Print Your Name Here: ______________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your name at the top of the *scratch* sheet *if it is to be graded*. No books or notes are allowed. A scientific calculator is OK—but not needed. The maximum total score is 100.

Part I: Short Questions. Answer 8 of the following 12 questions for 6 points each. **Circle** the numbers of the 8 questions listed below that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. True or False: If $f(x) = \cos x$ for all $x \in \mathbb{E}^1$ then $f(\mathbb{E}^1)$ an open set in $\mathbb{E}^1$?

$f(\mathbb{R}) = [-1,1]$ closed

2. Suppose $\mathbf{f} \in C^1(\mathbb{E}^2, \mathbb{E}^2)$ such that the matrix $[\mathbf{f}'(\mathbf{x})] = \begin{pmatrix} 1-\|x\| & -1 \\ 1 & 1+\|x\| \end{pmatrix}$. True or False: $\mathbf{f}(B_1(\mathbf{0}))$ is an *open* set in $\mathbb{E}^2$.

$\det\begin{pmatrix} 1-\|x\| & -1 \\ 1 & 1+\|x\| \end{pmatrix} = 1-\|x\|^2+1 = 2-\|x\|^2 > 0$

for $x \in B_1(0)$. See Theorem 10.4.2

3. The Magnification Theorem fails to guarantee local injectivity of $f : \mathbb{E}^1 \to \mathbb{E}^1$ defined by $f(x) = \begin{cases} 2x + 4x^2 \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ at $x = 0$. Which hypothesis of the Magnification Theorem is violated by this function?

$x \neq 0$, then $f'(x) = 2 + 8x\sin\frac{1}{x} - 4\cos\frac{1}{x}$, $f'(0) = 2$ so f' is not continuous at $x=0$.

4. Find all points at which $\dfrac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = 0$, if $\mathbf{f}(\mathbf{x}) = (x_1^2 - x_2^2, x_1^2 + x_2^2)$, $\mathbf{f} \in C^1(\mathbb{E}^2, \mathbb{E}^2)$.

$\det\begin{pmatrix} 2x_1 & 2x_1 \\ -2x_2 & 2x_2 \end{pmatrix} = 8x_1x_2 = 0 \iff x_1x_2 = 0.$

5. Let $f \in C^1(\mathbb{E}^4, \mathbb{E}^2)$ be such that the matrix $[\mathbf{f}'(\mathbf{x})]_{2\times 4} = \begin{bmatrix} x_2 & x_1 & x_3+x_4 & 1 \\ -x_1 & -1 & 1 & x_3-x_4 \end{bmatrix}$. Find the set of all points $\mathbf{x}_0 = (x_1, x_2, x_3, x_4)$ at which the implicit function theorem guarantees that there is a *local* C^1 solution at $\mathbf{x}_0$ to the equation $f(\mathbf{x}) = f(\mathbf{x}_0)$ for x_2 and x_3 in terms of the other two variables.

Use Thm 10.5.1 with $y_1 = x_2$ and $y_2 = x_3$:

$\det\begin{pmatrix} x_1 & x_3+x_4 \\ -1 & 1 \end{pmatrix} = x_1 + x_3 + x_4$. So the condition is $x_1 + x_3 + x_4 \neq 0$

6. True or False: If $D \subset \mathbb{E}^n$ and $S \subset \mathbb{E}^4$ are both open and if $\mathbf{f} : D \to S$ is a diffeomorphism of D onto S, then $n > 4$.

$n = 4$, see exercise 10.59

7. If $\mathbf{f} \in C^1(\mathbb{E}^3, \mathbb{E}^2)$ and $[\mathbf{f}'(\mathbf{0})] = \begin{pmatrix} 1 & 2 & 4 \\ 5 & 3 & 1 \end{pmatrix}$ express the matrix $[((a,b)\cdot \mathbf{f})'(\mathbf{0})]$ in terms of a and b.

$D((a,b)f)(0) = (a,b)\begin{pmatrix} 1 & 2 & 4 \\ 5 & 3 & 1 \end{pmatrix} = (a+5b, 2a+3b, 4a+b)$

8. Suppose $F \in C^1(\mathbb{E}^n, \mathbb{E}^1)$ such that for all $i = 1, \ldots, n$ we have $\frac{\partial F}{\partial x_i} \neq 0$. If $F(x_1, \ldots, x_n) = 0$, find the numerical value of the product $\prod_{i=1,\ldots,n} \frac{\partial x_i}{\partial x_{i+1}}$, where x_{n+1} means x_1.

$\underline{(-1)^n}$

We have

$f(x_1(x_2, \ldots, x_n), x_2, \ldots, x_n) = 0$

$f(x_1, x_2(x_1, x_3, \ldots), x_3, \ldots, x_n) = 0$

$\vdots$

$f(x_1, \ldots, x_n(x_1, \ldots, x_{n-1})) = 0$

Differentiate the first w.r.t. x_2, the next w.r.t. $x_3, \ldots,$ and the last one w.r.t. x_1. Then we get

see extra page

9. True or False: If $\mathbf{f} : D \to \mathbb{E}^n$ where D is open and connected in $\mathbb{E}^m$, and if $\|\mathbf{f}'(\mathbf{x})\| = 0$ for all $\mathbf{x} \in D$, then f is *uniformly continuous* on D.

True, because f is constant

10. Let $\mathbf{f} : \mathbb{E}^2 \to \mathbb{E}^2$ by $\mathbf{f}(\mathbf{x}) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2)$. Describe an open set $U \subseteq \mathbb{E}^2$ on which f is injective and such that $f(U) = \mathbb{E}^2 \setminus \{0\}$.

11. Suppose that $\mathbf{f} = (f_1, f_2) \in C^1(\mathbb{E}^3, \mathbb{E}^2)$ and $[\mathbf{f}'(\mathbf{x}_0)] = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \end{pmatrix}$. If $\mathbf{f}(x_1, x_2(x_1), x_3(x_1)) = 0$ with x_2 and x_3 being C^1 functions of x_1 in a neighborhood of $\mathbf{x}_0 = (a, b, c)$, find $x_2'(a)$ and $x_3'(a)$. (Hint: Apply the Chain Rule to differentiate $\mathbf{f}(x_1, x_2(x_1), x_3(x_1)) = \mathbf{0}$ with respect to x_1 on both sides.)

Differentiating $f(x_1, x_2(x_1), x_3(x_1)) = 0$ w.r.t. x_1 gives

$\begin{pmatrix} 2 + x_2' + 3x_3' \\ -1 + x_2' + 2x_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $x_2'(a) = 7$, $x_3'(a) = -3$.

12. Let $\mathbf{g} : \mathbb{E}^2 \to \mathbb{E}^2$ such that $\mathbf{g}(\mathbf{x}) = (2x_1 + 3x_2, 4x_1 + x_2)$. Let $K = g(B_1(\mathbf{0}))$ and find $\int_K 1\, dx$.

use Thm 11.5.1: $\int_K 1\, dx = \int_{g^{-1}(K)} 1(g(x)) \,|\det Dg(x)|\, dx$

$= 10 \int_{B(0)} dx = 10\pi.$

Part II: Proofs. *Prove carefully* 2 of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more then 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Suppose that $T \in \mathcal{L}(\mathbb{E}^m, \mathbb{E}^p)$ and that $\mathbf{f} : \mathbb{E}^n \to \mathbb{E}^m$ is differentiable.

(i) (16) Prove that $T \circ \mathbf{f}$ is differentiable at each $\mathbf{x} \in \mathbb{E}^n$ and that $(T \circ \mathbf{f})'(\mathbf{x}) \equiv T \circ \mathbf{f}'(\mathbf{x})$.

(ii) (10) If $\mathbf{a} \in \mathbb{E}^m$ is a constant vector, *use part* (i) to prove that $(\mathbf{a} \cdot \mathbf{f})'(\mathbf{x})$ exists and equals $\mathbf{a} \cdot \mathbf{f}'(\mathbf{x})$.

B. Let $\mathbf{f} : \mathbb{E}^2 \to \mathbb{E}^2$ by $\mathbf{f}(\mathbf{x}) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2)$.

(i) (20) Show that f is locally injective at each point $\mathbf{x} \in \mathbb{E}^2$. Be sure to cite the theorem you use and *show* that each hypothesis is satisfied.

(ii) (6) Show that $\mathbf{f}$ is not injective on $\mathbb{E}^2$.

C. The matrix $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ corresponds to a vector $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{E}^4$. Call X *orthogonal* if $XX^t = I$, the identity matrix, X^t being the transpose of X.

(i) (10) Prove that X is orthogonal if and only if $\mathbf{f}(\mathbf{x}) = 0 \in \mathbb{E}^3$, where $\mathbf{f}(\mathbf{x}) = (x_1x_3 + x_2x_4, x_1^2 + x_2^2 - 1, x_3^2 + x_4^2 - 1)$.

(ii) (16) Use the *Implicit Function Theorem carefully* to show that $\mathbf{f}(\mathbf{x}) = 0$ from part (i) has a solution for *three* of the four coordinates of $\mathbf{x}$ as C^1 functions of the remaining coordinate, in a neighborhood of $\mathbf{x} = (1, 0, 0, 1)$.

#8 cont.

the linear system

$$\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial x_2}+\frac{\partial f}{\partial x_2}=0$$

$$\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial x_3}+\frac{\partial f}{\partial x_3}=0$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial x_1}+\frac{\partial f}{\partial x_1}=0$$

we can write this as

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \frac{\partial x_n}{\partial x_1} \\ \frac{\partial x_1}{\partial x_2} & 1 & 0 & \cdots & 0 \\ 0 & \frac{\partial x_2}{\partial x_3} & 1 & 0 \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & & 0 \\ & & & \frac{\partial x_{n-1}}{\partial x_n} & 1 \end{pmatrix} \nabla f(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

As $\nabla f(\vec{x}) \neq \vec{0}$ it follows that

$$0 = \det \begin{pmatrix} 1 & \cdots & \frac{\partial x_n}{\partial x_1} \\ \frac{\partial x_1}{\partial x_2} & \ddots & \\ & \ddots & 1 \end{pmatrix} = 1 \cdot \det \begin{pmatrix} 1 & & \\ \frac{\partial x_2}{\partial x_3} & \ddots & 0 \\ & \ddots & \\ & \frac{\partial x_{n-1}}{\partial x_n} & 1 \end{pmatrix}$$

$$+ (-1)^{n+1} \det \begin{pmatrix} \frac{\partial x_1}{\partial x_2} & 1 & 0 & \\ 0 & \frac{\partial x_2}{\partial x_3} & 1 & \\ \vdots & & \ddots & 1 \\ & & & \frac{\partial x_{n-1}}{\partial x_n} \end{pmatrix} \frac{\partial x_n}{\partial x_1}$$

$$= 1 + (-1)^{n+1} \prod \frac{\partial x_i}{\partial x_{i+1}}. \text{ Thus } \prod \frac{\partial x_i}{\partial x_{i+1}} = (-1)^{n+2} = (-1)^n.$$

A. (a) You can do this in two different way

a-1) $T \in \mathcal{L}(\mathbb{E}^m, \mathbb{E}^p) \Rightarrow T$ is differentiable and $DT(x) = T$.
Hence, by the chain rule $T \circ f$ is differentiable and
$D(T\circ f)(x) = DT(f(x))\, Df(x) = T\, Df(x)$.

a-2) Directly: Write
$f(x+h) - f(x) = Df(x)h + \varepsilon(h)$. Then
$T\circ f(x+h) - T\circ f(x) = T\, Df(x) + T(\varepsilon(h))$.
The map $T \circ Df(x)$ is linear and

$$\frac{\|T(\varepsilon(h))\|}{\|h\|} \leq \|T\| \frac{\|\varepsilon(h)\|}{\|h\|} \longrightarrow 0 \text{ as } \|h\| \to 0.$$

(b) The map $L(x) = \vec{a}\cdot\vec{x}$ is linear $\mathbb{E}^m \longrightarrow \mathbb{E}^1$.
By (a) we get
$D(L\circ f)(x) = L \circ Df(x) = \vec{a}\, Df(x)$ where we think of $\vec{a}$ as a $1\times m$ matrix.

B) i) $f(\vec{x}) = e^{x_1}(\cos x_2, \sin x_2)$

$$Df(x) = \begin{pmatrix} e^{x_1}\cos x_2 & -e^{x_1}\sin x_2 \\ e^{x_1}\sin x_2 & e^{x_1}\cos x_2 \end{pmatrix}$$

$\det Df(x) = e^{2x_1} > 0$. It follows from Theorem 10.4.1 (The Magnification Theorem) or the inverse function theorem (Thm 10.4.3) that f is locally injective.

ii) $f(x_1, 2\pi n) = f(x_1, 0)$ for all $n \in \mathbb{Z}$.

c) i) writing out what XX^t is we get

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} = \begin{pmatrix} x_1^2+x_2^2 & x_1x_3+x_2x_4 \\ x_1x_3+x_2x_4 & x_3^2+x_4^2 \end{pmatrix}$$

Thus $XX^t = I \iff$

$$\begin{pmatrix} x_1^2+x_2^2 & x_1x_3+x_2x_4 \\ x_1x_3+x_2x_4 & x_3^2+x_4^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} x_1^2+x_2^2-1 & x_1x_3+x_2x_4 \\ x_1x_3+x_2x_4 & x_3^2+x_4^2-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

ii) $f(\vec{x}) = (x_1x_3+x_2x_4,\, x_1^2+x_2^2-1,\, x_3^2+x_4^2-1)$

$$Df(1,0,0,1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

As $\det\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \neq 0$ we can either solve for x_2 as function of x_1, x_3, x_4 or we can solve for x_3 as a function of x_1, x_2 and x_4.