Mathematics 2057 (McGehee) Notes on The Exterior Algebra April 8, 1998

Fix an integer $n \geq 1$. The *exterior algebra of differential forms in* $\mathbf{R}^n$ consists of (1) all scalar-valued functions (0-forms); (2) the constant forms; and (3) sums and products thereof.

The 1-form dx is the only constant form in $\mathbf{R}^1$. The most general 1-form in $\mathbf{R}^1$ is $f(x)dx$, where f is a 0-form.

In $\mathbf{R}^2$ there are two constant 1-forms, dx and dy, and one constant 2-form, $dx\,dy$. The most general 1-form may be written $P(x,y)dx + Q(x,y)dy$. The most general 2-form is $f(x,y)dx\,dy$.

Differential Forms in $\mathbf{R}^3$

There are three constant 1-forms, dx, dy, and dz; three constant 2-forms, $dx\,dy$, $dy\,dz$, and $dz\,dx$; and one constant 3-form, $dx\,dy\,dz$. The most general 1-form may be written

$$\omega = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz.$$

The most general 2-form may be written

$$\sigma = P(x,y,z)dy\,dz + Q(x,y,z)dz\,dx + R(x,y,z)dx\,dy.$$

Notice that a vector field in $\mathbf{R}^3$,

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k},$$

may be associated either with the 1-form ω or the 2-form σ. The most general 3-form is

$$f(x,y,z)dx\,dy\,dz,$$

where f is a 0-form.

Differential Forms in $\mathbf{R}^4$

There are four constant 1-forms: dx, dy, dz, dt; six constant 2-forms: $dx\,dy$, $dy\,dz$, $dz\,dt$, $dt\,dx$, $dx\,dz$, $dy\,dt$; four constant 3-forms: $dx\,dy\,dz$, $dy\,dz\,dt$, $dz\,dt\,dx$, $dt\,dx\,dy$; and one constant 4-form: $dx\,dy\,dz\,dt$.

The Algebraic Rules

Whenever du and dv are constant 1-forms, and f, f_1, f_2 are 0-forms, then

$$du(fdv) = fdudv; \quad (f_1+f_2)du = f_1du + f_2du; \quad f(du+dv) = fdu + fdv; \quad du\,dv = -dv\,du.$$

The operator d, the *exterior derivative*, maps k-forms to $(k+1)$-forms in accord with these rules: (1) If ω is a constant form, then $d\omega = 0$. (2) If ω_1 and ω_2 are k-forms, then $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$. (3) If ω is a k-form and f is a 0-form, then $d(f\omega) = df\omega + fd\omega$.

Example: Let $f = f(x,y)$ be a 0-form in $\mathbf{R}^2$. Then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Example: Let $\omega = Pdx + Qdy$. Then

$$d\omega = dPdx + dQdy = \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right)dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right)dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy.$$

The calculation of the last step uses the facts that $dxdx = dydy = 0$ and $dxdy = -dydx$. Question: If $\omega = Pdx + Qdy + Rdz$, an arbitrary 1-form in 3-space, find $d\omega$. What does it have to do with the curl?

Example: If $x = x(s,t), y = y(s,t)$, one may show that

$$dx\,dy = \left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right)ds\,dt.$$

A specific case: If $x = r\cos\theta, y = r\sin\theta$, then $dx\,dy = rdr\,d\theta$.

Math 4038 (McGehee) Remarks on Two of Maxwell's Equations

Considering B as a 2-form and E as a 1-form in $x, y, z-$space, we define the *electromagnetic field* to be this 2-form in $x, y, z, t-$space:

$$B + Edt = B_1 dydz + B_2 dzdx + B_3 dxdy + E_1 dxdt + E_2 dydt + E_3 dzdt. \quad (1)$$

To say that the 2-form $B + Edt$ is closed means that its differential is zero:

$$d(B + Edt) = 0. \quad (2)$$

We will now compute $d(B + Edt)$. It will become clear that (2) is equivalent to [redacted] two of Maxwell's Equations. The differential of the first summand on the right-hand side of (1) is as follows:

$$d(B_1 dydz) = dB_1 dydz = \left(\frac{\partial B_1}{\partial x}dx + \frac{\partial B_1}{\partial y}dy + \frac{\partial B_1}{\partial z}dz + \frac{\partial B_1}{\partial t}dt\right) dydz.$$

Recall the rules for algebraic operations with differential forms. Since $dydydz$ and $dzdydz$ are both zero, and since $dtdydz = dydzdt$, the result is

$$d(B_1 dydz) = \frac{\partial B_1}{\partial x}dxdydz + \frac{\partial B_1}{\partial t}dydzdt.$$

Similarly, one obtains

$$\begin{aligned} d(B_2 dzdx) &= \frac{\partial B_2}{\partial y}dxdydz + \frac{\partial B_2}{\partial t}dzdxdt \quad \text{and} \\ d(B_3 dxdy) &= \frac{\partial B_3}{\partial z}dxdydz + \frac{\partial B_3}{\partial t}dxdydt. \end{aligned}$$

The differential of the fourth summand on the right-hand side of (1) is as follows:

$$d(E_1 dxdt) = dE_1 dxdt = \left(\frac{\partial E_1}{\partial x}dx + \frac{\partial E_1}{\partial y}dy + \frac{\partial E_1}{\partial z}dz + \frac{\partial E_1}{\partial t}dt\right) dxdt.$$

Since $dxdxdt = dtdxdt = 0$ and $dydxdt = -dxdydt$, the result is

$$d(E_1 dxdt) = -\frac{\partial E_1}{\partial y}dxdydt + \frac{\partial E_1}{\partial z}dzdxdt.$$

Similarly, one obtains

$$\begin{aligned} d(E_2 dydt) &= \frac{\partial E_2}{\partial x}dxdydt - \frac{\partial E_2}{\partial z}dydzdt \quad \text{and} \\ d(E_3 dzdt) &= \frac{\partial E_3}{\partial y}dydzdt - \frac{\partial E_3}{\partial x}dzdxdt. \end{aligned}$$

Reorganizing, we obtain

$$\begin{aligned} d(B+Edt) = & \left(\frac{\partial B_1}{\partial x}+\frac{\partial B_2}{\partial y}+\frac{\partial B_3}{\partial z}\right) dxdydz \\ & + \left(\frac{\partial B_1}{\partial t}+\frac{\partial E_3}{\partial y}-\frac{\partial E_2}{\partial z}\right) dydzdt \\ & + \left(\frac{\partial B_2}{\partial t}+\frac{\partial E_1}{\partial z}-\frac{\partial E_3}{\partial x}\right) dzdxdt \\ & + \left(\frac{\partial B_3}{\partial t}+\frac{\partial E_2}{\partial x}-\frac{\partial E_1}{\partial y}\right) dxdydt, \end{aligned}$$

Of course this 3-form is zero if and only if each of the four coefficients is zero. The first coefficient is the divergence of B, and the other three are the components of the sum of the time derivative of B and the curl of E. It follows, then, that condition (2) is equivalent to the first two of Maxwell's Equations:

$$\nabla \cdot \mathrm{B} = 0 \tag{3}$$

and

$$\frac{\partial \mathrm{B}}{\partial t} + \nabla \times \mathrm{E} = 0. \tag{4}$$

Mathematics 2057 (McGehee) **Equation of Continuity, Etc.** **May 6, 1998**

Let $\mathbf{J} = J_1\mathbf{i} + J_2\mathbf{j} + J_3\mathbf{k}$ be current density. Each component is a function of x, y, z, and t. The units of $|\mathbf{J}|$ are coulombs per second per square meter, or amperes per square meter. Let ρ be charge density, a scalar-valued function of x, y, z, and t, in coulombs per cubic meter. Now let R be a bounded simply connected region in $\mathbf{R}^3$ with outward-oriented boundary ∂R. The net outflow of charge contained in R during the time interval $[t_0, t_1]$ equals

$$\int_{t_0}^{t_1} \left(\int_{\partial R} J_1\, dy\, dz + J_2\, dz\, dx + J_3\, dx\, dy \right) dt \quad = \quad \int_{t_0}^{t_1} \left(\iint_{\partial R} \mathbf{J} \cdot \mathbf{n}\, dA \right) dt, \tag{1}$$

and also equals

$$\int_R (\rho(x,y,z,t_0) - \rho(x,y,z,t_1))\, dx\, dy\, dz \quad = \quad \int_{R\times\{t_0\}} \rho\, dx\, dy\, dz - \int_{R\times\{t_1\}} \rho\, dx\, dy\, dz. \tag{2}$$

Each of the integrals (1) and (2) is the integral of a 3-form in $\mathbf{R}^4$ over part of the boundary of the four-dimensional object $R \times [t_0, t_1]$. That boundary is the union of $\partial R \times [t_0, t_1]$, $R \times \{t_0\}$, and $R \times \{t_1\}$. The integral of $J_1\, dy\, dz\, dt + J_2\, dz\, dx\, dt + J_3\, dx\, dy\, dt$ over $R \times \{t_0\}$ and $R \times \{t_1\}$ is zero, because "there's no dt" on those parts. The integral of ρ over $\partial R \times [t_0, t_1]$ is zero because it is only two-dimensional in terms of the space variables. Therefore both (1) and (2) may be viewed as integrals of their respective 3-forms over the whole (3-dimensional) boundary of $R \times [t_0, t_1]$, and we may write

$$(1) - (2) = \int_{\partial(R\times[t_0,t_1])} -J_1\, dy\, dz\, dt - J_2\, dz\, dx\, dt - J_3\, dx\, dy\, dt + \rho\, dx\, dy\, dz = 0. \tag{3}$$

We have shown that the 3-forms $dy\, dz\, dt$, $dz\, dx\, dt$, and $dx\, dy\, dt$ must be assigned negative orientation to make sense when used in $\mathbf{R}^4$ (space-time); that explains the sign reversals. It follows, incidentally, that $dx\, dy\, dz\, dt$ should also have negative orientation. Let

$$\mathcal{J} := \rho\, dx\, dy\, dz - J_1\, dy\, dz\, dt - J_2\, dz\, dx\, dt - J_3\, dx\, dy\, dt, \tag{4}$$

which is a 3-form in $\mathbf{R}^4$ called the total current. The integral (3) is the integral over $\partial(R \times [t_0, t_1])$ of $\mathcal{J}$. Applying the Generalized Fundamental Theorem of Calculus, we obtain

$$\int_{\partial(R\times[t_0,t_1])} \mathcal{J} = \int_{R\times[t_0,t_1]} d\mathcal{J}.$$

Since this equals zero for every R and every $[t_0, t_1]$, it must be that $d\mathcal{J} = 0$.

PROBLEM 1: Using the algebraic rules for forms and for the operator d, show that

$$d\mathcal{J} = -\left(\frac{\partial \rho}{\partial t} + \frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z} \right) dx\, dy\, dz\, dt. \tag{5}$$

For the 4-form in (5) to be zero means that the coefficient is identically zero, which gives us

$$\text{The continuity equation:} \quad \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{6}$$

Overlooking the fact that Maxwell did not think in the language of modern differential geometry, one may say: Maxwell, realizing that a form is exact if and only if it is closed (at least locally), looked around for a 2-form ω such that $d\omega = \mathcal{J}$, and found that the following proposal for ω was consistent with the known experiments:

$$d\left(\epsilon E - \frac{1}{\mu} B\, dt\right) = \mathcal{J}, \tag{7}$$

where E is the 2-form associated with the electric field intensity, namely

$$E = E_1\, dy\, dz + E_2\, dz\, dx + E_3\, dx\, dy, \tag{8}$$

and B is the 1-form associated with the magnetic field intensity, namely

$$B = B_1\, dx + B_2\, dy + B_3\, dz, \tag{9}$$

where all the E_k and B_k are functions of x, y, z, and t. Thus

$$\epsilon E - \frac{1}{\mu} B\, dt = \epsilon(E_1\, dy\, dz + E_2\, dz\, dx + E_3\, dx\, dy) - \frac{1}{\mu}(B_1\, dx\, dt + B_2\, dy\, dt + B_3\, dz\, dt) \tag{10}$$

PROBLEM 2: Using the algebraic rules for forms and for the operator d, show that

$$\begin{aligned} d\left(\epsilon E - \frac{1}{\mu} B\, dt\right) &= \epsilon \left(\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\right) dx\, dy\, dz \\ &+ \left(\epsilon \frac{\partial E_1}{\partial t} - \frac{1}{\mu}\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)\right) dy\, dz\, dt \\ &+ \left(\epsilon \frac{\partial E_2}{\partial t} - \frac{1}{\mu}\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)\right) dz\, dx\, dt \\ &+ \left(\epsilon \frac{\partial E_3}{\partial t} - \frac{1}{\mu}\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\right) dx\, dy\, dt. \end{aligned} \tag{11}$$

Matching up the coefficients on $dx\, dy\, dz$ from (4) and (11), we obtain Maxwell's first equation:

$$\epsilon \nabla \cdot \mathbf{E} = \rho \qquad \text{(Gauss's Law)} \tag{12}$$

Matching up the remaining triple of coefficients in (4) and (11), we obtain Maxwell's fourth equation, which improved Ampère's Law to cover the case when $\mathbf{E}$ is a function also of time:

$$\nabla \times \mathbf{B} = \mu \mathbf{J} + \mu\epsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{13}$$