Front face; F^2 , the half-turn of the Front face; and F^{-1} , the counterclockwise quarter-turn of the front face. And so forth.

The process is described in six steps. Step 1 is given in the greatest detail, but you will probably soon be able just to look at your cube and think out what to do. For the other steps, the description is adequate but needs to be fleshed out with practice and observation.

Step 1

In Step 1, the four down edge cubies are restored. You can hold the cube so that the one you are working on is the down front (df) cubie.

<u>How to restore the down front cubie</u>

There are 24 cases. If the df cubie is already in the df cubicle, we do nothing. The table below lists the other 23 cases and gives a solution for each one which does not move previously restored down edge cubies, if any. Examples: The first entry says: If the df cubie is in the rf cubicle, just rotate the Front face one quarter-turn clockwise. The fourth entry says: If the df cubie is in the ur cubicle, turn the Up face clockwise a quarter turn, then turn the Front face a half turn.

rf	F	dl	$L^2\ U^{-1}\ F^2$	fl	$D^{-1}\ L\ D$
uf	F^2	rd	$R F$		
lf	F^{-1}	ld	$L^{-1}\ F^{-1}$	fu	$F\ D\ R^{-1}\ D^{-1}$
ur	$U\ F^2$			fd	$F\ D^{-1}\ L\ D$
ub	$U^2\ F^2$	fr	$D\ R^{-1}\ D^{-1}$	bd	$B\ D\ R\ D^{-1}$
ul	$U^{-1}\ F^2$	br	$D\ R\ D^{-1}$		
		rb	$D^2\ B^{-1}\ D^2$	ru	$R^{-1}\ F\ R$
dr	$R^2\ U\ F^2$	lb	$D^2\ B\ D^2$	lu	$L\ F^{-1}\ L^{-1}$
db	$B^2\ U^2\ F^2$	bl	$D^{-1}\ L^{-1}\ D$	bu	$U\ R^{-1}\ F\ R$

Step 2

In Step 1, the four down edge cubies were restored. In Step 2, three of the down corner cubies will be restored without disturbing the four down edge cubies.

Take one of the down corner cubies that is not already restored. It must first be "prepared," which means placed in the upper layer with the down color NOT on the up face. If the cubie is not already prepared, two cases are possible:

(1) If the cubie is in the down layer, turn the cube so that the cubie is in the down front left cubicle. One of these will work:

$$FU^2F^{-1} \text{ or } L^{-1}U^2L$$

(2) If the cubie is in the upper layer with the down color on the up face, turn the cube so that the cubie is in the up back right cubicle. Do

$$FUF^{-1}.$$

When the cubie has been prepared, turn the cube so that the cubie's home cubicle is the down front left cubicle. Then do U or U^2, as needed, to put the cubie in the up back right cubicle. Then one of these will complete the task of restoring the cubie:

$$FU^2F^{-1} \text{ or } L^{-1}U^2L\ .$$

Step 3

Steps 1and 2 restored all four of the edge cubies that belong on the down face, and three of the four corner cubies that belong on the down face. Turn the cube so that all the down corner cubies other than the down back right cubie, called the "working corner," have been restored. Step 3 will restore the back left, front left, and front right edge cubies. In other words, it will restore all the middle-layer edge cubies except the back right one, which is the one above the working corner. It will do so while leaving the achievements of Steps 1 and 2 undisturbed.

Select one of the middle-layer edge cubies that needs to be restored.

If the cubie is in a middle-layer edge cubicle, it must first be placed in the upper layer. Turn the cube so that the cubie is in the left front cubicle. Then put the working corner underneath the cubie by means of D or D^2 or D^{-1} . Then do

$$FUF^{-1} \text{ or } L^{-1}UL,$$

either of which will put the cubie in the upper layer.

Next, turn the cube so that the cubie's home cubicle is the front left cubicle. Do D or D^2 or D^{-1} to put the working corner in the down front left cubicle. Do U or U^2 or U^{-1} to put the cubie in the right up cubicle. One of these,

$$FUF^{-1} \text{ or } L^{-1}U^2L,$$

will restore the cubie. Finally, do D^{-1} or D^2 or D to put the working corner back where it was.

Step 4

When steps 1, 2, and 3 have been done, all cubies other than those of the top layer, the back right edge (the working edge), and the down back right corner (the working corner) have been restored.

After step 4, all the edge cubies will have been restored. Every move in step 4 involves B and then B^{-1} , or else R^{-1} and then R , with various possible turns of the up layer coming before, between, and after; and nothing else. Therefore no move in Step 4 will disturb the achievements of Steps 1, 2, and 3.

This procedure is for the restoration of (1) the up left, up front, and up back edges, *in that order*; and then of (2) the up right and right back edges. The cubie that belongs in the left up edge cubicle may already be there, properly oriented (up color on up face, left color on left face). Perhaps you can make it so by a simple turn of the up layer.

(1) To get one of these three edge cubies into its home cubicle, first put it into the working edge (if it is not already there), without disturbing the others that are already restored. Then move it to its home cubicle.

How to move edge cubie x from the up layer to the working edge

One of these transformations will work, depending on whether x is in the up left, up front, up back, or up right edge:

$$B\,U\,B^{-1}\,U^{-1}\,,\quad B\,U^{2}\,B^{-1}\,U^{2},\quad U^{-1}\,B\,U\,B^{-1}\,,\ \text{or}\ B\,U^{-1}\,B^{-1}\,U$$

How to move edge cubie x from the working edge to its home cubicle

Suppose the up color of x is on the right face. If the home cubicle of x is the up back cubicle, do $B\,U^{-1}\,B^{-1}\,U$; otherwise: Turn the up layer so that the home cubicle of x is in the up back edge. Do B . Do the inverse of the up-layer turn that you did. Do B^{-1} .

Suppose the up color of x is on the back face: Turn the up layer if necessary so that the home cubicle of x is in the up right edge. Do R^{-1} . Do the inverse of the up-layer turn that you did. Do R .

(2) How to restore the up right and back right cubies

Case 1: They're already in place. Case 2: They're both in place but with the wrong orientation. Do

$$B\,U^{-1}\,B^{-1}\,U\,R^{-1}\,U\,R\,U^{-1}.$$

Case 3: They're in each other's places. Then the cubie that belongs in the up right cubicle is in the back right cubicle. If its up color is on the right face, do

$$V = B\,U\,B^{-1}\,U\,B\,U\,B^{-1}\,U^{2}.$$

If its up color is on the back face, do

$$W = U^{-1}R^{-1}U^{-1}RU^{-1}R^{-1}U^{-1}RU^{-1}$$

Step 5

When steps 1, 2, 3, and 4 have been done, all cubies other than the corner cubies of the top layer and the down back right corner (the working corner) have been restored. After Step 5, every corner cubie will be in its home cubicle. It may or may not have the correct orientation; if not, we'll twist it in Step 6.

Every move in Step 5 consists of the transformation

$$S = L\ D^2\ L^{-1} \quad (\text{or } T = F^{-1}\ D^2\ F)\ .$$

done twice, with various well-chosen turns of the up layer coming before, between, and after; and nothing else. Therefore no move in Step 5 will disturb the achievements of Steps 1 through 4.

Pick two corner cubies on the up face such that one of them--call it x--needs to be moved to the cubicle now occupied by the other one--call it y . Turn the up face so that x is in the up front left cubicle. (Make a mental note of where y is now.) Do S (or T) . Turn the up face so that y is in the up front left cubicle. Do S (or T) again. Invert (undo) the two turns of the up face that you have done.

Three applications of the preceding paragraph is all you need, if you are careful. When necessary, you can hold your cube differently (change the coordinate system!), so that a different face becomes the up face while applying the paragraph.

Step 6

When steps 1 through 5 have been done, all cubies are in place, and the only remaining problem is that some or all of the corner cubies in the up layer and the working (back right down) corner need to be twisted.

Let S and T be as defined for Step 5, noting that ST and TS are inverses. Every move in Step 6 consists of ST followed by TS , or else TS followed by ST, with various well-chosen turns of the up layer coming before, between, and after; and nothing else. Therefore no move in Step 6 will disturb the achievements of Steps 1 through 5.

Looking at a corner cubie that needs twisting, notice whether it needs to be twisted clockwise or counterclockwise. For example, if it is in the left front up cubicle, and the color facing left needs to be facing up, then the cubie needs to be twisted clockwise.

Find an up corner cubie--call it x -- that needs to be twisted clockwise. Pick another up corner cubie---call it y --that needs to be twisted; if possible, pick one that needs to be twisted counterclockwise. The following process will twist x clockwise and y counterclockwise. (The number of cubies that need twisting will then be reduced by one, maybe by two.)

Turn the up layer so that x is in the up left front corner cubicle. Make a mental note of where y is. Do ST. Turn the up layer so that y is in the up left front corner cubicle. Do TS . Turn the up layer so as to undo the two previous turns you did.

(If TS were done before ST , x would be twisted counterclockwise and y clockwise.)