Chapter 1

Fourier Series

1.1 Harmonic Functions on the Disk

In this section we discuss one of the problems that motivated the beginning of the theory of Fourier series and is close to Fourier's original work. Let $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_n)^2$ be the Laplace operator on $\mathbb{R}^n$. It is one of the most interesting differential operators on $\mathbb{R}^n$, in part because of the role it plays in partial differential equations arising in physics:

- **The heat equation**: $\Delta u = a^2 u_t$. Here $u(x,t)$ is a function of $n+1$ variables, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $t > 0$, and the subscript $_t$ denotes the partial derivative with respect to t.

- **The wave equation:** $a^2 \Delta u = u_{tt}$.

- **The Schrödinger's equation:** $\frac{1}{i}\Delta u = u_t$.

- **The Helmholtz's equation:** $-\Delta u = \lambda u$.

Fourier analysis is *one* of the main tools used to deal with the solutions to these equations; this will be discussed that later in this book. As a motivation we start with the equation $\Delta u = 0$ on the unit disc

$$D := \{z \in \mathbb{C} \mid |z| < 1\} = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\right\}$$

such that u takes prescribed values on the boundary. Thus we would like to solve the following *Dirichlet's problem:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (x,y) \in D \tag{1.1.1.1}$$

$$u(x,y,0) = f(x,y) \qquad x^2 + y^2 = 1\,. \tag{1.1.1.2}$$

Here f is a given $\mathbf{L}^2$-function on the boundary and we will assume that $u \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$. That is u is twice continuously differentiable in D and continuous on the closed domain $\bar{D} = \{(x,y) \mid x^2 + y^2 \leq 1\}$.

Definition 1.1.1. *Let Ω be an open subset of $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{C}$ is called **harmonic** on Ω if $\Delta u = 0$.*

Notice that a harmonic function can be viewed as a time independent solution to the heat equation. Let us rewrite (1.1.1.1) using polar-coordinates

$$x = r\cos(\theta)\,, \quad y = r\sin(\theta)\,.$$

The Laplacian becomes

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\right)$$

and $u(r,\theta)$ is periodic in θ with period 2π, i.e., $u(r,\theta+2\pi) = u(r,\theta)$. The Direchlet's problem (1.1.1.1) and (1.1.1.2) is now

$$\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0\,, \quad u(1,\theta) = f(\theta)\,. \tag{1.1.1.3}$$

One approach to this problem is to use *separation of variables*, that is start by looking for solutions of the form:

$$u(r,\theta) = F(r)G(\theta).$$

Then the Laplace equation (1.1.1.1) can be rewritten as:

$$\frac{1}{G(\theta)}\frac{d^2 G}{d\theta^2}(\theta) = -\frac{r}{F(r)}\frac{d}{dr}\left(r\frac{dF}{dr}(r)\right)\,.$$

The left hand side is independent of r and the right hand side is independent of θ. Hence there is a constant k such that

$$\frac{1}{G(\theta)}\frac{d^2 G}{d\theta^2}(\theta) = -\frac{r}{F(r)}\frac{d}{dr}\left(r\frac{dF}{dr}(r)\right) = k.$$

This gives two ordinary differential equations:

$$\frac{d^2G}{d\theta^2}(\theta) = kG(\theta)$$
$$r\frac{d}{dr}\left(r\frac{dF}{dr}(r)\right) = r^2F''(r) + rF'(r) = -kF(r)\,.$$

The general solution to these equations are:

$$G_k(\theta) = \begin{cases} a_0 + b_0\theta & \text{if} \quad k = 0; \\ a_k e^{\sqrt{k}\theta} + b_k e^{-\sqrt{k}\theta} & \text{if} \quad k \neq 0 \end{cases} \tag{1.1.1.4}$$

$$F_k(r) = \begin{cases} A_0 + B_0 \log(r) & \text{if} \quad k = 0; \\ A_k r^{\sqrt{-k}} + B_k r^{-\sqrt{-k}} & \text{if} \quad k \neq 0 \end{cases} \tag{1.1.1.5}$$

where we have indicated the dependence of F and G on the constant k by the index k. The function $G_k(\theta)$ has period 2π if and only if $k = -n^2 < 0$ or $k = 0$ and $b = 0$. The function F_k is defined on all of D if and only if $B_k = 0$ for all k. The same conclusion holds if we only assume that u is $\mathbf{L}^2$ on the disk. We hence have $F_k(r) = A_k r^n$. Concluding we obtain solutions:

$$u_n(r,\theta) = r^n\left(a_n e^{in\theta} + b_n e^{-in\theta}\right), \qquad n \in \mathbb{N}_0 = \{0,1,2,\dots\}$$

Writing $b_n = a_{-n}$ and noticing formally that sums of solutions are solutions, we can tentatively write a solution as:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta} \tag{1.1.1.6}$$

$$u(1,\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} = f(\theta)\,. \tag{1.1.1.7}$$

To make the step from this formal solution to an actual solution we still need to resolve issues of the following type:

(a) Is it possible to choose the constants a_n such that the given function f can be written as $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$?

(b) If the answer to (1) is yes, how can we actually find the constants a_n?

(c) In what sense (pointwise, in $L^p, \ldots$) does the series in 1.1.1.7 represent the function f?

(d) Does the equation (1.1.1.6) then give a smooth function on the disk such that $\lim_{r\to 1-} u(r,\theta) = f(\theta)$?

(e) Is the solution to our problem unique?

(f) Is every harmonic function in the disk given by a series as in 1.1.1.6?

To look for an answer, we make few more formal calculations. Later we will show that those calculations can be justified. First multiply f by $e^{-im\theta}$ and then integrate. We interchange the summation and integration and use

$$\int_0^{2\pi} e^{ik\theta}\, d\theta = \begin{cases} 2\pi & \text{if} \quad k = 0 \\ 0 & \text{if} \quad n \neq 0 \end{cases}$$

to obtain

$$\begin{aligned} \int_0^{2\pi} f(\theta) e^{-im\theta}\, d\theta &= \int_0^{2\pi} \sum_{n=-\infty}^{\infty} a_n e^{in\theta} e^{-im\theta}\, d\theta \\ &= \sum_{n=-\infty}^{\infty} a_n \int_0^{2\pi} e^{i(n-m)\theta}\, d\theta \\ &= 2\pi a_m\,. \end{aligned}$$

One of our first results on Fourier series says that if we set

$$a_m = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-im\theta}\, d\theta,$$

then $f(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ holds in $\mathbf{L}^2([0,2\pi])$. The constant function $\theta \mapsto 1$ is in $\mathbf{L}^2([0,2\pi], \frac{d\theta}{2\pi})$ with norm one. Hence by the Hölder's inequality for $\mathbf{L}^2$-functions one has

$$|a_n| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|\, d\theta \leq |f|_1 < \infty\,.$$

Note for $r \leq R < 1$,

$$\begin{aligned}
\left| \sum_{n\in\mathbb{Z}} a_n r^{|n|} e^{in\theta} \right| &\leq \sum_{n\in\mathbb{Z}} |a_n|\, R^{|n|} \\
&\leq |f|_1 \left(\sum_{n=1}^{\infty} R^n + \sum_{n=0}^{\infty} R^n \right) \\
&\leq \frac{2\,|f|_1}{1-R}\,.
\end{aligned}$$

Therefore the series defining $u(r,\theta)$ converges uniformly on compact subsets of D. The derivatives of this series can also be shown to converge uniformly on compact subsets of D. Thus the series defines a smooth function on the disk. To evaluate the limit $\lim_{r\to 1-} u(r,\theta)$, we rewrite $u(r,\theta)$ as an integral over $[0,2\pi]$. This will be done by formally interchanging summation and integration and using the following simple fact

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \sum_{n=0}^{\infty} \left(re^{i\theta}\right)^n + \sum_{n=0}^{\infty} \left(re^{-i\theta}\right)^n - 1 = \frac{1-r^2}{1-2r\cos(\theta)+r^2}\,. \tag{1.1.1.8}$$

The function

$$P(r,\theta) := \frac{1-r^2}{1-2r\cos(\theta)+r^2}\,, \quad 0 \leq r < 1,\ \theta \in \mathbb{R} \tag{1.1.1.9}$$

is called the *Poisson kernel* for the unit disk. We will point out some properties of the Poisson kernel in the following exercises. See figure 1.1 for the graph of the Poisson kernel for $r = 0.5$ (blue), $r = 0.7$ (green), and $r = 0.9$ (red).

Inserting the definition of a_n we get:

$$\begin{aligned}
u(r,\theta) &= \sum_{n\in\mathbb{Z}} a_n r^{|n|} e^{in\theta} \\
&= \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-in\phi}\, d\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) r^{|n|} e^{in(\theta-\phi)}\, d\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) P(r,\theta-\phi)\, d\phi\,.
\end{aligned} \tag{1.1.1.10}$$

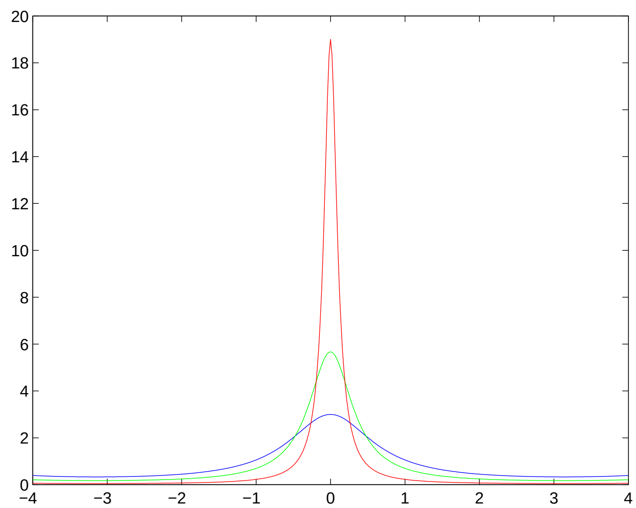

Figure 1.1: Poisson Kernels for $r = .5, \quad .7, \quad .9$

Hence u is given by convolving f with the Poisson kernel. This can be used to show that if f is continuous, then $u(r,\theta) \to f(\theta)$ uniformly.

Exercise Set 1:

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1. Prove equation (1.1.1.9): $\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos(\theta)+r^2}$.

2. Prove the following:

(a) $P(r,\theta) \geq 0$ and $\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta)\, d\theta = 1$ for all $r \geq 0$.

(b) The maximum of $\theta \mapsto P(r,\theta)$ occurs at $\theta = 0$ and $\max_\theta P(r,\theta) = \frac{1+r}{1-r}$. In particular $P(r,0) \to \infty$ as $r \to 1$.

(c) The function $\theta \mapsto P(r,\theta)$ takes its minimum at $\theta = \pi$. Evaluate $P(r,\pi)$.

3. Suppose that f is 2π-periodic and piecewise continuous. Show that

$$\lim_{r\to 1-} \frac{1}{2\pi}\int_0^{2\pi} f(\phi)P(r,\theta-\phi)\, d\phi = f(\theta)$$

if f is continuous at θ.

4. Write $u(x,y) = u(z)$ where $z = x + iy$. Suppose that f is continuous. Show that u is holomorphic on D if and only if $a_n = 0$ for all $n < 0$.

1.2 Periodic Functions

Definition 1.2.1. *A function* $f : \mathbb{R} \to \mathbb{C}$ *is periodic if there exists a* $L > 0$ *such that* $f(x + L) = f(x)$ *for all* $x \in \mathbb{R}$. *The number* L *is called a period of* f.

Let P_f be the set of periods of f. Then $P_f \neq \emptyset$ if and only if f is periodic. Note P_f need not have a smallest element for the characteristic function of the irrationals has all positive rationals for periods.

If f is periodic, let $L_f := \inf P_f \geq 0$. The next lemma states that $P_f - P_f$ is a subgroup of $\mathbb{R}$ invariant under multiplication by $k \in \mathbb{Z}$

Lemma 1.2.2. *Let* L *and* M *be a periods of* f. *Then* $f(x+jL+kM) = f(x)$ *for all* $j, k \in \mathbb{Z}$.

Proof. We have $f(x - L) = f((x - L) + L) = f(x)$. The other statements follows by induction. ∎

Lemma 1.2.3. *Suppose that* L_f *is a period for* f. *Then* $P_f = L_f\mathbb{N}$.

Proof. Obviously $L_f\mathbb{N} \subseteq P_f$. Let $L \in P_f$. Then we can choose $n \in \mathbb{N}$ such that

$$nL_f \leq L < (n + 1)L_f.$$

Hence $0 \leq L - nL_f < L_f$. If $L - nL_f > 0$, then Lemma 1.2.2 implies $L - nL_f$ is a smaller period than L_f which is clearly untrue. Hence $L_f\mathbb{N} = P_f$. ∎

Lemma 1.2.4. *If* $L_f = 0$ *and* f *is continuous, then* f *is constant.*

Proof. Fix x. Choose a sequence L_n of periods converging to 0. Choose integers k_n so that $k_nL_n \leq x < (k_n + 1)L_n$. Then $k_nL_n \to x$. Hence $f(k_nL_n) \to f(x)$. But $f(k_nL_n) = f(0 + k_nL_n) = f(0)$. Consequently, $f(0) = f(x)$ and f is a constant function. ∎

The functions $x \mapsto \cos(mx)$, $\sin(kx)$, and $e^{inx} = \cos(nx) + i\sin(nx)$ all have period 2π. We will show in a certain sense that "each" 2π–periodic function can be written as an infinite linear combination of e^{inx} and hence also of $\cos(nx)$ and $\sin(nx)$. If f has period $L > 0$, then $g(x) = f(Lx/2\pi)$ has period 2π. Hence f can be written as a linear combinations of functions of form $e^{2\pi inx/L}$. We therefore restrict ourselves to functions of period 2π. Let

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$$

be the one-dimensional *torus*. Then $\mathbb{T}$ is a closed and bounded subset of $\mathbb{C}$ and hence compact. Furthermore $\mathbb{T}$ is an abelian group under multiplication and the map

$$\mathbb{R} \ni x \overset{\kappa}{\mapsto} e^{ix} \in \mathbb{T}$$

is a surjective group homomorphism of $(\mathbb{R}, +)$ onto $(\mathbb{T}, \cdot)$ with kernel $2\pi\mathbb{Z}$. The torus $\mathbb{T}$ has a natural topology as a subset of $\mathbb{C}$.

Let $z, w, z_0, w_0 \in \mathbb{T}$. Then $|z| = |w| = |z_0| = |w_0| = 1$ and

$$|zw - z_0w_0| \leq |w|\,|z - z_0| + |z_0|\,|w - w_0| = |z - z_0| + |w - w_0|$$

and

$$\left|z^{-1} - z_0^{-1}\right| = |\bar{z} - \bar{z}_o| = |z - z_0| \ .$$

Hence it follows that both the multiplication and the inverse map are continuous maps in this topology. These are conditions defining a *topological group.*

Lemma 1.2.5. *The mapping $\kappa : \mathbb{R} \to \mathbb{T}$ is a continuous periodic open mapping from $\mathbb{R}$ onto $\mathbb{T}$ satisfying $\kappa(\theta + \phi) = \kappa(\theta)\kappa(\phi)$ for all θ and ϕ in $\mathbb{R}$. Moreover, every complex function f on $\mathbb{R}$ having period 2π has form $f = F \circ \kappa$ for a unique function F on $\mathbb{T}$, and f is continuous iff F is continuous.*

Proof. Clearly κ is continuous, onto, and has period 2π. Let $I = (a, b)$ be an open interval. Then if $b - a > 2\pi$, $\kappa(I)$ equals $\mathbb{T}$ and if $b - a \leq 2\pi$, then $\kappa(I)$ is an 'open arc' in $\mathbb{T}$ and thus is open in $\mathbb{T}$ in the relative topology from $\mathbb{C}$. Since every open subset of $\mathbb{R}$ is a countable union of open intervals, we see $\kappa(U)$ is open in $\mathbb{T}$ for any open subset U of $\mathbb{R}$.

Let f be a function on $\mathbb{R}$ with period 2π. Define $F(e^{ix}) = f(x)$. F is well defined and is clearly the only function with $F \circ \kappa = f$. Note f is continuous if F is continuous. If f is continuous and U is open, then $F^{-1}(U) = \kappa(f^{-1}(U))$ is an open set in $\mathbb{T}$, for κ is an open mapping. Thus F is continuous. ∎

1.3 Integration on the Torus

Let X be a topological space. Denote the space of complex valued continuous function on X by $C(X)$. The last lemma can be used to integrate and differentiate functions on $\mathbb{T}$. Define a regular Borel measure μ on $\mathbb{T}$ by

$$\mu(E) = \frac{1}{2\pi} m(\kappa^{-1}(E) \cap [0, 2\pi))$$

where m is Lebesgue measure on $\mathbb{R}$. Then $g \in L^1(\mathbb{T}, \mu)$ iff $g \circ \kappa \in L^1[0, 2\pi]$ and then

$$\int g(z)\, d\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} g(e^{ix})\, dx.$$

The measure μ is left and right invariant; i.e.,

$$\mu(aE) = \mu(Ea) = \mu(E)$$

for all Borel subsets E of $\mathbb{T}$ and $a \in \mathbb{T}$.

The left and right invariance of the measure μ implies

$$\int g(y^{-1}z)\, d\mu(z) = \int g(zy)\, d\mu(z) = \int g(z)\, d\mu(z)$$

for all $g \in L^1(\mathbb{T}, \mu)$ and $y \in \mathbb{T}$. The measure is also invariant under the inverse-mapping. Thus

$$\int g(z^{-1})\, d\mu(z) = \int g(z)\, d\mu(z)\,.$$

Indeed,

$$\int g(z^{-1})\, d\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} g(e^{-ix})\, dx = \frac{1}{2\pi} \int_0^{2\pi} g(e^{ix})\, dx.$$

Denote the corresponding linear space of p-integrable complex valued functions by $L^p(\mathbb{T})$. Recall the norm is given by

$$|f|_p = \left(\int |f(z)|^p \, d\mu(z) \right)^{\frac{1}{p}}.$$

This space is the same as the space of L^p functions on $[0, 2\pi]$ or the space of Lebesgue measurable functions on $\mathbb{R}$ that are 2π-periodic and p-integrable over $[0, 2\pi]$. Let us recall the following two well known facts on integration.

Theorem 1.3.1. *Let $1 \leq p < \infty$. Then $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$.*

Theorem 1.3.2 (Hölder Inequality). *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $p, q \geq 1$ satisfy $1/p + 1/q = 1$. Let $f \in L^p(X)$ and $g \in L^q(X)$. Then $fg \in L^1(X)$ and*

$$|fg|_1 \leq |f|_p \, |g|_q \ .$$

Lemma 1.3.3. *Suppose $f \in L^2(\mathbb{T})$ satisfies $\int f(z)\overline{g(z)}\, d\mu(z) = 0$ for all continuous functions g. Then $f = 0$.*

Proof. Since $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, we can choose a sequence $g_n \in C(\mathbb{T})$ with $|g_n - f|_2 \to 0$. Hence

$$\int f\bar{f} \leq \lim \left(\int |f(\bar{f} - \bar{g}_n)| \, d\mu + | \int f\bar{g}_n \, d\mu| \right) \leq \lim |f|_2 |f - g_n|_2 = 0.$$

Hence $\int f\bar{f}\, d\mu = |f|_2^2 = 0$, which implies that $f = 0$ a.e. ∎

Finally, by the Riesz–Fischer Theorem, we know the spaces $L^p(T)$ are complete if equipped with norm $f \mapsto |f|_p$. In particular, $L^2(T)$ is a Hilbert space with inner product

$$(f, g)_2 = \int_{\mathbb{T}} f(z)\bar{g}(z)\, d\mu(z).$$

The L^p spaces become smaller with larger p.

Theorem 1.3.4. *Let $1 \leq p \leq \infty$. Then $L^p(\mathbb{T}) \subset \mathbb{L}^1(\mathbb{T})$ and $|f|_1 \leq |f|_p$ for all $f \in L^p(\mathbb{T})$.*

Proof. Let q be such that $1/p + 1/q = 1$. Then the constant function $z \mapsto 1$ is in $L^q(\mathbb{T})$ as $\mu(\mathbb{T}) = 1 < \infty$. By Hölder's-inequality, one has

$$\int |f| \ 1 \, d\mu \leq |f|_p \, |1|_q = |f|_p \ .$$

∎

Definition 1.3.5. *Let $1 \leq p \leq \infty$. For $a \in \mathbb{T}$, define linear operators $L(a)$ and $R(a)$ on $L^p(T)$ by*

$$\begin{aligned} L(a)f(z) &= f(a^{-1}z) \\ R(a)f(z) &= f(za). \end{aligned}$$

Then L and R are called the left and right regular representations of $\mathbb{T}$ on $L^p(T)$.

Suppose f is a complex valued function on $\mathbb{T}$. Then $\check{f}$ will be the function defined by

$$\check{f}(z) = f(z^{-1}).$$

Lemma 1.3.6. *The mappings $a \mapsto L(a)$ and $a \mapsto R(a)$ are homomorphisms of $\mathbb{T}$ into the group of invertible linear isometries of $L^p(\mathbb{T})$. Moreover, $f \mapsto \check{f}$ is a linear isometry of $L^p(\mathbb{T})$ satisfying*

$$(L(a)f)\check{} = R(a)\check{f}.$$

Proof. Note

$$\begin{aligned} L(ab)f(x) &= f((ab)^{-1}x) = f(b^{-1}a^{-1}x) \\ &= L(b)f(a^{-1}x) = L(a)L(b)f(x). \end{aligned}$$

and thus $L(ab) = L(a)L(b)$ on L^p. Clearly $L(1) = I$; and since $L(a)L(a^{-1}) = L(1) = L(a^{-1})L(a)$, we have $L(a)^{-1} = L(a^{-1})$. Thus $a \mapsto L(a)$ is a group homomorphism.

Suppose $p = \infty$. Then $|L(a)f|_\infty = \text{essup}|f(az)| = \text{essup}|f(z)| = |f|_\infty$. For $1 \le p < \infty$, we have

$$|L(a)f|_p^p = \int |f(a^{-1}z)|^p \, d\mu(z) = \int |f(z)|^p \, d\mu(z)$$

and thus $|L(a)f|_p = |f|_p$.

Note

$$\begin{aligned} (L(a)f)\check{}(z) &= L(a)f(z^{-1}) \\ &= f(a^{-1}z^{-1}) \\ &= \check{f}(za) \\ &= R(a)\check{f}(z) \end{aligned}$$

and if $1 \le p < \infty$, then

$$|\check{f}|_p^p = \int |f(z^{-1})|^p \, d\mu(z) = \int |f(z)|^p \, d\mu(z) = |f|_p^p.$$

One easily notes $|\check{f}|_\infty = |f|_\infty$. Thus $f \mapsto \check{f}$ is a linear isometry and it is onto for $(\check{f})\check{} = f$. ∎

It is common to use $\lambda(a)$ for the representation $L(a)$ and $\rho(a)$ for the representation $R(a)$.

Definition 1.3.7 (Convolution). *Let f and g be in $L^1(\mathbb{T})$. The convolution $f * g$ of f and g is defined by*

$$f * g(x) = \int_{\mathbb{T}} f(y)g(y^{-1}x)\, d\mu(x).$$

Note $(y,x) \mapsto f(y)g(y^{-1}x)$ is a measurable function on $\mathbb{T} \times \mathbb{T}$ and

$$\int\int \left|f(y)g(y^{-1}x)\right|\, d\mu(x)d\mu(y) = |f|_1\,|g|_1 < \infty\,.$$

It follows that for almost all $x \in \mathbb{T}$, the function $y \mapsto f(y)g(y^{-1}x)$ is integrable and

$$\int |f * g(x)|\, d\mu(x) \leq \int\int |f(y)g(y^{-1}x)|\, d\mu(y)\, d\mu(x) = |f|_1\, |g|_1.$$

Hence $f * g \in L^1$ is in $L^1(\mathbb{T})$. As $L^p(\mathbb{T})$ and $L^q(\mathbb{T})$ are subspaces of $L^1(\mathbb{T})$ with larger norms, one has

$$|f * g|_1 \leq |f|_1 * |g|_1 \leq |f|_p|g|_q$$

whenever $f \in L^p$ and $g \in L^q$.

Lemma 1.3.8. *Let $f, g, h \in L^1(\mathbb{T})$. Then the following hold:*

(a) $f * g \in L^1(\mathbb{T})$ *and* $|f * g|_1 \leq |f|_1\, |g|_1$.

(b) $f * g = g * f$.

(c) $f * (g * h) = (f * g) * h$.

(d) $\lambda(a)f * g = f * \lambda(a)g = \lambda(a)(f * g)$

Proof. Note (a) was proved just before we stated the lemma.

To see (b), note

$$\begin{aligned}
f * g(z) &= \int f(y)g(y^{-1}z)\, d\mu(y) \\
&= \int f(zy)g((zy)^{-1}z)\, d\mu(y) \\
&= \int g(y^{-1})f(yz)\, d\mu(y) \\
&= \int g(y)f(y^{-1}z)\, d\mu(y)
\end{aligned}$$

where we have used invariance of integration under transformations $y \mapsto zy$ and $y \mapsto y^{-1}$.

For (c) we have

$$\begin{aligned}
(f * g) * h(z) &= \int (f * g)(y) h(y^{-1}z)\, d\mu(y) \\
&= \int \int f(x) g(x^{-1}y) h(y^{-1}z^{-1})\, d\mu(x)\, d\mu(y) \\
&= \int f(x) \int g(x^{-1}y) h(y^{-1}z)\, d\mu(y)\, d\mu(x) \\
&= \int f(x) \int g(x^{-1}xy) h((xy)^{-1}z)\, d\mu(y)\, d\mu(x) \\
&= \int f(x) \int g(y) h(y^{-1}x^{-1}z)\, d\mu(y)\, d\mu(x) \\
&= \int f(x) g * h(x^{-1}z)\, d\mu(x) \\
&= f * (g * h)(z)
\end{aligned}$$

where the changes in the order of integration follow by Fubini's theorem and we have used the invariance of the measure $d\mu$ under left translation by x^{-1}.

For (d) note

$$\begin{aligned}
\lambda(a) f * g(y) &= \int \lambda(a) f(x) g(x^{-1}y)\, d\mu(x) \\
&= \int f(a^{-1}x) g(x^{-1}y)\, d\mu(x) \\
&= \int f(a^{-1}ax) g((ax)^{-1}y)\, d\mu(x) \\
&= \int f(x) g(x^{-1}a^{-1}y)\, d\mu(x) \\
&= f * g(a^{-1}y) \\
&= \lambda(a)(f * g)(y) \\
&= \lambda(a)(g * f)(y) \\
&= \lambda(a) g * f(y) \\
&= f * \lambda(a) g(y)
\end{aligned}$$

where we have used the commutativity of convolution. ∎

Proposition 1.3.9. *Suppose* $1 \leq p \leq q \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $f * g \in C(\mathbb{T})$ *whenever* $f \in L^p(\mathbb{T})$ *and* $g \in L^q(\mathbb{T})$.

Proof. Note if $f \in C(\mathbb{T})$ and $g \in L^q(\mathbb{T})$ and $z_n \to z$ in $\mathbb{T}$, then

$$\begin{aligned} f * g(z_n) &= g * f(z_n) = \int g(x) f(x^{-1} z_n)\, d\mu(x) \\ &\to \int g(x) f(x^{-1} z)\, d\mu(x) \\ &= g * f(z) \\ &= f * g(z) \end{aligned}$$

as $n \to \infty$ by the Lebesgue dominated convergence theorem. Indeed, $x \mapsto g(x) f(x^{-1} z_n)$ is dominated pointwise by $|g|\,|f|_\infty$ which is in $L^1(\mathbb{T})$ and converges pointwise to $g(x) f(x^{-1} z)$. Hence $f * g \in C(\mathbb{T})$ if $f \in C(\mathbb{T})$.

Now suppose $f \in L^p(\mathbb{T})$ and $\epsilon > 0$. Let $z_n \to z$. Since $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$, we know we can choose $f_0 \in C(\mathbb{T})$ satisfying $|f - f_0| \leq \frac{\epsilon}{3(|g|_q + 1)}$. Choose N so that $|f_0 * g(z_n) - f_0 * g(z)| < \frac{\epsilon}{3}$ for $n \geq N$. Then for $n \geq N$, we have

$$\begin{aligned} |f * g(z_n) - f * g(z)| &\leq |(f - f_0) * g(z_n)| + |f_0 * g(z_n) - f_0 * g(z)| + |(f_0 - f) * g(z)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

since by Hölder's inequality

$$\begin{aligned} |(f - f_0) * g(y)| &\leq \int |(f - f_0)(x) g(x^{-1} y)|\, d\mu(x) \\ &\leq |f - f_0|_p |L(y)\check{g}|_q \\ &= |f - f_0|_p |g|_q \\ &\leq \frac{\epsilon}{3(|g|_q + 1)} \cdot |g|_q \\ &\leq \frac{\epsilon}{3} \end{aligned}$$

for all $y \in \mathbb{T}$. ∎

EXERCISE SET 2:

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1. Show if f is a function on $\mathbb{T}$ such that $f \circ \kappa$ is a simple measurable function on $\mathbb{R}$, then

$$\int f(z)\, d\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix})\, dx.$$

2. Show $L^p(\mathbb{T}) \subseteq L^q(\mathbb{T})$ and $||f||_p \geq ||f||_q$ for $p \geq q$.

3. For $w \in \mathbb{T}$ define $\lambda(w) : C(\mathbb{T}) \to C(\mathbb{T})$ by $[\lambda(w)f](z) = f(w^{-1}z)$. Show that $\lambda(w)$ is continuous with norm one.

4. Let $1 \leq p < \infty$. Let $\lambda(z)f(w) = f(z^{-1}w)$ for $f \in L^p(\mathbb{T})$. Show for each f and each $\epsilon > 0$, there is a $\delta > 0$ such that $|\lambda(z)f - f|_p < \epsilon$ if $|z - 1| < \delta$.

5. For $1 \leq p < q$, find $f \in L^p(\mathbb{T})$ such that $f \notin L^q(\mathbb{T})$.

6. Let $h \in L^1(\mathbb{T})$.

(a) Let $g \in L^2(\mathbb{T})$. Show that $\lambda(h)g := h * g$ is in $L^2(\mathbb{T})$. (**Hint:** Let $f \in L^2(\mathbb{T})$. Then $f\overline{\lambda(h)g}$ is integrable and $f \mapsto \int f(z)\overline{\lambda(h)g(z)}\, d\mu(z)$ is a continuous linear form on $L^2(\mathbb{T})$.

(b) Show that $\lambda(h) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a bounded linear map with $|\lambda(h)| \leq |h|_1$.

1.4 The Fourier Transform

The functions of the form

$$p(\theta) = \sum_{n=-N}^{M} a_n e^{in\theta}$$

on $\mathbb{R}$ are called *trigonometric polynomials.* The trigonometric polynomials are 2π-periodic and as functions on $\mathbb{T}$ they can simply be written as

$$p(z) = \sum_{n=-N}^{M} a_n z^n$$

The trigonometric polynomials form an algebra of continuous functions on $\mathbb{T}$ which separate points, contain the constants, and are closed under conjugation. By the Stone–Weierstrass Theorem, this algebra is dense in $C(\mathbb{T})$ under the $|\cdot|_\infty$ norm. Since $C(\mathbb{T})$ is dense in every $L^p(\mathbb{T})$ except $L^\infty(\mathbb{T})$, one can show the algebra of trigonometric polynomials is dense in every $L^p(\mathbb{T})$ where $1 \leq p < \infty$. We shall be interested in particular trigonometric polynomials; namely the partial sums of Fourier series.

Let $e_n(\theta) = e^{in\theta}$. On $\mathbb{T}$, this is the function $z \mapsto z^n$. If $f \in L^1(\mathbb{T})$, then the *Fourier transform* $\hat{f} : \mathbb{Z} \to \mathbb{C}$ of f is defined by

$$\hat{f}(n) := \int f(z) z^{-n}\, d\mu(z) = \frac{1}{2\pi} \int_I f(e^{i\theta})\, e^{-in\theta} d\theta \tag{1.4.1.1}$$

where I is any interval in $\mathbb{R}$ having length 2π. Note that $\hat{f}(n) = (f, e_n)$ for $f \in L^2(\mathbb{T})$. The corresponding *Fourier series* is

$$\sum \hat{f}(n)\, e^{in\theta} = \sum \hat{f}(n) z^n$$

or

$$\sum_{n=-\infty}^{\infty} (f, e_n) e_n .$$

Notice that

$$(e_n, e_m) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta}\, d\theta = \delta_{n,m} \, . \tag{1.4.1.2}$$

Hence $\{e_n \mid n \in \mathbb{Z}\}$ is a orthonormal subset of $L^2(\mathbb{T})$.

Definition 1.4.1. *Let $g \in C(\mathbb{T})$ and $r \in \mathbb{N}_0$. Then g is r-times continuously differentiable if the continuous periodic function $g \circ \kappa$ is r-times continuously differentiable on $\mathbb{R}$.*

We denote the space of r-times continuously differentiable functions by $C^r(\mathbb{T})$. The space of smooth functions $C^\infty(\mathbb{T})$ is $\cap_r C^r(\mathbb{T})$; i.e., the space of functions that are r-times continuously differentiable for all r. Define the first order differential operator D by

$$[Df](e^{ix}) = \frac{df \circ \kappa}{dx}(x) = \frac{d}{dx} f(e^{ix}) \, .$$

Notice that $Dz^n = inz^n$ for all n. Recall $\lambda(w) f(z) = f(w^{-1} z)$.

Lemma 1.4.2. *Let $f, g \in L^1(\mathbb{T})$. Let $p = \sum_{n=-N}^{M} a_n e_n$ be a trigonometric polynomial. Then the following hold:*

(a) $\left|\hat{f}(n)\right| \leq |f|_1$ *for all* $n \in \mathbb{Z}$.

(b) $\widehat{\lambda(w)f}(n) = w^{-n}\widehat{f}(n)$.

(c) Then $\widehat{f^{\vee}}(n) = \hat{f}(-n)$.

(d) $\lim_{n\to\infty} \hat{f}(n) = 0$ *(Lebesgue Lemma).*

(e) $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.

(f) $f * p = \sum_{n=-N}^{M} a_n \hat{f}(n) e_n$.

(g) Assume that $f \in C^r(\mathbb{T})$. *Then* $\widehat{D^r f}(n) = (in)^r \hat{f}(n)$.

Proof. To see (a), note $\left|\hat{f}(n)\right| = \left|\int f(z)z^{-n}\, d\mu(z)\right| \leq \int |f(z)|\, d\mu(z) = |f|_1$ because $|z| = 1$ for $z \in \mathbb{T}$. Thus (a) holds.

For (b), note

$$\begin{aligned}
\widehat{\lambda(w)f}(n) &= \int \lambda(w) f(z) z^{-n}\, d\mu(z) \\
&= \int f(w^{-1}z) z^{-n}\, d\mu(z) \\
&= \int f(z)[wz]^{-n}\, d\mu(z) \\
&= w^{-n} \int f(z) z^{-n}\, d\mu(z) \\
&= w^{-n}\hat{f}(n).
\end{aligned}$$

For (c) one has

$$\begin{aligned}
\widehat{f^{\vee}}(n) &= \int f(z^{-1}) z^{-n}\, d\mu(z) \\
&= \int f(z) z^{n}\, d\mu(z) \\
&= \hat{f}(-n)\,.
\end{aligned}$$

To do (d), we first do the case where $F = f \circ \kappa|_{[0,2\pi)} = C\chi_{[a,b]}$ with $0 \leq a < b < 2\pi$. Then

$$\begin{aligned}
\hat{f}(n) &= \frac{C}{2\pi}\int_a^b e^{-in\theta}\, d\theta \\
&= \frac{C}{2\pi i n}\left[e^{-inb} - e^{-ina}\right] \to 0 \qquad n \to \infty\,.
\end{aligned}$$

It follows that the claim holds for any F which is a step function on $[0, 2\pi)$. But these define f's which are dense in $L^1(\mathbb{T})$. Hence if f is in $L^1(\mathbb{T})$ and $\epsilon > 0$, one can choose a step function $F_0 \in L^2[0, 2\pi)$ satisfying

$$|f \circ \kappa - F_0|_1 < \frac{\epsilon}{2}.$$

Setting $f_0(e^{i\theta}) = F_0(\theta)$, we have

$$\begin{aligned} |\hat{f}(n)| &\leq |\hat{f}(n) - \hat{f}_0(n)| + |\hat{f}_0(n)| \\ &\leq |f - f_0| + |\hat{f}_0(n)| \\ &< \frac{\epsilon}{2} + |\hat{f}_0(n)| \\ &< \epsilon \end{aligned}$$

for n large.

For (e), by left invariance of the measure μ and Fubini's Theorem, we obtain:

$$\begin{aligned} \widehat{f * g}(n) &= \int f * g(z)\, z^{-n}\, d\mu(z) \\ &= \int \left[\int f(w) g(w^{-1} z)\, d\mu(w) \right] z^{-n}\, d\mu(z) \\ &= \int \int f(w) g(w^{-1} z) z^{-n}\, d\mu(z) d\mu(w) \\ &= \int f(w) \int g(z)\, [wz]^{-n}\, d\mu(z) d\mu(w) \\ &= \int f(w) w^{-n} \int g(z) z^{-n}\, d\mu(z) d\mu(w) \\ &= \widehat{f}(n) \widehat{g}(n)\,. \end{aligned}$$

To do (f), we have by the definition of $\hat{f}(n)$ that:

$$\begin{aligned} f * p(z) &= \sum a_n \int f(w)(w^{-1}z)^n\, d\mu(w) \\ &= \sum a_n z^n \int f(w) w^{-n}\, d\mu(w) \\ &= \sum a_n \hat{f}(n) e_n(z)\,. \end{aligned}$$

Finally for (g), note for $r = 1$ the statement follows immediately by integration by parts:

$$\widehat{Df}(n) = \frac{1}{2\pi}\int_0^{2\pi} (f\circ\kappa)'(\theta)\, e^{-in\theta}\, d\theta = \frac{-1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{d}{d\theta}e^{-in\theta}\, d\theta = in\hat{f}(n)$$

∎

Corollary 1.4.3. *For $f \in L^1(\mathbb{T})$, $f * e_n = \hat{f}(n)e_n$. Moreover, $e_m * e_n = \delta_{m,n}e_n$.*

Proof. Note e_n is a trigonometric polynomial. Hence by (f), $f * e_n = \hat{f}(n)e_n$. Since

$$\hat{e}_m(n) = \int e_m(z)z^{-n}\, d\mu(z) = \int e_m(z)\overline{e_n(z)}\, d\mu(z) = \delta_{m,n},$$

we see $e_m * e_n = \delta_{m,n}e_n$. ∎

Lemma 1.4.4. *Let $g \in C(\mathbb{T})$ and $\epsilon > 0$. Then there exists a trigonometric polynomial p such that*

$$|g - p|_\infty < \epsilon\,.$$

Proof. $\mathbb{T}$ is a compact Hausdorff space, and the space $\mathcal{A}$ of all trigonometric polynomials $p(z) = \sum_{n=-N}^{M} a_n z^n$ form an algebra of continuous functions on $\mathbb{T}$ which contain the constants and separate points. Moreover $\mathcal{A}$ is closed under conjugation for $\overline{z^n} = z^{-n}$. By the Stone–Weierstrass Theorem for continuous complex valued functions on a compact Hausdorff space, one has for each $f \in C(\mathbb{T})$ and each $\epsilon > 0$, there is a $p \in \mathcal{A}$ with $|f - p|_\infty < \epsilon$. ∎

Theorem 1.4.5 (Plancherel Theorem). *The set of functions $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$. In particular, if f is in $L^2(\mathbb{T})$, then*

(a) $f = \sum_{n=-\infty}^{\infty}(f, e_n)e_n$ *in* $L^2(\mathbb{T})$ *and*

(b) $|f|^2 = \sum_{n=-\infty}^{\infty} |(f, e_n)|^2$.

Proof. By equation 1.4.1.2, the set $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal. Let $\epsilon > 0$ and let $f \in L^2(\mathbb{T})$. Choose $g \in C(\mathbb{T})$ such that $|f - g|_2 < \epsilon/2$. By Lemma 1.4.4 there is a trigonometric polynomial p such that

$$|g - p|_\infty < \epsilon/2\,.$$

Thus

$$\begin{aligned} |g-p|_2^2 &= \int |g(z)-p(z)|^2 \, d\mu(z) \\ &\le \int |g-p|_\infty^2 \, d\mu(z) \\ &< (\epsilon/2)^2 \, . \end{aligned}$$

Thus $|g-p|_2 < \epsilon/2$. It follows that

$$|f-p|_2 \le |f-g|_2 + |g-p|_2 < \epsilon \, .$$

Thus e_n form a complete orthonormal basis and the theorem follows. ∎

Let ℓ^2 be the space of biinfinite complex sequences $\{a_n\}_{n=-\infty}^{\infty}$ which satisfy $\sum |a_n|^2 < \infty$. This space is a Hilbert space with inner product

$$(\{a_n\},\{b_n\}) = \sum a_n \bar{b}_n$$

and norm

$$|\{a_n\}| = \sqrt{\sum |a_n|^2}.$$

One can give a direct proof of this fact; however, it follows easily from measure theory. Namely, let ν be counting measure on $\mathbb{Z}$; thus every subset of $\mathbb{Z}$ is measurable, and $\nu(E)$ is the number of elements in E. Then ν is a measure, every function is measurable, and a function $a : \mathbb{Z} \to \mathbb{C}$ is in $L^2(\mathbb{Z},\nu)$ iff $\sum |a(n)|^2 = \int |a(n)|^2 \, d\nu(n) < \infty$.

We now easily reformulate the Plancherel Theorem:

Theorem 1.4.6. *The Fourier transform $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2$ is an isomorphism of Hilbert spaces.*

Proof. We have by the Plancherel Theorem that

$$\sum \left|\hat{f}(n)\right|^2 = |f|^2 < \infty \, .$$

Hence $\mathcal{F}f \in \ell^2$ and $\mathcal{F}$ is an isometry into ℓ^2. Let $A = \{a_n\} \in \ell^2$. Define a sequence

$$f_n(z) = \sum_{j=-n}^{n} a_n z^n \in L^2(\mathbb{T}).$$

Thus $f_n = \sum_{j=-n}^{n} a_n e_n$. For $m \geq n$ one has

$$|f_n - f_m|^2 = \sum_{n<|j|\leq m} |a_j|^2 \,.$$

But $\sum |a_j|^2 < \infty$. Hence if $\epsilon > 0$ then we can find an $N \in \mathbb{N}$ such that for all $n \geq N$

$$\sum_{n\leq |j|} |a_j|^2 < \epsilon.$$

But this obviously implies that $\{f_n\}$ is a Cauchy sequence in $L^2(\mathbb{T})$. Then $\sum a_j e_j$ converges in L^2 to an L^2 function f and

$$\begin{aligned} \hat{f}(n) &= (f, e_n) \\ &= \sum a_j (e_j, e_n) \\ &= a_n \,. \end{aligned}$$

Hence $\mathcal{F} f = A$ and $\mathcal{F}$ is surjective. ∎

Let $C^\infty(\mathbb{T})$ be the space of smooth function on $\mathbb{T}$, i.e., $C^\infty(\mathbb{T}) = \cap_{r\in\mathbb{N}} C^r(\mathbb{T})$. Define a vector space topology on $C^\infty(\mathbb{T})$ by the seminorms

$$\sigma_n(f) := |D^n f|_\infty \,.$$

We leave it as an exercise to show that $C^\infty(\mathbb{T})$ with this topology is a locally convex complete topological vector space. This topology is called the Schwartz topology on $C^\infty(\mathbb{T})$ and the space $C^\infty(\mathbb{T})$ with this topology is denoted by $\mathcal{D}(\mathbb{T})$. To find the image of $\mathcal{D}(\mathbb{T})$ under the Fourier transform, let $\mathcal{S}(\mathbb{Z})$ be the space of sequences $a = \{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that for each k

$$\rho_k(a) := \sup_n (1 + |n|)^k |a_n| < \infty \,.$$

The ρ_k are seminorms. With these seminorms $\mathcal{S}(\mathbb{Z})$ becomes a locally convex complete topological vector space. Sequences $\{a_n\}$ which satisfy $\rho_k(a) < \infty$ for all k are said to be rapidly decreasing. Notice that

$$\frac{|n|^k}{(1+|n|)^k} \to 1 \,, \quad n \to \infty \,.$$

Hence there are positive constants C_k such that for all $n \neq 0$,

$$C_k(1+|n|)^k\,|a_n|_k \leq |n|^k\,|a_n| \leq (1+|n|)^k\,|a_k|\ .$$

The topology on $\mathcal{S}(\mathbb{Z})$ can therefore also be defined by the seminorms ρ_k' where and

$$\rho_k'(a) := \sup |n|^k\,|a_n|\ , \quad k \neq 0\,.$$

In these formulas, expression 0^0 is given value 1. This topology can also be defined by using the seminorms

$$\rho_k''(a) = \sum |n|^k\,|a_n|\ .$$

Theorem 1.4.7. *The Fourier transform $f \mapsto \hat{f}$ is a topological isomorphism of $\mathcal{D}(\mathbb{T})$ onto $\mathcal{S}(\mathbb{Z})$.*

Proof. We have the Fourier transform of $D^k f$ is $n \mapsto (in)^k \hat{f}(n) \in \ell^2$. Thus $\sum |n|^{2k}|\hat{f}(n)|^2 < \infty$ for each k. Hence $\sup_n |n^{2k}\hat{f}(n)|^2$ is finite for all k. Consequently $\sup_n |n^k \hat{f}(n)| < \infty$ for all k. Thus $\hat{f} \in \mathcal{S}(\mathbb{T})$.

Clearly $f \mapsto \hat{f}$ is linear. It is one-to-one, for $\hat{f} = 0$ implies $f = 0$ in $L^2(\mathbb{T})$, and thus $f = 0$ in $\mathcal{D}(\mathbb{T})$.

We show this mapping is onto. Let $\{a_n\} \in \mathcal{S}(\mathbb{Z})$. For each k, define $g_k(z) = \sum (in)^k a_n z^n$. We note this series converges uniformly for each k. Indeed,

$$\begin{aligned}\sum |i^k n^k a_n z^n| &\leq \sum |n^k a_n| \\ &\leq \sum_{n\neq 0} |n^{-2}(1+|n|^{k+2})|a_n| \\ &\leq \rho_{k+2}(a) \sum |n^{-2}| \\ &< \infty.\end{aligned}$$

Thus each $g_k \in C(\mathbb{T})$. Hence $\sum (in)^k a_n e^{in\theta}$ converges uniformly on $\mathbb{R}$ and since

$$D(\sum_{n=-N}^{N} (in)^k a_n e^{in\theta}) = \sum_{n=-N}^{N} (in)^{k+1} a_n e^{in\theta}$$

converges uniformly to $g_{k+1}(e^{i\theta})$ on $\mathbb{R}$, we see $g_k(e^{i\theta})$ is differentiable and has derivative $g_{k+1}(e^{i\theta})$. Thus $g_0 \in \mathcal{D}(\mathbb{T})$ and $D^k g_0 = g_k$. Moreover, $\hat{g}_0 = a$, and we see $f \mapsto \hat{f}$ is onto.

To show $f \mapsto \hat{f}$ and $\hat{f} \mapsto f$ are continuous, it is sufficient to show they are continuous at 0.

Now $f \mapsto \hat{f}$ is continuous at 0 iff $f \mapsto \rho'_k(\hat{f})$ are continuous at 0. But

$$\begin{aligned}\rho'_k(\hat{f}) &= \sup |n^k \hat{f}(n)| \\ &= |\widehat{D^k f}|_\infty \\ &< \epsilon\end{aligned}$$

if $|D^k f|_\infty < \epsilon$, for $|\hat{g}(n)| \le |g|_1 \le |g|_\infty$. Hence $f \mapsto \hat{f}$ is continuous.

We finally show $\hat{f} \mapsto f$ is continuous at 0. Note

$$\begin{aligned}|D^k f|_\infty &= \sup_{|z|=1} |\sum_n (in)^k \hat{f}(n) z^n| \\ &\le \sum |n^k \hat{f}(n)| \\ &\le |\hat{f}(0)| + \sum_{n\ne 0} n^{-2} \cdot \sup(1+|n|)^{k+2} |\hat{f}(n)| \\ &\le \rho_0(\hat{f}) + \rho_{k+2}(\hat{f}) \cdot \sum_{n \ne 0} \frac{1}{n^2} \\ &< \epsilon\end{aligned}$$

if $\rho_0(\hat{f}) < \frac{\epsilon}{2}$ and $\rho_{k+2}(\hat{f}) < \frac{\epsilon}{2}\left(\sum_{n\ne 0} \frac{1}{n^2}\right)^{-1}$. ∎

Corollary 1.4.8. *Suppose f is a periodic C^∞ function on $\mathbb{R}$ having period 2π. Then the Fourier series $\sum \hat{f}(n) e^{in\theta}$ converges uniformly to f and the derivatives of these series converge uniformly to the derivatives of f.*

Exercise Set 3:

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1. Suppose σ_k are seminorms on vector space X and ρ_k are seminorms on vector space Y. Give X and Y the topological vector space topologies defined by these seminorms. Show a linear transformation $T : X \to Y$ is continuous iff $\rho_k \circ T$ is continuous at 0 for each k.

(Hint: Recall a subset U of X will be open in the topology defined by the seminorms σ_k if for each $p \in U$, there is an $\epsilon > 0$ and finitely many seminorms $\sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_n}$ so that if $\sigma_{k_i}(q-p) < \epsilon$ for $i = 1, 2, \ldots, n$, then $q \in U$.)

2. Show that $f \in L^2([0, 2\pi], dx)$ can be written in the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)\,.$$

Find an expression for a_n and b_n.

3. Let g be the function on the torus corresponding to $f(\theta) = |\theta|$, $\theta \in [-\pi, \pi)$.

(a) Find $\hat{g}(n)$.

(b) Show that the Fourier series converges uniformly.

4. Use the Fourier transform to evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

5. Let f be the periodic function corresponding to $\chi_{[-\pi,0)} - \chi_{[0,\pi)}$. Evaluate $\hat{f}(n)$.

6. A function f on the torus is even if $f(z) = f(z^{-1})$ and odd if $f(z) = -f(z^{-1})$. Suppose that $f \in C^2(\mathbb{T})$. Show the following:

(a) If f is even, then $f(e^{i\theta}) = \sum \hat{f}(n) \cos(n\theta)$;

(b) If f is odd, then $f(e^{i\theta}) = i \sum \hat{f}(n) \sin(n\theta)$.

7. Let $g \in C^1(\mathbb{T})$ and $f \in L^1(\mathbb{T})$. Then $f * g \in C^1(\mathbb{T})$ and $D(f*g) = f * Dg$.

8. Let $L > 0$. Let f be a L-periodic function such that $\int_0^L |f(t)|^2 \, dt < \infty$. Show that there are constants $a_n \in \mathbb{C}$ such that in $L^2([0, L))$ we have

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n t/L}\,.$$

Find an expression for a_n.

9. Show that the seminorms ρ_k, ρ'_k, and ρ''_k all define the same topology on $\mathcal{S}(\mathbb{Z})$.

10. Let $\mathbb{T}^k := \{\mathbf{z} = (z_1, \dots, z_k) \mid z_j \in \mathbb{T}\}$ with the product topology. For $\mathbf{z}, \mathbf{w} \in \mathbb{T}^k$, let $\mathbf{zw} = (z_1 w_1, \dots, z_k w_k)$. For $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$, let $e_{\mathbf{n}}(\mathbf{z}) = z_1^{n_1} \cdots z_k^{n_k}$. Show the following:

(a) $\mathbb{T}^k$ is a topological group, i.e., the map $\mathbb{T}^k \times \mathbb{T}^k \ni (\mathbf{z}, \mathbf{w}) \mapsto \mathbf{z}^{-1}\mathbf{w} \in \mathbb{T}^k$ is continuous.

(b) $e_{\mathbf{n}} : \mathbb{T}^k \to \mathbb{T}$ is a continuous homomorphism.

(c) IF $\chi : \mathbb{T}^k \to \mathbb{T}$ is a continuous homomorphism, then there exists a $\mathbf{n} \in \mathbb{Z}^k$ such that $\chi = e_{\mathbf{n}}$.

11. (Fourier series of $\mathbb{T}^k$) Let $\mu_k = \mu \times \cdots \times \mu$ be the product measure on $\mathbb{T}^k$. For $f \in L^1(\mathbb{T}^k)$, define $\hat{f} : \mathbb{Z}^k \to \mathbb{C}$ by

$$\hat{f}(\mathbf{n}) := \int_{\mathbb{T}^k} f(\mathbf{z})\, e_{-\mathbf{n}}(\mathbf{z})\, d\mu_k\,.$$

Show the following:

(a) If $f \in L^2(\mathbb{T}^k)$, then $|f|_2 = \sqrt{\sum_{\mathbf{n} \in \mathbb{Z}^k} \left|\hat{f}(\mathbf{n})\right|^2}$.

(b) If $f \in L^2(\mathbb{T}^k)$, then $f = \sum_{\mathbf{n} \in \mathbb{Z}^k} \hat{f}(\mathbf{n})\, e_{\mathbf{n}}$ in $L^2(\mathbb{T}^k)$.

12. Show that there is no differentiable function f on $\mathbb{T}$ such that $Df = 1$.

13. (Differential equations on the torus) Let $p(z) = \sum_{n=0}^k a_n z^n$ be a polynomial. Define $p(D) : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ by

$$p(D)f = \sum_{n=0}^{k} a_n D^n f\,.$$

Show that if $g \in C^\infty(\mathbb{T})$ is such that $\hat{g}(n) = 0$ if $p(in) = 0$, then the differential equation $p(D)f = g$ has a solution.

14. (The Hilbert transform on the torus) Let $f \in L^2(\mathbb{T})$. Show there exists a unique $g \in L^2(\mathbb{T})$ such that

$$\hat{g}(n) = -i\,\mathrm{sgn}(n)\,\hat{f}(n)\,, \quad \forall n \in \mathbb{Z}\,.$$

Define $Hf = g$. Then $H : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is linear. Prove the following:

(a) $|Hf|_2 = |f|_2$ if and only if $\int f\, d\mu = 0$.

(b) If $f \in C^\infty(\mathbb{T})$, then $Hf \in C^\infty(\mathbb{T})$.

(c) If $f \in C^\infty(\mathbb{T})$, then $Hf(1) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} f(e^{i\theta})\, \sin(n\theta)\, d\theta$.

1.5 Aproximate Units

Lemma 1.5.1. *Let $f \in L^1(\mathbb{T})$ and $g \in L^p(\mathbb{T})$ where $1 \le p \le \infty$. Then $f{*}g \in L^p(\mathbb{T})$ and $|f{*}g|_p \le |f|_1|g|_p$. Moreover, if $g \in C(\mathbb{T})$, then $f{*}g \in C(\mathbb{T})$.*

Proof. We have already seen this is true if $p = 1$. So we may assume $p > 1$. If $p = \infty$, $|f * g(x)| \le \int |f(y)|\,|g(y^{-1}x)|\, d\mu(y) \le |g|_\infty \int |f(y)|\, d\mu(y) = |f|_1|g|_\infty$. Suppose $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned}
|\langle f * g, h\rangle| &= \left|\int f * g(x) h(x)\, d\mu(x)\right| \\
&\le \int\int |f(y)g(y^{-1}x)h(x)|\, d\mu(y)\, d\mu(x) \\
&\le \int |f(y)| \int |\lambda(y)g(x)h(x)|\; d\mu(x)\, d\mu(y) \\
&\le \int |f(y)|\, |\lambda(y)g|_p |h|_q\, d\mu(y) \\
&= |g|_p \int |f(y)|\, d\mu(y)\, |h|_q \\
&= |f|_1|g|_p|h|_q.
\end{aligned}$$

This implies $f * g$ defines a bounded linear functional on L^q and consequently must be in L^p. Moreover, the norm of this bounded linear functional is at most $|f|_1|g|_p$. Thus $f * g \in L^p$ and $|f * g|_p \le |f|_1|g|_p$.

Finally suppose $g \in C(\mathbb{T})$ and $x_n \to x$ as $n \to \infty$. Then $g(y^{-1}x_n) \to g(y^{-1}x)$ for all y and $|f(y)g(y^{-1}x_n)| \le |f(y)|\,|g|_\infty$. Hence by the Lebesgue dominated convergence theorem, $\int f(y)g(y^{-1}x_n)\, d\mu(y) \to \int f(y)g(y^{-1}x)\, d\mu(x)$ as $n \to \infty$. Thus $f * g$ is continuous function. ∎

Definition 1.5.2. *An approximate unit in $L^1(\mathbb{T})$ will be a sequence ϕ_n of nonnegative measurable functions satsifying*

(a) $\int \phi_n\, d\mu = 1$ for each n

(b) if U is a neighborhood of 1, then $\sup_{x\notin U}\phi_n(x)\to 0$ as $n\to\infty$.

Proposition 1.5.3. *Let ϕ_n be an approximate unit in $L^1(\mathbb{T})$, and suppose $f\in L^p$ where $1\le p<\infty$. Then $\phi_n * f\to f$ in $L^p(\mathbb{T})$. Moreover, if $f\in C(\mathbb{T})$, then $\phi_n * f$ converges uniformly to f.*

Proof. We note by the Hahn–Banach Theorem that there is always an $h\in L^q$ satisfying $|h|_q=1$ and $\langle \phi_n * f - f, h\rangle = |\phi_n * f - f|_p$. Now by Hölder's inequality,

$$\begin{aligned}|\langle \phi_n * f - f, h\rangle| &\le \int\int \phi_n(y)|f(y^{-1}x)-f(x)|\,|h(x)|\,d\mu(y)\,d\mu(x)\\ &= \int \phi_n(y)\int |\lambda(y)f(x)-f(x)|\,|h(x)|\,d\mu(x)\,d\mu(y)\\ &\le \int \phi_n(y)\,|\lambda(y)f-f|_p|h|_q\,d\mu(y).\end{aligned}$$

Now if $\epsilon>0$, we can choose a neighborhood U of 1 in $\mathbb{T}$ such that $|\lambda(y)f - f|_p<\frac{\epsilon}{2}$ if $y\in U$. (See exercise 3 in exercise set 2.) But $\phi_n\to 0$ uniformly off U. Hence for large n, $\int_{T-U}\phi_n(y)\,d\mu(y)<\frac{\epsilon}{4|f|_p}$. Thus for large n, we have

$$|\langle \phi_n * f - f, h\rangle| \le \int_{\mathbb{T}-U}\phi_n(y)(2|f|_p)\,d\mu(y)+\int_U \frac{\epsilon}{2}\phi_n(y)\,dy<\epsilon$$

for any h with $|h|_q=1$. Consequently $|\phi_n * f - f|_p<\epsilon$ for large n.

Finally, suppose $f\in C(\mathbb{T})$. First choose a neighborhood U of 1 such that $|f(y^{-1}x)-f(x)|<\frac{\epsilon}{2}$ whenever $y\in U$, and then choose N such that $\sup_{x\in\mathbb{T}-U}\phi_n(x)<\frac{\epsilon}{4|f|_\infty}$ for $n\ge N$. Then if $n\ge N$, one has

$$\begin{aligned}|\phi_n * f(x) - f(x)| &\le \int \phi_n(y)|f(y^{-1}x)-f(x)|\,dy\\ &< \int_{\mathbb{T}-U}2\phi_n(y)|f|_\infty\,d\mu(y)+\int_U \phi_n(y)\frac{\epsilon}{2}\,d\mu(y)\\ &\le \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

∎

1.6 Convergence of the Fourier Series

We saw in a previous section that the Fourier transform converges in L^2 if f is an L^2-function. Also if f is smooth then the Fourier series converges uniformly to f. The proof actually shows that $f(z) = \sum \hat{f}(n) z^n$ uniformly if f is in $C^2(\mathbb{T})$. But in general it does not hold that the Fourier series $\sum a_n z^n$ converges to $f(z)$. In this section we will deal with the question how to recover the function from its Fourier series. We start by stating two negative results; however, first we have a few preliminaries.

Define the partial Fourier sum $s_N(f)$ to be

$$s_N(f) = \sum_{n=-N}^{N} \hat{f}(n) e_n \,.$$

Note $s_N(f) = f * D_N$ where $D_N = e_{-N} + e_{-N+1} + \cdots + e_{N-1} + e_N$. D_N is a trigonometric polynomial that is an idempotent under convolution. It is called the Dirichlet kernel.

Lemma 1.6.1. *Let $N \in \mathbb{N}$. Then the following hold:*

(a) $D_N(z) = \frac{z^{N+1/2} - z^{-(N+1/2)}}{z^{1/2} - z^{-1/2}}$ *if* $z \neq 1$ *and* $D_N(1) = 2N+1$.

(b) $D_N(e^{i\theta}) = \frac{\sin((N+1/2)\theta)}{\sin(\theta/2)}$ *if* $\theta \neq 0$ *and* $D_N(1) = 2N+1$

(c) $D_N(z) = D_N(z^{-1})$.

(d) $\int_{\mathbb{T}} D_N(z)\, d\mu(z) = 1$.

(e) $s_N(f)(z) = f * D_N(z) = D_N * f(z)$.

Proof. (a) We have

$$\begin{aligned} D_N(z) &= z^{-N} \sum_{n=0}^{2N} z^n \\ &= z^{-N} \frac{1 - z^{2N+1}}{1 - z} \\ &= \frac{z^{N+1/2} - z^{-(N+1/2)}}{z^{1/2} - z^{-1/2}} \,. \end{aligned}$$

Also $D_N(1) = \sum_{n=-N}^{N} 1 = 2N+1$.

(b) This follows immediately by using that $\sin(\psi) = (e^{i\psi} - e^{-i\psi})/2i$.

(c) This follows from (a).

(d) Using $\int_{\mathbb{T}} z^n \, d\mu = 0$ for $n \neq 0$ one has $\int_{\mathbb{T}} D_N(z) \, d\mu(z) = \sum_{n=-N}^{N} \int z^n \, d\mu(z) = \int 1 \, d\mu(z) = 1$.

(e) This follows by (f) of Lemma 1.4.2.

∎

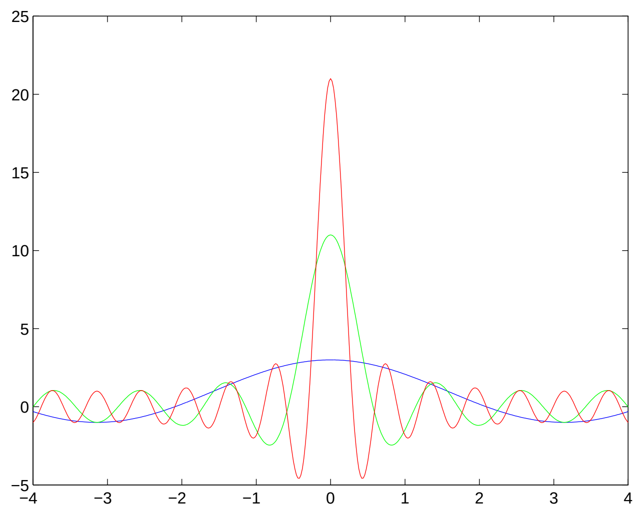

Figure 1.2: Dirichlet Kernels for $N = 1$(blue), 5(green), 10(red)

Figure 1.2 shows the functions D_N become more and more localized around $z = 1$ and then oscillates.

Lemma 1.6.2. *There exists a dense G_δ subset $D \subset L^1(\mathbb{T})$ such that the Fourier series does not converges in $L^1(\mathbb{T})$ for $f \in D$.*

Proof. Define linear transformations $\Lambda_N : L^1(\mathbb{T}) \to L^1(\mathbb{T})$ by $\Lambda_N f = D_N * f$. We note individually they are bounded for $||\Lambda_N|| \leq |D_N|_1$. They, however, are not uniformly bounded.

Indeed, we work on $L^2[-\pi, \pi]$. Set $f_k = \pi k \chi_{[-\frac{1}{k}, \frac{1}{k}]}$. Note each f_k has length 1 in $L^1(\mathbb{T})$. Moreover,

$$\hat{f}_k(n) = \frac{1}{2\pi} \int_{-\frac{1}{k}}^{\frac{1}{k}} k\pi e^{-in\theta} \, d\theta = -\frac{k}{2in} e^{-in\theta} |_{-\frac{1}{k}}^{\frac{1}{k}} = \frac{k}{2in} \left(e^{i\frac{n}{k}} - e^{-i\frac{n}{k}}\right) = \frac{k}{n} \sin(\frac{n}{k}).$$

Hence for fixed n, $\hat{f}_k(n) \to 1$ as $k \to \infty$. Thus $\Lambda_N(f_k) = f_k * D_N = \sum_{l=-N}^{N} \hat{f}_k(l) e_l \to \sum_{l=-N}^{N} e_l = D_N$ as $k \to \infty$.

Recall

$$D_N(e^{i\theta}) = \frac{\sin((N+\frac{1}{2})\theta)}{\sin\frac{\theta}{2}}.$$

Now $|D_N|_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{|\sin(N+1/2)\theta|}{\sin(\theta/2)}\,d\theta \to \infty$ as $N \to \infty$. Indeed,

$$\begin{aligned}
|D_N|_1 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{|\sin(N+\frac{1}{2})\theta|}{\sin(\frac{1}{2}\theta)}\,d\theta \\
&= \frac{1}{2\pi}\sum_{k=1}^{4N+2}\int_{(2N+1)^{-1}(k-1)\pi}^{(2N+1)^{-1}k\pi} \left|\frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\right| d\theta \\
&\geq \frac{1}{2\pi}\sum_{k=1}^{4N+2}\int_{(2N+1)^{-1}(k-1)\pi}^{(2N+1)^{-1}k\pi} \left|\frac{\sin(N+\frac{1}{2})\theta}{\frac{1}{2}\theta}\right| d\theta \\
&\geq \frac{1}{2\pi}\sum_{k=1}^{4N+2}\int_{(2N+1)^{-1}(k-1)\pi}^{(2N+1)^{-1}k\pi} \left|\frac{\sin(N+\frac{1}{2})\theta}{\frac{1}{2}(2N+1)^{-1}k\pi}\right| d\theta \\
&\geq \frac{2N+1}{\pi^2}\sum_{k=1}^{4N+2}\frac{1}{k}\int_{(2N+1)^{-1}(k-1)\pi}^{(2N+1)^{-1}k\pi} \left|\sin(N+\frac{1}{2})\theta\right| d\theta \ (\text{set } t = (N+\frac{1}{2})\theta) \\
&\geq \frac{2N+1}{\pi^2}\sum_{k=1}^{4N+2}\frac{1}{k}\int_{(k-1)\frac{\pi}{2}}^{k\frac{\pi}{2}} |\sin t|\,(N+\frac{1}{2})^{-1}\,dt \\
&= \frac{2}{\pi^2}\sum_{k=1}^{4N+2}\frac{1}{k}\int_0^{\frac{\pi}{2}} \sin t\,dt \\
&= \frac{2}{\pi^2}\sum_{k=1}^{4N+2}\frac{1}{k} \to \infty \text{ as } N \to \infty.
\end{aligned}$$

Hence the Λ_N are not uniformly bounded on the unit ball of $L^1(\mathbb{T})$. By the Banach–Steinhaus Theorem (i.e., the principle of uniform boundedness), there exists a dense G_δ set D such that

$$\sup_N \Big|\sum_{k=-N}^{N} \hat{f}(k) e_k\Big|_1 = \sup_N |\Lambda_N(f)|_1 = \infty \text{ for all } f \in D.$$

∎

Remark 1.6.3. We showed $|D_N|_1 \geq \frac{2}{\pi^2}\sum_{k=1}^{4N+2}\frac{1}{k}$. Thus $|D_N|_1 \geq \frac{2}{\pi^2}\ln(4N+3) \geq \frac{2}{\pi^2}\ln N$. Thus the D_N's are not bounded in $L^1(\mathbb{T})$. It is known that $|D_N|_1 = \frac{4}{\pi^2}\ln N + O(1)$. This is the central reason they do not form an approximate unit in $L^1(\mathbb{T})$.

The following shows convergence can be a problem even for continuous functions. We state it without proof.

Lemma 1.6.4. *Let* $\{z_k\}_{k=0}^{\infty}$ *be a sequence in* $\mathbb{T}$. *Then there exists a function* $f \in C(\mathbb{T})$ *such that* $\lim_{N\to\infty}|s_N(f)(z_k)| = \infty$ *for all* $k \in \mathbb{N}_0$.

Lemma 1.6.5 (Lebesgue Lemma). *Let* $g \in L^1[0,\pi]$. *Then*

$$\int_0^{\pi} g(x)\sin ax\,dx \to 0$$

as $a \to \infty$.

Sketch. This is similar to the argument for (d) in Lemma 1.4.2. First check it works for $g(x) = \chi_{[c,d]}$. Then show it works for step functions and then for any L^1 function. ∎

Theorem 1.6.6. *Suppose* $f(\theta) = F(e^{i\theta})$ *is a* 2π *periodic function on* $\mathbb{R}$ *and* f *is integrable on* $[-\pi,\pi]$. *Let* x *be a point where* $f(x+) = \lim_{\theta\to x+} f(\theta)$ *and* $f(x-) = \lim_{\theta\to x-} f(\theta)$ *exist. If there exist a* $K > 0$, *a* $\delta > 0$, *and an* $\alpha > 0$ *with* $|f(x+t) - f(x+)| \leq Kt^{\alpha}$ *and* $|f(x-t) - f(x-)| \leq Kt^{\alpha}$ *for* $0 < t < \delta$, *then* $D_N * F(e^{ix}) \to \frac{1}{2}(f(x+) + f(x-))$ *as* $N \to \infty$; *i.e., one has*

$$\sum_{k=-N}^{N}\hat{f}(k)e^{ikx} \to \frac{1}{2}(f(x+) + f(x-)).$$

Proof. Note

$$\begin{aligned} D_N * F(e^{ix}) &= \int D_N(z)F(z^{-1}e^{ix})\,d\mu(z) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(e^{it})F(e^{i(x-t)})\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(e^{it})f(x-t)\,dt. \end{aligned}$$

Hence $D_N * F(e^{ix}) = \frac{1}{2\pi}\int_0^\pi D_N(e^{it})f(x-t)\,dt + \frac{1}{2\pi}\int_{-\pi}^0 D_N(e^{-it})f(x-t)\,dt$ for $D_N(e^{-it}) = D_N(e^{it})$. Changing variables in the second integral gives

$$D_N * F(e^{ix}) = \frac{1}{2\pi}\int_0^\pi D_N(e^{it})(f(x+t)+f(x-t))\,dt.$$

Hence the result will follow if we show

$$\frac{1}{2\pi}\int_0^\pi D_N(e^{it})f(x+t)\,dt \to \frac{1}{2}f(x+)$$

and

$$\frac{1}{2\pi}\int_0^\pi D_N(e^{it})f(x-t)\,dt \to \frac{1}{2}f(x-)$$

as $N \to \infty$.

We show the first, for the second follows the same argument. Note $\frac{1}{2\pi}\int_0^\pi D_N(e^{it})\,dt = \frac{1}{2}\int D_N(z)\,d\mu(z) = \frac{1}{2}$. Hence

$$\begin{aligned}\frac{1}{2\pi}\int_0^\pi D_N(e^{it})f(x+t)\,dt - \frac{1}{2}f(x+) &= \frac{1}{2\pi}\int_0^\pi D_N(e^{it})(f(x+t)-f(x+))\,dt \\ &= \frac{1}{2\pi}\int_0^\pi \frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{1}{2}t)}(f(x+t)-f(x+))\,dt.\end{aligned}$$

By the Lebesgue Lemma, this will converge to 0 as $N \to \infty$ if the function

$$\psi(t) = \frac{f(x+t)-f(x+)}{\sin\frac{1}{2}t}$$

is integrable on $[0,\pi]$. Choose $\delta > 0$ so that $|f(x+t)-f(x+)| < Kt^\alpha$ and $\sin\frac{1}{2}t > \frac{1}{4}t$ for $0 < t < \delta$. Then

$$|\psi(t)| = \chi_{(0,\delta)}\frac{|f(x+t)-f(x+)|}{\sin\frac{1}{2}t} + \chi_{[\delta,\pi]}(t)\frac{|f(x+t)-f(x+)|}{\sin\frac{1}{2}t}.$$

The second of these two terms is clearly in $L^1[0,\pi]$, and the first is less than $\frac{4Kt^\alpha}{t}$ which is integrable on $[0,\pi]$ since $\alpha > 0$. ∎

Corollary 1.6.7. *Suppose $f(\theta)$ is periodic with period 2π and $f|_{[0,2\pi]} \in L^1[0,2\pi]$. If $f(x+)$ and $f(x-)$ exist and*

$$f'(x+) = \lim_{t\to 0+}\frac{f(x+t)-f(x+)}{t} \quad \textit{and} \quad f'(x-) = \lim_{t\to 0+}\frac{f(x-t)-f(x-)}{t}$$

exist, then

$$\lim_{N\to\infty}\sum_{k=-N}^{N}\hat{f}(k)e^{ikx}=\frac{1}{2}(f(x+)+f(x-)).$$

Proof. We can choose $\delta>0$ so that $\frac{|f(x\pm t)-f(x\pm)|}{t}<|f'(x\pm)|+1$ if $0<t<\delta$. Hence there is a K such that $|f(x\pm t)-f(x\pm)|<Kt$ for $0<t<\delta$. ∎

Definition 1.6.8. *A sequence $\{a_n\}_{n=0}^{\infty}$ is **Cesáro summable to L** if the average $1/(N+1)\sum_{k=0}^{N}s_k$ of the partial sums $\sum_{n=0}^{k}a_k$ converges to L. It is **Abel summable to L** if $\sum_{n=0}^{\infty}a_kr^k$ exists for all $0\le r<1$ and $\lim_{r\to 1-}\sum a_nr^n=L$.*

To recover $f\in L^1(\mathbb{T})$ from its Fourier transform one uses the average of partial sums, i.e., Cesáro summability. Define

$$\sigma_N(f)(z)=\frac{1}{N+1}\sum_{n=0}^{N}s_n(f)(z)\,.$$

Let

$$\Sigma_N(z)=\frac{1}{N+1}\sum_{n=0}^{N}D_N(z).$$

Then

$$\sigma_N(f)=f*\Sigma_N\,.$$

Lemma 1.6.9. *One has*

(a) $\Sigma_N(e^{i\theta})=\frac{1}{N+1}\left[\frac{\sin((N+1)\theta/2)}{\sin(\theta/2)}\right]^2$ *if* $e^{i\theta}\neq 1$ *and* $\Sigma_N(1)=N+1$. *In particular we have* $\Sigma_N\ge 0$.

(b) $\Sigma(z)=\Sigma(z^{-1})$.

(c) $\int_{\mathbb{T}}\Sigma_N\,d\mu=1$.

Proof. (a) Using $D_n(z) = (z^{n+1} - z^{-n})/(z-1)$, we see that

$$\begin{aligned}
\sum_{n=0}^{N} D_n(z) &= \frac{1}{z-1}\left[\frac{z^{N+2}-z}{z-1} - \frac{z^{-N-1}-1}{z^{-1}-1}\right] \\
&= \frac{1}{(z-1)^2}\left[z^{N+2} - z + z^{-N} - z\right] \\
&= \frac{z}{(z-1)^2}\left(z^{N+1} - 2 + z^{-N-1}\right) \\
&= \frac{1}{(z^{1/2} - z^{-1/2})^2}\left(z^{(N+1)/2} - z^{-(N+1)/2}\right)^2 \\
&= \left[\frac{z^{(N+1)/2} - z^{-(N+1)/2}}{z^{1/2} - z^{-1/2}}\right]^2 .
\end{aligned}$$

Hence if $z = e^{i\theta}$, one has

$$\Sigma_N(e^{i\theta}) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(e^{i\theta}) = \frac{1}{N+1}\left[\frac{\sin((N+1)\theta/2)}{\sin(\theta/2)}\right]^2 .$$

The claim for $z = 1$ follows either by continuity or by

$$\frac{1}{N+1}\sum_{n=0}^{N} 2n + 1 = N + 1 .$$

(b) This follows by the fact that sin is odd.

(c) By lemma 1.6.1 we have

$$\begin{aligned}
\int \Sigma_N(z)\, d\mu(z) &= \frac{1}{N+1}\sum_{n=0}^{N}\int D_n(z)\, d\mu(z) \\
&= \frac{1}{N+1}\sum_{n=0}^{N} 1 \\
&= 1
\end{aligned}$$

∎

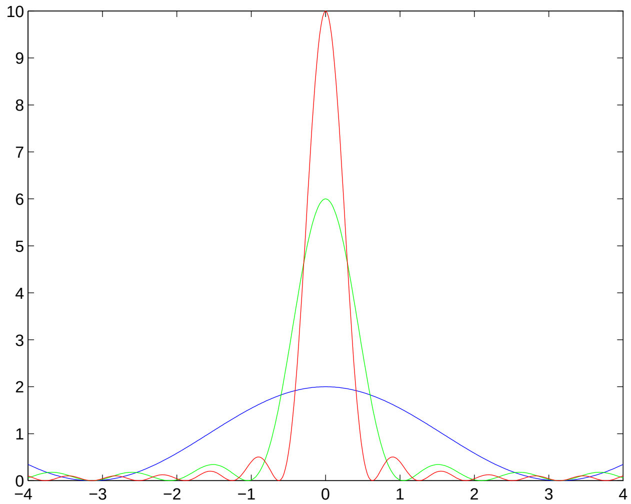

Figure 1.3: Fejer Kernels for $N = 1$(blue), 5(green), 9(red)

Note that the integral of Σ_N is concentrated more and more around $z = 1$ as $N \to \infty$. Figure 1.3 shows the graphs of the Fejer kernels Σ_N for $N = 1$, $N = 5$, and $N = 9$.

Lemma 1.6.10. *Let $0 < \delta < \pi$. Then there exists a constant $C = C(\delta)$ independent of N such that $\left|\Sigma_N(e^{i\theta})\right| \leq \frac{C}{N+1}$ for $\delta \leq |\theta| \leq \pi$.*

Proof. Choose $C > 0$ such that $|\sin(\theta/2)| \geq 1/\sqrt{C}$ for $\delta < |\theta| \leq \pi$. As $|\sin(N+1)\theta/2| \leq 1$ it follows that

$$\left|\Sigma_N(e^{i\theta})\right| \leq \frac{C}{N+1}\,.$$

∎

Theorem 1.6.11 (Fejér). *Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{T})$. Then $f * \Sigma_N$ is a trigonometric polynomial and*

$$\lim_{N\to\infty} |f * \Sigma_N - f|_p = 0\,.$$

If $f \in C(\mathbb{T})$, then

$$\lim_{N\to\infty} |f * \Sigma_N - f|_\infty = 0\,.$$

In particular the sequence $a_0 := \hat{f}(0)$, $a_n := \hat{f}(n)z^n + \hat{f}(-n)z^{-n}$ is uniformly Cesáro summable to $f(z)$.

Proof. By Lemmas 1.6.9 and 1.6.10, the Fejer kernels Σ_N form an approximate unit in $L^1(\mathbb{T})$. Hence the statements follow from Proposition 1.5.3. ∎

1.7 The Poisson kernel

In Section 1.1, we showed that by separation of variables using polar coordinates on the unit disc $|z| \leq 1$, Laplace's equations $\Delta u = 0$ produced solutions of form

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}.$$

We note if $r = 1$, the resulting function would be a Fourier series and should represent the boundary condition $u(r, 1) = f(\theta)$. Hence we would hope

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta} = f(\theta)$$

in some sense, i.e., pointwise, uniformly, in L^2, etc.

Associated with this decomposition is the function $P(r, \theta)$ where we take all the $a_n's$ in the function $u(r, \theta)$ equal to one. As can be seen in the next chapter, $P(1, \theta)$ is the 'Fourier series' of the Dirac function δ and conceivably $P(r, \theta)$ is close in some sense to δ for r near one. The function $P(r, \theta)$ is called the Poisson kernel. Hence

$$P(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

This series converges uniformly on any subset S of $[0, 1] \times \mathbf{R}$ for which $\sup\{r : (r, \theta) \in S\} < 1$.

As seen in Section 1.1,

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r \cos\theta + r^2}.$$

Define $P_r(e^{i\theta}) = P(r, e^{i\theta})$ for $0 \leq r < 1$.

Lemma 1.7.1. *P_r satisfy the following conditions.*

(a) $P_r(z) \geq 0$ for all $z \in \mathbb{T}$.

(b) $\int P_r(z)\,d\mu(z) = 1$ *for* $0 \leq r < 1$.

(c) *if* U *is a neighborhood of* 1 *in* $\mathbb{T}$, *then* $\sup_{z\notin U} |P_r(z)| \to 0$ *as* $r \to 1-$.

Proof. Clearly we have (a) and since $\sum r^{|n|} e_n(e^{i\theta})$ converges uniformly on $\mathbb{T}$, $\int P_r(z)\,d\mu(z) = \sum r^{|n|} \int e_n(z)\,d\mu(z) = r^0 \int e_0(z)\,dz = 1$. Thus (b) holds.

For (c), choose $a > 0$ with $1 - \cos\theta < \frac{1}{2}$ if $|\theta| < a$. Then for any δ with $0 < \delta < a$, one has

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} \leq \frac{1-r^2}{1-r+r^2}$$

if $\delta \leq |\theta| < \pi$. Since $\frac{1-r^2}{1-r+r^2} \to 0$ as $r \to 1-$, (c) follows. ∎

This lemma shows $P_r(z)$ form an 'approximate unit' in $L^1(\mathbb{T})$ and the argument in the proof of Proposition 1.5.3 shows the following are true:

(1) if $f \in L^p(\mathbb{T})$ where $1 \leq p < \infty$, $P_r * f \to f$ in $L^p(\mathbb{T})$ as $r \to 1-$.

(2) if $f \in C(\mathbb{T})$, then $P_r * f \to f$ uniformly on $\mathbb{T}$ as $r \to 1-$.

Theorem 1.7.2 (Poisson Theorem).

(a) *Let* $f \in L^1(\mathbb{T})$, *then the function*

$$u(r,\theta) = \frac{1}{2\pi}\int P(r,\theta-\phi) f(e^{i\theta})\,d\theta$$

is harmonic on the open disk $|z| < 1$.

(b) *If* $f \in L^p(\mathbb{T})$ *where* $1 \leq p < \infty$, *then*

$$u(r,\theta) \to f(e^{i\theta}) \text{ in } L^p(\mathbb{T}) \text{ as } r \to 1-.$$

(c) *If* $f \in C(\mathbb{T})$, *then*

$$u(r,\theta) \to f(e^{i\theta}) \text{ uniformly on } \mathbb{T} \text{ as } r \to 1-.$$

Proof. We note we already have (b) and (c). For (a), we need only note since $u(r,\theta) = \sum \hat{f}(n) r^{|n|} e^{in\theta}$, that $|\hat{f}(n)| \leq |f|_1$ for all n; and thus both the series and the series for the r and θ derivatives of any order converge uniformly on any disk $|r| < a$ where $a < 1$. ∎

Exercise Set 4:

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1. Show that the sequence $a_n = (-1)^n$ is Abel and Cesáro summable to $1/2$.

2. Show that if $\sum a_n$ converges to L, then $\{a_n\}$ is Abel and Cesáro summable to L.

3. Let $\delta_0 \geq \delta_1 \geq \cdots \geq 0$ be a decreasing sequence with $\lim \delta_n = 0$. Define $a_0 := \delta_0$ and $a_n := \delta_{n-1} - \delta_n$ for $n \geq 1$. Let $g(z) := \sum a_n z^n$. Show that g is continuous, $g * D_N$ converges uniformly to g, and $|g * D_N - g|_\infty \leq \delta_N$.

4. Let f be the function on $\mathbb{T}$ such that $f \circ \kappa|_{[-\pi,\pi)}(t) = t$.

(a) Evaluate $\hat{f}(n)$.

(b) Evaluate $\lim_{t\to\pi} \sum \hat{f}(n)e^{int}$.

1.8 Applications

In this section we discuss some applications of the Fourier transform. The first two are examples illustrating how one uses the Fourier transform to solve differential equations, and the last is an example of its application to geometry.

1.8.1 The Wave Equation

The general form of the wave equation is

$$\partial_x^2 u(x,t) = a^2 \partial_t^2 u(x,t)\,, \quad u(x,0) = f(x)\,,\ u_t(x.0) = g(x)\,. \tag{1.8.1.1}$$

Here $x \in [0,L]$, f, g are functions on $[0,L]$, and $a > 0$ is a constant. For simplicity we will assume that $L = \pi$. In the case where $u(0,t) = u(\pi,t) = 0$ for all t and f and g are real valued, this equation describes the vibration of a homogeneous string, fastened at both ends and starting at position $u(x,0) = f(x)$ with initial velocity $u_t(x,t) = g(x)$. The constant $a = T/\rho$ is given by the tension T and the linear density ρ.

As we are motivated by the vibration of a string, let us assume that $f, g \in C^2((0,\pi))$ and $f(0) = f(\pi) = g(0) = g(\pi) = 0$. Let us look for a smooth solution $x \mapsto u(x,t)$. First we extend $u(\cdot,t)$, f and g to an odd function on $[-\pi,\pi]$ by $f(-x) = -f(x)$, $g(-x) = -g(x)$, and $u(-x,t) = -u(x,t)$. Then f and g are continuous on $[-\pi,\pi]$ and smooth on $(-\pi,\pi)$. Taking the Fourier transform in the x-variable and denoting it by $\hat{u}(n,t)$, and using that all functions are real valued and odd, one has:

$$\begin{aligned}
\hat{u}(n,t) &= \overline{\hat{u}(-n,t)} = \frac{i}{2\pi}\int_{-\pi}^{\pi} u(\theta,t)\sin(n\theta)\,d\theta\,;\\
\hat{f}(n) &= \overline{\hat{f}(-n)} = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta\,;\\
\hat{g}(n) &= \overline{\hat{g}(-n)} = \frac{i}{2\pi}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta\,.
\end{aligned}$$

Next we notice that differentiation in the t-variable commutes with taking the Fourier transform in the x-variable. Thus for all $n \in \mathbb{Z}$:

$$-n^2\,\hat{u}(n,t) = a^2\,\hat{u}_{tt}(n,t)\,, \quad \hat{u}(n,0) = \hat{f}(n), \quad \hat{u}_t(n,0) = \hat{g}(n)\,.$$

This is an ordinary second order initial value problem for the function $t \mapsto \hat{u}(n,t)$ with unique solution

$$\hat{u}(n,t) = \hat{f}(n)\cos\left(\frac{n}{a}t\right) + a\hat{g}(n)\frac{\sin\left(\frac{n}{a}t\right)}{n}\,.$$

Summing up we conclude

$$u(x,t) = 2i\sum_{n=1}^{\infty}\left(\hat{f}(n)\cos\left(\frac{n}{a}t\right) + a\hat{g}(n)\frac{\sin\left(\frac{n}{a}t\right)}{n}\right)\sin(nx)\ . \qquad (1.8.1.2)$$

This solution also fulfills $u(0,t) = u(\pi,t) = 0$ for all $t \in \mathbb{R}$. We notice that this solution is periodic in t with period $2\pi a$.

If $L \neq \pi$, we replace $x \mapsto u(x,t)$ by $v(x,t) := u\left(\frac{L}{\pi}x\right)$ and similarly for f and g. Then the new Cauchy problem is:

$$\partial_x^2 v(x,t) = a_1^2 u_{tt}(x,t)\,, \quad v(x,0) = f_1(x),\ v_t(x,0) = g_1(x)$$

with $f_1(x) = f(Lx/\pi)$, $g_1(x) = g(Lx/\pi)$ and $a_1 = \pi a/L$. Using the above result for v we have:

$$u(x,t) = 2i\sum_{n=1}^{\infty}\left(\hat{f}(n)\cos\left(\frac{Ln}{a\pi}t\right) + \frac{\pi a}{L}\hat{g}(n)\frac{\sin\left(\frac{Ln}{a\pi}t\right)}{n}\right)\sin\left(\frac{\pi}{L}nx\right)\ .$$

1.8.2 The Heat Equation

In this section we discuss the heat equation

$$a^2\partial_x^2 u(x,t) = u_t(x,t)\,, \quad u(x,0) = f(x)\,,\; 0 \le x \le L\,.$$

In the case where f is real valued, this is the differential equation describing the heat flow in a homogeneous cylindrical rod of length L, whose lateral surface is insulated from the surrounding medium and where the initial temperature at the point $x \in [0, L]$ is $f(x)$. The constant a in this case is given by $a^2 = K/c\rho$ where K is the *thermal conductivity* of the material from which the rod is made, c is the *heat capacity*, and ρ is the *density*. We will only consider solutions that are fixed by the same constant at the endpoints $x = 0$ and $x = L$. We can then assume that $u(0,t) = u(L,t) = 0$ for all t. As for the wave equation, we will assume that $L = \pi$ and will accordingly extend all functions depending on the variable x to odd functions on the interval $[-\pi, \pi]$. Taking the Fourier transform in the x-variable we obtain:

$$-a^2n^2\hat{u}(n,t) = \hat{u}_t(n,t)\,, \quad \hat{u}(n,0) = \hat{f}(n)\,.$$

Thus $\hat{u}(n,t) = \hat{f}(n)e^{-a^2n^2t}$. Now using that f and $u(\cdot,t)$ are odd and real valued we have:

$$u(x,t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx-a^2n^2t} = 2i\sum_{n=1}^{\infty} \hat{f}(n)e^{-n^2a^2t}\sin(nx)\,.$$

Similarly for general $L > 0$, one obtains

$$u(x,t) = 2i\sum_{n=1}^{\infty} \hat{f}(n)e^{-n^2\pi^2a^2t/L^2}\sin\left(\frac{n\pi}{L}x\right)$$

where

$$\hat{f}(n) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{n\pi x}{L}}\,dx.$$

1.8.3 The Isoperimetric Problem

Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be a continuous smooth simple and closed curve. Then $C := \gamma([0, 2\pi])$ defines a bounded domain $\Omega(\gamma) \subset \mathbb{C}$. Assume for simplicity that the length of C is one. Question: For which curve is the area $a(\Omega(\gamma))$ is maximum?

Theorem 1.8.1 (Hurwitz). *We have $a(\Omega(\gamma)) \leq \frac{1}{4\pi}$ and $a(\Omega(\gamma)) = \frac{1}{4\pi}$ if and only if $\Omega(\gamma)$ is a circle.*

Proof. Let $\Omega = \Omega(\gamma)$. Set $G(e^{i\theta}) = \gamma(\theta) = x(\theta) + iy(\theta)$. Then G is a smooth function on the torus. Hence

$$G(z) = \sum_{n=-\infty}^{\infty} \widehat{G}(n) z^n$$

and if $z = e^{i\theta}$,

$$\gamma'(\theta) = DG(z) = i \sum_{n=-\infty}^{\infty} n\widehat{G}(n) z^n .$$

We may assume the curve is parameterized such that

$$|\gamma'(\theta)|^2 = x'(\theta)^2 + y'(\theta)^2 = \frac{1}{2\pi} .$$

Then, by the Plancherel formula

$$\begin{aligned} \frac{1}{4\pi^2} &= |DG|_2^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} |\gamma'(\theta)|^2 \, dt \\ &= \sum_{n=-\infty}^{\infty} n^2 |\widehat{G}(n)|^2 . \end{aligned}$$

Or

$$\pi \sum_{n=-\infty}^{\infty} n^2 \left| \widehat{G}(n) \right|^2 = \frac{1}{4\pi} .$$

Now applying Stoke's Theorem to the boundary of Ω and using $xx'+yy'=0$ and the Plancherel Theorem, one obtains

$$\begin{aligned}
a(D(\gamma)) &= \frac{1}{2}\int\int_{\Omega} d(x\,dy - y\,dx) \\
&= \frac{1}{2}\int_{\partial\Omega} x dy - y dx \\
&= \frac{1}{2}\int_0^{2\pi} x(\theta)y'(\theta) - y(\theta)x'(\theta)\,d\theta \\
&= \frac{-1}{2i}\int_0^{2\pi} \gamma(\theta)\overline{\gamma'(\theta)}\,d\theta \\
&= \pi \sum_{n=-\infty}^{\infty} n\widehat{G}(n)\overline{\widehat{G}(n)} \\
&= \pi \sum_{n=-\infty}^{\infty} n|\widehat{G}(n)|^2 .
\end{aligned}$$

It follows now that

$$\frac{1}{4\pi} - a(D(\gamma)) = \pi \sum_{n=-\infty}^{\infty} n(n-1)|\widehat{G}(n)|^2 .$$

Next note that $n(n-1) > 0$ for all $n \neq 0, 1$. Hence

$$\frac{1}{4\pi} - a(\Omega) = \pi \sum_{n=-\infty}^{\infty} n(n-1)|\widehat{G}(n)|^2 \geq 0$$

and

$$\frac{1}{4\pi} - a(\Omega) = 0$$

if and only if $\widehat{G}(n) = 0$ for $n \neq 0$ or $n \neq 1$. But in that case

$$G(e^{i\theta}) = \widehat{G}(0) + \widehat{G}(1)e^{-i\theta}$$

which is a circle. ∎