## Print Your Name Here:

Show all work in the space provided. Indicate clearly if you continue on the back. Write your name at the top of the scratch sheet if it is to be graded. No books or notes are allowed. A scientific calculator is OK-but not needed. The maximum total score is 200.

Part I: Short Questions. Answer 12 of the following 18 questions for 8 points each. Circle the numbers of the 12 questions listed below that you want counted-no more than 12! Detailed explanations are not required, but they may help with partial credit and are risk-free! Maximum score: 96 points. All matrices on this test refer to the standard basis for Euclidean space E<sup>n</sup>.

1. Give an example of a sequence of open sets  $O_n \subset \mathbb{E}^p$  such that  $\bigcap_{n \in \mathbb{N}} O_n$  is not open.

2. True or calse: Every subset of  $\mathbb{E}^n$  can be expressed as a union of closed sets.

3. True or Give a Counterexample: If a sequence  $x^j \in \mathbb{E}^n$  is bounded then  $\{x^j \mid j \in \mathbb{N}\}$  is a compact

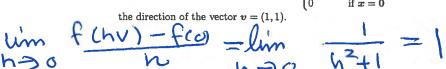
4. True on False: The graph  $G_f = \{(x, f(x)) \mid x \in [0, 1]\}$  is a connected subset of  $\mathbb{E}^2$  if and only if f is

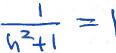
$$f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

5. Let  $f: \mathbb{E}^2 \to \mathbb{E}^1$  be defined by  $f(x) = \begin{cases} \frac{x_1 x_2^2}{x_1^4 + x_2^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$  True or False: f is continuous at 0.

$$f(r\cos\theta_3 r\sin\theta) = r \frac{\cos\theta \sin^2\theta}{v^2 \cos^4\theta + \sin^2\theta} \rightarrow 0$$

6. If  $f: \mathbb{E}^2 \to \mathbb{E}^1$  is given by  $f(x) = \begin{cases} \frac{x_1^2 x_2}{x_1^4 + x_2^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$  find the directional derivative  $D_v f(0,0)$ , in





7. True or False: If  $f \in \mathcal{C}(\mathbb{E}^n, \mathbb{E}^m)$  then  $\{x \mid f(x) = 0\}$  is a closed set.

8. Find a bijection  $f:[0,1) \xrightarrow{\text{onto}} S^1$ , the unit circle, with  $f \in C[0,1)$  but  $f^{-1} \notin C[S^1)$ .

$$f(t) = (\cos(\alpha \pi t), \sin(\alpha \pi t))$$

f(b) = (-1, -1), then there exists a point  $c \in \mathbb{E}^2$  such that f(c) = 0.

9. True of False: If  $f \in \mathcal{C}(\mathbb{E}^2, \mathbb{E}^2)$  such that there are a and  $b \in \mathbb{E}^2$  with f(a) = (1,1) and

f(x,y) = ex(cos(2Try), sin(2Try))

10. True or False: The set  $\mathcal{GL}(n,\mathbb{R})=\{T\in\mathcal{L}(\mathbb{E}^n)\mid \det T\neq 0\}$  is an open subset of the normed vector det is combinuous, so det (R. Lo3) is apen 11. True or false: If  $f: \mathbb{E}^n \to \mathbb{E}^m$  is such that  $\frac{\partial f_i}{\partial x_j}$  exists for all i, j, then f is differentiable have to exist and be continuous Del 12. Suppose  $f:[0,1]\times \mathbb{R} \to \mathbb{E}^2$  by  $f(x)=(c^{x_1}\cos x_2,\,c^{x_2}\sin x_2)$  True or False: For all a and  $b\in \mathbb{E}^2$  we have  $\|f(b)-f(a)\|\leq e\|b-a\|$ . True because sup (x,y) = 10 f (x,y) 11 = 17 < 00 Use Thm. 10.3.2-13. Which hypothesis of the Magnification Theorem fails to be satisfied at x=0 by  $f:\mathbb{E}^1\to\mathbb{E}^1$  defined by  $f(x) = \begin{cases} 2x + 4x^2 \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$  $\mathcal{P}_{l}$ hat continuous 14. Let  $f \in \mathcal{C}^1(\mathbb{E}^4, \mathbb{E}^1)$  be such that the matrix  $[f'(x)]_{1\times 4} = [x_2^2 - 1, x_1x_3, x_4^2 - 1, x_2x_3]$ . Find all  $x_0 = (x_1, x_2, x_3, x_4)$  at which the implicit function theorem guarantees that there is a local  $\mathcal{C}^1$  solution of the equation  $f(x) = f(x_0)$  for  $x_1$  in terms of the other three variables 15. Suppose that  $\mathbf{f} = (f_1, f_2) \in \mathcal{C}^1\left(\mathbb{E}^3, \mathbb{E}^2\right)$  and  $[\mathbf{f}'(\mathbf{x}_0)] = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \end{pmatrix}$ . If  $\mathbf{f}(x_1(x_2), x_2, x_3(x_2)) = 0$ with  $x_1$  and  $x_3$  being  $C^1$  functions of  $x_2$  in a neighborhood of  $x_0 = (a, b, c)$ , find  $x_1'(b)$  and  $x_3'(b)$ . (Hint: Apply the Chain Rule to differentiate  $f(x_1(x_2), x_2, x_3(x_2)) = 0$  with respect to  $x_2$  on both sides erentiate  $f(x_1(x_2), x_2, x_3(x_2)) = 0$  with respect to  $x_2$  on both sides  $2\frac{\partial x_1}{\partial x_2} + 3\frac{\partial x_2}{\partial x_2} = -1$   $-\frac{\partial x_1}{\partial x_2} + 2\frac{\partial x_3}{\partial x_2} = 1$   $\frac{\partial x_1}{\partial x_2} + \frac{\partial x_3}{\partial x_2} = -\frac{3}{7}$ We get the suplem 16. Let  $f: \mathbb{E}^2 \to \mathbb{E}^2$  by  $f(x) = (x_1 \cos x_2, x_1 \sin x_2)$ . Find the matrix  $[(f^{-1})'(0,1))]$  of the local inverse at (0,1) in the standard basis. Hint:  $f(1,\frac{\pi}{2}) = (0,1)$ . Use the Inverse Function Theorem. We always have  $f(x,x) = \int f(x,x) dx = \int f(x,x) dx = \int f(x,x) dx$ . De(1, 1/2) = (0-1) so De(0,1) = (01)

17. Let  $A \in \mathcal{L}(\mathbb{E}^4)$  such that AA' = 2I, where  $A = A^i$ , the transpose of the matrix of A. Find all possible values of  $\det A$ .

(detA)=24=16 so detA=±4

By the change of variable formula:

Soldx = Sldet Aldx =  $5 \cdot 2^4 = 86$ ACB)

Part II: Proofs. Prove carefully 4 of the following 6 theorems for 26 points each. Circle the letters of the 4 proofs to be counted in the list below—no more than 2! You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.

- A. Define the closure  $\bar{S}$  of  $S \subseteq \mathbb{E}^n$  to be the intersection of all closed sets that contain S. Explain why  $\bar{S}$  is a closed set, and prove that  $\bar{S} = S \cup C$ , where C is the set of all cluster points of S.
- B. Let f be a real-valued function defined on a closed finite interval [a,b] in  $\mathbb{E}^1$ . Define the graph  $G_f = \{(x_1,x_2) \in \mathbb{E}^2 \mid x_2 = f(x_1), x_1 \in [a,b]\}.$ 
  - (i) Prove: If  $f \in \mathcal{C}[a,b]$ , then  $G_f$  is a compact subset of  $\mathbb{E}^2$ . (Hint: Use the Heine-Borel Theorem.)
  - (ii) Let  $g(x_1) = \begin{cases} \sin \frac{\pi}{x_1} & \text{if } x_1 \in \{0, 1\}, \\ 0 & \text{if } x_1 = 0. \end{cases}$  Is the graph  $G_q$  a compact subset of  $\mathbb{E}^2$ ? Prove your conclusion.
- C. A subset  $D \subseteq \mathbb{E}^n$  is called *convex* if and only if for each pair of points a and  $b \in D$  the straight-line segment  $S(a,b) = \{\phi(t) \mid \phi(t) = a + t(b-a), \ t \in [0,1]\}$  is contained in D. Prove that every convex subset of  $\mathbb{E}^n$  is connected.
- $\text{D. Define } f: \mathbb{E}^2 \to \mathbb{E}^1 \text{ by } f(\boldsymbol{x}) = \begin{cases} \frac{x_1 \cdot x_2}{x_1^2 + x_2^2} & \text{if } \boldsymbol{x} \in \mathbb{E}^2 \setminus \{0\}, \\ 0 & \text{if } \boldsymbol{x} = 0. \end{cases}$ 
  - (i) Prove:  $\frac{\partial f}{\partial x_1}$  and  $\frac{\partial f}{\partial x_2}$  exist at every point  $x \in \mathbb{E}^2$ .
  - (ii) Prove: f is not differentiable at 0.
- E. Let  $SL(2,\mathbb{R})$  denote the set of all two-by-two real matrices  $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$  for which det X = 1. If  $X \in SL(2,\mathbb{R})$ , show by applying the Implicit Function Theorem that there are at least two of the four coordinates  $x_1, x_2, x_3, x_4$ , either of which can be expressed locally as a continuously differentiable function of the three other coordinates.
- F. Define  $\mathcal{O}(2) = \{X \in \mathcal{L}\left(\mathbb{R}^2\right) \mid XX^t = I\}$  where  $X^t$  denotes the transpose of X and I is the identity transformation. Denote the matrix  $[X] = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_1 \end{pmatrix}$  so that X is determined uniquely by the corresponding vector  $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{E}^4$ .
  - (i) Prove that  $X \in \mathcal{O}(2) \Leftrightarrow f(x) = 0 \in \mathbb{E}^3$ , where  $f : \mathbb{E}^4 \to \mathbb{E}^3$  is given by  $f(x) = (x_1x_3 + x_2x_4, x_1^2 + x_2^2 1, x_3^2 + x_4^2 1)$  for each  $x \in \mathbb{E}^4$ .
  - (ii) Show that  $[X] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{O}(2)$  and prove that in a neighborhood of X there exists at least 1 coordinate  $x_i$  of x such that the three remaining coordinates  $x_j, j \neq i$ , are locally  $\mathcal{O}^1$ -functions of  $x_i$ .