1[6P]) Which of the following functions \( y(t) \) is a solution (S)/not a solution (N) to the differential equation \( y' + 2ty = t \)?

<table>
<thead>
<tr>
<th>( y(t) )</th>
<th>Y</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( e^t + 1/2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2e^{-t^2} + 1/2 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2[9P]) Determine and mark with Y for yes, and N for no, if each of the following differential equation is separable (S), linear (L), and/or homogeneous (H). Note, that in each case, more than one might be correct.

<table>
<thead>
<tr>
<th>Equation</th>
<th>S</th>
<th>L</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t^2y' = y^2 + ty )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y' - t^2y = t^3y )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (y^2 + 2tyy' = ty )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3) Solve the following differential equation:

a[10P]) \( y' + 2ty = 0 \). Solution \( y(t) = \) __________
b[10P]) \ y' + y = ty^2. \ \textbf{Solution} \ y(t) = \underline{\text{__________}}

5) Solve the following initial value problems.

a[15P]) \ y' - \frac{1}{2}ty = t^3e^t, \ y(0) = 3. \ \textbf{Solution} \ y(t) = \underline{\text{__________}}

b[15P]) \ t^2y' = -y^2, \ y(1) = 1. \ \textbf{Solution} \ y(t) = \underline{\text{__________}}
3) A tank contains 100 gal of brine made by dissolving 40 lb of salt in water. A brine solution containing 10 grams salt per liter flows into the container at a rate of 4 liters per minute. The well-stirred mixture runs out at the same rate. Denote by \( y(t) \) the amount of salt in the tank at time \( t \).

(a[8P]) Write an initial value problem that \( y(t) \) must satisfy.

Solution: 

(b[10P]) Solve the initial value problem. Solution: \( y(t) = \) 

(c[5P]) How much salt is in the tank after 10 min? Solution: 

4[12P]) Apply Picard's to compute the approximations \( y_0(t) \), \( y_1(t) \), and \( y_2(t) \) to the solution of the initial value problem \( y' = (y + 1)^2 \), \( y(0) = 0 \).
1[6P]) Which of the following functions $y(t)$ is a solution (S)/not a solution (N) to the differential equation $y' = y - t$?

<table>
<thead>
<tr>
<th>$y(t)$</th>
<th>Y</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t + 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e^t + 1 + t$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2e^t - t - 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2[9P]) Determine and mark with Y for yes, and N for no, if each of the following differential equation is separable (S), linear (L), and/or homogeneous (H). Note, that in each case, more than one might be correct.

<table>
<thead>
<tr>
<th>Equation</th>
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<th>L</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y' = \frac{y - t}{y + t}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y' - ty = t^3y$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(y^2 + 2t^2) + 2tyy' = 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3) Solve the following differential equation:

a[10P]) $y' = -2yt$. Solution $y(t) =$
b[10P]) \ y' - y = ty^2. \ Solution \ y(t) = \underline{\hspace{1cm}} \\

5) Solve the following initial value problems.

a[15P]) \ y' - \frac{3}{t} y = t^3 e^t, \ y(1) = 3. \ Solution \ y(t) = \underline{\hspace{1cm}} \\

b[15P]) \ t^2 y' = -y^2 + yt, \ y(1) = 1. \ Solution \ y(t) = \underline{\hspace{1cm}}
3) A tank contains 100 gal of brine made by dissolving 40 lb of salt in water. Pure water runs into the tank at the rate of 4 gal/min, and the mixture, which is kept uniform by stirring, runs out at the same rate. Denote by $y(t)$ the amount of salt in the tank at time $t$.

(a) Write an initial value problem that $y(t)$ must satisfy.

Solution: ________________

(b) Solve the initial value problem. Solution: $y(t) = ___________

(c) How much salt is in the tank after 10 min? Solution: __________

4) Apply Picard's to compute the approximations $y_0(t)$, $y_1(t)$, and $y_2(t)$ to the solution of the initial value problem $y' = y^2 + 1$, $y(0) = 0$. 
MATH 2065-4, Fall 2010

Test 1, Thursday, Sept 23, 2010
. For partial credit, show all your work!

1[6P]) Which of the following functions \( y(t) \) is a solution (S)/not a solution (N) to the differential equation \( y' = y - t \)?

<table>
<thead>
<tr>
<th>( y(t) )</th>
<th>Y</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t+1 )</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>( e^{t+1} )</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>( 2e^t-t-1 )</td>
<td>✓</td>
<td></td>
</tr>
</tbody>
</table>

\( y = t+1, \ y' = 1, \ y - t = t+1 - t = 1 \neq y' \)

\( y = e^{t+1}, \ y' = e^{t+1}, \ y - t = e^{t+1} \)

\( y = 2e^t-t-1, \ y' = 2e^t-1, \ y - t = 2e^t-2t-1 \neq y' \)

2[9P]) Determine and mark with Y for yes, and N for no, if each of the following differential equation is separable (S), linear (L), and/or homogeneous (H). Note, that in each case, more than one might be correct.

<table>
<thead>
<tr>
<th>Equation</th>
<th>S</th>
<th>L</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y' = \frac{y-t}{y+t} )</td>
<td></td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>( y' = ty - t^2y )</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (y^2 + 2t^2) + 2tyy' = 0 )</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( y' = \frac{y-t}{y+t} = (\frac{y}{y+t}) - 1 \neq \frac{y-1}{y+1} \neq \frac{V-1}{V+1} \) H.

\( y' = t^3y + ty = (t^3 + t)y \) separable

\( y - (t + t^3)y = 0 \), linear

3) Solve the following differential equation:

a[10P]) \( y' = -2yt \). Solution \( y(t) = Ce^{-t^2} \)

you can use that this is separable and also linear \( y' + 2ty = 0 \).

separable

\[ \frac{dy}{y} = -2t \, dt, \quad \ln |y| = -t^2 - C \]

\[ y = Ce^{-t^2} \]
b[10P])\ y'-y=ty^2. Solution y(t) = \frac{-t+1+Ce^{-t}}{y}

Bernoulli equation with n = 2, y^{-2}y' - \frac{1}{y} = b.

z = \frac{1}{y}, z' = -\frac{1}{y^2}y'

z' + z = -t, p = 1, P(t) = t, \mu(t) = e^t

\int e^t \, dt = te^t + e^t + C, Z = \frac{1}{\mu} (te^t + e^t + C) = t + 1 + Ce^{-t}

y = \frac{1}{Z} = \frac{1}{-t + 1 + Ce^{-t}}

5) Solve the following initial value problems.

a[15P])\ y' - \frac{3}{2}y = t^3e^t, y(1) = 3. Solution y(t) = \frac{t^3(e^t + 2 - e)}{e^{3ln(t)} = t^3}

\text{Linear: } \mu = e^{-3ln(t)} = t^{-3}

\int e^t \, dt = e^t + C

y(t) = t^3(e^t + C)

y(1) = e + C = 3, C = 3 - e

b[15P])\ t^2y' = -y^2 + yt, y(1) = 1. Solution y(t) = \frac{t}{\ln(t) + 1}

This is a homogeneous equation

V' = -\left(\frac{v}{t}\right)^2 + \frac{v}{t} = -V^2 + V \quad \text{if } V = \frac{y}{t}

tv' + v = -V^2 + V \quad \text{or} \quad tv' = -V^2

- \frac{dv}{v^2} = \frac{dr}{r}, \quad \frac{1}{V} = \ln(t) \quad (t > 0)

v = \frac{1}{\ln(t) + C}, \quad y = tv = \frac{t}{\ln(t) + C}

y(1) = 1 = \frac{1}{r} \quad \text{so} \quad C = 1
3) A tank contains 100 gal of brine made by dissolving 40 lb of salt in water. Pure water runs into the tank at the rate of 4 gal/min, and the mixture, which is kept uniform by stirring, runs out at the same rate. Denote by \( y(t) \) the amount of salt in the tank at time \( t \).

a[8P]) Write an initial value problem that \( y(t) \) must satisfy.

Solution: \( \frac{dy}{dt} = -\frac{1}{25} y \) and \( y(0) = 40 \)

\( \frac{dy}{dt} \) = input - output. There is no salt in the input so, output \( \frac{4}{100} y(t) \)

\( y' = -\frac{1}{25} y \). It is given, that at time \( t=0 \) the amount of salt is \( 40 \) lb.

b[10P]) Solve the initial value problem. Solution: \( y(t) = \frac{40}{e^{-\frac{t}{25}}} \)

\( \frac{dy}{y} = -\frac{dt}{25} \), \( \ln y = -\frac{t}{25} + C \) \( y > 0 \)

\( y = Ce^{-\frac{t}{25}} \). Taking \( t=0 \), \( y(0) = 40 = C \)

\[ c[5P]) \text{How much salt is in the tank after 10 min? Solution: } \frac{40}{e^{-\frac{10}{25}}} \approx 26.8 \]

\( y(10) = 40e^{-\frac{10}{25}} = 40 e^{-\frac{2}{5}} \approx 26.8 \) lb

4[12P]) Apply Picard's to compute the approximations \( y_0(t) \), \( y_1(t) \), and \( y_2(t) \) to the solution of the initial value problem \( y' = y^2 + 1 \), \( y(0) = 0 \).

\( y_0 = 0 \)

\( y_1 = 0 + \int_0^t u^2 + 1 \, du = \int_0^t du = t \)

\( y_2 = 0 + \int_0^t u + 1 \, du = \frac{1}{3} t^3 + t \).
1[24P]) Compute the Laplace transform of each of the following functions:

a) \( \mathcal{L}(t^2e^{-2t})(s) = \)

b) \( \mathcal{L}(t\sin(4t) + e^{-3t}\cos(t))(s) = \)

c) \( \mathcal{L}((2te^t + 1)(t^3 + 2))(s) = \)

2[24P]) Find the partial fraction decomposition of each of the following rational functions. You may use the convolution formula if you prefer.

a) \( \frac{7s + 9}{(s - 1)(s + 3)} = \)
b) \[ \frac{1}{(s-1)(s^2+1)} = \]

c) \[ \frac{1}{s(s-1)} = \]

3[16P]] Evaluate the following convolutions. You may use the Laplace transform if you prefer.

a) \[ t^2 * t = \]

b) \[ t * e^{-4t} = \]

3[36P]] Compute the inverse Laplace transform for each of the following functions:

a) \[ \mathcal{L}^{-1} \left( \frac{5}{s^2 + 6s + 9} \right) = \]
b) \( \mathcal{L}^{-1} \left( \frac{2}{(s - 1)(s^2 + 1)} \right) = \)

c) \( \mathcal{L}^{-1} \left( \frac{s}{s^2 + 4s + 5} \right) (t) = \)

d) \( \mathcal{L}^{-1} \left( \frac{s + 1}{((s + 1)^2 + 1)^2} \right) = \)
A short table of Laplace transforms and inverse Laplace transform

\[
\mathcal{L}(af(t) + bg(t))(s) = aF(s) + bG(s)
\]
\[
\mathcal{L}(e^{at} f(t))(s) = F(s - a)
\]
\[
\mathcal{L}(-t f(t))(s) = \frac{d}{ds} F(s)
\]
\[
\mathcal{L}(1)(s) = \frac{1}{2}
\]
\[
\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}
\]
\[
\mathcal{L}(e^{at})(s) = \frac{1}{s - a}
\]
\[
\mathcal{L}(\cos(bt))(s) = \frac{s}{s^2 + b^2}
\]
\[
\mathcal{L}(\sin(bt))(s) = \frac{b}{s^2 + b^2}
\]
\[
\mathcal{L}(\sin(bt))(s) = \frac{b}{s^2 + b^2}
\]
\[
\mathcal{L}(f'(t))(s) = sF(s) - f(0)
\]
\[
\mathcal{L}(f * g(t))(s) = F(s)G(s)
\]
\[
\mathcal{L}^{-1}\left(\frac{1}{(s^2 + 1)^2}\right)(t) = \frac{1}{2}(\sin(t) - t \cos(t))
\]
\[
\mathcal{L}^{-1}\left(\frac{s}{(s^2 + 1)^2}\right)(t) = \frac{1}{2}t \sin(t)
\]
1[24P]) Compute the Laplace transform of each of the following functions:

a) \( \mathcal{L}(t^3e^{-2t})(s) = \frac{6}{(s+2)^4} \)

b) \( \mathcal{L}(t\sin(4t) + e^{-3t}\cos(t))(s) = \frac{8s}{(s^2+16)^2} + \frac{s+3}{(s+3)^2+1} \)

c) \( \mathcal{L}((2te^t+1)(t^3+2))(s) = \frac{48}{(s-1)^5} + \frac{6}{s^4} + \frac{4}{(s-1)^2} + \frac{2}{s} \)

2[24P]) Find the partial fraction decomposition of each of the following rational functions. You may use the convolution formula if you prefer.

a) \( \frac{7s + 9}{(s-1)(s+3)} = \frac{4}{s-1} + \frac{3}{s+3} \)

\[
\frac{7s + 9}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3} \quad \text{Then}
\]

\[(A + B)s + 3A - B = 7s + 9 \quad \text{or}
\]

\[
A + B = 7 \\
3A - B = 9
\]

\[
4A = 16 \quad \text{or} \quad A = 16/4 = 4
\]

\[
B = 7 - A = 7 - 4 = 3
\]
b) \( \frac{1}{(s-1)(s^2+1)} = \frac{1}{2} \left[ \frac{1}{s-1} - \frac{s+1}{s^2+1} \right] \)

while
\[
\frac{1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+1}{s^2+1}.
\]
Then
\[
\begin{cases}
A + B = 0 \\
-B + C = 0 \\
A + B = 1
\end{cases}
\]
Thus \( B = C \) and \( A + B = 0 \) or \( A = 1/2 \), \( B = C = -1/2 \).

c) \( \frac{1}{s(s-1)} = -\frac{1}{S} + \frac{1}{s-1} \)

3[16P]) Evaluate the following convolutions. You may use the Laplace transform if you prefer.

a) \( t^2 \ast t = \frac{1}{12} t^4 \).

Use the Laplace transform: \( \mathcal{L}(t^2 \ast t) = \frac{2}{s^3} \cdot \frac{1}{s^2} = \frac{2}{s^5} \)

\[ \frac{1}{12} \cdot \frac{1}{s^5} \]

b) \( t \ast e^{-4t} = \frac{1}{4} t - \frac{1}{16} + \frac{1}{16} e^{-4t} \)

\[
\int_0^t u e^{-4(t-u)} du = e^{-4t} \left[ \int_0^t u e^{-4u} du \right] = e^{-4t} \left[ \left. \frac{1}{4} u e^{-4u} \right|_0^t - \frac{1}{4} \int_0^t e^{-4u} du \right] = \frac{1}{4} e^{-4t} \left[ \frac{1}{4} t e^{-4t} - \frac{1}{16} e^{-4t} \right] = \frac{1}{4} t - \frac{1}{16} + \frac{1}{16} e^{-4t} \]

3[36P]) Compute the inverse Laplace transform for each of the following functions:

a) \( \mathcal{L}^{-1} \left( \frac{5}{s^2 + 6s + 9} \right) = 5t e^{-3t} \)
b) \( \mathcal{L}^{-1} \left( \frac{2}{(s-1)(s^2+1)} \right) = e^t - \cos t - \frac{1}{2} \sin t \)

\[
\frac{2}{(s-1)(s^2+1)} = -\frac{s+1}{s^2+1} + \frac{1}{s-1} = -\frac{s}{s^2+1} + \frac{1}{s-1} + \frac{1}{s-1}
\]

(use problem #2-b)

c) \( \mathcal{L}^{-1} \left( \frac{s}{s^2+4s+5} \right) = e^{-2t} \left[ \cos t - \sin t \right] \)

\[
\frac{s}{s^2+4s+5} = \frac{s}{(s+2)^2+1} = \frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}
\]

d) \( \mathcal{L}^{-1} \left( \frac{s+1}{((s+1)^2+1)^2} \right) = \frac{t e^{-t} \sin t}{2} \)

(use the last formula in the table)
A short table of Laplace transforms and inverse Laplace transform

\[ \mathcal{L}(af(t) + bg(t))(s) = aF(s) + bG(s) \]
\[ \mathcal{L}(e^{at}f(t))(s) = F(s - a) \]
\[ \mathcal{L}(-tf(t))(s) = \frac{d}{ds}F(s) \]
\[ \mathcal{L}(1)(s) = \frac{1}{2} \]
\[ \mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}} \]
\[ \mathcal{L}(e^{at})(s) = \frac{1}{s - a} \]
\[ \mathcal{L}(\cos(bt))(s) = \frac{s}{s^2 + b^2} \]
\[ \mathcal{L}(\sin(bt))(s) = \frac{b}{s^2 + b^2} \]
\[ \mathcal{L}(f'(t))(s) = sF(s) - f(0) \]
\[ \mathcal{L}(f * g(t))(s) = F(s)G(s) \]
\[ \mathcal{L}^{-1}\left(\frac{1}{(s^2 + 1)^2}\right)(t) = \frac{1}{2}(\sin(t) - t\cos(t)) \]
\[ \mathcal{L}^{-1}\left(\frac{s}{(s^2 + 1)^2}\right)(t) = \frac{1}{2}t\sin(t) \]
1[12P]) For each of the following differential equations determine if it is linear (Y=linear) or not (N=not):

<table>
<thead>
<tr>
<th>( L(y) )</th>
<th>Y</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y'' + y' + y = \sin(t) )</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>( y' + y y = t )</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>( y'' + \cos(y) = \cos(t) )</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>( y'' + y' + \cos(t) = y )</td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>

2[6P]) Show that \( y(t) = t^{1/2} \) is a solution to the differential equation \( 4t^2y'' + y = 0 \).

\[
\begin{align*}
    y &= t^{1/2} \\
    y' &= \frac{1}{2} t^{-1/2} \\
    y'' &= -\frac{1}{4} t^{-3/2}
\end{align*}
\]

\[4t^2y'' + y = 0 \]

3[7]) For the linear operator \( L = D^2 + 4D + 4 \) determine \( L(te^{-t}) = \)

\[
\begin{align*}
    y &= te^{-t} \\
    y' &= -te^{-t} + e^{-t} \\
    y'' &= te^{-t} - 2e^{-t}
\end{align*}
\]

\[
\begin{align*}
    L(te^{-t}) &= te^{-t} - 2e^{-t} + 4te^{-t} + 4e^{-t} \\
    &= te^{-t} + 2e^{-t}
\end{align*}
\]

4[45P]) Find the general solution to each of the following differential equations:

a) \( y'' - 3y' + 2y = e^t \). Solution \( y(t) = c_1 e^t + c_2 e^{2t} - te^t \)

\[
\begin{align*}
    s^2 - 3s + 2 &= (s - 1)(s - 2) \\
    y_n &= c_1 e^t + c_2 e^{2t} \\
    y_p &\leftrightarrow \frac{1}{(s-1)^2(s-2)} = \frac{-1}{(s-1)^2} + \frac{1}{(s-1)(s-2)} \\
    y_p &= -te^t
\end{align*}
\]

b) \( y'' - 4y' + 4y = e^t \). Solution \( y(t) = c_1 e^{2t} + c_2 te^t + e^t \)

\[
\begin{align*}
    s^2 - 4s + 4 &= (s - 2)^2 \\
    y_p &= A e^t, \quad y_p' = A e^t, \quad y_p'' = A e^t
\end{align*}
\]
c) \( y'' - Ay' + 5y = 1 \). Solution \( y(t) = e^{2t} (c_1 \cos(t) + c_2 \sin(t)) + \frac{1}{5} \)

\[ s^2 - 4s + 5 = (s - 2)^2 + 1 \]

5[15P]) Find the general solution to the differential equation \( y'' + y' - 2y = te^t \).

Solution: \( y(t) = \)

\[ s^2 + s - 2 = (s + 2)(s - 1) \]

\[ y_h(t) = c_1 e^{-2t} + c_2 e^t \]

\[ y_p(t): \text{ Write } y_p = c_1 (t) e^{-2t} + c_2 (t) e^t. \text{ Then} \]

\[ c_1 e^{-2t} + c_2 e^t = 0 \]

\[ -2c_1 e^{-2t} + c_2 e^t = te^t \]

\[ 3c_1 e^t - te^t \text{ on } c_2 = \frac{1}{3} t \]

\[ c_2 = \frac{1}{2} \int t e^t dt = -\frac{1}{6} t^2 \]

\[ 3c_1 e^{-2t} = -te^t \]

\[ c_1 = -\frac{1}{3} t e^t \]

\[ y_p = -\frac{1}{4} t e^t + \frac{1}{27} e^t \]

\[ y = c_1 e^{-2t} + c_2 e^t + \left(\frac{1}{6} t^2 - \frac{1}{9} t + \frac{1}{27}\right) e^t \]

6[15P]) Solve the initial value problem \( t^2 y'' + ty' - 4y = 0, y(1) = 1, y'(1) = 1 \).

Solution: \( y(t) = \frac{1}{4} (3t^2 + t^{-2}) \)

Euler equation: \( y'' - 4y = 0, y(t) = c_1 e^t + c_2 e^{-2t} \)

\[ y(1) = c_1 t^2 + c_2 t^{-2} \]

\[ \begin{align*}
  c_1 + c_2 &= 1 \\
  2c_1 - 2c_2 &= 1
\end{align*} \]
1[10P]) Let $L = D^2 + 3tD + 2$. What is $L(e^t + t) =$

2[48P]) Find the general solution to each of the following differential equation:

a) $y' = -2ty$. Solution: $y(t) =$

b) $y' = \frac{3y - t}{2ty}$. Solution: $y(t) =$

b) $y'' + y' - 6y = e^{2t}$. Solution: $y(t) =$

e) $y'' - 6y' + 10y = 0$. Solution: $y(t) =$
3[45P]) Solve each of the following initial value problems.

a) $y' - \frac{2}{t}y = t^2 \cos(t)$, $y(\pi) = 3$. Solution: $y(t) =$

b) $y'' + y' - 2y = 1$, $y(0) = 0$, $y'(0) = 1$. Solution $y(t) =$

c) $y' - y = \begin{cases} 
1 & \text{if } 0 \leq t < 2 \\
-1 & \text{if } 1 \leq t < \infty
\end{cases}$ $y(0) = 0$. Solution: $y(t) =$

4[8P]) Compute the Laplace transform $\mathcal{L}(t \cos(t))(s) =$

5[16P]) Compute the inverse Laplace transform for each of the following functions:
a) \( \mathcal{L}^{-1} \left( \frac{7s + 9}{s^2 + 2s - 3} \right)(t) = \)

b) \( \mathcal{L}^{-1} \left( \frac{s}{(s - 1)(s^2 + 1)} \right)(t) = \)

5[8P]) Evaluate the convolution \( t \ast e^t = \)

6[15P]) A tank contains 100 gal of brine made by dissolving 80 lb of salt in water. Pure water runs into the tank at the rate of 2 gal/min, and the mixture, which is kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time \( t \).
1[10P]) Let $L = D^2 + 4tD + 1$. Show that $\sin(t)$ is a solution to the differential equation $L(y) = 4t \cos(t)$.

2[48P]) Find the general solution to each of the following differential equation:

a) $y' = -2 \cos(t)y$. Solution: $y(t) =$

b) $y' = \frac{3y^2 - t^2}{2ty}$. Solution: $y(t) =$

b) $y'' - 3y' + 2y = e^t$. Solution: $y(t) =$
e) $y'' - 6y' + 10y = 0$. Solution: $y(t) =$

3(P45)

Solve each of the following initial value problems.

a) $y' - \frac{3}{t}y = t^3e^t$, $y(1) = 3$. Solution: $y(t) =$

b) $y'' + y' - 2y = 1$, $y(0) = 0$, $y'(0) = 1$. Solution $y(t) =$

c) $y' - y = \begin{cases} 1 & \text{if} \ 0 \leq t < 2 \\ -1 & \text{if} \ 1 \leq t < \infty \end{cases}$, $y(0) = 0$. Solution: $y(t) =$
4[8P]) Compute the Laplace transform $\mathcal{L}(t\cos(t))(s) =$

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5[8P]) Evaluate the convolution $t \ast e^t =$

6)[15P]) A tank contains 100 gal of brine made by dissolving 80 lb of salt in water. Pure water runs into the tank at the rate of 4 gal/min, and the mixture, which is kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time $t$. 