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Integral Bilinear Forms with a Group Action

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0. INTRODUCTION

Let Γ be a finitely generated group. By a Γ-form over $\mathbb{Z}$ we mean a non-degenerate bilinear form $b\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, either symmetric or skew-symmetric, together with a representation $\rho\colon \Gamma \to GL_n(\mathbb{Z})$ which verifies $b(\rho(\gamma)x, \rho(\gamma)y) = b(x, y)$ for all x, y in $\mathbb{Z}^n$ and all γ in Γ.

We do not expect to obtain a general classification, up to isomorphism, of Γ-forms over $\mathbb{Z}$. Nevertheless, we can fruitfully develop an arithmetic theory of Γ-forms, and most classical results on integral quadratic forms, for instance the Siegel mass formula, can be generalized to this context.

Section 1 is concerned with finiteness questions. We show that for a given nonzero integer d and a given semi-simple complex representation ρ_0 of Γ there are, up to isomorphism, only finitely many Γ-forms (b, ρ) over $\mathbb{Z}$ such that $\rho \simeq \rho_0$ over $\mathbb{C}$ and $\operatorname{disc}(b) = d$.

In Section 2 we compute the Tamagawa number of the group of automorphisms of a Γ-form and use it to establish the generalization of the classical Siegel mass formula.

Finally, in Section 3, we consider the special case in which Γ is a finite abelian group, and we compute the local densities for a unimodular Γ-form (b, ρ) such that ρ is isotypic over $\mathbb{Q}$.

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1. A FINITENESS THEOREM

Let Γ be a finitely generated group and R a commutative ring. A Γ-form over R will be an $R\Gamma$-module M, projective and finitely generated over R, together with an ε-symmetric non-degenerate bilinear form $b\colon M \to M^* :=$

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$\mathrm{Hom}_R(M, R)$ which is Γ-equivariant. Two Γ-forms (M, b) and (M', b') are isomorphic if there is an $R\Gamma$-isomorphism $\phi: M \to M'$ such that $\phi^* b' \phi = b$.

1.1. THEOREM. *Let V be a semi-simple $\mathbb{C}\Gamma$-module and d a nonzero integer. Then there are only finitely many isomorphism classes of Γ-forms (M, b) over $\mathbb{Z}$ such that $M \otimes_{\mathbb{Z}} \mathbb{C} \cong V$ and* $\mathrm{disc}(b) = d$.

Proof. We know, by a classical theorem, that there are only finitely many equivalence classes of integral bilinear forms of given rank and discriminant. Thus we can assume that we are dealing with a fixed ε-symmetric matrix $B \in M_n(\mathbb{Z})$ of determinant equal to d. We will show that there are only finitely many ways in which Γ can act on $\mathbb{Z}^n$ so as to induce a given semi-simple representation of Γ in $\mathbb{C}^n$ and preserve B.

Let $\langle x_1, ..., x_m \mid \text{relations} \rangle$ be a presentation of Γ. We can present the group algebra $\mathbb{C}\Gamma$ as a quotient of the free (noncommutative) algebra $\mathbb{C}\{x_1, ..., x_m, y_1, ..., y_m\}$ modulo $x_i y_i - 1$ and all the relations which arise from those defining Γ. We denote by (*) this set of relations. The set $\mathscr{R}_n$ of all n-dimensional complex representations of Γ can be viewed as the subset of $M_n(\mathbb{C})^{2m}$ consisting of $2m$-tuples $(X_1, ..., X_m, Y_1, ..., Y_m)$ of matrices satisfying (*), which is clearly a closed algebraic subset defined by integral equations. The group $GL_n(\mathbb{C})$ acts in an obvious way on $\mathscr{R}_n$ and its orbits correspond to isomorphism classes of representations. We know by a theorem of H. Kraft see [Kr, Chap. II, Sect. 7]) that a representation $\rho \in \mathscr{R}_n$ is semi-simple if and only if its orbit $GL_n(\mathbb{C})\rho$ is closed in $\mathscr{R}_n$.

Let $\mathscr{R}_n^0$ be the set of orthogonal representations (with respect t the matrix B) of Γ in $\mathbb{C}^n$, i.e., $\mathscr{R}_n^0 = \{(X_1, ..., X_m, Y_1, ..., Y_m) \in \mathscr{R}_n \colon X_i^t B X_i = B\}$. $\mathscr{R}_n^0$ is of course closed in $\mathscr{R}_n$. The orthogonal group of B, $O_n(\mathbb{C}, B)$, acts on $\mathscr{R}_n^0$, and its orbits can be interpreted as classes of Γ-forms over $\mathbb{C}$. It is easy to see that if $\rho \in \mathscr{R}_n^0$ is semi-simple, then $GL_n(\mathbb{C})\, \rho \cap \mathscr{R}_n^0$ contains only one $O_n(\mathbb{C}, B)$-orbit (this is the geometric translation of the fact that there is only one class of Γ-forms over $\mathbb{C}$ with a given underlying semisimple $\mathbb{C}\Gamma$-module). It follows that $O_n(\mathbb{C}, B)\rho$ is closed. By applying a general theorem of Borel and Harish–Chandra (see [Bo-HC], theorem 6.9) on closed orbits of reductive algebraic groups, we conclude that the intersection $O_n(\mathbb{C}, B)\rho \cap M_n(\mathbb{Z})^{2m}$ contains only finitely many $O_n(\mathbb{Z}, B)$-orbits. This is exactly what we wanted. ∎

1.2. COROLLARY. *Let (M, b) a Γ-form over $\mathbb{Z}$ such that $M \otimes_{\mathbb{Z}} \mathbb{C}$ is a semi-simple $\mathbb{C}\Gamma$-module. Then there are only finitely many isomorphism classes in the genus of (M, b).*

Proof. The discriminant of b and the class of $M \otimes_{\mathbb{Z}} \mathbb{C}$ are invariants of the genus of (M, b). Thus Corollary 1.2 follows directly from the theorem. ∎

Remark. Theorem 1.1 has been proved by E. Bayer and F. Michel (see [B–M]) for Γ cyclic.

More generally, Theorem 1.1 for $\mathbb{Z}\Gamma$-lattices in a semi-simple $\mathbb{Q}\Gamma$-module whose simple self-dual components have commutative endomorphism ring is a consequence of H. G. Quebbemann results (see [Q, 1.4–1.5]) together with the Jordan–Zassenhaus Theorem (see, e.g., [R, 26.4]).

2. The Mass Formula

We assume from now on that Γ is a *finite* group. Let (V, b) be an ε-symmetric Γ-form over $\mathbb{Q}$. Let G be the group of automorphisms of (V, b), considered as an algebraic group defined over $\mathbb{Q}$. The group G is reductive but not semi-simple in general. We will determine the Tamagawa number of the connected component, G^0, of the identity in G. We will show in particular, that $\tau(G^0)$ does not depend on the form b but only on the $\mathbb{Q}\Gamma$-module structure of V.

The field $\mathbb{Q}$ is ordered, hence each $\mathbb{Q}\Gamma$-module is selfdual. In particular, the isotypic (or homogeneous) components of V are all self-dual. Therefore the restriction of b to an isotypic component must be non-degenerate. Hence (V, b) splits canonically as an orthogonal sum:

$$(V, b) = (V_1, b_1) \perp \cdots \perp (V_r, b_r),$$

where the V_i are the isotypic components of V. The group G splits over $\mathbb{Q}$ as the product of the automorphism groups of the isotypic components (V_i, b_i). The Tamagawa number is multiplicative. Therefore it will be enough to compute it in the isotypic case.

Assume now that V is isotypic an let S be its simple component. We take first any form $c: S \to S^*$, symmetric or skew-symmetric, and call ι the adjoint involution on $D_S := \mathrm{End}_{\mathbb{Q}\Gamma}(S)$. We will say that S is of the *first kind* if the restriction of ι to the centre of D_S is trivial, and of the *second kind* otherwise (remark that this definition does not depend on the choice of c; every form c will induce the same automorphism of the center of D_S).

We fix once for all a form c_S on each simple $\mathbb{Q}\Gamma$-module S with the following conventions:

(i) If S is of the first kind and D_S is a quaternion algebra, we choose a Γ-form c_S on S in such a way that it induces the standard quaternion involution on D_S (this is possible by applying the Skolem–Noether theorem). Such a form is unique up to a central factor and in particular its sign ε_S is uniquely determined.

(ii) In all other cases we choose c_S to be positive definite.

$\mathrm{Hom}_{\mathbb{Q}\Gamma}(S, V)$ has a natural structure as a right vector space over D_S. We define an $\varepsilon\varepsilon_S$-hermitian D_S-valued form h on $\mathrm{Hom}_{\mathbb{Q}\Gamma}(S, V)$ by $h(f, g) = c_S^{-1} f^* bg$ (this is a particular case of the general "transfer" construction in [Q–S–S]).

Let E be the centre of D_S and $F \subset E$ the fixed field of the involution. Let U be the unitary group of h, viewed as an algebraic group defined over F. It is easy to check that the group G of automorphisms of (V, b) is obtained by applying the restriction functor $R_{F/\mathbb{Q}}$ to U. Hence G and U have the same Tamagawa number.

Let SU be the subgroup of U consiting of all elements with reduced norm 1. We have the following table of values for $\tau(SU)$ (see [W] and [M]):

First kind:

D	Commutative field		Quaternion algebra	
$\varepsilon\varepsilon_S$	$+1$	-1	$+1$	-1
$\tau(SU)$	2 $(rk(h) \geqslant 2)$	1	1	2

Second kind:

D_S	Commutative field	Skew-field
$\tau(SU)$	1	1

In the case of an involution of the first kind, SU is actually the connected component of 1 in U.

In the other case U is connected and we have a short exact sequence of algebraic groups over F:

$$1 \to SU \to U \xrightarrow{\mathrm{Nrd}} \mathrm{Ker}\, N_{E/F} \to 1,$$

where Nrd denotes the reduced norm. By Proposition 2.2.1 in Ono's paper $[O_1]$ we have $\tau(U) = \tau(SU)\,\tau(\mathrm{Ker}\, N_{E/F})$. Now $\mathrm{Ker}\, N_{E/F}$ is the special orthogonal group of a quadratic form of rank 2, and hence we have $\tau(\mathrm{Ker}\, N_{E/F}) = 2$ by the Siegel–Tamagawa theorem. Therefore $\tau(U) = 2$.

2.1. THEOREM. *Let (V, b) be any Γ-form over $\mathbb{Q}$ and G its automorphism group, considered as an algebraic group defined over $\mathbb{Q}$. The Tamagawa number of G^0 is:*

$$\tau(G^0) = 2^{p+q+r},$$

where the numbers p, q and r are defined by

$p = 0$ *if b is skew-symmetric. If b is symmetric, then p is the number of distinct simple components S of V of the first kind such that*:

(a) D_S *is a commutative field*,

(b) *S has multiplicity at least* 2 *in V*;

$q =$ *number of distinct simple components S of V of the first kind such that*:

(a) D_S *is a quaternion algebra*,

(b) $\varepsilon\varepsilon_S = -1$;

$r =$ *number of distinct simple components of V of the second kind.*

Proof. The Tamagawa number is multiplicative and remains unchanged under restriction of scalars, so we may apply the above known results on Tamagawa numbers of unitary groups. ∎

2.2. COROLLARY. $\tau(G^0)$ *depends only on the* $\mathbb{Q}\Gamma$*-module structure of V and on the sign* ε *of b.* ∎

DEFINITION. We say that a Γ-form (V, b) over $\mathbb{Q}$ is *definite* if the group $G(\mathbb{R})$, the group of real points of G, is compact.

Let (V, b) a definite Γ-form over $\mathbb{Q}$. The adelized group $G(A)$ acts on the set of Γ-stable lattices in V in the following way: for a lattice M and an adèle $\sigma = (\sigma_p) \in G(A)$, σM is the lattice defined by $(\sigma M)_p = \sigma_p(M_p)$ for all p. The isomorphism classes of lattices in V which are in the genus of (M, b) are in one-to-one correspondence with the set of double cosets $G(A)_M \backslash G(A)/G(\mathbb{Q})$, where $G(A)_M$ is the stabilizer of M in $G(A)$. Let $M_1, ..., M_k$ be representatives of the classes of lattices in V that belong to the genus of M. There are only finitely many of them by Corollary 1.2. Denote by w_i the order of the finite group $G(A)_{M_i} \cap G(\mathbb{Q})$ (which is the group of automorphisms of the Γ-form (M_i, b)). With these notations we have the familiar formula:

$$\sum_{i=1}^{k} \frac{1}{w_i} = \operatorname{vol}(G(A)/G(\mathbb{Q})) \cdot \operatorname{vol}(G(A)_M)^{-1}, \tag{1}$$

where vol is any invariant measure on $G(A)$.

A word of caution. The set of classes in the whole genus of M will in general be bigger than $\{M_1, ..., M_k\}$, because the Hasse Principle may not hold.

QUESTION (Hasse Principle). Let (V, b) and (V', b') be two Γ-forms over $\mathbb{Q}$ which are isomorphic everywhere locally. Are they isomorphic over $\mathbb{Q}$?

To answer this question, it is enough to consider isotypic $\mathbb{Q}\Gamma$-modules. Let (V, b) a Γ-form, where V is isotypic with simple component S. After choosing a Γ-form c_S on S as above, we get a (± 1)-hermitian form h on the right D_S-vector space $\mathrm{Hom}_{\mathbb{Q}\Gamma}(S, V)$, where the involution on D_S is the adjoint involution of c_S. It is easy to verify that the Hasse Principle holds for (V, b) iff it holds for $(\mathrm{Hom}_{\mathbb{Q}\Gamma}(S, V), h)$.

We know (see Kneser [K]) that the Hasse Principle is true for $(+1)$-hermitian forms over (skew) fields. But the Hasse Principle may fail for (-1)-hermitian forms over a quaternion division algebra D. (see [K, Sect. 5.10]). More precisely: if $\sim$ denotes the equivalence relation "being isomorphic everywhere locally," each equivalence class with respect to $\sim$ contains exactly 2^{m-2} isomorphism classes, where m is the number of places of the centre E of D where D does not split.

The following example shows that we cannot avoid this (-1)-hermitian situation, even for symmetric Γ-forms.

Let D be a quaternion algebra with centre $\mathbb{Q}$ which splits at infinity. By a theorem of M. Benard and K. L. Fields (see [Be]) there exists a finite group Γ and a simple $\mathbb{Q}\Gamma$-module S such that $\mathrm{End}_{\mathbb{Q}\Gamma}(S) \simeq D$. It follows from the assumption $D \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$ that a Γ-form c_S on S which induces the standard quaternion involution on D must be skew-symmetric. Any symmetric Γ-form b on a S-isotypic module V will rise to a (-1)-hermitian form on $\mathrm{Hom}_{\mathbb{Q}\Gamma}(S, V)$.

To interpret the term $\mathrm{vol}(G(A)_M)$ of formula (1) in terms of "local densities" as in Siegel's classical formula, we need some preparatory lemmas.

2.3. LEMMA. *Let e be the exponent of Γ and E the field of eth-roots of 1 over Q. Then E is a splitting field for G^0 (in the sense that all characters of G^0 are defined over E).*

Proof. By representation theory (see [Se, 12.3]), $V \otimes_{\mathbb{Q}} E$ is decomposed as a direct sum of absolutely simple $E\Gamma$-modules. Using the isotypic orthogonal decomposition of $V \otimes_{\mathbb{Q}} E$, we see that there is an isomorphism defined *over* E:

$$G \simeq G_1 \times \cdots \times G_r \times GL_{m_1} \times \cdots \times GL_{m_s},$$

where the G_i are orthogonal or symplectic groups over E (according as b is symmetric or skew-symmetric) and the GL_{m_j} are general linear groups over E. ∎

2.4. LEMMA (Landau). *Let E be a cyclotomic field and χ a nontrivial irreducible character of* $\mathrm{Gal}(E/\mathbb{Q})$. *Then the product* $\prod_p (1-\chi(p)p^{-1})^{-1}$ *converges to* $L(1, \chi; E/\mathbb{Q})$, *provided we take the primes in increasing order.*

Proof. see Landau [L, Sect. 109]. ∎

2.5. LEMMA. *Let ψ be the character of the Galois module $\hat{G}^0$ $(=\hat{G}^0_E$ by Lemma 2.3). Let $L(s, \psi; E/\mathbb{Q})_p$ the p-component of the L-series $L(s, \psi; E/\mathbb{Q})$. Then the product $\prod_p L(1, \psi; E/\mathbb{Q})_p$ converges to $L(1, \psi; E/\mathbb{Q})$ (provided we take the primes by increasing order).*

Proof. By hypothesis $G^0(\mathbb{R})$ is compact. Hence G^0 has no nontrivial characters defined over $\mathbb{R}$ and a fortiori over $\mathbb{Q}$, i.e., the Galois module $\hat{G}^0$ has no nonzero fixed points. Therefore ψ is either zero or a sum of nontrivial irreducible characters. We conclude the proof by applying Lemma 2.4.

2.6. LEMMA. *Let ω be a gauge-form on G^0 defined over Q. The product:*

$$\prod_p \int_{G^0(\mathbb{Z}_p)} |\omega|_p$$

is convergent (provided we take the primes in increasing order).

Proof. We know (see Ono [O_1]) that $\{L(1, \psi; E/Q)_p\}$ is a system of convergence factors for G^0. Thus Lemma 2.6 follows from Lemma 2.5. ∎

Now we are ready to express $\mathrm{vol}(G(A)_M)$ in terms of local densities,

Let M be a Γ-stable lattice in (V, b) and $M^\# = \{x \in V: b(x, M) \subset \mathbb{Z}\}$ its dual lattice, which is also Γ-stable. The free abelian subgroup $\mathrm{Hom}_{\mathbb{Z}\Gamma}(M, M^\#)$ of $\mathrm{End}_{\mathbb{Q}\Gamma}(V)$ is preserved by the adjoint involution. The subgroup of all self-adjoint homomorphisms in $\mathrm{Hom}_{\mathbb{Z}\Gamma}(M, M^\#)$ will be denoted by $\mathrm{Hom}_{\mathbb{Z}\Gamma}(M, M^\#)^+$.

Sometimes it will be useful to view G as a group scheme over $\mathbb{Z}$, rather than an algebraic group over an universal domain, for instance when we want to consider the points of G over a finite ring.

For any commutative ring R, we denote by $\mathscr{E}(R)$ the R-algebra $\mathrm{End}_{\mathbb{Z}\Gamma}(M) \otimes_{\mathbb{Z}} R$ and by $\mathscr{E}^+(R)$ the free R-module $\mathrm{Hom}_{\mathbb{Z}\Gamma}(M, M^\#)^+ \otimes_{\mathbb{Z}} R$. Let $f_R: \mathscr{E}(R) \to \mathscr{E}^+(R)$ be the map defined by $\sigma \to \bar{\sigma}\sigma$. The functor G is defined by $G(R) = f_R^{-1}(1)$.

For a finite prime p, we provide $\mathscr{E}(\mathbb{Z}_p)$ and $\mathscr{E}^+(\mathbb{Z}_p)$ with invariant measures of total mass 1. For the prime at infinity, $\mathscr{E}(\mathbb{R})$ and $\mathscr{E}^+(\mathbb{R})$ are provided with the Lebesgue measures giving total mass 1 to the tori $\mathscr{E}(\mathbb{R})/\mathscr{E}(\mathbb{Z})$ and $\mathscr{E}^+(\mathbb{R})/\mathscr{E}^+(\mathbb{Z})$.

2.7. PROPOSITION. *There exists a gauge form ω on G, defined over $\mathbb{Q}$, which is relatively invariant with respect to some character $\phi \in \hat{G}_{\mathbb{Q}}$ and such that*

$$\int_{G(\mathbb{Z}_p)} |\omega|_p = \lim_{U \to 1} \frac{\operatorname{vol} f_p^{-1}(U)}{\operatorname{vol}(U)} \tag{2}$$

for all primes p, including the prime at infinity. For finite primes p the right-hand side of (2) *is equal to* $|G(\mathbb{Z}/p^v\mathbb{Z})|\, p^{-v \dim G}$ *if v is sufficiently large.*

The limit is taken over a fundamental system of compact neighborhoods of 1 *in* $\mathscr{E}^+(\mathbb{Z}_p)$ *and vol denotes the normalized measure in* $\mathscr{E}(\mathbb{Z}_p)$ *or* $\mathscr{E}^+(\mathbb{Z}_p)$. *For simplicity we denote by* f_p *the map* $f_{\mathbb{Z}_p}$.

Proof. Let α, resp α^+, be a generator of the exterior power $\det \mathscr{E}(\mathbb{Z})$ (resp. $\det \mathscr{E}^+(\mathbb{Z})$). Put $d = \dim G$. It is easy to see that there exists a d-differential form Ω over $\mathscr{E}^\times$, the group of units of $\mathscr{E}$, such that (i) $l_x^* \Omega = (\det l_x)\Omega$ for all $x \in G$, where $l_x : \mathscr{E} \to \mathscr{E}$ denotes the left translation by x, and (ii) $\Omega \wedge f^*(\alpha^+) = \alpha$.

We claim that the form $\omega = \Omega|_G$ has the required properties. Indeed, it follows from (i) that it is relatively invariant with respect to $\phi(x) = \det l_x$. Furthermore, if we denote by ω_t the restriction of ω to $f^{-1}(t)$, it is a consequence of Fubini's theorem that

$$\operatorname{vol} f_p^{-1}(U) := \int_{f_p^{-1}(U)} |\alpha|_p = \int_U \left(\int_{f_p^{-1}(t)} |\omega_t|_p \right) |\alpha^+|_p. \tag{3}$$

We get formula (2) by shrinking U to 1 in (3). ∎

2.8. LEMMA. *For almost all p the canonical map* $G(\mathbb{Z}_p) \to (\pi_0 G)(\mathbb{Z}_p)$ *is surjective* ($\pi_0 G$ *denotes the quotient group scheme* G/G^0).

Proof. It is enough to prove the lemma in the case where V is an isotypic module. In this case $G = R_{F/\mathbb{Q}}(U)$, where U is some unitary group defined over a number field F. Furthermore, we can suppose that U is the unitary group of a skew-hermitian form h over a quaternion algebra, the lemma being trivial in all other cases. Let Nrd: $U \to \mu_2$ be the reduced norm, $U^0 = \operatorname{Ker} \mathrm{Nrd}$ the connected component of 1 in U. We may assume that U is defined over O_F, the ring of integers of F. Let X be the subscheme of U defined by $\mathrm{Nrd}(u) = -1$. It is easy to see that for almost all primes $\mathfrak{p}$ of F, the scheme X has points over the residue field $O_F/\mathfrak{p}$ (for almost all $\mathfrak{p}$ the reduction of U modulo $\mathfrak{p}$ is an ordinary orthogonal group over $O_F/\mathfrak{p}$). By Hensel's lemma X has points over $O_{F_\mathfrak{p}}$, i.e., the map Nrd: $U(O_{F_\mathfrak{p}}) \to \mu_2(F_\mathfrak{p}) = \{\pm 1\}$ is surjective. Moreover, for almost all p, $G(\mathbb{Z}_p) = \prod_{\mathfrak{p}|p} U(O_{F_\mathfrak{p}})$ and $(\pi_0 G)(\mathbb{Z}_p) = \prod_{\mathfrak{p}|p} \mu_2(F_\mathfrak{p})$. Thus the homomorphism $G(\mathbb{Z}_p) \to (\pi_0 G)(\mathbb{Z}_p)$ must be surjective, except for a finite number of places p. ∎

2.9. COROLLARY. *Let* $j: G \to \pi_0 G$ *the canonical projection and* $j_A: G(A) \to (\pi_0 G)(A)$ *be the induced adele map. Then* $\operatorname{Im} j_A$ *has finite index in* $(\pi_0 G)(A)$.

Proof. Since $\pi_0 G$ is a finite scheme, we have $(\pi_0 G)(\mathbb{Z}_p) = (\pi_0 G)(\mathbb{Q}_p)$ and hence $(\pi_0 G)(A) = \prod_p (\pi_0 G)(\mathbb{Z}_p)$. Thus the corollary follows immediatly from the lemma. ▮

DEFINITION. Let ω be a (relatively) invariant gauge-form on G. We define the Tamagawa measure on $G(A)$ to be the restricted product:

$$\mu := \prod_p \frac{1}{[G(\mathbb{Q}_p):G^0(\mathbb{Q}_p)]} |\omega|_p.$$

This definition makes sense. Indeed

$$\frac{1}{[G(\mathbb{Q}_p):G^0(\mathbb{Q}_p)]} \int_{G(\mathbb{Z}_p)} |\omega|_p = \frac{[G(\mathbb{Z}_p):G^0(\mathbb{Z}_p)]}{[G(\mathbb{Q}_p):G^0(\mathbb{Q}_p)]} \int_{G^0(\mathbb{Z}_p)} |\omega|_p$$

and, by Lemma 2.8,

$$\frac{[G(\mathbb{Z}_p): G^0(\mathbb{Z}_p)]}{[G(\mathbb{Q}_p):G^0(\mathbb{Q}_p)]} = 1$$

for almost all p. Hence the product defining μ is convergent provided we take the primes in increasing order.

2.10. PROPOSITION. $\mu(G(A)/G(\mathbb{Q})) = \tau(G^0)/[G(\mathbb{Q}):G^0(\mathbb{Q})]$. *We will denote by* $\tau(G)$ *this number.*

Proof. From Lemma 2.8 we see that μ can be characterized as the unique invariant measure on $G(A)$ which is compatible with the exact sequence $1 \to G^0(A) \to G(A) \to \operatorname{Im} j_A \to 1$, after $G^0(A)$ is provided with the usual Tamagawa measure and the compact group $\operatorname{Im} j_A$ with the measure of total mass equal to 1. It follows from this characterization of μ that

$$\begin{aligned} \tau(G) = \mu(G(A)/G(\mathbb{Q})) &= \operatorname{vol}(G^0(A)/G^0(\mathbb{Q})) \operatorname{vol}(\operatorname{Im} j_A/j(G(\mathbb{Q})) \\ &= \tau(G^0) \frac{1}{[G(\mathbb{Q}):G^0(\mathbb{Q})]}. \end{aligned}$$ ▮

2.11. COROLLARY. $\tau(G)$ *depends only on the* $\mathbb{Q}\Gamma$*-module structure of* V *and on the sign* ε *of* b.

Proof. We saw already that $\tau(G^0)$ depends only on the module V and on the sign ε of b. It is easy to verify that the index $[G(\mathbb{Q}):G^0(\mathbb{Q})]$ depends only on V and ε. ▮

1.12. THEOREM. *Let $(M_1, b_1),..., (M_h, b_h)$ be representatives of the classes in the genus of (M, b), and denote by w_i the order of the automorphism group of (M_i, b_i). Let n be the number of classes of Γ-forms (V', b') over $\mathbb{Q}$ which are isomorphic everywhere locally to (V, b). We have the mass formula*:

$$\sum_{i=1}^{h} \frac{1}{w_i} = n\tau(G) \prod_p \delta_p(M, b)^{-1},$$

where $\delta_p(M, b)$ is the "local density" defined by

$$\delta_p(M, b) := \frac{1}{[G(\mathbb{Q}_p):G^0(\mathbb{Q}_p)]} \lim_{U \to 1} \frac{\operatorname{vol} f_p^{-1}(U)}{\operatorname{vol} U}.$$

For p finite and v large enough, one can also write

$$\delta_p(M, b) = \frac{1}{[G(\mathbb{Q}_p):G^0(\mathbb{Q}_p)]} \frac{|G(\mathbb{Z}/p^v\mathbb{Z})|}{p^{v \dim G}}.$$

The product is taken over all places p of $\mathbb{Q}$ in increasing order.

Proof. Let $\{(V^j, b^j)\}_{j=1,...,n}$ be a set of representatives of the classes of Γ-forms over $\mathbb{Q}$ which are isomorphic to (V, b) everywhere locally. We denote by G^j the automorphism group of (V^j, b^j). On applying formula (1) for a lattice (M^j, b^j) in (V^j, b^j) which belongs to the genus of (M, b), we get

$$\sum_{i=1}^{k_j} \frac{1}{w_i^j} = \tau(G^j) \operatorname{vol}(G^j(A)_{M^j})^{-1} \tag{3}$$

(vol is now the Tamagawa measure on $G^j(A)$).

The underlying $\mathbb{Q}\Gamma$-modules V^j are all isomorphic to V; hence, by Corollary 2.11., $\tau(G^j) = \tau(G)$ for all j. By Proposition 2.7 $\operatorname{vol}(G^j(A)_{M^j}) = \prod_p \delta_p(M^j, b^j)$. Clearly, the local densities $\delta_p(M^j, b^j)$ depend only on the genus. Thus $\delta_p(M^j, b^j) = \delta_p(M, b)$ for all j. We get the announced formula by summing (3) over all j and by renaming the w_i^j's. ∎

Remark. The number n which appears in the mass formula can be computed in the following way: let $V_1,..., V_s$ the isotypic "skew-hermitian" components of V, i.e., V_i has a simple component S_i of the first kind, $D_i := \operatorname{End}_{\mathbb{Q}\Gamma}(S_i)$ is a quaternion algebra over its centre, and the form c_i on S_i which induces the standard involution on D_i is $(-\varepsilon)$-symmetric.

Let P_i be the finite set of places of $F_i :=$ centre of D_i for which D_i does not split. We know, by a theorem of M. Kneser (see [K]), that the number of classes of skew-hermitian forms which are locally everywhere isomorphic to a given one is equal to $2^{|P_i|-2}$. It follows that $n = 2^{\Sigma(|P_i|-2)}$.

3. An Example

We keep the notations of Section 2. We assume from now on that Γ is abelian and V is an isotypic $\mathbb{Q}\Gamma$-module. The Γ-form (V, b) will be symmetric and definite. The integral lattice (M, b) will be unimodular. In this situation, the centre E of the simple algebra $\mathrm{End}_{\mathbb{Q}\Gamma}(V)$ is a cyclotomic field $\mathbb{Q}(\xi_m)$. We will assume moreover that m is not a power of 2. Being a quotient of the group algebra $\mathbb{Q}\Gamma$, E can also be considered as a $\mathbb{Q}\Gamma$-module and is in fact the simple component of V. There is a canonical choice of Γ-form on E: the trace form $(x, y) \mapsto \mathrm{Tr}_{E/\mathbb{Q}}(x\bar{y})$, where $y \mapsto \bar{y}$ is complex conjugation on E. Let $O_E = \mathbb{Z}[\xi_m]$ be the ring of integers of E and O'_E the co-different of $E/\mathbb{Q}$; the map $(\mathrm{Tr}_{E/\mathbb{Q}})_*: \mathrm{Hom}_{O_E}(M, O'_E) \to \mathrm{Hom}(M, \mathbb{Z})$ is a Γ-equivariant isomorphism. We can associate to b the unique hermitian form $h: M \to \mathrm{Hom}_{O_E}(M, O'_E)$ defined by $\mathrm{Tr}_{E/\mathbb{Q}} \circ h = b$. Let F be the maximal real subfield of E and denote by U the unitary group of h, which is defined over O_F. By construction we have

$$G(\mathbb{Z}/p^v\mathbb{Z}) = \prod_{\mathfrak{p}|p} U(O_F/\mathfrak{p}^{ev}),$$

where e is the ramification index of p in F, and v is any positive integer. Then, to compute local densities, it will suffice to find the order of $U(O_F/\mathfrak{p}^k)$ for large k.

3.1. Lemma. *For $k \geqslant 1$ we have a short exact sequence*:

$$0 \to (\mathrm{Lie}\, U)(O_F/\mathfrak{p}) \xrightarrow{i} U(O_F/\mathfrak{p}^{k+1}) \xrightarrow{j} U(O_F/\mathfrak{p}^k) \to 1,$$

where $(\mathrm{Lie}\, U)(O_F/\mathfrak{p})$ *denotes the* $O_F/\mathfrak{p}$*-module* $\{u \in \mathrm{End}_{O_E}(M/\mathfrak{p}M) \colon u + \bar{u} = 0\}$, *which is also, as suggested by the notation, the group of points over $O_F/\mathfrak{p}$ of the Lie algebra* Lie U *of U. The homomorphism j is induced by the canonical projection $O_F/\mathfrak{p}^{k+1} \to O_F/\mathfrak{p}^k$, and i is defined by $i(u) = 1 + \pi^k u$, where $\pi \in \mathfrak{p}$ is any uniformising element.*

Proof. One sees easily that $1 + \pi^k u$ belongs to $U(O_F/\mathfrak{p}^{k+1})$ if and only if $u + \bar{u} = 0$ modulo $\mathfrak{p}$. Therefore $\mathrm{Ker}\, j = \mathrm{Im}\, i$.

Now E/F has no dyadic ramification, since m is not a power of 2. Thus the trace map $\mathrm{Tr}_{E/F}: O_E \to O_F$ is surjective. We choose $a \in O_E$ such that $a + \bar{a} = 1$. Let $u \in \mathrm{End}_{O_E}(M)$ be such that $\bar{u}u = 1 \bmod \mathfrak{p}^k$ and define $v = u + a(u - \bar{u}^{-1})$ in the ring $(\mathrm{End}_{O_E} M)_{(\mathfrak{p})}$ localized at $\mathfrak{p}$. An easy calculation shows that v verifies $\bar{v}v = 1 \bmod \mathfrak{p}^{k+1}$. Hence j is surjective. ∎

3.2. Corollary. $|U(O_F/\mathfrak{p}^k)| = |U(O_F/\mathfrak{p})|\ N(\mathfrak{p})^{(k-1)\dim U}$ *for $k \geqslant 1$.*

3.3. PROPOSITION. *Let χ be the non trivial character of E/F. For every finite prime $\mathfrak{p}$ of F we have*

$$|U(O_F/\mathfrak{p})|\, N(\mathfrak{p})^{-\dim U} = \prod_{2 \leqslant k \text{ even} \leqslant r} (1 - N(\mathfrak{p})^{-k}) \prod_{1 \leqslant k \text{ odd} \leqslant r} (1 - \chi(\mathfrak{p})\, N(\mathfrak{p})^{-k}), \tag{4}$$

where r is the rank of M over O_E.

Proof. (a) Assume that $\mathfrak{p}$ splits in E, i.e., $\mathfrak{p} = \mathfrak{p}\bar{\mathfrak{p}}$, $\mathfrak{p}$ prime of E, $\mathfrak{p} \neq \bar{\mathfrak{p}}$. Then $U(O_F/\mathfrak{p})$ may be identified with the general linear group of the $O_E/\mathfrak{p}$-vector space $M/\mathfrak{p}M$. The equality (4) follows from the well known formula for the order of $GL_r(O_E/\mathfrak{p})$. In this case $\chi(\mathfrak{p}) = 1$.

(b) Suppose $\mathfrak{p}$ remains prime when extended to O_E. Then $U(O_F/\mathfrak{p})$ is the unitary group of a hermitian form over a finite field. In this case formula (4) is just the standard formula for the order of such a group (now $\chi(\mathfrak{p}) = -1$).

(c) The extension E/F ramifies at some finite place if and only if m is a power of a prime number p (see, e.g., [Wa]). In this case the co-different O'_E is a principal ideal and we can choose a generator α of O'_E such that $\bar{\alpha} = -\alpha$. We define a skew-hermitian form $g\colon M \to \mathrm{Hom}_{O_E}(M, O_E)$ by $g = \alpha h$. By construction g is unimodular. The unique ramified prime $\mathfrak{p}$ of E is generated by $(\xi_m - 1)$, where ξ_m is a primitive mth-root of 1. Put $\mathfrak{p} = \mathfrak{p} \cap O_F$. The ring $O_E/\mathfrak{p}O_E$ is isomorphic to $\mathbb{F}_p[T]/(T^2)$, where T corresponds to $\xi_m - 1$. It is easy to see that the corresponding involution on $\mathbb{F}_p[T]/(T^2)$ is given by $T \mapsto -T$. Let $\tilde{M}$ be $\mathbb{F}_p[T]/(T^2)$-module $M/\mathfrak{p}M$. Since 2 is a unit in $\mathbb{F}_p$ (we assumed that there is no dyadic ramification) and the reduction $\tilde{g}$ of g modulo $\mathfrak{p}$ is unimodular, $\tilde{M}$ has a $\mathbb{F}_p[T]/(T^2)$-basis $e_1, ..., e_r$ such that

$$(\tilde{g}(e_i, e_j)) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} := S.$$

We can identify $U(O_F/\mathfrak{p})$ with the subgroup of $GL_r(\mathbb{F}_p[T]/(T^2))$ consisting of all matrices X verifying $X^*SX = S$, where X^* denotes the transposed conjugate of X. The matrics X in $GL_r(\mathbb{F}_p[T]/(T^2))$ can be written in the form $X = X_0 + TX_1$, where $X_i \in M_r(\mathbb{F}_p)$. The homomorphism $j\colon GL_r(\mathbb{F}_p[T]/(T^2)) \to GL_r(\mathbb{F}_p)$ given by $j(X) = X_0$ sends $U(O_F/\mathfrak{p})$ onto the symplectic group $\mathrm{Sp}_r(\mathbb{F}_p)$. The kernel of the restriction of j to $U(O_F/\mathfrak{p})$ can be identified with the subspace $\{X_1 \in M_r(\mathbb{F}_p)\colon X_1^t S = SX_1\}$, which has dimension $r(r-1)/2$ over $\mathbb{F}_p$. Hence $|U(O_F/\mathfrak{p})| = |\mathrm{Sp}_r(\mathbb{F}_p)|\, p^{r(r-1)/2}$ and the equality (4) follows from the formula for the order of the symplectic group over a finite field ($\chi(\mathfrak{p}) = 0$ if $\mathfrak{p}$ is ramified). ∎

3.4. COROLLARY.

$$\prod_{p \text{ finite}} \delta_p(M, b) = \prod_{\substack{2 \leqslant k \leqslant r \\ k \text{ even}}} \zeta_F(k)^{-1} \prod_{\substack{1 \leqslant k \leqslant r \\ k \text{ odd}}} L(k, \chi; E/F)^{-1}$$

Proof. $\delta_p(M, b) = |G(\mathbb{Z}/p^v\mathbb{Z})|\ p^{-v \dim G} = \prod_{\mathfrak{p} \mid p} |U(O_F/\mathfrak{p}^{ev})|\ N(\mathfrak{p})^{-ev \dim U}$ for v large enough. Now apply Corollary 3.2 and Proposition 3.3. Now we have to compute the density at infinity $\delta_\infty(M, b)$. The calculations for this are rather long and tedious but elementary. We shall only sketch the main steps, stating them as lemmas without proof. We refer to [Mo] for details. ▮

3.5. LEMMA. *We define a real scalar product* $\langle\, ,\rangle$ *on the algebra* $\mathscr{E}(\mathbb{R})$ by $\langle \sigma, \tau \rangle = \operatorname{Trace}_{\mathbb{R}}(\bar{\sigma}\tau)$, *where* $\operatorname{Trace}_{\mathbb{R}}(\sigma)$ *means the trace of* σ *as an* $\mathbb{R}$-*linear endomorphism of* $V \otimes_{\mathbb{Q}} \mathbb{R}$. *The subspace* $\mathscr{E}^+(\mathbb{R})$ *is provided with the restriction of* $\langle\, ,\rangle$. *We denote by* vol (*resp.* vol^+) *the Lebesgue measure given by* $\langle\, ,\rangle$ *on* $\mathscr{E}(\mathbb{R})$ (*resp. on* $\mathscr{E}^+(\mathbb{R})$).

With these notations we have:

(i) $\text{vol}(\mathscr{E}(\mathbb{R})/\mathscr{E}(\mathbb{Z})) = \Delta_E^{r^2/2}$,

(ii) $\text{vol}^+(\mathscr{E}^+(\mathbb{R})/\mathscr{E}^+(\mathbb{Z})) = 2^{r^2[F:\mathbb{Q}]/2}\, \Delta_F^{r^2/2} N(\mathscr{D}_{E/F})^{r(r+1)/4}$,

where Δ_E (*resp.* Δ_F) *is the absolute discriminant of* E (*resp.* F) *and* $\mathscr{D}_{E/F}$ *is the different of* E/F.

3.6. LEMMA. *The scalar product on* $\mathscr{E}(\mathbb{R})$ *induces a Riemannian metric on* $G(\mathbb{R})$, *which is actually invariant by left* (*or right*) *translations in* $G(\mathbb{R})$. *Let* vol *be the aasociated invariant measure on* $G(\mathbb{R})$. *We have*:

$$\text{vol}\ G(\mathbb{R}) = (\text{vol}\ U_r)^{[F:\mathbb{Q}]},$$

where Ur is the standard unitary group in $M_r(\mathbb{C})$ *and* vol U_r *is the volume of* U_r *with respect to the Riemannian metric on* U_r *given by the scalar product* $\langle X, Y \rangle = \operatorname{Tr}_{\mathbb{C}/\mathbb{R}}(\operatorname{Tr}(X^* Y))$ *on* $M_r(\mathbb{C})$.

Hint. $G(\mathbb{R})$ can be identified in a natural way with the product $(U_r)^{[F:\mathbb{Q}]}$. We need only verify that this identification is metric-preserving.

3.7. LEMMA. $\text{vol}\ U_r = 2^{r^2/2} \text{vol}\ S^1\ \text{vol}\ S^3 \cdots \text{vol}\ S^{2r-1}$ (*here* $\text{vol}\ S^k$ *is the volume of* S^k *with respect to the standard scalar product of* $\mathbb{R}^{k+1}$).

Hint. Consider the fibration $U_{r-1} \to U_r \to S^{2r-1}$ and proceed by induction on r.

3.8. PROPOSITION. *The value of* $\delta_\infty(M, b)$ *is*

$$N(\mathscr{D}_{E/F})^{r(r+1)/4}(\Delta_F/\Delta_E)^{r^2/2} \prod_{k=1}^{r} \left[\frac{2\pi^k}{(k-1)!}\right]^{[F:Q]}$$

Proof. Let ω be the gauge-form on G given by Proposition 2.7. We have

$$\mathrm{vol} = \frac{\mathrm{vol}(\mathscr{E}(\mathbb{R})/\mathscr{E}(\mathbb{Z}))}{\mathrm{vol}^+(\mathscr{E}^+(\mathbb{R})/\mathscr{E}^+(\mathbb{Z}))} \cdot |\omega|_\infty.$$

Hence the proposition follows from Lemmas 3.5, 3.6, and 3.7, together with the fact that vol $S^{2k-1} = 2\pi^k/(k-1)!$. ▮

3.9. THEOREM. *The following mass formula holds*:

$$\sum_{i=1}^{h} \frac{1}{w_i} = 2N(\mathscr{D}_{E/F})^{-r(r+1)/4}(\Delta_E/\Delta_F)^{r^2/2} \prod_{k=1}^{r} (2\pi^k/(k-1)!)^{-[F:Q]}$$

$$\prod_{\substack{2 \leqslant k \leqslant r \\ k \text{ even}}} \zeta_F(k) \prod_{\substack{1 \leqslant k \leqslant r \\ k \text{ odd}}} L(k, \chi; E/F).$$

Proof. We apply Theorem 2.12, Corollary 3.4, and Proposition 3.8. In this case G is connected and $\tau(G) = 2$. The number n in theorem 2.12 is equal to 1 because, Γ being abelian, $\mathbb{Q}\Gamma$ cannot have any quaternion component. ▮

3.10. COROLLARY. *If* V *is* simple (*i.e.*, $r = 1$), *the class number* h *of* (M, b) *is given by*

$$h = h_E/h_F,$$

where h_E (*resp.* h_F) *is the ideal class number of* E (*resp.* F).

Proof. In this case $G = \mathrm{Ker}\, N_{E/F}$ and hence $w_1 = \cdots = w_h = w =$ number of roots of 1 in E. The rank of M over O_E is odd (in fact equal to 1); hence E/F has no ramified finite primes. Therefore $\mathscr{D}_{E/F} = (1)$. The mass formula of Theorem 3.9 becomes

$$h/w = \; = 2(\Delta_E/\Delta_F)^{1/2}(2\pi)^{-[F:Q]}L(1, \chi; E/F).$$

This formula happens to be the classical relation for the relative class number of cyclotomic fields (see, e.g., [Wa]). ▮

Remark. Corollary 3.10 can also be obtained without using the mass formula, by direct considerations and some class field theory, as in E. Bayer's paper (see [B]).

References

[B] E. Bayer, Unimodular hermitian and skew-hermitian forms, *J. Algebra* **74** (1982), 341–373.

[B–M] E. Bayer and F. Michel, Finitude du nombre des classes d'isomorphisme des structures isométriques entières, *Comm. Math. Helv.* **54** (1979), 378–396.

[Be] M. Benard, Quaternion constituents of group algebras, *Proc. Amer. Math. Soc.* **30** (1971).

[Bo–HC] A. Borel and Harish–Chandra, Arithmetic subgroups of algebraic groups, *Ann. of Math.* **75** (1962).

[K] M. Kneser, Lectures on Galois Cohomology of Classical Groups," Tata Institute Lecture Notes, Vol. 47, 1969.

[Kr] H. Kraft, Geometrische Methoden in der Invariantentheorie, Lecture notes, Bonn 1977–1978.

[L] E. Landau, "Handbuch des Lehre der Verteilung der Primzahlen I," Teubner, Leipzig/Berlin, 1909.

[M] J. G. M. Mars, The Tamagawa number of ${}^2A_n$, *Ann. of Math.* **89** (1969).

[Mo] J. Morales, "Formule de masse pour les structures isométriques entières," Ph. D. thesis, Université de Genève, 1983.

[O1] T. Ono, On the relative theory of Tamagawa numbers, *Ann. of Math.* **82** (1965).

[O2] T. Ono, On Tamagawa numbers, *in* "AMS Proc. of Symposia in Pure Math.," Vol. IX, Amer. Math. Soc., Providence, R. I., 1966.

[Q] H. G. Quebbemann, Ein Endlichkeitssatz für Klassenzahlen invarianter Formen, *Comm. Math. Helv.* **55** (1980).

[Q–S–S] H. G. Quebbeman, W. Schrlau, and M. Schulte, Quadratic and hermitian forms in additive and abelian categories, *J. Algebra* **59** (1979), 264–289.

[R] I. Reiner, "Maximal Orders", Academic Press, London/Nex York 1975.

[Se] J.-P. Serre, "Linear Representations of Finite Groups," Graduate Texts in Math., Springer-Verlag, Berlin/New York, 1977.

[Wa] L. C. Washington, "Introduction to Cyclotomic Fields" Graduate Texts in Math., Springer-Verlag, Berlin/New York, 1982.

[W] A. Weil, "Adeles and Algebraic Groups," new ed., Birkhäuser-Verlag, 1982.