Math 7370, Exercises # 1. Fall 2013

We will discuss the solution to those problems in class on Tuesday, Sept 23. Please turn in the solution to **TWO** of the problems in class on Thursday, Sept 19.

1) Let $\mathrm{SL}(n,\mathbb{R}) = \{g \in \mathrm{GL}(n,\mathbb{R}) \mid \det g = 1\}$. Show that $\mathrm{SL}(n,\mathbb{R})$ is a Lie group and its Lie algebra is given by

$$\mathfrak{sl}(n,\mathbb{R}) = \{X \in \mathrm{M}(n,\mathbb{R}) \mid \mathrm{Tr}X = 0\}\,.$$

2) Show that any matrix in $\mathrm{SL}(2,\mathbb{R})$ is conjugate to a multiple of a matrix of the following form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(The first give the elliptic orbits, the second the hyperbolic orbits and the third the nilpotent orbits.) Use that to show that the image of $\exp : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ is the set of matrices

$$\{a \in \mathrm{SL}(2,\mathbb{R}) \mid \mathrm{Tr}a > -2\} \cup \{-\mathrm{I}\}\,.$$

3) Classify all Lie algebras of dimensions 1, 2 and 3.

4) Let $\mathfrak{s}_n$ be the space of symmetric $n \times n$-matrices and let P_n^+ denote the space of positive definite $n \times n$-matrices. Thus $a \in P_n^+$ if and only if for all $u \in \mathbb{R}^n$, $u \neq 0$, $(a(u), u) > 0$ (note that it follows that a is symmetric). Show that $\exp : \mathfrak{s}_n \to P_n^+$ is a diffeomorphsim.

5) Let $G \subset \mathrm{GL}(n,\mathbb{R})$ be a closed subgroup with Lie algebra $\mathfrak{g}$. The center $Z(G)$ of G is

$$Z(G) = \{a \in G \mid (\forall b \in G)\, ab = ba\}\,.$$

The center of the Lie algebra $\mathfrak{z}(\mathfrak{g})$ is given by

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid (\forall Y \in \mathfrak{g})\, [X, Y] = \{0\}\}\,.$$

(1) Assume that G is connected. Show that $Z(G)$ is a closed subgroup with Lie algebra $\mathfrak{z}(\mathfrak{g})$.
(2) Find $\mathfrak{z}(\mathfrak{gl}(n,\mathbb{R})$ and $Z(\mathrm{GL}(n,\mathbb{R})$.
(3) Show that by an example that the statement in (1) is not necessarily correct if G is not connected.