Student Statement
Neal D. Livesay

1. Current research

My research focuses on studying the relationship between representation theory and geometry. In particular, my primary research problem focuses on the construction of moduli spaces of flat $G$-bundles. This problem has been studied extensively in recent years due to its connection to the geometric Langlands correspondence, a collection of far-reaching and influential conjectures connecting seemingly unrelated areas of mathematics. Recent work has been done to construct some of these moduli spaces for $G = GL_n$; my current work further refines these constructions for $G = Sp_{2n}$ with the hope of gaining insight for constructing these moduli spaces for general reductive groups. Below I have outlined some of the history of my problem, my role in the problem, my proposed timeline for the next year, and finally a short list of citations.

1.1. Some historical context. Suppose we have a linear differential equation defined in some connected open set $U$ in the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. It is well-known that a solution to the differential equation in $U$ can be analytically continued along a loop around a singularity to get new, linearly independent solutions. The data encoding the changes in a solution associated to loops around a given singularity is called the monodromy group associated to that singularity. Hence a linear differential equation determines a set of singularities and a monodromy group at each of those singularities.

Conversely, suppose we have a specified set of singularities and associated groups. Is there a linear differential equation with those singularities and specified groups as monodromy groups? This is the celebrated Riemann-Hilbert problem. A modern refinement of this problem concerns the existence of moduli spaces of meromorphic connections on $\mathbb{P}^1$ (or equivalently, flat $G$-bundles) with specified singular points and associated formal types (which captures the “local behavior” at that singular point, including the monodromy group).

1.2. Connections and leading terms. Connections have nice explicit formulations that lend themselves exceedingly well to systematic analysis. Let us consider a rank $n$ meromorphic connection $\nabla = d + A\frac{dz}{z}$ on $\mathbb{P}^1$, with $A \in \mathfrak{gl}_n(C(z))$. Fixing a local trivialization near a singular point $b$, the connection has a Laurent series expansion

$$\nabla = d + (M_{-r}z^{-r} + M_{-r+1}z^{-r+1} + \ldots)\frac{dz}{z}$$

with $M_i \in \mathfrak{gl}_n(\mathbb{C})$, $M_{-r} \neq 0$, and $r \geq 0$. The leading term $M_{-r}$ for a given trivialization can give surprisingly useful information about the connection. For instance, if there exists a local trivialization where $M_{-r}$ is non-nilpotent, then the Laurent series expansion at $b$ for any trivialization must have leading term with degree at most $-r$. In this case, $r$ is an invariant of the connection called the slope. Furthermore, if there exists a local trivialization such that $M_{-r}$ is regular diagonalizable, then $\nabla$ can diagonalized by a gauge change so

$$\nabla = d + (D_{-r}z^{-r} + D_{-r+1}z^{-r+1} + \ldots + D_0)\frac{dz}{z}$$

with $D_i$ diagonal. To reiterate, diagonalizability (and regularity) of the leading term implies diagonalizability of the entire connection and termination at 0. We call this diagonalized one-form a formal type of $\nabla$ at $b$.

In the case that all singularities of a meromorphic connection on $\mathbb{P}^1$ are of this form, Boalch has constructed well-behaved moduli spaces ([Boa01]).

1.3. Shortcomings of the theory, recent work. Many interesting connections have leading terms that are nilpotent, but not regular diagonalizable. For instance, consider the local trivialization of an Airy-type connection

$$\nabla = d + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{dz}{z} = d + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}z^{-1}\frac{dz}{z} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}z\frac{dz}{z}$$

(1)

at the irregular singular point 0 (a specialization of the $G$-bundle considered by Frenkel and Gross [FG09]). The leading term is nilpotent. In fact, the slope of this connection turns out to be $\frac{1}{2}$, not 1. Furthermore, the leading term is not diagonalizable. The techniques described in 1.2 fail to bear fruit with this example.

In a recent series of papers (for instance, [BS13]), Bremer and Sage introduced a new notion of leading term by considering refinements of the normal, naive degree filtration on the loop algebra. This new theory generalizes many aspects of the classical theory, allowing for quick computations of rational slopes, as well as having an analogous “simultaneous diagonalization” into a formal type for connections with regular
semisimple (rather than merely diagonalizable) leading term. To roughly outline the idea in the $GL_2$ case (which includes equation (1)), notice that \((\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix})^2 = tI_2\). It is possible to put a filtration on the loop algebra so that \((\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix})\) can be viewed as a typical element of degree \(\frac{1}{2}\). We can “diagonalize” the matrix of one-forms into a filtration of the loop algebra, and write the resulting one-form as a series in this “root of $t$.” The resulting formal type is amenable to many systematic analyses analogous to the classical methods.

Furthermore, Bremer and Sage, following Boalch, have constructed explicit, well-behaved moduli spaces for these “regular” connections.

1.4. **Current and future work.**

1.4.1. **Current work.** Bremer and Sage developed their theory for connections associated to the general linear group $GL_n$. My current research involves proving the analogous theory for the symplectic group $Sp_{2n}$. There are two main components to my project:

1. Local theory: Determine the appropriate refinements of the symplectic loop algebra filtration and work out the details of formal types at a given singularity.
2. Global theory: Construct the moduli space.

Component (1) has presented the primary challenge in my project. Not only are the filtrations associated to the symplectic group more complicated, but I am trying to develop the theory in a way that most resembles the general linear case so as to better elucidate future generalizations. I have completed Component (1) over the past two years. For Component (2), we confidently conjecture that the moduli spaces should be a symplectic reduction by $sp_{2n}$ of a product of symplectic manifolds which encode the local data.

We expect that this work will be completed this semester and that a preprint will be submitted for publication by August 2016.

1.4.2. **Future work.** There are currently two natural next-step projects:

1. Work out the theory for the remaining classical groups.
2. Expand on certain technical details of the theory that have only been rigorously developed for uniform semisimple formal types, and not the more general regular semisimple formal types.

Principles developed in the current project for deriving filtrations associated to the symplectic group should have analogous principles for the other classical groups. The development of the theory for all the classical groups should shed light on the general theory for arbitrary reductive groups. Project (2) is another necessary ingredient for the further development of this theory.

**References**

