We will discuss the solution to those problems in class on Tuesday, Sept 23. Please turn in the solution to \textbf{TWO} of the problems in class on Thursday, Sept 19.

1) Let $\text{SL}(n, \mathbb{R}) = \{ g \in \text{GL}(n, \mathbb{R}) \mid \det g = 1 \}$. Show that $\text{SL}(n, \mathbb{R})$ is a Lie group and its Lie algebra is given by
\[ \mathfrak{sl}(n, \mathbb{R}) = \{ X \in \text{M}(n, \mathbb{R}) \mid \text{Tr}X = 0 \}. \]

2) Show that any matrix in $\text{SL}(2, \mathbb{R})$ is conjugate to a multiple of a matrix of the following form
\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\] (The first give the elliptic orbits, the second the hyperbolic orbits and the third the nilpotent orbits.) Use that to show that the image of $\exp : \mathfrak{sl}(2, \mathbb{R}) \to \text{SL}(2, \mathbb{R})$ is the set of matrices
\[ \{ a \in \text{SL}(2, \mathbb{R}) \mid \text{Tr}a > -2 \} \cup \{-1\}. \]

3) Classify all Lie algebras of dimensions 1, 2 and 3.

4) Let $\mathfrak{g}_n$ be the space of symmetric $n \times n$-matrices and let $P^n_+ \subset \mathbb{R}^n$ denote the space of positive definite $n \times n$-matrices. Thus $a \in P^n_+$ if and only if for all $u \in \mathbb{R}^n$, $u \neq 0$, $(a(u), u) > 0$ (note that it follows that $a$ is symmetric). Show that $\exp : \mathfrak{g}_n \to P^n_+$ is a diffeomorphism.

5) Let $G \subset \text{GL}(n, \mathbb{R})$ be a closed subgroup with Lie algebra $\mathfrak{g}$. The center $Z(G)$ of $G$ is
\[ Z(G) = \{ a \in G \mid (\forall b \in G) ab = ba \}. \]

The center of the Lie algebra $\mathfrak{z}(\mathfrak{g})$ is given by
\[ \mathfrak{z}(\mathfrak{g}) = \{ X \in \mathfrak{g} \mid (\forall Y \in \mathfrak{g}) [X, Y] = \{0\} \}. \]

(1) Assume that $G$ is connected. Show that $Z(G)$ is a closed subgroup with Lie algebra $\mathfrak{z}(\mathfrak{g})$.

(2) Find $\mathfrak{z}(\mathfrak{gl}(n, \mathbb{R}))$ and $Z(\text{GL}(n, \mathbb{R}))$.

(3) Show that by an example that the statement in (1) is not necessarily correct if $G$ is not connected.