Math 7370, Exercises #2. Due in class October 10

1) Let $H \subset \mathrm{GL}(n, \mathbb{R})$ a closed subgroup. Let $G = H \rtimes \mathbb{R}^n$ where the product is defined by the inclusion in the group of bijections on $\mathbb{R}^n$ by

$$(h, v)(x) = h(x) + v\,.$$

a) Work out the product and the inverse in G.
b) Identify $\mathbb{R}^n$ with the affine subspace of $\mathbb{R}^{n+1}$

$$\mathbb{R}^n \simeq \{(x, 1)^T \mid x \in \mathbb{R}^n\}\,.$$

Show that G is isomorphic to the subgroup of $\mathrm{GL}(n+1, \mathbb{R})$ given by

$$\left\{ \begin{pmatrix} h & v \\ 0 & 1 \end{pmatrix} \,\middle|\, h \in H \text{ and } v \in \mathbb{R}^n \right\}.$$

2) Let $\mathfrak{g}$ be a Lie algebra (ower $\mathbb{R}$ or $\mathbb{C}$) Define

$$\mathcal{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] := \{[X, Y] \mid X, Y \in \mathfrak{g}\}\,.$$

a) Show that $\mathcal{D}\mathfrak{g}$ is an ideal in $\mathfrak{g}$ and $\mathfrak{g}/\mathcal{D}\mathfrak{g}$ is abelian.
b) Show that if $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{a}$ is abelian, then $\mathcal{D} \subseteq \mathfrak{a}$.
c) Let $\mathfrak{h}_n$ be the Heisenberg Lie algebra

$$\mathfrak{h}_n = \left\{ \begin{pmatrix} 0 & x & t \\ 0_n^T & 0_{n,n} & y^T \\ 0 & 0_n & 0 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R}^n,\ t \in \mathbb{R} \right\}.$$

Here 0_n stands for the zero vector in $\mathbb{R}^n$ and $0_{n,n}$ is the zero $n \times n$ matrix.
Find $\mathcal{D}\mathfrak{h}_n$.
d) Show that $\mathcal{D}\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{R})$ and $\mathcal{D}\mathfrak{gl}(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{R})$.

3) Let G_j, $j \in J$ be a family of linear Lie groups. Denote the Lie algebra of G_j by $\mathfrak{g}_j$. Show that $G := \bigcap_{j \in J}$ is a linear Lie group and that the Lie algebra of G is $\bigcap_{j \in J} \mathfrak{g}_j$.