Contents

CHAPTER 1

Basic Analysis

In this chapter we recall the basic tools from analysis needed for these lecture notes. The main purpose is to fix the notation and collect the most important statements that we will need. The reader can easily skip this part for the first reading and come back to it as needed.

1. Vector spaces

An Euclidean vector space is a vector over $\mathbb{R}$ with an inner product written as $\langle v\,,\,w\rangle_V$ or simply $\langle v\,,\,w\rangle$. The corresponding norm is $\|v\|_V = \sqrt{\langle v\,,\,v\rangle_V}$ or $\|v\|$. In inner product space is a vector space over the reals or the complex numbers with an inner product $\langle u\,,\,v\rangle$ which, in case $\mathbb{K} = \mathbb{C}$ is assumed linear in the first factor and conjugate linear in the second variable.

Let V and W be real or complex finite dimensional inner product spaces and let $T : V \to W$ be a linear linear map. We write Tv or $T(v)$ for T evaluated applied to the vector $v \in V$. We denote by T^* the linear map $T^* : W \to V$ given by

$$\langle T^*w\,,\,v\rangle_V = \langle w\,,\,Tv\rangle_W\,, \quad w \in W,\ v \in V. \tag{1.1}$$

The operator T^* is called *the adjoint* of T. If V is finite dimensional and $T : V \to W$ is a linear map into a vector space W then T is continuous. Denote by $\mathrm{B}_{\mathbb{K}}(V,W)$, or simply $\mathrm{B}(V,W)$ if it is clear what field we are using, the space of $\mathbb{K}$-linear continuous maps $V \to W$. Note also that if $T, S \in \mathrm{B}_{\mathbb{K}}(V,W)$ then $T^*S : V \to V$ is linear and hence continuous. Define $(S,T)_2 := \mathrm{Tr}(T^*S)$. Then $\langle\cdot\,,\,\cdot\rangle$ is an inner product on $\mathrm{B}(V,W)$. Hence $\mathrm{B}(V,W)$ is an inner product space.

We can identify V with $\mathbb{R}^n$, $n = \dim V$ by choosing an orthonormal basis $v_1, \ldots, v_n$, of V and define a map $V \to \mathbb{R}^n$ by

$$x_1v_1 + \ldots + x_nv_n \mapsto (x_1, \ldots, x_n)^T\,.$$

Then the inner product on V becomes the standard inner product on $\mathbb{R}^n$:

$$\langle v\,,\,w\rangle_V = x_1y_1 + \ldots + x_ny_n,\ \text{where } v = \sum_{j+1}^{n} x_jv_j \text{ and } w = \sum_{j=1}^{n} y_jv_j$$

but mostly we try to work without reference to a basis and coordinate free.

If V is a topological vector space over $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers, then the *dual* of V is the vector space V^* of continuous linear maps $\lambda : V \to \mathbb{K}$. If V is finite dimensional then every linear map $\varphi : V \to \mathbb{K}$ is continuous, so V^* is identical to the algebraic dual. We will use $\lambda(v)$ and $\langle \lambda, v\rangle$ for the evaluation of of $\lambda \in V^*$ at $v \in V$. If W is another topological vector space and $T : V \to W$ is a continuous linear map, then we define $T^t : W^* \to V^*$ by

$$\langle T^t(\lambda), v\rangle := \langle \lambda, T(v)\rangle\,, \quad v \in V, \lambda \in W^*\,.$$

Assume now that V is an Euclidean vector spaces then $V \simeq V^*$. For each $v \in V$ we define $\lambda_v \in V^*$ by $\lambda_v(u) = \langle u\,,\, v\rangle$. The map $v \mapsto \lambda_v$ is a linear bijection. We denote the inverse by $\lambda \mapsto v_\lambda$. We define an inner product on V^* by $\langle \lambda\,,\, \mu\rangle = \langle v_\lambda\,,\, v_\mu\rangle$.

If $v_1, \ldots, v_n$ is a basis for V then there exists a basis $\lambda_1, \ldots, \lambda_n$ for V^* such that $\lambda_i(v_j) = \delta_{ij}$. The basis $\lambda_1, \ldots, \lambda_n$ is called the *dual basis*

If W is another Euclidean vector space and $T : V \to W$ linear, then for $\lambda \in W^*$ and $v \in V$ we get

$$\begin{aligned}
\langle T^t(\lambda), v\rangle &= \langle \lambda, T(v)\rangle \\
&= \langle v_\lambda\,,\, T(v)\rangle_W \\
&= \langle T^*(v_\lambda)\,,\, v\rangle_V\,.
\end{aligned}$$

Thus $v_{T^t(\lambda)} = T^*(v_\lambda)$, ie., the map T^t corresponds to T^* if we identify the space with its dual as above. We note that $(ST)^t = T^tS^t$ and $(ST)^* = T^*S^*$.

Linear maps between vector spaces corresponds to matrices. Let V respectively W be a n respectively m dimensional vector spaces. Fix a basis $v_1, \ldots, v_n$ for V and $w_1, \ldots, w_m$ for W. Let $T : V \to W$ be linear and define a matrix (t_{ij}) by $T(v_j) = \sum_{i=1}^m t_{ij}w_i$. The correspondence $T \mapsto (t_{ij})$ depends on the basis and is therefore not natural but can be useful for some calculations.

2. Smooth and Analytic Functions

Most functions that we need to consider are the continuous functions, smooth functions and the analytic functions. We start by recalling some standard multi-index notations. Then we introduce smooth and analytic functions. Then we recall the definition of a manifold. We start with few comments on Banach space valued smooth functions on V. We will later discuss smooth functions with values in a Fréchet space.

2.1. Multi-Index Notation. For a multi-indic $\alpha \in (Z^+)^n$ let

$$\begin{aligned}
|\alpha| &= \alpha_1 + \ldots + \alpha_n \\
\alpha! &= \alpha_1! \cdots \alpha_n!
\end{aligned}$$

If $\alpha, \beta \in (\mathbb{Z}^+)^n$ are two multi-indices, then we write $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all $i = 1, \ldots, n$. We write $\alpha > \beta$ if $\alpha \geq \beta$ and there exists an i such that $\alpha_i > \beta_i$. Furthermore, if $\alpha \geq \beta$, then

$$\begin{aligned} \alpha - \beta &= (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n) \quad \text{and} \\ \binom{\alpha}{\beta} &= \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n} = \frac{\alpha!}{\beta!(\alpha - \beta)!} \quad (\alpha \geq \beta) \end{aligned}$$

Suppose that V is a finite dimensional vector space. Fix a basis $v_1, \ldots, v_n$ for V. For $\alpha \in (\mathbb{Z}^+)^n$ and $v = \sum_{j=1}^n x_j v_j$ let

$$v^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} .$$

Note that v^α depends on the basis $v_1, \ldots, v_n$.

2.2. Differentiable and Analytic Functions. Let W be a real or complex Banach space. Suppose $\emptyset \neq \Omega \subseteq V$ is an open set. A function $f : \Omega \to W$ is *differentiable* at $x \in \Omega$ if there exists an $\mathbb{R}$-linear transformation $Df(x) : V \to W$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Df(x)h\|_W}{\|h\|_V} = 0 .$$

The function f is differentiable on Ω if f is differentiable at all points $x \in \Omega$. If f is differentiable, then f is continuous.

Assume that $\dim W < \infty$. Then any linear map is continuous.

The function

$$Df : \Omega \to \mathrm{B}_{\mathbb{R}}(V, W) , \quad x \mapsto Df(x)$$

is called the *derivative* or the *total derivative* of f. The function f is *continuously differentiable* if $Df : \Omega \to \mathrm{B}_{\mathbb{R}}(V, W)$ is continuous. Using induction, we say that f is r-times continuously differentiable, $r \geq 2$, if $D^{r-1} f$ exists and is continuously differentiable. We say that f is *smooth* if f is r-times continuously differentiable for all $r \in \mathbb{Z}^+$.

We denote by $C^r(\Omega)$ the space of r-times continuously differentiable functions and by $C^\infty(\Omega)$ the space of smooth functions on Ω. Then

$$C^\infty(\Omega) = \bigcap_{r \in \mathbb{Z}^+} C^r(\Omega) .$$

Let again W be a Banach space. Let I be a countable index set and $\{b_\alpha\}_{\alpha \in I}$ a sequence in in W. We say that $\sum_{\alpha \in I} b_\alpha = b \in W$ if for each $\epsilon > 0$ there exists a finite subset $F \subset I$ such that for all finite subsets $E \subset \mathbb{Z}^+$, $F \subseteq E$, we have

$$\|b - \sum_{\alpha \in E} b_\alpha\| < \epsilon .$$

A function $f : \Omega \to W$ is *analytic* if for every $x \in \Omega$ there exists an open ball $B_r(x) = \{y \in V \mid \|x - y\| < r\}$ with radius $r > 0$ and center x, and for each multi-index $\alpha \in \mathbb{Z}^+$ there exist elements $a_\alpha \in W$ such that $B_r(x) \subseteq \Omega$ and

$$f(y) = \sum_{\alpha \in (\mathbb{Z}^+)^n} a_\alpha (y - x)^\alpha \,. \tag{2.1}$$

We note that the right hand side in (2.1) depends on a choice of a basis for V. But if such an expression exists for one basis, then it exists for any choice of a basis. Hence the definition of an analytic function is independent of the basis used in (2.1).

We denote by $C^\omega(\Omega)$ the space of analytic functions on Ω. Note that $C^\omega(\Omega) \subset C^\infty(\Omega)$.

If $\mathcal{V} \subseteq V$ and $\mathcal{W} \subset W$ are open, then a *diffeomorphism* of $\mathcal{V}$ onto $\mathcal{W}$ is a map $g : \mathcal{V} \to \mathcal{W}$ such that g is bijective and g and g^{-1} are smooth. g is an *analytic diffeomorphism* if g and g^{-1} are both analytic.

A topological space $\mathcal{M}$ is an n-dimensional *manifold* if for each $x \in \mathcal{M}$, there exists an open neighborhood U_x of x, an open set $V_x \subset \mathbb{R}^n$, and a homeomorphism $\varphi_x : U_x \to V_x$ such that if $x, y \in M$ and $U_x \cap U_y \neq \emptyset$, then the map

$$\varphi_x \circ \varphi_y^{-1} : \varphi_y(U_x \cap U_y) \to \mathbb{R}^n$$

is smooth. (U_x, φ_x) is a *chart* around x. In this text all manifolds are smooth, have at most countable many components and are paracompact. Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds.

A function $f : \mathcal{M} \to \mathcal{N}$ is r-times continuously differentiable if for all $x \in \mathcal{M}$ and there exist a chart (U, φ) around x and a chart (V, ψ) around $f(p)$ such that the function

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^{\dim \mathcal{N}}$$

is r-times continuously differentiable. $C^r(\mathcal{M}, \mathcal{N})$ denotes the space of r-times continuously differentiable functions from $\mathcal{M}$ to $\mathcal{N}$. If $r = 0$ we simply write $C(\mathcal{M}, \mathcal{N})$. If $\mathcal{N} = \mathbb{C}$ then we write $\mathbb{C}^r(\mathcal{M})$ instead of $\mathcal{C}^r(\mathcal{M}, \mathbb{C})$. We set

$$\mathcal{C}^\infty(\mathcal{M}, \mathcal{N}) = \bigcap_{r=0}^{\infty} \mathcal{C}^r(\mathcal{M}, \mathcal{N}) \,.$$

If the coordinate changes are analytic functions, then $\mathcal{M}$ is an *analytic manifold*. A function $f : \mathcal{M} \to \mathcal{N}$ is then *analytic* if the local expression of f is analytic. If $\mathcal{M}$ and $\mathcal{N}$ are smooth/analytic manifolds, then $g : \mathcal{M} \to \mathcal{N}$ is a diffeomorphis respectively analytic diffeomorphism if g is bijective and g and g^{-1} are differentiable respectively analytic.

Let $\mathcal{M}$ be a manifold, a family $\{U_\alpha\}$ of subset of $\mathcal{M}$ is a *covering* of $\mathcal{M}$ if $\mathcal{M} = \bigcup U_\alpha$, it is an *open covering* if it is a covering and all U_α are open.

The covering $\{U_\alpha\}$ is *a locally finite covering* if for each $p \in \mathcal{M}$ there exists an open neighborhood U_p of p such that $\{\alpha \mid U_p \cap U_\alpha \neq \emptyset\}$ is finite.

DEFINITION 2.1. Let $\{U_\alpha\}$ be a locally finite covering of the manifold $\mathcal{M}$. A family $\{\psi_\alpha\}$ of smooth function on $\mathcal{M}$ is a *partition of unity subordinate to* $\{U_\alpha\}$ if

(1) $0 \leq \psi_\alpha \leq 1$.
(2) $\mathrm{supp}(\psi_\alpha) \subset U_\alpha$.
(3) $\sum_\alpha \psi_\alpha = 1$.

THEOREM 2.2 (Partition of Unity). *Let $\mathcal{M}$ be a manifold and $(U_\alpha, \varphi_\alpha)$ an atlas for $\mathcal{M}$, such that $\{U_\alpha\}$ is locally finite. Then there exists a partition of unity subordinate to $\{U_\alpha\}$.*

PROOF. See [?] Theorem 1.11, page 10. □

Let $f : \Omega \to W$. For $v \in V$ we denote by $\partial_v f : \Omega \to W$ the *directional derivative*

$$\partial_v(f)(x) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

if the limit exists. If f is differentiable, then $\partial_v f(x)$ exists for all $v \in V$ and $x \in \Omega$ and

$$\partial_v(f)(x) = Df(x)v\,.$$

Fix an orthonormal basis $v_1, \ldots, v_m$ of V and $w_1, \ldots, w_n$ for W. Define $f^i : \Omega \to \mathbb{R}$, $i = 1, \ldots, m$, by

$$f = \sum_{i=1}^{n} f^i w_i\,.$$

Let $\partial_j = \partial_{v_j}$. Then the linear map $Df(x)$ is given by the matrix

$$D(f)(x) = (\partial_j f^i(x))_{i=1,\ldots,m; j=1,\ldots,n}\,.$$

A *polynomial function* on V^* is a function $P : V^* \to \mathbb{C}$, such that with respect to a fixed orthonormal basis $v_1, \ldots, v_n$, we have

$$P(\lambda) = \sum_{|\alpha| \leq N} a_\alpha \lambda^\alpha \tag{2.2}$$

for some $a_\alpha \in \mathbb{C}$. Let $\partial_j = \partial_{v_j}$ and set

$$\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}\,.$$

For a polynomial function P as in (5.6) let

$$P(\partial) := \sum_{|\alpha| \leq N} a_\alpha \partial^\alpha\,.$$

$P(\partial)$ is a constant coefficient differential operator. We will see in the Section **??** (see Remark **??**) that the definition of $P(\partial)$ is independent of the choice of basis.

2.3. Inverse and Implicit Function Theorem. In this subsection V and W are finite dimensional Euclidean vector spaces. The proof of the following, except some of the statements that a function is smooth or analytic can be found in the book by Spivak, [**?**], p. 34–39, and p. 40–43. Ω will always stands for a nonempty open subset of V.

THEOREM 2.3 (Inverse Function Theorem). *Suppose $f : \Omega \to V$ is a continuously differentiable function on Ω. If $x \in \Omega$ and $\det Df(x) \neq 0$, then there exists an open set $U \subseteq \Omega$ containing x and an open subset $W \subseteq V$ containing $f(x)$ such that $f : U \to W$ is bijective and the function $f^{-1} : W \to U$ is differentiable. Furthermore,*

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$$

for all $y \in W$. If f is smooth, then the function f^{-1} is smooth. If f is analytic, then f^{-1} is analytic as well.

Let $\Omega_V \subseteq V$, and $\Omega_W \subseteq W$ be open and nonempty. For a function $f : \Omega_V \times \Omega_W \to W$, and $a \in \Omega_V$, define $f_a : \Omega_W \to W$ by $f_a(b) := f(a, b)$.

THEOREM 2.4 (Implicit Function Theorem). *Let $\Omega_V \subseteq V$, and $\Omega_W \subseteq W$ be open and nonempty. Suppose $f : \Omega_V \times \Omega_W \to W$ is continuously differentiable. Let $(a, b) \in \Omega_V \times \Omega_W$ such that $f(a, b) = 0$. If $Df_a(b) : W \to W$ is an isomorphism, then there exists*

1. *an open set $\mathcal{V} \subseteq \Omega_V$ containing a and an open set $\mathcal{W} \subseteq \Omega_W$ containing b,*
2. *a differentiable function $g : \mathcal{V} \to \mathcal{W}$*

such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in \mathcal{V}$. If f is smooth respectively analytic, then g can be chosen smooth respectively analytic.

We will mainly need the following consequence of the implicit function theorem. For a subspace $W_1, W_2 \subset V$ we say that V is the direct sum of W_1, W_2, denoted by $V = W_1 \oplus W_2$, if each vector $v \in V$ has an unique expression as $v = w_1 + w_2$ for some $w_j \in W_j$. Let

$$W^{\perp} := \{v \in V \mid (\forall w \in W)\ (v, w) = 0\}.$$

Then $V = W \oplus W^{\perp}$ and the sum is orthogonal. A linear map $P : V \to V$ is a *projection* if $P^2 = P$. It is an *orthogonal projection* if $P^2 = P^* = P$. We have $V = \mathrm{Im}(P) \oplus \mathrm{Ker}(P)$, and this sum is orthogonal if and only if P is an orthogonal projection. If $W \subseteq V$ is a subspace, then P_W denotes the orthogonal projection onto W.

THEOREM 2.5. *Let $W \subseteq V$ be a subspace of V. Let $\Omega \subset V$ be open and nonempty, and let $f : \Omega \to W$ be a smooth map. Suppose that $a \in \Omega$ is such that $f(a) = 0$, and $Df(a) : V \to W$ is surjective. Then there exists nonempty open sets $\mathcal{V}, \mathcal{W} \subseteq \Omega$ and a diffeomorphism $h : \mathcal{V} \to \mathcal{W}$ such that $a \in \mathcal{V}$ and*
$$f \circ h(v) = P_W(v)$$
for all $v \in \mathcal{V}$. If f is analytic, then h can be chosen to be analytic. *ImpFctTh2*

THEOREM 2.6. *Let $\Omega \subseteq V$ be open and nonempty set. Let $g : \Omega \to W$ be a smooth function such that $Dg(x)$ has rank* $\dim W$ *whenever $g(x) = 0$, $x \in \Omega$. Then $g^{-1}(0)$ is an* $(\dim V - \dim W)$*-dimensional manifold in V. If g is analytic, then $g^{-1}(0)$ is an analytic manifold.*

PROOF. Note first, that the assumption on the rank of $Dg(x)$ implies that $\dim W \leq \dim V$. If $\dim W = \dim V$, then g is locally a diffeomorphism. Hence $g^{-1}(0)$ is a discrete union of points. We can therefore assume that $\dim W < \dim V$. Then we may as well assume that $W \subset V$. The statement then follows from Theorem 5.9. $\square$

EXAMPLE 2.7 (Spheres). Let $a \in \mathbb{R}^n$ and let $R > 0$. Define $g : \mathbb{R}^n \to \mathbb{R}$ by ex-sphers
$$g(x) = \|x - a\|^2 - R^2 .$$
Then g is analytic. In this case $Dg(x)$ is an $n \times 1$-matrix which we view as a column vector. Then
$$Dg(x) = 2(x - a) \neq 0$$
for all $x \in \mathbb{R}^n$ such that $g(x) = 0$. Hence $S_R^{n-1}(a)$ is an $(n-1)$-dimensional analytic submanifold of $\mathbb{R}^n$. Note, if $x = a$, then $Dg(x) = 0$ and $g^{-1}(0) = \{a\}$ is not an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. $\diamondsuit$

2.4. Compactly Supported Functions. Recall that the *support* of a continuous vector-valued or complex-valued function on a manifold (or topological space) $\mathcal{M}$ is the set
$$\text{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}\,, \tag{2.3}$$
eq-support1

where $^-$ stands for the *closure* of a set. If μ is a Borel or Radon measure on $\mathcal{M}$ and f is a measurable function on $\mathcal{M}$, then the support of f is the complement of the union of all open subset $U \subseteq \mathcal{M}$ such that $f(x) = 0$ for almost all $x \in U$:
$$\text{supp}(f) := \mathcal{M} \setminus \bigcup_{\substack{U \subset \mathcal{M},\ \text{open}, \\ f(x) = 0 \\ \text{for almost all } x \in U}} U\,.$$
Then clearly supp(f) is closed. If f is continuous, then this new definition agrees with the usual one in (2.3).

If E is a space of measurable functions, then the subscript $_c$ will indicate the subspace of compactly supported functions in E. For a non-empty compact set $K \subseteq \mathcal{M}$ let

$$C_K^\infty(\mathcal{M}) := \{f \in C_c^\infty(\mathcal{M}) \mid \mathrm{supp}(f) \subseteq K\}.$$

Now, assume that V is an Euclidean space as before and that $\Omega \subseteq V$ is a nonempty open set. For $K \subset \Omega$ compact define a family of seminorms on $C_K^\infty(\Omega)$ by

$$\tau_{K,P}(f) := \|P(\partial)f\|_\infty, \quad P \text{ a polynomial}. \tag{2.4}$$

Then $C_K^\infty(\mathcal{M})$ is a complete, locally convex topological vector space. The topology can also be defined by the increasing family of seminorms

$$\widetilde{\tau}_{K,n}(f) := \sup_{\deg P \leq n} \|P(\partial)f\|_\infty. \tag{2.5}$$

It is therefore clear, that $C_K^\infty(\Omega)$ is a Fréchet space in this topology. For a manifold $\mathcal{M}$ and $K \subseteq \mathcal{M}$, we define a topology on $C_K^\infty(\mathcal{M})$ in a similar way by using local charts.

We make $C_c^\infty(\Omega)$ into a topological vector space by considering it as the inductive limit of the space $C_K^\infty(\Omega)$. Thus a sequence $\{f_n\}_n$ converges to f in $C_c^\infty(\Omega)$ if and only if

(1) there exists a compact set K such that $\mathrm{supp}(f_n) \subseteq K$ for all n,
(2) if P is a polynomial, then $P(\partial)f_n \to P(\partial)f$ uniformly on K.

The space $C_c^\infty(\Omega)$ is locally convex and complete. The elements of $C_c^\infty(\Omega)$ are called *test functions* on Ω.

Lemma 2.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be topological spaces. Let μ be a Radon measure on $\mathcal{M}$. Suppose that $f : \mathcal{M} \times \mathcal{N} \to \mathbb{C}$ is such that:*

(1) *The function $x \mapsto f(m,x)$ is continuous for all $m \in \mathcal{M}$.*
(2) *There exists a non-negative function $g \in L^1(\mathcal{M}, \mu)$ such that*

$$|f(m,x)| \leq g(m)$$

for all $x \in \mathcal{N}$.

Then $m \mapsto f(m,x)$ is integrable for all $x \in \mathcal{N}$ and the function

$$F(x) := \int_{\mathcal{M}} f(m,x)\, d\mu(m)$$

is continuous. Furthermore, if $x_o \in \mathcal{N}$, then

$$\lim_{x \to x_o} F(x) = \int_{\mathcal{M}} \lim_{x \to x_o} f(m,x)\, d\mu(m).$$

Proof. This follows from Lebesgue Dominated Convergence (LDC) Theorem. $\square$

LEMMA 2.9. *Let $\mathcal{M}$ be a topological space and μ a Radon measure on $\mathcal{M}$. Let $\emptyset \neq I \subseteq \mathbb{R}$ be an open interval. Suppose that $f : \mathcal{M} \times I \to \mathbb{C}$. Assume the following:* le-intDiff

(1) *The exists an $s \in I$ such that $m \mapsto f(m,s)$ is in $L^1(\mathcal{M},\mu)$.*
(2) *The function $t \mapsto f(m,t)$ is differentiable for all $m \in \mathcal{M}$.*
(3) *For all $t_o \in I$ there exists an open interval J containing t_o and a non-negative function $g \in L^1(\mathcal{M},\mu)$ such that*

$$(\forall t \in J) \quad |\partial_t f(m,t)| \leq g(m)\,.$$

Then $m \mapsto f(m,t)$ is integrable for all $t \in I$ and the function

$$F(t) := \int_{\mathcal{M}} f(m,t)\, d\mu(m)$$

is differentiable on I. Furthermore,

$$\frac{dF}{dt}(t) = \int_{\mathcal{M}} \partial_t f(m,t)\, d\mu(m)\,.$$

PROOF. For $t_o \in I$ let J be as in (3). Let $t \in J$ and let $m \in \mathcal{M}$. Then there exists t^* between t_o and t such that

$$\frac{f(m,t) - f(m,t_o)}{t - t_o} = \partial_t f(m,t^*)\,.$$

In particular,

$$\left|\frac{f(m,t) - f(m,t_o)}{t - t_o}\right| = |\partial_t f(m,t^*)| \leq g(m) \tag{2.6}$$

independent of t and t_o. Taking $t_o = s$ as in (1) we get

$$|f(m,t)| \leq g(m)|t-s| + |f(m,s)|$$

and it follows that $m \mapsto f(m,t)$ is integrable for all t. By (2.6) we can apply the LDC Theorem to get

$$\begin{aligned}
\lim_{t\to t_o} \frac{F(t) - F(t_o)}{t - t_o} &= \lim_{t\to t_o} \int_{\mathcal{M}} \frac{f(m,t) - f(m,t_o)}{t - t_o}\, d\mu(m) \\
&= \int_{\mathcal{M}} \lim_{t\to t_o} \frac{f(m,t) - f(m,t_o)}{t - t_o}\, d\mu(m) \\
&= \int_{\mathcal{M}} \partial_t f(m,t)\, d\mu(m)\,,
\end{aligned}$$

In particular the limit exists and $dF/dt(t) = \int_{\mathcal{M}} \partial_t f(m,t)\, d\mu(m)$ as stated in the theorem. $\square$

2.5. Vector Valued Functions. In this subsection we discuss differentiability of functions with values in an infinite-dimensional topological vector space. Here the differentiability might depend on the topology.

DEFINITION 2.10. Let W be a complete Hausdorff locally convex topological vector space. Let $\emptyset \neq I \subset \mathbb{R}$ be an open interval. A map $F : I \to W$ is *strongly differentiable* if for all $t_o \in I$ the limit

$$\lim_{t \to t_0} \frac{F(t) - F(t_o)}{t - t_o}$$

exists in the topology of W. F is *weakly differentiable* if for all $\varphi \in W^*$ the function $t \mapsto \varphi(F(t))$ is differentiable.

If F is strongly differentiable, then the limit is unique and we denote it by $\partial_t F(t)$.

LEMMA 2.11. *If $F : I \to W$ is strongly differentiable, then F is weakly differentiable and*

$$\frac{d}{dt}\varphi(F(t)) = \varphi(\partial_t F)$$

for all $\varphi \in W^*$.

PROOF. As φ is continuous and the limit exists in the topology of W we get

$$\begin{aligned} \lim_{t \to t_o} \frac{\varphi(F(t)) - \varphi(F(t_o))}{t - t_o} &= \varphi\left(\lim_{t \to t_o} \frac{F(t) - F(t_o)}{t - t_o}\right) \\ &= \varphi(\partial_t F(t))\,. \end{aligned}$$

$\square$

3. Distributions

Let $\emptyset \neq \Omega \subseteq V$. The *dual* of $C_c^\infty(\Omega)$ is denoted by $\infty_c^{-\infty}(\Omega)$. The elements of $C_v^{-\infty}(\Omega)$ are called *distributions* on Ω. Recall that the topology on $C_c^\infty(V)$ is the relative topology from $S(V)$. Hence, if $\varphi \in S(V)^*$, then

$$\mathrm{Res}(\varphi) := \varphi|_{C_c^\infty(V)} \in C_c^{-\infty}(V)$$

Assume that $\varphi|_{C_c^\infty(V)} = 0$. As $C_c^\infty(V)$ is dense in $S(V)$ it follows that $\varphi = 0$. Thus we can view $S(V)^*$, as a subspace of $C_c^{-\infty}(V)$. The elements of $S(V)^*$ are the *tempered distributions*. Let V be a complete locally convex Hausdorff topological vector space with dual V^*. The weakest topology on V^* such that all the linear maps $\lambda \mapsto \lambda(v)$, $v \in V$, are continuous, is the *weak$*$-topology* on V^*. If nothing else is said then this is the topology that we will use on V^*. In particular this is the topology on $C_c^{-\infty}(V)$ and $S(V)^*$.

Let $f \in L^1_{\mathrm{loc}}(V)$. Then f can be viewed as a distribution φ_f given by

$$\varphi_f(g) = \int_V g(x) f(x)\, dx\,, \quad g \in C_c^\infty(V)\,.$$

If φ is a distribution and there exists a function f such that $\varphi = \varphi_f$, then we say that φ is a function. We say that a distribution φ is a continuous function, smooth function, or an integrable function if φ is of the the form φ_f with the function f continuous, smooth, or integrable.

We note that φ_f is not necessarily tempered. As an example take $f(x) = e^{x^k}$ for some $k \in \mathbb{N}$. Then $\varphi_f \in C_c^{-\infty}(V)$, but not in $S(V)^*$. A function f on V is of *polynomial growth* if there exists a $k \in \mathbb{N}$ such that $\omega^{-k} f$ is bounded. The following is obvious:

LEMMA 3.1. *If f is of polynomial growth then $\varphi_f \in S(V)^*$.*

The natural way to define natural operations like differentiation to distributions, one makes sure that they agree with the old operation for functions viewed as distributions. As an example let $f \in C^\infty(V)$. Let $g \in C_c^\infty(V)$. Then

$$\begin{aligned}
\varphi_{\partial^\alpha f}(g) &= \int_V g(x) \partial^\alpha f(x)\, dx \\
&= (-1)^{|\alpha|} \int_V [\partial^\alpha g(x)] f(x)\, dx \\
&= (-1)^{|\alpha|} \varphi_f(\partial^\alpha g)
\end{aligned}$$

and the last expression makes sense for arbitrary distributions.

Let $\varphi \in C_c^{-\infty}(V)$, $g \in C_c^\infty(V)$, $v \in V$, $t > 0$ and $1 \le p \le \infty$. Define

(1) $(\partial^\alpha \varphi)(g) := (-1)^{|\alpha|} \varphi(\partial^\alpha g)$.
(2) $(L(v)\varphi)(g) := \varphi(L(-v)g)$.
(3) $(D_p(t)\varphi)(g) := \varphi(D_q(1/t)g)$, where $1/p + 1/q = 1$.

It follows from the above definition, that every distribution can be differentiated as many times as we please. In particular, $f \in L_{\mathrm{loc}}(V)$ and P is a polynomial function on V, then $P(\partial)\varphi_f$ is well-defined. $P(\partial)\varphi_f$ is called the *distributional derivative of f*.

For $f : V \to W$ let $f^\vee(x) = f(-x)$. If $f \in S(V)$ and $v \in V$, then $L(v) f^\vee = (L(-v)f)^\vee \in S(V)$. For $\varphi \in S(V)^*$ and $f \in S(V)$ we define the *convolution* of φ with f by

$$\varphi * f(v) := \varphi((L(v)f)^\vee) = \varphi(L(-v)f^\vee)\,. \tag{3.1}$$

LEMMA 3.2. *Let $f \in S(V)$, $\varphi \in S(V)^*$, and let P be a polynomial. Then the following holds:*

(1) *$\varphi * f \in C^\infty(V)$ and $P(\partial)(\varphi * f) = \varphi * (P(\partial) f)$.*

(2) *$P(\partial)(\varphi * f)$ is of polynomial growth.*
(3) $\varphi * f \in S(V)^*$

PROOF. The first part follows from the fact that for $v \in V$

$$\frac{f(\cdot + tv) - f}{t} \xrightarrow{t\to 0} \partial_v f$$

in the $S(V)$ topology.

The second part follows from Lemma **??** and the third part is a consequence of (2). □

LEMMA 3.3. *Let $g \in S(V)$ be such that $\|g\|_1 = 1$ and $g \geq 0$. Let $g_t = D_1(t)g$. Suppose $\varphi \in S(V)^*$. Then $\varphi * g_t \to \varphi$ in $S(V)^*$.* *le-appIdeTempDist*

PROOF. Let $f \in S(V)$. One shows that

$$\varphi * g_t(f) = \varphi(g_t * f)\,.$$

The claim then follows from $g_t * f \to f$ in $S(V)$, see Lemma **??**. □ clarify later

4. Fourier Analysis

In this section we discuss the Fourier transform of functions and distributions on V. In our notation the Fourier transform $\widehat{f} = \mathcal{F}f$ of a function on V is a function on the dual V^*. We show how to fix the normalization of the Lebesgue measure on V^* such that the Plancherel Formula holds. Here July 1

5. Holomorphic Functions

This section contains the main results on holomorphic functions that will be needed later. We refer to [**?**] for proofs and a more detailed treatment. Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be the *complexification of* V. Thus $V_{\mathbb{C}}$ is a complex vector space containing V as a real vector subspace and $V_{\mathbb{C}} = V + iV$.

We extend the inner product on V to a Hermitian inner product on $V_{\mathbb{C}}$. If $V = \mathbb{R}^n$ then $V_{\mathbb{C}} = \mathbb{C}^n$ and the extension of the canonical inner-product on $\mathbb{R}^n$ is the Hermitian form

$$(z, w) = \sum_{j=1}^{n} z_j \overline{w_j}\,.$$

If $e_1, \ldots, e_n$ is an orthonormal basis for V, then $e_1, \ldots, e_n$ is also an orthonormal basis for $V_{\mathbb{C}}$. Furthermore the map $\mathbb{C}^n \to V_{\mathbb{C}}$, $(z_1, \ldots, z_n) \mapsto z_1e_1 + \ldots + z_ne_n$ is a $\mathbb{C}$-linear isomorphism and

$$(z_1e_1 + \ldots + z_ne_n, w_1e_1 + \ldots + w_ne_n) = z_1\overline{w}_1 + \ldots + z_n\overline{w}_n\,.$$

We fix an orthonormal basis from now on and use coordinates $z = \sum_{j=1}^{n} z_je_j$ when needed.

If $r_j > 0$, $j = 1, \ldots, n$, let $\mathbf{r} := (r_1, \ldots, r_n)$. If $r_1 = r_2 = \ldots = r_n = r > 0$, then we write $\mathbf{r} = (r, r, \ldots, r)$. For $z \in V_{\mathbb{C}}$ let

$$P_{\mathbf{r}}(z) = \{w \in V_{\mathbb{C}} \mid \forall j \ : \ |w_j - z_j| \leq r_j\} \tag{5.1}$$

be the *polydisc* with center z and "radius" $\mathbf{r}$. Set

$$p_{\mathbf{r}}(z) = \{w \in V_{\mathbb{C}} \mid \forall j \ : \ |w_j - z_j| = r_j\}\,. \tag{5.2}$$

Note that $p_{\mathbf{r}}(z)$ is contained in the topological boundary

$$\partial P_{\mathbf{r}}(z) = P_{\mathbf{r}}(z) \setminus P_{\mathbf{r}}(z)^o = \{w \in V_{\mathbb{C}} \mid \exists j \ : \ |w_j - z_j| = r_j\}\,.$$

In general $p_r(z)$ is smaller than the topological boundary.

Definition 5.1. Let $\Omega \subset V_{\mathbb{C}}$ be open and non-empty. Let $F : \Omega \to \mathbb{C}$.

(1) F is *holomorphic* on Ω if for all $z_o \in \Omega$ there exists $\mathbf{r} > 0$ and a convergent power series $\sum_\alpha a_\alpha (z - z_o)^\alpha$ on $P_{\mathbf{r}}(z_o)$ such that
$$F(z_1 e_1 + \ldots + z_n e_n) = \sum_\alpha a_\alpha (z - z_o)^\alpha\,, \quad z \in P_{\mathbf{r}}(z_o)^o\,.$$
(2) F is *weakly-holomorphic* on Ω if for all $z_o \in \Omega$ and all $v \in V_{\mathbb{C}}$ the function of one variable $h \mapsto F(z_o + hv)$ is holomorphic in a neighborhood of 0 in $\mathbb{C}$.
(3) F is *complex differentiable* in Ω if for every $z_o \in \Omega$ there exists a complex linear map $DF(z_o) : V_{\mathbb{C}} \to V_{\mathbb{C}}$ and a function φ so that for $z \in \Omega$
$$F(z) = F(z_o) + DF(z_o)(z - z_o) + \varphi(z)$$
and
$$\lim_{z \to z_o} \frac{|\varphi(z)|}{\|z - z_o\|} = 0\,.$$

If F is weakly holomorphic then we set

$$\partial_v F(z_o) := \lim_{h \to 0} \frac{F(z_o + hv) - F(z_o)}{h}\,. \tag{5.3}$$

If F is complex differentiable on Ω, then $z \mapsto DF(z)$ is a holomorphic functions with values in $\mathrm{M}(n, \mathbb{C})$, the space of $n \times n$-matrices. We note that if F is complex differentiable, then F is weakly differentiable and

$$\partial_v F(z_o) = DF(z_o)v\,.$$

If $F : \Omega \to \mathbb{C}$ is continuous and $z_o \in \Omega$ let $r > 0$ be so that $P_{\mathbf{r}}(z_o) \subset \Omega$. Define $C_F : P_{\mathbf{r}}(z_o)^o \to \mathbb{C}$ by

$$C_F(z) = \left(\frac{1}{2\pi i}\right)^n \int_{p_{\mathbf{r}}(z_o)} \frac{F(\zeta)}{(\zeta - z)^{\mathbf{1}}}\, d\zeta\,. \tag{5.4}$$

For $z = z_o$ (5.4) is

$$C_F(z_o) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} F(z_o + \sum_{j=1}^n r_j e^{i\theta_j} e_j)\, d\theta_1 \ldots d\theta_n\,.$$

The equivalence of the first two statements in the following theorem is Osgood's Theorem.

THEOREM 5.2. *Let $\Omega \neq \emptyset$ be an open subset in $V_{\mathbb{C}}$ and $F : \Omega \to \mathbb{C}$ continuous. Then the following are equivalent:* *Osgood*

1. *F is holomorphic on Ω.*
2. *F is weakly holomorphic on Ω.*
3. *F is complex differentiable.*
4. *For all $z \in \Omega$ we have $C_F(z) = F(z)$.*

PROOF. See [**?**], p. 18 and 19. □

A *domain* in $V_{\mathbb{C}}$ is a non-empty, open and pathwise connected subset of $V_{\mathbb{C}}$. In the rest of this section D denotes a domain in $V_{\mathbb{C}}$. Denote by $\mathcal{O}(D)$ the space of holomorphic functions on D. The topology on $\mathcal{O}(D)$ is defined by the seminorms

$$\nu_K(F) = \sup_{z\in K} |F(z)| \tag{5.5}$$

where $\emptyset \neq K \subset D$ is compact. Thus, the topology is that of *uniform convergence on compact subsets.*

LEMMA 5.3 (Weierstrass Convergence Theorem). *Let $D \subset V_{\mathbb{C}}$ be a domain, and $\{F_n\}$ a sequence of holomorphic functions on D that converges uniformly to a function F. Then F is holomorphic.*

PROOF. Let $z \in D$ and let $r > 0$ be such that $P_{\mathbf{r}}(z) \subset \Omega$. Let $w \in P_{\mathbf{r}}(z)^o$. As $F_n \to F$ uniformly on compact sets it follows that F is continuous. In particular, $A = \sup_{w\in P_{\mathbf{r}}(z)} |F(w)| < \infty$. Let $N \in \mathbb{N}$ be such that

$$\sup_{w\in P_{\mathbf{r}}(z)} |F_k(w) - F(w)| < 1$$

for all $k \geq N$. We then let

$$B := \max\{\|F_1|_{P_{\mathbf{r}}}\|_\infty, \ldots, \|F_{N-1}|_{P_{\mathbf{r}}}\|_\infty, A+1\}$$

Then

$$|F_n(w)| \leq B$$

for all $w \in P_{\mathbf{r}}(z)$ and all $n \in \mathbb{N}$. Finally,

$$\left|\frac{F_n(\zeta)}{\zeta - w}\right| \leq \frac{C}{d(w, p_{\mathbf{r}}(z))} < \infty$$

for $w \in P_{\mathbf{r}}(z)$ and $n \in \mathbb{N}$. As $p_{\mathbf{r}}(z)$ is compact, we can interchange the integration and the limit process in the following

$$\begin{aligned} F(w) &= \lim_{n\to\infty} F_n(w) \\ &= \lim_{n\to\infty} \left(\frac{1}{2\pi i}\right)^n \oint_{p_{\mathbf{r}}(z)} \frac{F_n(\zeta)}{(\zeta - w)^{\mathbf{1}}}\, d\zeta \\ &= \left(\frac{1}{2\pi i}\right)^n \oint_{p_{\mathbf{r}}(z)} \lim_{n\to\infty} \frac{F_n(\zeta)}{(\zeta - w)^{\mathbf{1}}}\, d\zeta \\ &= \left(\frac{1}{2\pi i}\right)^n \oint_{p_{\mathbf{r}}(z)} \frac{F(\zeta)}{(\zeta - w)^{\mathbf{1}}}\, d\zeta\,. \end{aligned}$$

Thus $C_F(w) = F(w)$ and hence F is holomorphic by Theorem 5.2. □

THEOREM 5.4. *The space $\mathcal{O}(D)$ is a a Fréchet space.*

PROOF. This follows from Lemma 5.3. □

We denote by $C^\omega(\Omega)$ the space of analytic functions on Ω. Note that $C^\omega(\Omega) \subset C^\infty(\Omega)$.

If $\mathcal{V} \subseteq V$ and $\mathcal{W} \subset W$ are open, then a *diffeomorphism* of $\mathcal{V}$ onto $\mathcal{W}$ is a map $g : \mathcal{V} \to \mathcal{W}$ such that g is bijective and g and g^{-1} are smooth. g is an *analytic diffeomorphism* if g and g^{-1} are both analytic.

A topological space $\mathcal{M}$ is an n-dimensional *manifold* if for each $x \in \mathcal{M}$, there exists an open neighborhood U_x of x, an open set $V_x \subset \mathbb{R}^n$, and a homeomorphism $\varphi_x : U_x \to V_x$ such that if $x, y \in M$ and $U_x \cap U_y \neq \emptyset$, then the map

$$\varphi_x \circ \varphi_y^{-1} : \varphi_y(U_x \cap U_y) \to \mathbb{R}^n$$

is smooth. (U_x, φ_x) is a *chart* around x. In this text all manifolds are smooth, have at most countable many components and are paracompact. Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds.

A function $f : \mathcal{M} \to \mathcal{N}$ is r-times continuously differentiable if for all $x \in \mathcal{M}$ and there exist a chart (U, φ) around x and a chart (V, ψ) around $f(p)$ such that the function

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^{\dim \mathcal{N}}$$

is r-times continuously differentiable. $C^r(\mathcal{M}, \mathcal{N})$ denotes the space of r-times continuously differentiable functions from $\mathcal{M}$ to $\mathcal{N}$. If $r = 0$ we simply write $C(\mathcal{M}, \mathcal{N})$. If $\mathcal{N} = \mathbb{C}$ then we write $\mathbb{C}^r(\mathcal{M})$ instead of $\mathcal{C}^r(\mathcal{M}, \mathbb{C})$. We set

$$\mathcal{C}^\infty(\mathcal{M}, \mathcal{N}) = \bigcap_{r=0}^{\infty} \mathcal{C}^r(\mathcal{M}, \mathcal{N})\,.$$

If the coordinate changes are analytic functions, then $\mathcal{M}$ is an *analytic manifold*. A function $f : \mathcal{M} \to \mathcal{N}$ is then *analytic* if the local expression of

f is analytic. If $\mathcal{M}$ and $\mathcal{N}$ are smooth/analytic manifolds, then $g : \mathcal{M} \to \mathcal{N}$ is a diffeomorphis respectively analytic diffeomorphism if g is bijective and g and g^{-1} are differentiable respectively analytic.

Let $\mathcal{M}$ be a manifold, a family $\{U_\alpha\}$ of subset of $\mathcal{M}$ is a *covering* of $\mathcal{M}$ if $\mathcal{M} = \bigcup U_\alpha$, it is an *open covering* if it is a covering and all U_α are open. The covering $\{U_\alpha\}$ is *a locally finite covering* if for each $p \in \mathcal{M}$ there exists an open neighborhood U_p of p such that $\{\alpha \mid U_p \cap U_\alpha \neq \emptyset\}$ is finite.

DEFINITION 5.5. Let $\{U_\alpha\}$ be a locally finite covering of the manifold $\mathcal{M}$. A family $\{\psi_\alpha\}$ of smooth function on $\mathcal{M}$ is a *partition of unity subordinate to* $\{U_\alpha\}$ if

(1) $0 \leq \psi_\alpha \leq 1$.
(2) $\operatorname{supp}(\psi_\alpha) \subset U_\alpha$.
(3) $\sum_\alpha \psi_\alpha = 1$.

THEOREM 5.6 (Partition of Unity). *Let $\mathcal{M}$ be a manifold and $(U_\alpha, \varphi_\alpha)$ an atlas for $\mathcal{M}$, such that $\{U_\alpha\}$ is locally finite. Then there exists a partition of unity subordinate to $\{U_\alpha\}$.*

PROOF. See [?] Theorem 1.11, page 10. □

Let $f : \Omega \to W$. For $v \in V$ we denote by $\partial_v f : \Omega \to W$ the *directional derivative*

$$\partial_v(f)(x) = \lim_{t\to 0} \frac{f(x + tv) - f(x)}{t}$$

if the limit exists. If f is differentiable, then $\partial_v f(x)$ exists for all $v \in V$ and $x \in \Omega$ and

$$\partial_v(f)(x) = Df(x)v\,.$$

Fix an orthonormal basis $v_1, \ldots, v_m$ of V and $w_1, \ldots, w_n$ for W. Define $f^i : \Omega \to \mathbb{R}$, $i = 1, \ldots, m$, by

$$f = \sum_{i=1}^{n} f^i w_i\,.$$

Let $\partial_j = \partial_{v_j}$. Then the linear map $Df(x)$ is given by the matrix

$$D(f)(x) = (\partial_j f^i(x))_{i=1,\ldots,m;j=1,\ldots,n}\,.$$

A *polynomial function* on V^* is a function $P : V^* \to \mathbb{C}$, such that with respect to a fixed orthonormal basis $v_1, \ldots, v_n$, we have

$$P(\lambda) = \sum_{|\alpha|\leq N} a_\alpha \lambda^\alpha \tag{5.6}$$

for some $a_\alpha \in \mathbb{C}$. Let $\partial_j = \partial_{v_j}$ and set

$$\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}\,.$$

For a polynomial function P as in (5.6) let

$$P(\partial) := \sum_{|\alpha|\leq N} a_\alpha \partial^\alpha .$$

$P(\partial)$ is a constant coefficient differential operator. We will see in the Section **??** (see Remark **??**) that the definition of $P(\partial)$ is independent of the choice of basis.

5.1. Inverse and Implicit Function Theorem. In this subsection V and W are finite dimensional Euclidean vector spaces. The proof of the following, except some of the statements that a function is smooth or analytic can be found in the book by Spivak, [**?**], p. 34–39, and p. 40–43. Ω will always stands for a nonempty open subset of V.

Theorem 5.7 (Inverse Function Theorem). *Suppose $f : \Omega \to V$ is a continuously differentiable function on Ω. If $x \in \Omega$ and* $\det Df(x) \neq 0$*, then there exists an open set $U \subseteq \Omega$ containing x and an open subset $W \subseteq V$ containing $f(x)$ such that $f : U \to W$ is bijective and the function $f^{-1} : W \to U$ is differentiable. Furthermore,*

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$$

for all $y \in W$. If f is smooth, then the function f^{-1} is smooth. If f is analytic, then f^{-1} is analytic as well.

Let $\Omega_V \subseteq V$, and $\Omega_W \subseteq W$ be open and nonempty. For a function $f : \Omega_V \times \Omega_W \to W$, and $a \in \Omega_V$, define $f_a : \Omega_W \to W$ by $f_a(b) := f(a, b)$.

Theorem 5.8 (Implicit Function Theorem). *Let $\Omega_V \subseteq V$, and $\Omega_W \subseteq W$ be open and nonempty. Suppose $f : \Omega_V \times \Omega_W \to W$ is continuously differentiable. Let $(a, b) \in \Omega_V \times \Omega_W$ such that $f(a, b) = 0$. If $Df_a(b) : W \to W$ is an isomorphism, then there exists*

(1) *an open set $\mathcal{V} \subseteq \Omega_V$ containing a and an open set $\mathcal{W} \subseteq \Omega_W$ containing b,*
(2) *a differentiable function $g : \mathcal{V} \to \mathcal{W}$*

such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in \mathcal{V}$. If f is smooth respectively analytic, then g can be chosen smooth respectively analytic.

We will mainly need the following consequence of the implicit function theorem. For a subspace $W_1, W_2 \subset V$ we say that V is the direct sum of W_1, W_2, denoted by $V = W_1 \oplus W_2$, if each vector $v \in V$ has an unique expression as $v = w_1 + w_2$ for some $w_j \in W_j$. Let

$$W^\perp := \{v \in V \mid (\forall w \in W)\ (v, w) = 0\} .$$

Then $V = W \oplus W^\perp$ and the sum is orthogonal. A linear map $P : V \to V$ is a *projection* if $P^2 = P$. It is an *orthogonal projection* if $P^2 = P^* = P$.

We have $V = \mathrm{Im}(P) \oplus \mathrm{Ker}(P)$, and this sum is orthogonal if and only if P is an orthogonal projection. If $W \subseteq V$ is a subspace, then P_W denotes the orthogonal projection onto W.

THEOREM 5.9. *Let $W \subseteq V$ be a subspace of V. Let $\Omega \subset V$ be open and nonempty, and let $f : \Omega \to W$ be a smooth map. Suppose that $a \in \Omega$ is such that $f(a) = 0$, and $Df(a) : V \to W$ is surjective. Then there exists nonempty open sets $\mathcal{V}, \mathcal{W} \subseteq \Omega$ and a diffeomorphism $h : \mathcal{V} \to \mathcal{W}$ such that $a \in \mathcal{V}$ and* ImpFctTh2

$$f \circ h(v) = P_W(v)$$

for all $v \in \mathcal{V}$. If f is analytic, then h can be chosen to be analytic.

THEOREM 5.10. *Let $\Omega \subseteq V$ be open and nonempty set. Let $g : \Omega \to W$ be a smooth function such that $Dg(x)$ has rank $\dim W$ whenever $g(x) = 0$, $x \in \Omega$. Then $g^{-1}(0)$ is an $(\dim V - \dim W)$-dimensional manifold in V. If g is analytic, then $g^{-1}(0)$ is an analytic manifold.*

PROOF. Note first, that the assumption on the rank of $Dg(x)$ implies that $\dim W \leq \dim V$. If $\dim W = \dim V$, then g is locally a diffeomorphism. Hence $g^{-1}(0)$ is a discrete union of points. We can therefore assume that $\dim W < \dim V$. Then we may as well assume that $W \subset V$. The statement then follows from Theorem 5.9. $\square$

EXAMPLE 5.11 (Spheres). Let $a \in \mathbb{R}^n$ and let $R > 0$. Define $g : \mathbb{R}^n \to \mathbb{R}$ by ex-sphers

$$g(x) = \|x - a\|^2 - R^2 .$$

Then g is analytic. In this case $Dg(x)$ is an $n \times 1$-matrix which we view as a column vector. Then

$$Dg(x) = 2(x - a) \neq 0$$

for all $x \in \mathbb{R}^n$ such that $g(x) = 0$. Hence $S_R^{n-1}(a)$ is an $(n-1)$-dimensional analytic submanifold of $\mathbb{R}^n$. Note, if $x = a$, then $Dg(x) = 0$ and $g^{-1}(0) = \{a\}$ is not an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. $\diamondsuit$

LEMMA 5.12. *Let (X, μ) be a measure space open and non-empty and let μ be a Radon measure on Ω. Assume that $\emptyset \neq \Omega \subset V_{\mathbb{C}}$ is open and that $f : X \to \mathbb{C}$ is such that*

1. *For all $z \in \Omega$ the function $x \mapsto f(x, z)$ is integrable.*
2. *For all $x \in X$ the function $z \mapsto f(x, z)$ is holomorphic.*
3. *For each $z_o \in \Omega$ there exists an open neighborhood $W \subset \Omega$ of z_o and a non-negative function $g \in L^1(\Omega, d\mu)$ such that for all $z \in W$*

$$|f(x, z)| \leq g(x) .$$

Let $F(z) := \int_X f(x, z)\, d\mu(x)$. Then F is holomorphic and for all $v \in V_{\mathbb{C}}$ and $z_o \in \Omega$ we have

$$\partial_w F(z_o) = \int_X \partial_{2,v} f(x, z_o)\, d\mu(x)$$

where $\partial_{2,v}$ refers to the directional derivative in the second variable.

PROOF. For $z_o \in \Omega$ let W be as in (3). By replacing f by $(x,z) \mapsto f(x, z+z_o)$ we can assume that $z_o = 0$. Fix $r > 0$ such that $P_{2\mathbf{r}}(0) \subset W$. Let $v \in V_{\mathbb{C}}$, $v \neq 0$. Let $s := r/\|v\|$. Then $tv \in W$ for all $|t| < 2s$. Let

$$P^1_{2s}(0) := \{z \in \mathbb{C} \mid |z| < 2s\}\,.$$

Suppose that $|z| < s$. Let γ be the path $\gamma := \{\zeta \in \mathbb{C} \mid |\zeta| = 2s\}$ with the positive orientation. Then $d(z,\gamma) \geq s$. By the one-dimensional Cauchy Integral Theorem we get

$$\begin{aligned}
\left|\frac{f(x,zv)-f(x,0)}{z}\right| &= \left|\frac{1}{2\pi i}\frac{1}{z}\oint_\gamma \left(\frac{f(x,\zeta v)}{\zeta - z} - \frac{f(x,\zeta v)}{\zeta}\right) d\zeta\right| \\
&= \left|\frac{1}{2\pi i}\oint_\gamma \frac{f(x,\zeta v)}{\zeta(\zeta - z)}\, d\zeta\right| \\
&\leq \frac{1}{2\pi}\oint_\gamma \frac{|f(x,\zeta v)|}{|\zeta(\zeta - z)|}\,|d\zeta| \\
&\leq \frac{1}{2\pi}\oint_\gamma \frac{g(x)}{2s^2}\,|d\zeta| \\
&= g(x)/s\,.
\end{aligned}$$

Hence Lebesgue Dominated Convergence Theorem allows us to move the limit inside the integral to get

$$\begin{aligned}
\lim_{z\to 0}\frac{F(zv)-F(0)}{z} &= \lim_{z\to 0}\int_X \frac{f(x,zv)-f(x,0)}{z}\, d\mu(x) \\
&= \int_X \lim_{z\to 0}\frac{f(x,zv)-f(x,0)}{z}\, d\mu(x) \\
&= \int_X \partial_{2,v} f(x,0)\, d\mu(x)
\end{aligned}$$

and the claim follows. $\square$

Let $\mathcal{M}$ be a $2n$-dimensional manifold. Then $\mathcal{M}$ is a *complex manifold* if there is an atlas $(U_\alpha, \varphi_\alpha)_{\alpha,A}$ such that for all $\alpha \in A$ we have $\varphi_\alpha(U_\alpha) =: V_\alpha \subseteq \mathbb{C}^n$ and for $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ the map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \mathbb{C}^n$$

is holomorphic. We then say that $(U_\alpha, \varphi_\alpha)$ is a complex atlas for $\mathcal{M}$.

If $\mathcal{M}$ is a complex manifold and $(U_\alpha, \varphi_\alpha)$ a complex atlas, then $F : \mathcal{M} \to \mathbb{C}$ is *holomorphic* if F is continuous and $F \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{C}$ is holomorphic. Denote by $\mathcal{O}(\mathcal{M})$ the space of holomorphic functions on $\mathcal{M}$. The topology on $\mathcal{O}(\mathcal{M})$ is that of uniform convergence on compact subsets of $\mathcal{M}$.

THEOREM 5.13. *The space $\mathcal{O}(\mathcal{M})$ is complete.*

Proof. Suppose that $F_n \to F$ in $\mathcal{O}(\mathcal{M})$. Let $(U_\alpha, \varphi_\alpha)$ be a complex atlas. Fix $(U_\alpha, \varphi_\alpha)$. Let $K \subset V_\alpha$ be compact. Then $\varphi_\alpha^{-1}(K) \subset U_\alpha$ is compact. Hence $F_n \circ \varphi_\alpha^{-1} \to F \circ \varphi_\alpha^{-1}$ uniformly on $\varphi_\alpha(K)$. By Weierstrass's Convergence Theorem, Theorem 5.3, the limit function $F \circ \varphi_\alpha^{-1}$ is holomorphic on V_α. Hence F is holomorphic and $\mathcal{O}(\mathcal{M})$ is complete. □

6. Spectral Theory for Unbounded Operators

We assume that the reader is familiar with the basic Hilbert space theory, in particular the spectral theory for bounded operators. For fixing the notation as well as for motivational purposes, we review the basic facts and then discuss the spectral theory for unbounded operators. Our main example later on will be the Laplacian ∂^2 which is an unbounded self-adjoint operator on $L^2(V)$. We discuss semigroups with unbounded generators, and how to take a square root of a positive operator. We use the book [**?, ?, ?**] as reference.

In this section $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces with inner product $(\cdot,\cdot)_\mathcal{H}$, respectively $(\cdot,\cdot)_\mathcal{K}$. If it is clear in which Hilbert space we are working, then we leave out the subscript. We assume that the Hilbert spaces are defined over the field of complex numbers.

6.1. Bounded Operators. As before $\mathrm{B}(\mathcal{H},\mathcal{K})$ denotes the Banach space of bounded linear operators $T : \mathcal{H} \to \mathcal{K}$. If $\mathcal{K} = \mathcal{H}$, then we write $\mathrm{B}(\mathcal{H})$. For $T \in \mathrm{B}(\mathcal{H},\mathcal{K})$ define $T^* \in \mathrm{B}(\mathcal{K},\mathcal{H})$ by

$$(T(u), v)_\mathcal{K} = (u, T^*(v))_\mathcal{H} \quad \text{for all } \ u \in \mathcal{H},\, v \in \mathcal{K}\,.$$

The operator T^* is the *adjoint* of T. The following holds:

(1) $T \mapsto T^*$ is conjugate linear,
(2) $(TS)^* = S^*T^*$, $T^{**} = T$,
(3) $\|T\| = \|T^*\|$, and $\|TT^*\| = \|T^*T\| = \|T\|^2$.

Thus $\mathrm{B}(\mathcal{H})$ is a Banach $*$-algebra.

Another important topology on $\mathrm{B}(\mathcal{H},\mathcal{K})$ is the *strong operator topology* which is given by the family of seminorms

$$\sigma_v(T) = \|T(v)\|_\mathcal{K}\,, \quad v \in \mathcal{H}\,. \tag{6.1}$$

If $\mathcal{L} \subseteq \mathcal{H}$ then

$$\mathcal{L}^\perp := \{u \in \mathcal{H} \mid (\forall v \in \mathcal{H})\ (u, v) = 0\}\,.$$

It is the orthogonal complement of $\mathcal{L}$. $\mathcal{L}^\perp$ is a closed subspace of $\mathcal{H}$ and $\overline{\mathcal{L}} = (\mathcal{L}^\perp)^\perp$.

Lemma 6.1. *Suppose that* $T \in \mathrm{B}(\mathcal{H},\mathcal{K})$*, Then* $\ker(T) = \mathrm{Im}(T^*)^\perp$ *and* $\overline{\mathrm{Im}(T)} = \ker(T^*)^\perp$.

PROOF. The first statement follows immediately from $(v, T^*(u)) = (T(v), u)$. We then have

$$\overline{\operatorname{Im}(T)} = (\operatorname{Im}(T)^\perp)^\perp = \ker(T^*)^\perp .$$

□

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An operator $T \in B(\mathcal{H}, \mathcal{K})$ is *positive* if $(Tu, u) \geq 0$ for all $u \in \mathcal{H}$. As $(T^*T(u), u) = (T(u), T(u)) = \|T(u)\|^2$ it follows that T^*T is positive operator on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is *normal* if T and T^* commutes and *self-adjoint* if $T^* = T$. $T \in B(\mathcal{H}, \mathcal{K})$ is *unitary* if $T^*T = \mathrm{id}_{\mathcal{H}}$ and $TT^* = \mathrm{id}_{\mathcal{K}}$. Thus T is bijective and

$$(T(u), T(v)) = (T^*T(u), v) = (u, v) \quad \text{for all} \quad u, v \in \mathcal{H} . \tag{6.2}$$

If (6.2) holds, then T is a *partial isometry*. It is equivalent to $\|T(u)\| = \|u\|$ for all $u \in \mathcal{H}$. We denote by $B(\mathcal{H}, \mathcal{K})_+$ the space of positive operators and $U(\mathcal{H}, \mathcal{K})$ the space of unitary operators. If $\mathcal{H} = \mathcal{K}$ then we write $B(\mathcal{H})_+$ respectively $U(\mathcal{H})$.

Let I be an index set, $1 \leq p \leq \infty$. Then

$$\ell^p(I) = \{(a_i)_{i \in I} \mid \|(a_i)_{i\in I}\|_p := (\sum_{i \in I} |a_i|^p)^{1/p} < \infty\} \tag{6.3}$$

is a Banach space and a Hilbert space for $p = 2$. Let $I = \mathbb{Z}^+$ and $k \in \mathbb{N}$ define $T : \ell^2(\mathbb{Z}^+)) \to \ell^2(\mathbb{Z}^+)$ by

$$(T((a_i)))_j = \begin{cases} 0 & \text{for} \quad j = 0, 1, \ldots, k-1 \\ a_{j-1} & \text{for} \quad j = k, k+1, \ldots \end{cases} .$$

Then T is a partial isometry which is not an unitary isomorphism. In finite dimension, every partial isometry is also an unitary isometry.

For $T \in B(\mathcal{H})$ define the *spectrum* of T by

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \mathrm{I} \text{ is not invertible } \} \tag{6.4}$$

Assume that $|\lambda| > \|T\|$. Then $\|\lambda^{-1}T\| = |\lambda|^{-1}\|T\| < 1$ and hence $\sum_{n=0}^\infty (\lambda^{-1}T)^n(u)$ converges for all $u \in \mathcal{H}$ and

$$\|\sum_{n=0}^\infty (\lambda^{-1}T)^n(u)\| \leq \sum_{n=0}^\infty \|\lambda^{-1}T\|^n \|u\| = \frac{\|u\|}{1 - \|\lambda^{-1}T\|} .$$

It follows that $u \mapsto \sum_{n=0}^\infty (\lambda^{-1}T)^n(u)$ defines a continuous linear map, which we will denote by $(1-\lambda^{-1}T)^{-1}$. It is easy to see that the operator $-\lambda^{-1}(1-\lambda^{-1}T)^{-1}$ is the inverse to $T - \lambda\mathrm{I}$ and hence $\lambda \notin \sigma(T)$. As the set of invertible elements in $B(\mathcal{H})$ is open in $B(\mathcal{H})$ it follows that $\sigma(T)$ is closed and hence $\sigma(T)$ is compact and contained in the closed ball of radius $\|T\|$. If T is unitary then $\sigma(T) \subseteq \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, and if T is self-adjoint. Then $\sigma(T) \subset \mathbb{R}$. Finally, if T is positive, then $\sigma(T) \subset \mathbb{R}^+ \cup \{0\}$.

6.2. The Spectral Theorem for Bounded Operators. Recall that a map $P : \mathcal{H} \to \mathcal{H}$ is a projection if $P^2 = P$ and that P is orthogonal projection if $P^2 = P = P^*$. Denote by $P(\mathcal{H})$ the set of orthogonal projections on $\mathcal{H}$. Let X be a topological space and $\mathcal{B} = \mathcal{B}(X)$ the Borel sigma algebra on X. A *resolution of the identity* is a mapping $E : \mathcal{B} \to P(\mathcal{H})$ such that

(1) $E(\emptyset) = 0$ and $E(X) = \mathrm{id}$,
(2) $E(A \cap B) = E(A)E(B)$,
(3) If $A \cap B = \emptyset$ then $E(A \cup B) = E(A) + E(B)$,
(4) If $v, w \in \mathcal{H}$ then $E_{v,w} : A \mapsto (E(A)v, w)$ is a regular complex Borel measure on X.

Let E be a resolution of the identity and $v \in \mathcal{H}$. Then $E_{v,v}(A) = (E(A)v, v) = (E(A)^2 v, v) = \|E(A)v\|^2 \geq 0$ so $E_{v,v}$ is a (positive) measure with $E_{v,v}(X) = \|v\|^2 < \infty$. If P and Q are two orthogonal projections, then we write $P \leq Q$ if $P(\mathcal{H}) \subseteq Q(\mathcal{H})$. Then $\leq$ defines a partial ordering on $P(\mathcal{H})$. By (2) and (3) we have $E(A \cap B) \leq E(A), E(B)$ and, if $\{A_j\}_{j\in\mathbb{N}}$ is a family of disjoint sets

$$E(\bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} E(A_j) \leq \sum_{j=1}^{\infty} E(A_j)\,.$$

The series on the right converges in the strong operator topology, but not in the norm topology, except all but finitely many $E(A_j)$ are zero. This is clear because if $I \subset \mathbb{N}$ is finite, then $\sum_{j\in I} E(A_j)$ is as $E(A_j)$ and $E(A_k)$ commutes by (2) and $\|\sum_{j\in I} E(A_j)\| = 1$.

Let $f : X \to \mathbb{C}$ be a measurable function. We say that $f(x) = 0$ E-almost every where, if

$$E(\{x \in X \mid f(x) \neq 0\}) = 0\,.$$

f is *essentially bounded* if there exists a E-zero set such that $f\chi_E$ is bounded. Let $N \subset \mathbb{C}$ be the largest open subset of $\mathbb{C}$ such that $E(f^{-1}(N)) = 0$. The *essential range* of f is $\mathbb{C} \setminus N$. Hence, f is essentially bounded if and only if $\mathbb{C} \setminus N$ is bounded. If f is essentially, bounded set

$$\|f\|_\infty = \sup\{|\lambda| \mid \lambda \in \mathbb{C} \setminus N\} = \sup_{x\in X} |(f\chi_{f^{-1}(N)})(x)|\,.$$

Let $L^\infty(X)$ be the space of essentially bounded measurable functions on X with functions agreeing E-almost every where identified. With the point wise multiplication as a product and conjugation given by $f^*(x) = \overline{f(x)}$, the space $L^\infty(X)$ becomes an abelian Banach $*$-algebra.

Let $f \in L^\infty(E)$ and $v, w \in \mathcal{H}$. Then, as $E_{v,w}$ is a bounded complex measure, it follows that

$$\int_X |f(x)| d|E_{v,w}|(x) \leq \|f\|_\infty \|v\| \|w\|\,.$$

It therefore exists a unique $T = T_f \in B(\mathcal{H})$ such that

$$(T_f(v), w) = \int_X f(x) dE_{v,w}(x)\,. \tag{6.5}$$

We write this as

$$T_f = \int_X f\, dE\,. \tag{6.6}$$

As $E(X) = \mathrm{I}$ we have

$$\|T_f\| = \|f\|_\infty\,. \tag{6.7}$$

THEOREM 6.2. *Suppose that E is a resolution of the identity. Then the map $L^\infty(X) \to B(\mathcal{H})$, $f \mapsto T_f$, is a $*$-isomorphism onto a closed abelian $*$-Banach subalgebra of $B(\mathcal{H})$. Furthermore,*

1. *If $v \in \mathcal{H}$, then $\|T_f(v)\| = \int_X |f(x)|\, dE_{v,v}(x)$.*
2. *An operator $S \in B(\mathcal{H})$ commutes with all $E(A)$ if and only if S commutes with all T_f, $f \in L^\infty(X)$.*

PROOF. The idea of the proof is to start with simple step functions of the form $f = \sum_{j=1}^N \alpha_j \chi_{A_j}$. Then

$$T_f = \sum_{j=1}^N \alpha_j E(A_j)\,.$$

Furthermore

$$T_f^* = \sum_{j=1}^n \overline{\alpha_j} E(A_j) = T_{f^*}\,.$$

A simple calculation also show that

$$T_{fg} = \sum_{i,j \text{ finite}} \alpha_i \beta_j E(A_j \cap B_j) = T_f T_g$$

if $g = \sum_{\text{finite}} \beta_j \chi_{B_j}$. The general statement can then be proven by approximating an arbitrary function $f \in L^\infty(X)$ by step functions. We refer to [**?**], p. 319-320, for details. □

THEOREM 6.3 (Spectral Theorem for Bounded Opeators). *Let $T \in B(\mathcal{H})$ be normal. Then there exists a unique resolution of the identity E on $\sigma(T)$ such that*

1. $T = \displaystyle\int_{\sigma(T)} \lambda\, dE(\lambda)$.
2. *$S \in B(\mathcal{H})$ commutes with T if and only if S commutes with all $E(A)$, $A \subseteq \sigma(T)$ a Borel set.*

Furthermore the following holds true:

(3) *If* $P(z,w) = \sum a_{j,k} z^j w^k$ *is a polynomial then*

$$P(T,T^*) = \sum a_{j,k} T^j T^{*k} = \int P(\lambda, \overline{\lambda})\, dE(\lambda)\,.$$

(4) *if* T *is positive, then*

$$\sqrt{T} = \int_{\sigma(T)} \sqrt{\lambda}\, dE(\lambda)$$

is well defined.

PROOF. See [**?**], p. 324. □

CHAPTER 2

Lie Groups, Representations, and Homogeneous Spaces

In this chapter we discuss some basic facts about Lie groups and homogeneous spaces. To simplify the matter we will mostly deal with linear Lie groups. In the next chapter we specialize this discussion to the sphere and the hyperbolic space.

1. Lie Groups and Homogeneous Manifolds

Recall, a (finite dimensional) *Lie group* G is a group and an analytic manifold and those two structures are connected by the requirement that the map

$$G \times G \to G\,, \quad (x, y) \mapsto xy^{-1} \tag{1.1}$$

is analytic.

It is the strength of the interplay between algebra (group) and analysis (manifold) that it is enough to assume that G is locally Euclidean and the map (1.1) is continuous. This is Hilbert's fifth problem, which was solved by A. Gleason, D. Montgomery and L. Zippin in the 1950. A good reading on this exciting problem is the book by I. Kaplansky [**?**]. We refer to the book by S. Helgason [**?**] or the book by V. S. Varadaranjan [**?**] for information about Lie groups.

We will only consider finite dimensional Lie group, and hence simply call them Lie groups. As for manifolds, we will always assume that our Lie groups are separable. In particular, they have at most countable many connected components.

In this chapter $\mathbb{F}$ will always stand for the field of real or complex numbers. One can develop similar theory for the shew field $\mathcal{H}$, but we will not do so. $V \simeq \mathbb{F}^n$ will be a n-dimensional vector space over $\mathbb{F}$. $\mathrm{M}(V)$ will stand for the space of $\mathbb{F}$-linear maps $V \to V$. After choosing a basis it is isomorphic to the space $\mathrm{M}(n, \mathbb{F})$ of $n \times n$-matrices. The isomorphism is given by

$$T(e_j) = \sum_{i=1}^{n} t_{ij} e_i\,.$$

This way $\mathrm{M}(V)$ becomes a n^2-dimensional vector space isomorphic to $\mathbb{F}^{n^2}$.

Our basic examples of Lie groups will be *linear Lie groups*, e.g. closed subgroups of the *general linear group*

$$\mathrm{GL}(n, \mathbb{F}) = \mathrm{GL}(V) := \{A \in \mathrm{M}(V) \mid \det A \neq 0\}$$

where $\mathbb{F}$ stands for the field of real or complex numbers. As $\det : \mathrm{M}(V) \to \mathbb{F}$ is a polynomial functions and hence analytic it follows that $\mathrm{GL}(V)$ is an open subset of $\mathrm{M}(V)$ and hence a n^2-dimensional manifold. The coordinate maps are simply $x_{ij}((a_{\nu\mu})_{\nu,\mu}) = a_{ij}$. The multiplication of two matrices is given by

$$[(x_{ij})(y_{ij})]_{\nu\mu} = \sum_{j=1}^{n} x_{\nu j} y_{j\mu}$$

and those are polynomial functions. By Cramer's rule the inverse $g \mapsto g^{-1}$ is a rational map in the coordinates. In particular, both multiplication and taking the inverse are analytic maps. Hence $\mathrm{GL}(V)$ is a Lie group. Our aim is to show

(1) If $H \subset \mathrm{GL}(V)$ is a closed subgroup, then H is a Lie group;
(2) If $H \subset G \subseteq \mathrm{GL}(V)$ are closed subgroups, then G/H is an analytic manifold.

The main tool towards those goals is the exponential map which we will now discuss.

Definition 1.1. (1) Let $(\mathfrak{g}, [\cdot,\cdot])$ be a vector space over $\mathbb{F}$ with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then $(\mathfrak{g}, [\cdot,\cdot])$, or simply $\mathfrak{g}$, is a *Lie algebra* if
   (a) (anti commutativity) $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$,
   (b) (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.
(2) If $\mathfrak{g}$ and $\mathfrak{h}$ are two Lie algebra over $\mathbb{F}$. A *Lie algebra homomorphism* $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a $\mathbb{F}$ linear map such that $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ for all $X, Y \in \mathfrak{g}$.
(3) The Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic if there exists a linear isomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ which is also a Lie algebra homomorphism.
(4) A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal if $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$.

Note that if $\mathfrak{h}$ is an ideal, then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Furthermore, if $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism then

$$\varphi(\varphi^{-1}([X, Y])) = [X, Y] = \varphi([\varphi^{-1}(X), \varphi^{-1}(Y)]) \,.$$

As φ is injective, it follows that φ^{-1} is also a Lie algebra homomorphism.

Example 1.2. Let A be an associative algebra. Define $[\,,\,] : A \times A \to A$ by

$$[a, b] := ab - ba \,.$$

Then a simple calculation shows that $(A, [\,,\,])$ is a Lie algebra. In particular, $\mathrm{M}(V)$ is a Lie algebra usually denoted by $\mathfrak{gl}(V)$ or $\mathfrak{gh}(n, \mathbb{F})$. ♢

Let us discuss here briefly the construction of the tangent bundle of a manifold and vector fields. We will discuss vector bundle and sections more generally in a moment.

Let $\mathcal{M}$ be a manifold and $T(\mathcal{M}) = \bigcup_{m\in\mathcal{M}} T_m(\mathcal{M})$ its tangent bundle with projection $\pi : T(\mathcal{M}) \to \mathcal{M}$. Let $(U_\alpha, \varphi_\alpha)$ be coordinates. Recall the tangent vectors $\dfrac{\partial}{\partial x_i}$ on U_α defined by

$$\left.\frac{\partial}{\partial x_i}\right|_p (f) := \partial_i(f\circ\varphi_\alpha^{-1})(\varphi_\alpha(p))\,, \quad p\in U_\alpha,\ f\in C^\infty(U_\alpha)\,.$$

We will simply denote those partial derivatives by $\partial_i|_p$. The tangent vectors $\partial_1|_p, \ldots, \partial_m|_p$ form a basis for each $T_p(\mathcal{M})$, $p\in U_\alpha$. We make $T(\mathcal{M})$ into a m-dimensional manifold by defining local coordinates by

$$\mathcal{W}_\alpha := \pi^{-1}(U_\alpha) \ni (p, \sum_{i=1}^m t_i\partial_i|_p) \mapsto (\varphi_\alpha(p), t_1, \ldots, t_m) \in \mathbb{R}^m\times\mathbb{R}^m\,.$$

If $\mathcal{M}$ in an analytic manifold, then $T(\mathcal{M})$ becomes an analytic manifold also.

A (smooth) *vector field* on $\mathcal{M}$ is a smooth map $X : \mathcal{M}\to T(\mathcal{M})$ such that $X(p)\in T_p(\mathcal{M})$ for all $p\in\mathcal{M}$. such that $X(m)\in T_m(\mathcal{M})$ for all $m\in\mathcal{M}$. If $\mathcal{M}$ is an analytic manifold, then we define an analytic vector filed in the same way. We often write X_p for $X(p)$.

In local coordinates we have

$$X(p) = \sum_{j=1}^n a_j(p)\ \left.\frac{\partial}{\partial x_j}\right| |_p\,. \tag{1.2}$$

X is smooth respectively analytic if and only if the functions a_j are smooth respectively analytic. We denote the space of smooth vector fields by $C^\infty(T(\mathcal{M})) = \Gamma^\infty(\mathcal{M})$ and the space of analytic vector fields by $C^\omega(T(\mathcal{M})) = \Gamma^\omega(\mathcal{M})$.

Let X, Y be vector fields. Define $XY(f) = X(Y(f))$ and

$$[X,Y] = XY - YX\,.$$

Lemma 1.3. *Let* $X, Y\in\Gamma^\infty(\mathcal{M})$. *Then* $[X,Y]\in\Gamma^\infty(\mathcal{M})$. $(\Gamma^\infty(\mathcal{M}), [\,,\,])$ *is Lie algebra and* $\Gamma^\omega(\mathcal{M})$ *a subalgebra.*

Proof. Write locally $X = \sum_j = a_j\partial_j$ and $Y = \sum_j = b_j\partial_j$. Then

$$XY = \sum_i a_i\sum_j \partial_i b_j\partial_j + \sum_{ij} a_i b_j\partial_i\partial_j$$

and similarly for YX. It follows that

$$XY - YX = \sum_i\left(\sum_j a_j\partial_j b_i - b_j\partial_j a_i\right)\,\partial_i$$

and it follows that $[X, Y]$ is a vector field which is clearly smooth, respectively analytic, if X and Y are smooth, respectively both analytic. The rest of the statement is a simple calculation which follows from the fact that $X(YZ) = (XY)Z$ and is left to the reader. $\square$

For G a Lie group let $\mathfrak{g} := \Gamma^\infty(G)^G$ be the Lie algebra of (left) invariant vector fields. $\mathfrak{g}$ is a Lie algebra and it is called the *Lie algebra of* G. It is standard to denote Lie groups by capital Latin letters, where as the space $\Gamma(G)^G$ is denoted by the corresponding German letter.

EXAMPLE 1.4 (Invariant vector fields on $\mathbb{R}^n$). Let $G = \mathcal{M} = \mathbb{R}^n$. Then a vector field X can be written globally as $X = \sum_{j=1}^n a_j \partial_j$. Let $x, y \in \mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n)$. As $\partial_j(f \circ \ell_y)(x) = \partial_j f(x + y)$.

$$X(f \circ \ell_y)(x) = \sum_{j=1}^n a_j(x)\partial_j(f \circ \ell_y)(x) = \sum_{j=1}^n a_j(x)\partial_j f(x + y)$$

and

$$(X(f) \circ \ell_y)(x) = X(f)(x + y) = \sum_{j=1}^n a_j(x + y)\partial_j(f)(x + y).$$

Hence X is invariant if and only if the maps a_j are constants. $\diamondsuit$

LEMMA 1.5. *The map* $\mathfrak{g} \to T_e(G)$*,* $X \mapsto X(e)$*, is a linear isomorphism. In particular,* $\dim \mathfrak{g} = \dim G < \infty$*.*

PROOF. The map $T : \mathfrak{g} \to T_e(G)$, $T(X) = X(e)$ is clearly linear. If X is invariant, then $X(g) = (d\ell_g)_e(X(e))$. In particular, if $X(e) = 0$, then $X = 0$ and T is injective. If $v \in T_e(G)$, define $X(g) := (d\ell_g)_0(v)$. Then $X \in \mathfrak{g}$ and $T(X) = v$, so T is surjective. $\square$

DEFINITION 1.6. Let G be a topological group. A *one-parameter subgroup* is a continuous homomorphism $\gamma : \mathbb{R} \to G$.

Note that, if γ is a one-parameter subgroup, then the image $\gamma(\mathbb{R})$ is a subgroup of G. It need not to be closed.

Let $I \subseteq \mathbb{R}$ be an open interval, $0 \in I$. Let $\mathcal{M}$ be a manifold and $\gamma : I \to \mathcal{M}$ at least once differentiable. We use the notation $\dot\gamma(t) = (d\gamma)_t(d/dt|_t)$. If $\mathcal{M}$ is a finite dimensional vector space, then the tangent space at each point is isomorphic to V and $\dot\gamma$ is the usual derivative $\dot\gamma = \frac{d\gamma}{dt}$.

2. The Exponential Function

The matrix or operator exponential map $\exp : \mathrm{M}(V) \to \mathrm{M}(V)$ is given by the power series

$$\exp(X) = \sum_{j=0}^\infty \frac{X^n}{n!} \tag{2.1}$$

To show that the series converges, consider the operator norm

$$\|X\| := \sup_{\|u\|=1} \|X(u)\|$$

on $\mathrm{M}(V)$. Note that $\mathrm{M}(V)$ is a finite dimensional vector space and hence all topologies that make $\mathrm{M}(V)$ into a Hausdorff topological vector space, ie., addition and scalar multiplications are continuous, are the same. Thus $\|\cdot\|$ defines the standard topology on $\mathrm{M}(V)$. What makes the operator norm so useful is that $\|XY\| \leq \|X\|\,\|Y\|$ for all $X, Y \in \mathrm{M}(V)$. In particular $\|X^n\| \leq \|X\|^n$ for all $n \in \mathbb{N}$.

THEOREM 2.1. *The sum defining the exponential map converges uniformly on each closed ball $\overline{B}_R(0) = \{X \in \mathrm{M}(V) \mid \|X\| \leq R\}$. Furthermore, $\|\exp X\| \leq e^{\|X\|}$.*

PROOF. Let $\epsilon > 0$. For $R > 0$ let $N \in \mathbb{N}$ be such that $\sum_{k=n}^{m} |x|^k/k! < \epsilon$ for all $N \leq n < m$ and all $x \in \mathbb{F}$ with $|x| \leq R$. Assume that $\|X\| \leq R$. Then

$$\left\| \sum_{k=n}^{m} \frac{X^k}{k!} \right\| \leq \sum_{k=n}^{m} \frac{\|X\|^k}{k!} < \epsilon\,.$$

It follows that the series on the right hand side of (2.1) is uniformly Cauchy on the closed ball $\bar{B}_R(0)$ and hence converges uniformly on $\bar{B}_R(0)$.

We have

$$\|e^X\| = \lim_{N\to\infty} |\sum_{j=0}^{N} \frac{X^j}{j!}\| \leq \lim_{N\to\infty} \sum_{j=0}^{N} \frac{\|X\|^j}{j!} \leq e^{\|X\|}$$

as claimed. □

It follows that $\exp : \mathrm{M}(V) \to \mathrm{M}(V)$ is an analytic map.

For $x = (x_1, \ldots, x_n)^t \in \mathbb{F}^n$, denote by $\mathrm{diag}(x_1, \ldots, x_n)$ the diagonal matric with diagonal elements $x_1, \ldots, x_n$, i.e.,

$$\mathrm{diag}(x) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_n \end{pmatrix}.$$

THEOREM 2.2. *Let $X, Y \in \mathrm{M}(V)$ and $g \in \mathrm{GL}(V)$.*

(1) *If X and Y commutes, then $e^{X+Y} = e^X e^Y$.*
(2) *If $g \in \mathrm{GL}(V)$ then $g e^X g^{-1} = \exp(gXg^{-1})$,*
(3) *If*

$$X = \begin{pmatrix} \lambda_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_n \end{pmatrix}$$

then e^X is the upper triangular matric

$$e^X = \begin{pmatrix} e^{\lambda_1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & e^{\lambda_n} \end{pmatrix}$$

(4) $(e^X)^t = e^{X^t}$ *and* $(e^X)^* = e^{X^*}$.
(5) *If X is a lower triangular matric, then so is e^X.*
(6) $\det(e^X) = e^{\mathrm{Tr}(X)}$. *In particular $e^X \in \mathrm{GL}(V)$.*

PROOF. (1) We have $(X+Y)^k = \sum_{j=1}^{k} \binom{k}{j} X^j Y^{k-j}$ as X and Y commutes. Thus

$$\begin{aligned} e^{X+Y} &= \sum_{k=0}^{\infty} \frac{1}{k!}(X+Y)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} X^j Y^{k-j} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{X^j}{j!} \frac{Y^{k-j}}{(k-j)!} \\ &= \sum_{k=0}^{\infty} \frac{X^k}{k!} \sum_{j=0}^{\infty} \frac{Y^j}{j!} \\ &= e^X e^Y . \end{aligned}$$

(2) This follows by considering first the finite sums

$$g \left(\sum_{j=0}^{N} \frac{X^j}{j!} \right) g^{-1} = \sum_{j=0}^{N} \frac{gX^j g^{-1}}{j!} = \sum_{j=0}^{N} \frac{(gXg^{-1})^j}{j!}$$

and then take the limit $N \to \infty$.

(3) follows by the fact that for all $j \in \mathbb{N}$

$$\begin{pmatrix} \lambda_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_n \end{pmatrix}^j = \begin{pmatrix} \lambda_1^j & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_n^j \end{pmatrix} .$$

For (4) consider finite partial sums and use that $(X^j)^t = (X^t)^j$ and $(X^j)^* = (X^*)^j$.

(5) follows from (3) and (4).

For (6) we note first that by (3) the statement is clear for upper triangular matrices. For the general case, chose $g \in \mathrm{GL}(n, \mathbb{C})$ so that $Y = g^{-1}Xg$ is upper triangular. Then

$$\mathrm{Tr}Y = \mathrm{Tr}(g^{-1}Xg) = \mathrm{Tr}(Xgg^{-1}) = \mathrm{Tr}(X)$$

and, as $e^X = ge^Y g^{-1}$

$$\det(e^X) = \det(ge^Y g^{-1}) = \det(e^Y) = e^{\mathrm{Tr}Y} = e^{\mathrm{Tr}X}\,.$$

□

EXAMPLE 2.3. Let $X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Induction shows that

$$X^{2j} = (-1)^j \mathrm{I} \quad \text{and} \quad X^{2j+1} = (-1)^j X\,.$$

Hence

$$\begin{aligned} \exp(tX) &= \sum_{j=0}^{\infty} (-1)^j \frac{t^{2j}}{(2j)!}\mathrm{I} + \sum_{j=0}^{\infty} (-1)^j \frac{t^{2j+1}}{(2j+1)!}\,X \\ &= \cos(t)\mathrm{I} + \sin(t)X \\ &= \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}. \end{aligned}$$

This shows in particular that $\exp : \mathrm{M}(V) \to \mathrm{GL}(V)$ is not injective.

Assume that X is real and symmetric, $X^t = X$. Then there is $g \in \mathrm{GL}(V)$ such that $gXg^{-1} = \mathrm{diag}(x_1, \dots, x_n)$. Hence

$$e^X = g^{-1}\mathrm{diag}(e^{x_1}, \dots, e^{x_n})g$$

and it follows easily that exp is a bijection from $\mathrm{Sym}(V)$, the space of symmetric matrices (operators), onto the set of positive definite matrices (operators). ♢

As $\mathrm{GL}(V)$ is an open subset of the vector space $\mathrm{M}(V)$ we can identify the tangent space $T_g(\mathrm{GL}(V))$ with $\mathrm{M}(V)$. The identification is given by

$$Xf(g) = \left.\frac{d}{dt}\right|_{t=0} f(g + tX) = Df(g)X = \partial_X f(g)\,.$$

We identify the tangent space of $\mathrm{M}(V)$ with $\mathrm{M}(V)$. Then $(D\exp)(0)$ is a linear map $\mathrm{M}(V) \to \mathrm{M}(V)$.

LEMMA 2.4. $(D\exp)(0) = \mathrm{id}$.

PROOF. This is a direct consequence of Lemma 3.1, part 1. Let $f \in C^\infty(\mathrm{GL}(V))$ and $X \in \mathrm{M}(V)$. Then by the chain rule:

$$(D\exp)(0)(X)f = \left.\frac{d}{dt}\right|_{t=0} f(e^{tX}) = Df(\mathrm{I})D\exp(0)X = Df(\mathrm{I})X\,.$$

□

Now the Inverse Function Theorem, Theorem 5.7 implies the following theorem:

THEOREM 2.5. *There exists an open neighborhood* $0 \in U \subset \mathrm{M}(V)$ *an open neighborhood* $\mathrm{I} \in W \subset \mathrm{GL}(V)$ *such that* $\exp : U \to W$ *is a diffeomorphism.*

So how big can we choice U and W? A partial answer is given by the following.

LEMMA 2.6. *If* $\|A - \mathrm{I}\| < 1$ *then* $A \in \mathrm{GL}(V)$ *and* $A^{-1} = \sum_{j=0}^{\infty} (\mathrm{I} - A)^j$.

PROOF. Assume that $Au = 0$. If $u \neq 0$ then

$$\|u\| = \|(A - I)(u)\| \leq \|A - I\| \|u\| < \|u\|$$

which is impossible. Thus A is injective and hence an isomorphism. The series $B := \sum_{j=0}^{\infty} (\mathrm{I} - A)^j$ converges as $\|\mathrm{I} - A\| < 1$. We have $(\mathrm{I} - A)B = \sum_{j=1}^{\infty} (\mathrm{I} - A)^j = B - \mathrm{I}$. Similarly, $B(\mathrm{I} - A) = B - \mathrm{I}$. Thus $AB = BA = \mathrm{I}$ and the claim follows. □

Using the power series for $\log(t)$ which converges for $|1 - t| < 1$ we get

LEMMA 2.7. *If* $\|A - I\| < 1$ *then the power series*

$$\log(A) := \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(A - \mathrm{I})^k}{k} \tag{2.2}$$

converges uniform on every closed ball

$$\overline{B}_r(\mathrm{I}) := \{A \in \mathrm{GL}(V) \mid \|A - \mathrm{I}\| \leq r\}, \quad 0 < r < 1$$

and

(1) $e^{\log A} = A$,
(2) $\log(e^X) = X$.

The condition $\|A - \mathrm{I}\| < 1$ is only sufficient and not always necessary for the convergence of the power series defining $\log A$.

DEFINITION 2.8. A matrix X is nilpotent if there exists k such that $X^k = 0$. It is unipotent if $X - \mathrm{I}$ is nilpotent.

Note, if a is unipotent then $a \in \mathrm{GL}(V)$. In fact, assume that $au = 0$. Then $(a - \mathrm{I})u = u$ and hence, with k such that $(a - \mathrm{I})^k = 0$, $u = (a - \mathrm{I})^k u = 0$.

If a is unipotent, then clearly the power series (2.2) reduces to a polynomial and hence converges independent of the norm of $a - \mathrm{I}$. This norm can be arbitrary big as can be seen by the matrix

$$n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

As $(n_x - \mathrm{I})e_2 = xe_1$ it follows that $\|n_x - \mathrm{I}\| \geq |x|$ which can be arbitrary big. But as a consequence of this we get:

THEOREM 2.9. *Let*

$$N = \{\mathrm{I} + X \mid X \in \mathrm{GL}(V)\,,\ X_{ij} = 0 \textit{ for } i > j\}$$

be the group of upper triangular matrices with 1 *on the diagonal and*

$$\mathfrak{n} = \{X \mid X \in \mathrm{GL}(V)\,,\ X_{ij} = 0 \textit{ for } i > j\}\,.$$

Then $\exp : \mathfrak{n} \to N$ *is an analytic diffeomorphism.*

3. The exponential map and one parameter subgroups

For $X \in \mathrm{M}(V)$ define $\gamma_X : \mathbb{R} \to \mathrm{GL}(V)$ by $\gamma_X(t) = e^{tX}$. Then $\gamma_X : \mathbb{R} \to \mathrm{GL}(V)$ is a one-parameter subgroup.

THEOREM 3.1 (Characterization of the *exp*). *The following holds:*

(1) γ_X *is analytic and* $\dot{\gamma}_X(t) = X\gamma_X(t) = \gamma_X(t)X$.
(2) *If* $\epsilon > 0$ *and* $\gamma : (-\epsilon, \epsilon) \to \mathrm{GL}(V)$ *is differentiable such that* $\dot{\gamma}(t) = \gamma(t)X$ *or* $\dot{\gamma}(t) = X\gamma(t)$ *for some* $X \in \mathrm{M}(V)$, *then there exists* $X \in \mathrm{M}(V)$ *such that* $\gamma = \gamma(0)\gamma_X$. *In particular,* γ *extends to a analytic function on* $\mathbb{R}$.
(3) *If* $\gamma : \mathbb{R} \to \mathrm{GL}(V)$ *is a one-parameter subgroup, then* γ *is analytic and there exists an unique* $X \in \mathrm{M}(V)$ *such that* $\gamma = \gamma_X$.

PROOF. (1) γ_X is differentiable and

$$\frac{\gamma_X(t+h) - \gamma_X(t)}{t} = \gamma_X(t)\frac{\gamma_X(h) - \gamma_X(0)}{h} = \frac{\gamma_X(h) - \gamma_X(0)}{h}\gamma_X(t)\,.$$

It therefore suffice to show that

$$\lim_{h\to 0} \frac{\gamma_X(h) - \gamma_X(0)}{h} = X\,.$$

For that we have

$$\begin{aligned}
\left\|\frac{\gamma_X(h) - \gamma_X(0)}{h} - X\right\| &= \left\|\frac{1}{h}\sum_{k=1}^{\infty}\frac{h^k}{k!}X^k - X\right\| \\
&\leq \left\|\sum_{k=2}^{\infty}\frac{X^k}{k!}h^{k-1}\right\| \\
&= |h|\sum_{k=0}^{\infty}\frac{\|X\|^{k+2}}{(k+2)!}|h|^k \\
&\leq |h|\|X\|^2 e^{|h|\|X\|} \\
&\overset{h\to 0}{\longrightarrow} 0
\end{aligned}$$

(2) Assume that $\dot\gamma(t) = X\gamma(t)$. Define $F : (-\epsilon, \epsilon) \to \mathrm{GL}(V)$ by $F(t) := \exp(-tX)\gamma(t) = \gamma_X(t)^{-1}\gamma(t)$. Then F is differentiable and

$$F'(t) = -\gamma_X(t)^{-1}X\gamma(t) + \gamma_X(t)^{-1}X\gamma(t) = 0\,.$$

Hence $F(t)$ is constant and the claim follows as $F(0) = \gamma(0)$. In case $\gamma'(t) = \gamma(t)X$ we define $F(t) = \gamma(t)\gamma_X(t)^{-1}$ and the rest of the proof is the same.

(3) First we note that for $h \in \mathbb{R}$

$$\frac{\gamma(t+h) - \gamma(t)}{h} = \frac{\gamma(h) - \mathrm{I}}{h}\gamma(t) \tag{3.1}$$

so we only have to show that γ is differentiable at $t = 0$. It then follows from (**??**) that

$$\dot\gamma(t) = \dot\gamma(0)\gamma(t)$$

and hence by (2)

$$\gamma(t) = e^{tX} \qquad \text{where } X = \dot\gamma(0)\,.$$

For $h > 0$ and $t \in \mathbb{R}$ define

$$F(t) := \frac{1}{h}\int_t^{h+t} \gamma(u)\,du\,.$$

Then F is differentiable as γ is continuous. The change of variable $v = u - t$ gives

$$F(t) = \frac{1}{h}\int_0^h \gamma(v+t)\,dv = \frac{1}{h}\int_0^h \gamma(v)\,dv\,\gamma(t)\,.$$

As γ is continuous it follows that

$$\lim_{h\to 0} \frac{\int_0^h \gamma(v)\,dv}{h} = \gamma(0) = \mathrm{I}\,.$$

Thus, there exists $\epsilon > 0$ such that

$$\|\frac{1}{h}\int_0^h \gamma(v)\,dv - \mathrm{I}\| < 1$$

for all $0 < h < \epsilon$. Fix such an h and define $A := h^{-1}\int_0^h \gamma(v)\,dv$. Then A is invertible by Lemma 2.6 and $\gamma(t) = A^{-1}F(t)$. Hence γ is differentiable. $\square$

We now know that $T_{\mathrm{I}}(\mathrm{GL}(V)) \simeq \mathrm{M}(V)$ using either that $\mathrm{GL}(V)$ is an open subset of $\mathrm{M}(V)$ or by using the exponential map as local coordinates. The above connection with one-parameter subgroups also shows that the Lie algebra $\mathfrak{gl}(V)$ of $\mathrm{GL}(V)$ is isomorphic to $\mathrm{M}(V)$ as a vector space. Here the isomorphism is given by

$$\mathrm{M}(V) \to \mathfrak{gl}(V) \quad X \mapsto D_X$$

where

$$D_X f(a) = \left.\frac{d}{dt}\right|_{t=0} f(ae^{tX}) = \left.\frac{d}{dt}\right|_{t=0} f(a + taX)\,.$$

Thus we now have two Lie algebra structures on $\mathrm{M}(V)$, once coming from the algebra structure on $\mathrm{M}(V)$ and the second coming from $\mathrm{M}(V)$ as the Lie algebra of left-invariant vector fields on $\mathrm{GL}(V)$. The following shows that those structures are the same.

LEMMA 3.2. *Let* $X, Y \in \Gamma^\infty(\mathcal{M})^G$. *Then* $[X, Y] \in \Gamma^\infty(\mathcal{M})^G$. *Thus* $\Gamma^\infty(\mathcal{M})^G$ *is a Lie algebra.*

PROOF. Let $a \in G$. Then $XY(f \circ \ell_a) = X(Y(f) \circ \ell_a) = (XY(f)) \circ \ell_a$. Thus $[X, Y](f \circ \ell_a) = ([X, Y](f)) \circ \ell_a$ and the claim follows from (4.2). □

For $G = \mathrm{GL}(V)$ we have now two Lie algebras canonically associated to G. First of all $\mathfrak{g}$, the Lie algebra of left invariant vector fields on G, and secondly the Lie algebra $\mathfrak{gl}(V)$. We will now show that those are isomorphic as Lie algebras. For that define a map $X \mapsto \widetilde{X}$, $\mathfrak{gl}(V) \to \mathfrak{g}$, by

$$\widetilde{X} f(g) := \frac{d}{dt}\bigg|_{t=0} f(ge^{tX}) = (d\ell_g)_{\mathrm{I}}(X) f\,.$$

LEMMA 3.3. *Let* $X, Y \in \mathrm{M}(V)$. *Then*

$$\widetilde{[X, Y]} = [\widetilde{X}, \widetilde{Y}]\,.$$

In particular, $\mathrm{M}(V)$ *with the standard commutator product,* $[X, Y] = XY - YX$, *is isomorphic to the Lie algebra* $\mathfrak{gl}(V)$ *of* $\mathrm{GL}(V)$.

PROOF. We only have to show that

$$D_{[X,Y]} = [D_X, D_Y]\,.$$

As both sides are left invariant it is enough to show that this holds at the identity.

We have

$$D_X f(e^{sY}) = \frac{d}{dt}\bigg|_{t=0} f(e^{sY}e^{tX} = Df(e^{sY})e^{sY}X\,.$$

Hence

$$D_Y D_X f(\mathrm{I}) = D^2 f(\mathrm{I})(X, Y) + Df(\mathrm{I})YX\,.$$

As $D^f(\mathrm{I})$ is a symmetric bilinear form (the Hessian) it follows that

$$[D_X, D_Y] = Df(\mathrm{I})(XY - XY) = D_{[X,Y]}$$

and the lemma follows. □

4. Further properties of the exponential map

THEOREM 4.1 (Properties of exp). *Let* $X, Y, X_1, \ldots, X_s \in \mathrm{M}(V)$ *and* $t \in \mathbb{R}$ *such that* $|t|$ *is sufficiently small.*

(1) *Let* $X_1, \ldots, X_s \in \mathrm{M}(V)$ *and* $t \in \mathbb{R}$. *Then*

$$e^{tX_1} \cdots e^{tX_s} = \exp\left[t\sum_{j=1}^{s} X_j + \frac{t^2}{2}\sum_{1\le i<j\le s}[X_i, X_j] + O(t^3)\right].$$

(2) $e^{tX}e^{tY}e^{-tX} = \exp(tY + t^2[X, Y] + O(t^3))$.
(3) $e^{tX}e^{tY}e^{-tX}e^{-tY} = \exp(t^2[X, Y] + O(t^3))$.
(4) $e^{X+Y} = \lim_{m\to\infty}(\exp(X/m)\exp(Y/m))^m$.
(5) $e^{[X,Y]} = \lim_{m\to\infty}\left(e^{X/m}e^{Y/m}e^{-X/m}e^{-Y/m}\right)^{m^2}$.

PROOF. (1) Let U and V be as in Lemma 2.5. Assume first that $s = 2$. Set $X = X_1$ and $Y = X_2$. Then

$$\begin{aligned} e^{tX}e^{tY} &= (I + tX + \frac{t^2}{2}X^2 + O(t^3))(\mathrm{I} + tY + \frac{t^2}{2}Y^2 + O(t^3)) \\ &= \mathrm{I} + t(X+Y) + \frac{t^2}{2}(X^2 + Y^2 + 2XY) + O(t^3) \\ &= \mathrm{I} + t(X+Y) + \frac{t^2}{2}((X+Y)^2 + [X, Y]) + O(t^3). \end{aligned}$$

On the other hand

$$\exp(t(X+Y)+\frac{t^2}{2}[X,Y]+O(t^3)) = \mathrm{I}+t(X+Y)+\frac{t^2}{2}[X,Y]+\frac{t^2}{2}(X+Yr)+O(t^3).$$

Let $\mathrm{I} \in V$ be open, symmetric and such that $V^2 \subseteq W$. Let $\epsilon > 0$ be so that $tX, tY, t(X+Y) + \frac{t^2}{2}[X, Y] + O(t^3) \in W$ for all $|t| < \epsilon$. The claim then follows as $\exp : W \to V$ is a diffeomorhism. The general case follows in the same way by induction.

(2) and (3) follows from (1) by taking respectively $s = 3$ and $X_1 = X$, $X_2 = Y$, and $X_3 = -X$, respectively $s = 4$ and $X_1 = X$, $X_2 = Y$, $X_3 = -X$, and $X_4 = -Y$.

Let $t = 1/m$. Then (1) implies for m big enough that

$$\begin{aligned} \left(e^{X/m}e^{Y/m}\right)^m &= \left(\exp\left(\frac{1}{m}(X+Y) + O\left(\frac{1}{m^2}\right)\right)\right)^m \\ &= \exp\left(X + Y + O\left(\frac{1}{m}\right)\right). \end{aligned}$$

Hence, as $\exp : \mathrm{M}(V) \to \mathrm{GL}(V)$ is continuous,

$$\lim_{m\to\infty} \left(e^{X/m}e^{Y/m}\right)^m = \exp(X+Y)\,.$$

(5) follows in the same way by using (3) instead of (1). □

THEOREM 4.2. *Let* $\mathfrak{s}_1, \ldots, \mathfrak{s}_k \subseteq \mathrm{M}(V)$ *be subspaces such that* $\mathrm{M}(V) = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_k$. *Define* $\Psi : \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_k \to \mathrm{GL}(V)$ *by*

$$\Psi(X_1, \ldots, X_k) := \exp(X_1)\cdots\exp(X_k)\,.$$

Then there exists $0 \in V_j \subseteq \mathfrak{s}_j$ *open and* $\mathrm{I} \in U \subseteq \mathrm{GL}(V)$ *open such that* $\Psi : V_1 \times \cdots \times V_k \to U$ *is an analytic diffeomorphsim.*

PROOF. It follows from Theorem 4.1, part 1, that

$$(D\Psi)(0, \ldots, 0)(X_1, \ldots, X_k) = X_1 + \ldots + X_k\,.$$

Thus $(D\Psi)(0, \ldots, 0) : \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_k \to \mathrm{M}(V)$ is an linear isomorphism and the claim follows from the Inverse Function Theorem. □

4.1. Closed Subgroups and Their Lie Algebra. The main topic of this section is to show that a closed subgroup of $\mathrm{GL}(V)$ with the relative topology is a Lie group. We determine its Lie algebra. But first we discuss the Lie algebra of invariant vector fields. As before $\mathbb{F}$ denotes the field of real or complex numbers.

Let $\mathcal{M}$ be a set and let G be a group. G *acts* on $\mathcal{M}$ if there is a map $m : G \times \mathcal{M} \to \mathcal{M}$ such that

(1) $m(e, x) = x$ for all $x \in \mathcal{M}$.
(2) $m(ab, x) = m(a, m(b, x))$ for all $a, b \in G$ and all $x \in \mathcal{M}$.

We say that $\mathcal{M}$ is a G-space if G acts on $\mathcal{M}$.

We often write $a \cdot x$ or $\ell_a(x)$ for $m(a, x)$ and m_x for $m(a, x)$. (2) says then that $\ell_{ab} = \ell_a \circ \ell_b$. (1) implies, that $\ell_a : \mathcal{M} \to \mathcal{M}$ is a bijection with inverse $\ell_{a^{-1}}$. Thus, $a \mapsto \ell_a$ is a group homomorphism from G into the group of bijections on $\mathcal{M}$.

Suppose G acts on $\mathcal{M}$ and $\mathcal{N}$. A map $\varphi : \mathcal{M} \to \mathcal{N}$ is a *G-map* if for all $x \in \mathcal{M}$ and all $g \in G$

$$\varphi(g \cdot x) = g \cdot \varphi(x)\,.$$

We say that $\mathcal{M}$ are *G-isomorphic* if there exists a bijective G-map $\mathcal{M} \to \mathcal{N}$. If $\varphi : \mathcal{M} \to \mathcal{N}$ is a bijective G-map, then

$$\begin{aligned} \varphi(\varphi^{-1}(g \cdot x)) &= g \cdot x \\ &= g \cdot \varphi(\varphi^{-1}(x)) \\ &= \varphi(g \cdot \varphi^{-1}(x))\,. \end{aligned}$$

As φ is injective it follows that

$$\varphi^{-1}(g \cdot x) = g \cdot \varphi^{-1}(x)\,.$$

Hence $\varphi^{-1} : \mathcal{N} \to \mathcal{M}$ is also a G-map.

If $x \in \mathcal{M}$ then $G \cdot x = \{g \cdot x \mid g \in G\}$ is called the *orbit* or G-orbit through x. The orbit through x will often be denoted by $\mathcal{O}_x$. The set $\mathcal{M}$ is a disjoint union of orbits

$$\mathcal{M} = \dot{\bigcup}_{x \in \mathcal{M}} \mathcal{O}_x x\,.$$

G acts *transitively* on $\mathcal{M}$ if for every pair $x, y \in \mathcal{M}$ there exists $a \in G$ such that $a \cdot x = y$. Thus, for every $x \in \mathcal{M}$ the orbit $G \cdot x$ through x in the whole space.

Define an equivalence relation $x \sim y$ if $\mathcal{O}_x = \mathcal{O}_y$. Thus $x \sim y$ if and only if there exists $g \in G$ such that $g \cdot x = y$. The set of equivalence classes is denoted by $G \backslash \mathcal{M}$. Thus G acts transitively if and only if $\# G \backslash \mathcal{M} = 1$.

A subset $\mathcal{N} \subseteq \mathcal{M}$ is *invariant* or G-invariant, if $a \cdot \mathcal{N} \subseteq \mathcal{N}$ for all $a \in G$.

If $\mathcal{M}$ is a topological space and G a topological group (which will always be assumed to be locally compact and Hausdorff) then we will always assume that the map $m : G \times \mathcal{M} \to \mathcal{M}$ is *separately continuous*, i.e. $\ell_a : \mathcal{M} \to \mathcal{M}$ and $m_x : G \to \mathcal{M}$ are continuous for all $a \in G$ respectively all $x \in \mathcal{M}$. If $\mathcal{M}$ is smooth or analytic and G is a Lie group, then we assume that the above maps are smooth respectively analytic.

Suppose that the Lie group G acts smoothly on the manifold $\mathcal{M}$. For $a \in G$ and $p \in \mathcal{M}$ denote by $(d\ell_a)_p : T_p(\mathcal{M}) \to T_{a \cdot p}(\mathcal{M})$ the differential of the map $\ell_a ; \mathcal{M} \to \mathcal{M}$ at the point p. Thus $(d\ell_a)_p(v)(f) = v(f \circ \ell_a)$ for $v \in T_p(\mathcal{M})$. G acts on $\Gamma^\infty(\mathcal{M})$ by

$$a \cdot X(p) = (d\ell_a)_{a^{-1} \cdot p}(X(a^{-1} \cdot p)) \tag{4.1}$$

and if $\mathcal{M}$ is analytic then $\Gamma^\omega(\mathcal{M})$ is invariant. A vector field in *G-invariant* if $a \cdot X = X$ for all $a \in G$. Thus, X is G-invariant if and only if for all $a \in G$ and $p \in \mathcal{M}$ we have

$$X(f \circ \ell_a) = (d\ell_a)_p(X)(p) = X(\ell_a(p))\,. \tag{4.2}$$

This follows from (4.1) by replacing p by $\ell_a(p)$. We denote the space of invariant vector fields by $\Gamma^\infty(\mathcal{M})^G$.

If $H \subseteq \mathrm{GL}(V)$ is a closed subgroup, then we consider H as a topological space with the relative topology. Thus, a set $U \subseteq H$ is open if and only if there exists an open set $V \subseteq \mathrm{GL}(V)$ such that $U = H \cap V$. Set

$$\begin{aligned} \mathfrak{h} \;&:=\; \{X \in \mathfrak{gl}(V) \mid e^{tX} \in H \quad \text{for all} \quad t \in \mathbb{R}\} \\ &=\; \{X \in \mathfrak{gl}(V) \mid \gamma_X(\mathbb{R}) \subseteq H\}\,. \end{aligned}$$

THEOREM 4.3. *Let $H \subseteq \mathrm{GL}(V)$ be a closed subgroup. Then $\mathfrak{h}$ is a Lie algebra over $\mathbb{R}$, called the Lie algebra of H.*

PROOF. It is clear from the definition, that $\mathbb{R}\mathfrak{h} \subset \mathfrak{h}$. Let $X, Y \in \mathfrak{h}$ and $t \in \mathbb{R}$. By Theorem 4.1

$$\exp(t(X+Y)) = \lim_{m\to\infty} \left(\exp(\frac{t}{m}X)\exp(\frac{t}{m}Y)\right)^m .$$

We have $\exp(\frac{t}{m}X), \exp(\frac{t}{m}Y) \in H$ for all t and m according to the definition of $\mathfrak{h}$. As H is a group,

$$\left(\exp(\frac{t}{m}X)\exp(\frac{t}{m}Y)\right)^m \in H .$$

Finally, as H is closed, it follows that the limit is in H. Using Theorem 4.1, part 5, one shows in the same way that $\exp(t[X,Y]) \in H$ for all $t \in \mathbb{R}$. Thus $\mathfrak{h}$ is a Lie algebra. □

THEOREM 4.4. *Let $H \subseteq \mathrm{GL}(V)$ be a closed subgroup. Then there exists an open zero neighborhood $U_{\mathfrak{h}} \subseteq \mathfrak{h}$, and an open set $\mathrm{I} \in U_H \subseteq H$ such that $\exp : U_{\mathfrak{h}} \to U_H$ is a homeomorphism.*

PROOF. Let $\mathfrak{s} \subset \mathfrak{gl}(V)$ be a complement to $\mathfrak{h}$ in $\mathfrak{gl}(V)$, i.e. $\mathfrak{gl}(V) = \mathfrak{s} \oplus \mathfrak{h}$. Then choice $U_{\mathfrak{s}} \subset \mathfrak{s}$, $U_{\mathfrak{h}} \subset \mathfrak{h}$ and U_G as in Lemma 4.2. We can assume that $U_{\mathfrak{s}}$ is a ball

$$U_{\mathfrak{s}} = \{X \in \mathfrak{s} \mid \|X\| < r\}$$

for some $r > 0$. We claim that it is possible to find $U_{\mathfrak{s}}$ such that if $X \in U_{\mathfrak{s}}$ and $e^X \in H$, then $X = 0$. Assume that this is not possible. Then for each $n \in \mathbb{N}$, then there exists $X_n \in \mathfrak{s}$ such that $0 < \|X_n\| < r/n$ and $\exp(X_n) \in H$. Let $s_n \in \mathbb{N}$ be such that $s_n X_n \in U_{\mathfrak{s}}$ and $(s_n+1)X_n \notin U_{\mathfrak{s}}$. Then

$$s_n\|X_n\| \leq r \leq (s_n+1)\|X_n\| < r(s_n+1)/n . \tag{4.3}$$

Hence $s_n \to \infty$. As $\overline{U_{\mathfrak{s}}}$ is compact and $s_n X_n \in \overline{U_{\mathfrak{s}}}$ we can, by going over to a subsequence if necessary, assume that $X := \lim s_n X_n$ exists. As $X_n \to 0$ (4.3) implies that $\|X\| = r$. Thus $X \neq 0$.

For $t \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$ let $q_n \in \mathbb{Z}$ be such that $|q_n - ts_n| < 1$. Let $r_n := q_n - ts_n$. Then $|r_n| < 1$ and $ts_n = q_n - r_n$. Now

$$\begin{aligned}
\exp(tX) &= \lim_{n\to\infty} \exp(ts_nX_n) \\
&= \lim_{n\to\infty} \exp(q_nX_n)\exp(-r_nX_n) \\
&= \lim_{n\to\infty} (\exp X_n)^{q_n}\exp(-r_nX_n) .
\end{aligned}$$

As $|r_nX_n| < r/n$ it follows that $\exp(-r_nX_n) \to \mathrm{I}$. As $\exp X_n \in H$ for all $n \in \mathbb{N}$ it follows that $(\exp X_n)^{q_n} \in H$. Thus, as H is closed, it follows that $\exp tX \in H$. As t was arbitrary $X \in \mathfrak{h}$. Hence $X \in \mathfrak{s} \cap \mathfrak{h} = \{0\}$ contradicting the fact that $X \neq 0$.

We assume from now on that $U_{\mathfrak{s}}$ is so that $\exp U_{\mathfrak{s}} \cap H = \{\mathrm{I}\}$. If $U \subset U_{\mathfrak{h}}$ is open, then $\exp(U_{\mathfrak{s}}) \exp(U)$ is open in $\mathrm{GL}(V)$ and

$$\exp(U_{\mathfrak{s}}) \exp(U) \cap H = \exp(U)\,,$$

showing that $\exp(U)$ is open in H.

Let $W \subset \exp(U_{\mathfrak{h}})$ be open. Then there exists $V \subseteq \mathrm{GL}(V)$ open such that $V \cap H = W$. But then $V \cap U_G$ is open in U_G. Hence there exists $V_{\mathfrak{s}} \subset U_{\mathfrak{s}}$, $V_{\mathfrak{h}} \subset U_{\mathfrak{h}}$ open, such that

$$V \cap U_G = \exp(V_{\mathfrak{s}}) \exp(V_{\mathfrak{h}})\,.$$

But then $W = \exp(V_{\mathfrak{h}})$. It follows that $\exp : U_{\mathfrak{s}} \to \exp(U_{\mathfrak{s}}) = H \cap U_G$ is a homeomorphism. □

Let $H \subseteq \mathrm{GL}(V)$ be a closed subgroup. Let $U_{\mathfrak{h}}$ and U_H be as in Theorem 4.3. Using translates hW of open subsets $W \subseteq U_H$ we make H into a Lie group using the same ideas as we sketched for $\mathrm{GL}(V)$ earlier in this chapter. We have $\Gamma^\infty(H)^H \simeq T_{\mathrm{I}}(H) \simeq \mathfrak{h}$ just as for $\mathrm{GL}(V)$. The result is:

THEOREM 4.5. *Let $H \subseteq \mathrm{GL}(V)$ be a closed subgroup. Then H, with the induced topology, is a Lie group. Its Lie algebra is isomorphic to*

$$\mathfrak{h} = \{X \in \mathfrak{gl}(n, \mathbb{F}) \mid (\forall t \in \mathbb{R})\ e^{tX} \in H\}\,.$$

LEMMA 4.6. *Let $H \subseteq \mathrm{GL}(V)$ be a closed subgroup and $\gamma : \mathbb{R} \to H$ a one-parameter subgroup. Then there exists an unique $X \in \mathfrak{h}$ such that $\gamma = \gamma_X$.*

PROOF. As $H \subseteq \mathrm{GL}(V)$ and the topology is the induced topology from $\mathrm{GL}(V)$ we can view γ as a one-parameter subgroup of $\mathrm{GL}(V)$. By Theorem 3.1 it follows that there exist an unique $X \in \mathfrak{gl}(V)$ such that $\gamma = \gamma_X$. But then $\exp(\mathbb{R}X) \subseteq H$ and hence $X \in \mathfrak{h}$. □

5. Homomorphisms

In this section we show that a continuous homomorphism φ between Lie groups G and H is analytic and determined by a Lie algebra homomorphism $\dot{\varphi} : \mathfrak{g} \to \mathfrak{h}$ such that

$$\varphi(\exp_G X) = \exp_H(\dot{\varphi}(X))$$

for all $X \in \mathfrak{g}$.

THEOREM 5.1. *Let G and H be two linear Lie group and $\varphi : G \to H$ a continuous group homomorphism. Then φ is analytic and there exists a Lie algebra homomorphism $\dot{\varphi} : \mathfrak{g} \to \mathfrak{h}$ such that for all $X \in \mathfrak{g}$*

$$\varphi(e^X) = e^{\dot{\varphi}(X)}\,.$$

PROOF. For $X \in \mathfrak{g}$ let $\sigma_X(t) = \varphi(e^{tX})$. Then $\sigma_X : \mathbb{R} \to H$ is a one-parameter subgroup in H. By Lemma 4.6 there exists a unique $Y \in \mathfrak{h}$ such that $\sigma_X(t) = \gamma_Y(t) = e^{tY}$. Define $\dot\varphi(X) := Y$. Note that by definition

$$\dot\varphi(X) = \left.\frac{d}{dt}\right|_{t=0} \varphi(e^{tX})\,. \tag{5.1}$$

We show first that $\dot\varphi$ is linear. It is clear from (5.1) and the chain rule that $\dot\varphi(\lambda X) = \lambda\dot\varphi(X)$. Let $X, Y \in \mathfrak{g}$. Then by Theorem 4.1, the fact that φ is continuous, and that $\dot\varphi(\lambda X) = \lambda\dot\varphi(X)$:

$$\begin{aligned}
\exp(t\dot\varphi(X+Y)) &= \varphi(\exp(t(X+Y))) \\
&= \varphi(\lim_{k\to\infty} (\exp(tX/k)\exp(tY/k))^k) \\
&= \lim_{k\to\infty} \varphi((\exp(tX/k)\exp(tY/k))^k) \\
&= \lim_{k\to\infty} (\varphi(\exp(tX/k))\varphi(\exp(tY/k)))^k \\
&= \lim_{k\to\infty} (\exp(t\dot\varphi(X)/k)\varphi\exp(t\dot\varphi(Y)/k))^k \\
&= \exp(t(\dot{\varphi(X)} + \dot\varphi(Y)))\,.
\end{aligned}$$

Thus $\dot\varphi(X+Y) = \dot\varphi(X) + \dot\varphi(Y)$.

We now show that $\dot\varphi([X,Y]) = [\dot\varphi(X), \dot\varphi(Y)]$. We use again Theorem 4.1, but this time part 5:

$$\begin{aligned}
\exp(t\dot\varphi([X,Y])) &= \varphi(\exp(t[X,Y])) \\
&= \varphi\left(\lim_{k\to\infty}\left(e^{tX/k}e^{Y/k}e^{-tX/k}e^{tY/k}\right)^{k^2}\right) \\
&= \lim_{k\to\infty}\varphi\left(\left(e^{tX/k}e^{tY/k}e^{-tX/k}e^{tY/k}\right)^{k^2}\right) \\
&= \lim_{k\to\infty}\left(\varphi(e^{tX/k})\varphi(e^{tY/k})\varphi(e^{-tX/k})\varphi(e^{tY/k})\right)^{k^2} \\
&= \lim_{k\to\infty}\left(e^{t\dot\varphi(X)/k}e^{t\dot\varphi(Y)/k}e^{-t\dot\varphi(X)/k}e^{t\dot\varphi(Y)/k}\right)^{k^2} \\
&= \exp(t[\dot\varphi(X), \dot\varphi(Y)])\,.
\end{aligned}$$

Differentiating at $t = 0$ shows that $\dot\varphi([X,Y]) = [\dot\varphi(X), \dot\varphi(Y)]$ and hence $\dot\varphi$ is a Lie algebra homomorphism.

That φ is analytic follows now from $\varphi(ge^X) = \varphi(g)e^{\dot\varphi(X)}$. $\square$

EXAMPLE 5.2. Not all important Lie groups are viewed naturally as closed subgroups of $\mathrm{GL}(V)$ for some V. Let $H \subseteq \mathrm{GL}(n, \mathbb{F})$ be a closed Lie subgroup and $\Gamma \subseteq \mathbb{F}^n$ a closed subgroup such that $A\Gamma \subseteq \Gamma$ for all $A \in H$. Let $G = \Gamma \rtimes H$ and make G into a manifold using the product structure. We define the multiplication by

$$(\gamma_1, h_1)\cdot(\gamma_2, h_2) := (\gamma_1 + h_1(\gamma_2), h_1h_2)$$

and the inverse by

$$(\gamma, h)^{-1} = (-h^{-1}(\gamma), h^{-1}) .$$

Then G is not a linear Lie group. But it is in fact isomorphic to the closed subgroup of $\mathrm{GL}(n+1, \mathbb{F})$ given by

$$H' = \left\{ \begin{pmatrix} A & \gamma \\ 0 & 1 \end{pmatrix} \middle| \gamma \in \Gamma, A \in H \right\} .$$

But the first realization is the natural one as we consider G as acting on $\mathbb{F}^n$ and not $\mathbb{F}^{n+1}$. ♢

We finish this section with a remark, showing that as long as we are not looking at Lie group with complex structure, we can concentrate the discussion on closed subgroups of $\mathrm{GL}(n, \mathbb{R})$. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the map

$$T(x_1 + iy_1, \ldots, x_n + iy_n)^t = (x_1, \ldots, x_n, y_1, \ldots, y_n)^t . \tag{5.2}$$

Write elements of $\mathrm{GL}(2n, \mathbb{R})$ as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{where} \quad A, B, C, D \in \mathrm{M}(n, \mathbb{R}) .$$

The linear that corresponds to the matrix X is complex linear if and only if it commutes with multiplication by i. Using (5.2) we see that multiplication by i corresponds to the matrix

$$J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

from above. Thus X is $\mathbb{C}$ linear if and only if $XJ_n = J_nX$. Writing this out we get

$$\begin{pmatrix} B & -A \\ D & -C \end{pmatrix} = \begin{pmatrix} -C & -D \\ A & B \end{pmatrix}$$

which happens if and only if

$$X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} .$$

6. Examples

In this section we discuss some standard examples. V is a finite dimensional vector space over $\mathbb{F}$ and can if needed be identified with $\mathbb{F}^n$ for $n = \dim V$.

6.1. SL(V)**.** Let

$$\mathrm{SL}(V) = \{g \in \mathrm{GL}(V) \mid \det g = 1\} .$$

As $\det : \mathrm{GL}(V) \to \mathbb{F}$ is a continuous homomorphism and $\mathrm{SL}(V) = \ker(\det)$ it follows that $\mathrm{SL}(V)$ is a closed subgroup of $\mathrm{GL}(n, \mathbb{F})$ and hence a Lie group. Denote its Lie algebra by $\mathfrak{sl}(V)$. Then, by Theorem 13.10, $X \in \mathfrak{sl}(V)$ if and only if

$$(\forall t \in \mathbb{R}) \quad \det(e^{tX}) = e^{t\mathrm{Tr}(X)} = 1 . \tag{6.1}$$

Thus

$$\mathfrak{sl}(V) = \{X \in \mathfrak{gl}(V) \mid \mathrm{Tr}(X) = 0\} .$$

6.2. Orthogonal Groups. Let $(\ ,\ )$ be an inner product on V. Let $A \in \mathrm{GL}(V)$ be a self-adjoint or skew symmetric ($A^* = -A$) matrix. Define a bilinear form β_A by

$$\beta_A(x, y) = (Ax, y) .$$

If for some $x \in V$ we have $\beta_A(x, y) = 0$ for all $y \in V$, then $Ax = 0$. As A is injective it follows that $x = 0$. Hence β_A is nondegenerate. If $\mathbb{F} = \mathbb{R}$, then A is symmetric, $A^t = A$ and β_A is a symmetric bilinear form on $\mathbb{R}^n$. If $\mathbb{F} = \mathbb{C}$, then $A^* = (\overline{A})^t = A$ and $\beta_A(x, y) = \overline{\beta_A(y, x)}$. Thus β_A is Hermitian. If $A^* = -A$ then β_A is shew-symmetric, $\beta_A(x, y) = -\overline{\beta_A(y, x)}$.

Define

$$\mathrm{O}(A) := \{g \in \mathrm{M}(V) \mid (\forall x, y \in V)\ \beta_A(g(x), g(y)) = \beta_A(x, y)\} .$$

Assume that for some $x \in V$ and $y \in V$

$$\beta_A(g(x), g(y)) = \beta_A(x, y) \text{ and } g(x) = 0 . \tag{6.2}$$

Then, $\beta_A(x, y) = 0$ for all $y \in V$ and hence $x = 0$ as β_A is non-degenerated. It follows that $\mathrm{O}(A) \subseteq \mathrm{GL}(V)$. Let $\mathrm{SO}(V) := \mathrm{O}(A) \cap \mathrm{SL}(V)$.

We can also describe $\mathrm{O}(A)$ in the following way:

Lemma 6.1. $\mathrm{O}(A) = \{g \in \mathrm{GL}(V) \mid g^*Ag = A\}$.

Proof. We have $\beta_A(g(x), g(y)) = \beta_A(x, y)$ if and only if

$$(Ag(x), g(y)) = (g^*Ag(x), y) = (A(x), y)$$

for all $x, y \in V$. This happen if and only if $g^*Ag = A$. □

Lemma 6.2. *$\mathrm{O}(A)$ and $\mathrm{SO}(A)$ are Lie groups. The Lie algebra of $\mathrm{O}(A)$ is*

$$\mathfrak{o}(V) = \{X \in \mathfrak{gl}(V) \mid X^*A = -AX\}$$

and the Lie algebra $\mathfrak{so}(V)$ of $\mathrm{SO}(V)$ is

$$\mathfrak{so}(V) = \{X \in \mathfrak{gl}(V) \mid X^*A = -AX \text{ and } \mathrm{Tr}(X) = 0\} .$$

PROOF. The map $\mathrm{GL}(V) \to \mathrm{M}$, $g \mapsto F(g) := g^*Ag - A$, is continuous. Thus $\mathrm{O}(A) = F^{-1}(0)$ is closed. If $a, b \in \mathrm{O}(A)$. Then

$$\beta_A(ab(x), ab(y)) = \beta_A(a(b(x)), a(b(x))) = \beta_A(b(x), b(y)) = \beta_A(x, y)\,.$$

Hence $ab \in \mathrm{O}(A)$. Similarly,

$$\beta_A(a^{-1}(x), a^{-1}(y)) = \beta_A(a(a^{-1}(x)), a(a^{-1}(y))) = \beta_A(x, y)$$

showing that $a^{-1} \in \mathrm{O}(A)$. Hence $\mathrm{O}(A)$ is a Lie group.

We have $X \in \mathfrak{o}(A)$ if and only if for all $t \in \mathbb{R}$

$$e^{tX^*} A e^{tA} = A\,. \tag{6.3}$$

Differentiating (6.3) at $t = 0$ shows that $X^*A + AX = 0$. If $X^*A + AX = 0$, then $AXA^{-1} = -X^*$ and

$$\begin{aligned} e^{tX^*} A e^{tX} &= e^{tX^*} e^{tAXA^{-1}} A \\ &= e^{tX^*} e^{-tX^*} A \\ &= A \end{aligned}$$

showing that $X \in \mathfrak{g}(A)$. The statements about $\mathrm{SO}(V)$ follows in the same way using Example 6.1. □

Assume that $V = \mathbb{R}^n$. Then $\det(g^tAg) = \det(g)^2A = \det A$ implies that $\det(g) = \pm 1$. In particular, $\mathfrak{o}(V) = \mathfrak{so}(V)$ and $\#\mathrm{O}(V)/\mathrm{SO}(V) = 2$. If $A = \mathrm{I}$ then we also use the notations $\mathrm{O}(A) = \mathrm{O}(n)$ and $\mathrm{SO}(n)$. If $p, q \in \mathbb{N}_0$, $p + q = n$, and A is the matrix

$$A = \mathrm{I}_{p,q} = \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & \mathrm{I}_q \end{pmatrix}$$

then

$$\beta_A(x, y) = x_1y_1 + \ldots + x_py_p - x_{p+1}y_{p+1} - \ldots - x_ny_n$$

and the groups $\mathrm{O}(A)$ and $\mathrm{SO}(A)$ are denoted by $\mathrm{O}(p, q)$ respectively $\mathrm{SO}(p, q)$. Note that $\mathrm{O}(0, n) = \mathrm{O}(n, 0) = \mathrm{O}(n)$. As every symmetric matrix in $\mathrm{GL}(V)$ is conjugate to one of the matrices $\mathrm{I}_{p,q}$ then $\mathrm{O}(A)$ is conjugate to $\mathrm{SO}(p, q)$ for some p and q.

If $A^t = -A$ then $n = 2m$ is even and $\beta_A(x, y) = -\beta_A(y, x)$. Hence β_A is a *symplectic form* on $\mathbb{R}^n$ we can assume that

$$A = J_m = \begin{pmatrix} 0 & -\mathrm{I}_m \\ \mathrm{I}_m & 0 \end{pmatrix}.$$

Thus

$$\beta_A(x, y) = x_1y_{m+1} + \ldots + x_my_n - x_{m+1}y_1 - \ldots - x_ny_m\,.$$

In this case the group $\mathrm{SO}(A)$ is denoted by $\mathrm{Sp}(n, \mathbb{R})$. We will often denote this symplectic form by ω. Define a homomorphism $\varphi : \mathrm{GL}(m, \mathbb{R}) \to$

$\mathrm{GL}(n,\mathbb{R})$ by

$$\varphi(a) = \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix} .$$

Write $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $x_j, y_j \in \mathbb{R}^m$. Then, with $\theta(a) = (a^{-1})^t$:

$$\begin{aligned} \omega(\varphi(a)x, \varphi(a)y) &= \omega\left(\begin{pmatrix} ax_1 \\ \theta(a)x_2 \end{pmatrix}, \begin{pmatrix} ay_1 \\ \theta(a)y_2 \end{pmatrix}\right) \\ &= (ax_1, \theta(a)y_2) - (\theta(a)x_2, ay_1) \\ &= (x_1, y_2) - (x_2, y_1) \\ &= \omega(x, y) . \end{aligned}$$

Hence $\varphi(\mathrm{GL}(n,\mathbb{R})) \subseteq \mathrm{Sp}(n,\mathbb{R})$.

If $V = \mathbb{C}^n$ and A is self adjoint, then we can assume that $A = \mathrm{I}$ as every self adjoint matrix is conjugate to I. In this case $\mathrm{O}(A)$ is denoted by $\mathrm{U}(n)$ and $\mathrm{SO}(A)$ is denoted by $\mathrm{SU}(n)$. If $A^* = -A$ then iA is self adjoint. As $\mathrm{O}(A) = \mathrm{O}(iA)$ it follows that we again get the group $\mathrm{U}(n)$.

We can identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ by $(x_1, \ldots, x_n)^t \mapsto (x_1, \ldots, x_m)^t + i(x_{m+1}, \ldots, x_n)^t$. Then the symplectic form above can then be written as $\omega(x, y) = -\operatorname{Im}(x, y)$ where $(\ ,\ )$ denotes the standard inner product on $\mathbb{C}^m$. As $(Ax, Ay) = (x, y)$ for all $A \in \mathrm{U}(m)$ it follows as we can view $\mathrm{U}(m)$ as a subgroup of $\mathrm{Sp}(n,\mathbb{R})$.

7. Homogeneous Spaces

Let G be a topological group and $H \subset G$ a closed subgroup. Let $\mathcal{M} = G/H$ and let $\kappa : G \to \mathcal{M}$, $g \mapsto gH$, be the quotient map. The *quotient topology* on $\mathcal{M}$ is defined as the finest topology on $\mathcal{M}$ such that κ is continuous. This is achieved by defining $U \subseteq \mathcal{M}$ to be open if and only if $\kappa^{-1}(U)$ open in G. This topology makes $\mathcal{M}$ into a locally compact Hausdorff topological space.

LEMMA 7.1. *Let $H, G \subseteq \mathrm{GL}(V)$ be closed subgroups with Lie algebra $\mathfrak{h}$ respectively $\mathfrak{g}$. Let $\mathfrak{q} \subseteq \mathfrak{g}$ be a vector space such that $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$. Then there exists open zero neighborhoods $U_{\mathfrak{q}} \subset \mathfrak{q}$ and $U_{\mathfrak{h}} \subseteq \mathfrak{h}$ and an open neighborhood U_G of $\mathrm{I} \in G$ such that $U_{\mathfrak{q}} \times U_{\mathfrak{h}} \to U_G$, $(X, Y) \mapsto \exp(X)\exp(Y)$, is an analytic diffeomorphism and if $X \in U_{\mathfrak{q}}$ then $\exp X \in H$ if and only if $X = 0$.*

PROOF. Let $\mathfrak{s} \subseteq \mathfrak{gl}(V)$ be such that $\mathfrak{gl}(V) = \mathfrak{s} \oplus \mathfrak{h}$. Let $U_{\mathfrak{s}}$ and $U_{\mathfrak{h}}$ be as in the proof of Theorem 4.4. It is then easy to see that the Lemma holds with $U_{\mathfrak{q}} := U_{\mathfrak{s}} \cap \mathfrak{q}$. $\square$

THEOREM 7.2. *Let G be a Lie group and H a closed subgroup. Then there exists an unique analytic structure on $\mathcal{M} = G/H$ such that κ is analytic. Furthermore the map*

$$G \times \mathcal{M} \to \mathcal{M}\,, \quad (a, bH) \mapsto m(a, bH) := abH$$

is smooth and G acts analytically on $\mathcal{M}$.

PROOF. Let $m_o = \kappa(\mathrm{I})$ be the base point. Let $\mathfrak{g}$ denote the Lie algebra of H. Let $\mathfrak{q}$, $U_\mathfrak{q}$, $U_\mathfrak{h}$, and U_G be as in Lemma 7.1. Let $V_\mathfrak{q} \subset U_\mathfrak{q}$ be a open symmetric zero neighborhood such that $\overline{V_\mathfrak{q}}$ is compact and $(\overline{\exp(V_\mathfrak{q})})^2 \subseteq U_G$. Then $\kappa^{-1}(\exp(V_\mathfrak{q}) \cdot m_o) = \exp(V_\mathfrak{q})H = \exp(V_\mathfrak{q})\exp(U_\mathfrak{h})H$ which is open. Hence $\kappa(\exp(V_\mathfrak{q}))$ is open. Let $V = \kappa(V_\mathfrak{q})$. Let $\psi V_\mathfrak{q} \to V$ be the map $X \mapsto \kappa(\exp X))$. We claim that ψ is a homeomorphism. Clearly $X \to \kappa(\exp(X))$ is a continuous and surjective map from $\overline{V_\mathfrak{q}}$ onto the compact set $\overline{V}$. Assume that $X, Y \in \overline{V_\mathfrak{q}}$ are such that $\kappa(\exp(X)) = \kappa(\exp(Y))$. Then there exists a $h \in H$ such that $\exp(X) = \exp(Y)h$. But then $\exp(-Y)\exp(X) \in \overline{(\exp(V_\mathfrak{q}))^2} \cap H = \{\mathrm{I}\}$. Hence $\exp(X) = \exp(Y)$ which implies that $X = Y$. It follows that $\kappa \circ \exp : \overline{V_\mathfrak{q}} \to \overline{V}$ is a homeomorphism. Hence ψ is a homeomorphism as claimed.

Now the open sets $V_a := aV$, $a \in G$ form a open covering of $\mathcal{M}$ and $\psi_a = \psi^{-1}\ell_{a^{-1}} : V_a \to V_\mathfrak{q}$ is a homomorphism. As before for $\mathrm{GL}(V)$ we see that $V_a \cap V_b \not\emptyset$ then $\psi_a \circ \psi_b^{-1} : \psi_b^{-1} \cap \psi_a^{-1}(V_a \cap V_b) \to \mathfrak{q}$ is analytic. Hence $\{(V_a, \psi_a) \mid a \in G\}$ is an analytic atlas for $\mathcal{M}$. It is clear from the definition of the covering that all the maps $\ell_a : \mathcal{M} \to \mathcal{M}$ are analytic (in fact analytic isomorphism $V_b \to V_{ab}$ for all $b \in G$). □

REMARK 7.3. We remark for later use, that the map $s : V \to \exp(V_\mathfrak{q}) \subseteq G$, $\psi(X) \mapsto \exp(X)$ is a local section, ie., s is analytic, injective, and $\kappa \circ s = \mathrm{id}_V$.

The tangent space $T_a(G)$ is isomorphic to $\mathfrak{g}$ for every $a \in G$. Consider the map $(d\kappa)_a : \mathfrak{g} \to T_{a \cdot m_o}(G/H)$ where $m_o = \kappa(\mathrm{I}) = \{H\}$. We have

$$(d\kappa)_a(X)f = \left.\frac{d}{dt}\right|_{t=0} f(\kappa(ae^{tX})) = \left.\frac{d}{dt}\right|_{t=0} f(ae^{tX} \cdot m_o)\,.$$

This shows that $\mathfrak{h} \subseteq \ker((d\kappa)_a)$. Let $V_\mathfrak{q}$ be as above and let $X \in \mathfrak{q}$, $X \neq 0$. Then there exists $\epsilon > 0$ such that $tX \in V_\mathfrak{q}$ for all $|t| < \epsilon$. Let $\varphi \in C_c^\infty(\mathfrak{q})$ be such that $\varphi \equiv 1$ in a neighborhood of 0 and $\mathrm{Supp}(\varphi) \subset V_\mathfrak{q}$. Let $(\ ,\ )$ be an inner product on $\mathfrak{q}$. Then $f(\exp Y \cdot m_o) := (Y, X)\varphi(Y)$ and $f \equiv 0$ outside of $\kappa(\exp(V_\mathfrak{q}))$. Then $f \in C_c^\infty(G/H)$ and

$$(d\kappa)_a(X)f = \|X\|^2 > 0\,.$$

We have thus proved the following lemma:

LEMMA 7.4. *Let $a \in G$. Then $(d\kappa)_a : \mathfrak{q} \to T_{a \cdot m_o}(G/H)$ is a linear isomorphism.*

Note that this identification $\mathfrak{q} \simeq T_{a\cdot m_o}(G/H)$ depends on a. Assume that $a \cdot m_o = b \cdot m_o$. Then there exists $h \in H$ such that $a = bh$ and

$$\begin{aligned}(d\kappa)_a(X)f &= \left.\frac{d}{dt}\right|_{t=0} f(ae^{tX} \cdot m_o)\\ &= \left.\frac{d}{dt}\right|_{t=0} f(ghe^{tX} \cdot m_o)\\ &= \left.\frac{d}{dt}\right|_{t=0} f(ae^{t\mathrm{Ad}(h)X} \cdot m_o)\\ &= (d\kappa)_b(\mathrm{Ad}(h)X)f\,.\end{aligned}$$

We will discuss this in more details in the section on vector bundle.

LEMMA 7.5. *Let the notation be as in Theorem 7.2. Let $\mathcal{N}$ be smooth or analytic manifold and $f : \mathcal{M} \to \mathcal{N}$ continuous. Then f is smooth, respectively analytic, if and only if $f \circ \kappa : G \to \mathcal{N}$ is smooth, respectively analytic.*

PROOF. Let $\tilde{f} = f\circ\kappa$. Let $W_a := a \exp V_{\mathfrak{q}} \exp V_{\mathfrak{h}}$, and $\varphi_a = (\exp|_{V_{\mathfrak{q}}\oplus V_{\mathfrak{h}}})^{-1}$. Then $(W_a, \varphi_a)_{a\in G}$ is an analytic atlas on G with the property that $\kappa(W_a) = V_a$. Denote the projection $\mathfrak{q}\oplus\mathfrak{h} \to \mathfrak{q}$ by pr_1. Then in those local coordinates we have $f \circ (\psi_a^{-1} \circ \mathrm{pr}_1) = \tilde{f} \circ \varphi_a^{-1}$ and the claim follows. $\square$

Assume that that the G acts analytically on the manifold $\mathcal{M}$. For m_o in $\mathcal{M}$ let

$$G^{m_o} := \{a \in G \mid a \cdot m_o = m_o\}\,.$$

G^{m_o} is a closed subgroup of G and hence a Lie group. It is called the *stabilizer* of m_o. By Theorem 7.2 G/G^{m_o} is an analytic G-space and the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\kappa"] \arrow[dr, "a\mapsto a\cdot m_o"'] & G/G^{m_o} \arrow[d, "\exists!\,\varphi \text{ analytic}"] \\ & \mathcal{M} \end{tikzcd} .$$

The map φ is given by $aG^{m_o} \mapsto a \cdot m_o$. It is injective by the definition of G^{m_o} and analytic by Theorem 7.2. φ is surjective if and only if G acts transitively.

THEOREM 7.6. *Assume that the Lie group G acts transitively on the analytic manifold $\mathcal{M}$. Then φ is an analytic diffeomorphism.*

PROOF. Let $H = G^{m_o}$ and $x_o = \kappa(\mathrm{I})$. We show first that φ is a homeomorphism. For that we only need to show that φ is open. That holds if and only if $\tilde{\varphi} := \varphi \circ \kappa$ is open. Let $\emptyset \neq V$ be open, let $a \in V$ and let $m = \tilde{\varphi}(a)$. Now choose a compact symmetric I-neighborhood U such that $aU^2 \subseteq V$.

There exists $\{a_j\}_{j\in J}$ a finite or countable infinite sequence of elements in G such that $G = \bigcup_{j\in J} a_j U$.

Let $W := \tilde{\varphi}(U)$ and $W_j = a_j \cdot W = \tilde{\varphi}(a_j U)$. Then, as $\tilde{\varphi}$ is continuous, each W_j is compact and hence closed. We also have $\mathcal{M} = \bigcup_j W_j$ as the G-action is transitive. Hence, by Baire Category Theorem, there exists a j such that $W_j^o \neq \emptyset$. But then each W_k has a non-empty interior as $W_k = \ell_{a_k a_j^{-1}} W_j$ and $\ell_{a_k a_j^{-1}}$ is a diffeomorphism. In particular, $W^o \neq \emptyset$. Let $z = u \cdot m_o \in W^o$. Then $m_o \in u^{-1}W^o \subset U^2 \cdot m_o$ and $u^{-1}W^o$ is open. It follows that

$$\tilde{\varphi}(a) = a \cdot m_o \in au^{-1}W^o \subseteq aU^2 \cdot m_o \subseteq \tilde{\varphi}(V)\,.$$

As a was arbitrary, it follows that $\tilde{\varphi}(V)$ is open.

Now the topological invariance of dimension shows that $\dim G/H = \dim \mathcal{M}$. It therefore suffice to show that $(d\varphi)_x : T_x(G/H) \to T_{\varphi(x)}(\mathcal{M})$ is injective for $x \in G/H$. Denote by $\ell_a : \mathcal{M} \to \mathcal{M}$ the map $m \mapsto a \cdot m$ and $\tau_a : G/H \to G/H$ the map $bH \mapsto abH$. Then both ℓ_a and τ_a are analytic diffeomorphism and $\varphi(\tau_a(x)) = \ell_a(\varphi(x))$. In particular, $(d\varphi)_{a\cdot x_o} = (d\ell_a)_{x_o} \circ (d\varphi)_{x_o}$. It is therefore enough to show that $(d\varphi)_{x_o} : \mathfrak{q} \simeq T_{x_o}(G/H) \to T_{m_o}(\mathcal{M})$ is injective. Let $X \in \mathfrak{q}$ and assume that $X \neq 0$. Let $V_\mathfrak{q} \subseteq \mathfrak{q}$ be a zero neighborhood as in the proof of Theorem 7.2. As $\varphi : \exp(V_\mathfrak{q}) \cdot x_o \to \exp(V_\mathfrak{q}) \cdot m_o$ it follows that $t \mapsto \gamma(t) := \exp(tX) \cdot m_o$ is injective for t small. In particular $0 \neq \dot{\gamma}(0) = (d\varphi)_{x_o}(X)$. □

8. Integration on Lie Groups and Homogeneous Spaces

9. Basic Representation Theory

If the group G acts on a vector space V, then the action is *linear* if ℓ_a is a linear map for all $a \in G$. In that case we use Greek lower case letters like π, ρ, etc. for ℓ and say that π is a *representation* of G in V. If π is given, then we write V_π for the vector space on which π acts.

If G is a topological group and π a representation of G in a topological vector space V, then we will always assume that π is continuous. By that we mean that $\pi(a)$ is a continuous linear map and for all $v \in V$

$$G \to V\,, \quad a \mapsto \pi(a)v$$

is continuous. A subspace $W \subseteq V$ is *invariant* if $\pi(G)W \subseteq W$. V is *irreducible* if the only closed invariant subspaces are $\{0\}$ and V. If π and ρ are representations of G, then a linear G-map $T : V_\pi \to W_\rho$ is called an *intertwining operator*. If nothing else is said, then we will assume T to be bounded. We denote by $\mathrm{B}_G(\pi, \rho)$ or $\mathrm{Hom}_G(V_\pi, V_\rho)$ the set of bounded intertwining operators.

In this section $\mathbb{F}$ again denotes the field of real or complex numbers and $G \subset \mathrm{GL}(n, \mathbb{F})$ a Lie group. H will always denote a closed subgroup of G

if nothing else is said. Let $\mathcal{M}$ be a smooth (analytic) G-space and assume that the action is transitive. Fix a base point $x_o \in \mathcal{M}$ and let

$$H = G^{x_o} := \{g \in G \mid g \cdot x_o = x_o\}\,.$$

Then H is a closed subgroup of G. Define a map from $G \to \mathcal{M}$ by $\varphi(g) = g{\cdot}x_o$. Then φ is differentiable and surjective. Furthermore, it factors through H:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\kappa"'] & \mathcal{M} \\ G/H \arrow[ur, "\exists!\widetilde{\varphi}"'] & \end{tikzcd}$$

where $\kappa : G \to G/H$ is the canonical map $g \mapsto gH$. We will now make G/H into a manifold such that $\widetilde{\varphi}$ such that the action $m(g, aH) = (ga)H$ is smooth and $\widetilde{\varphi}$ is a G-isomorphism.

Let K be a closed subgroup of G. Let $G/K := \{aH \mid a \in G\}$ and let $\kappa : G \to G/K$ be the quotient map $a \mapsto aH$. Then G/H becomes a topological space be declaring $U \subset G/K$ for open if and only if $\kappa^{-1}(U) \subset G$ is open. κ is then an open map and $f : \mathcal{M} \to Y$ is continuous if and only if $f \circ \kappa : G \to Y$ is continuous. Here Y is any topological space. Finally G acts continuously on G/K by $m(a, bK) = (ab)K$.

Let $\mathfrak{k}$ denote the Lie algebra of K and let $\mathfrak{s} \subset \mathfrak{g}$ be a complementary subspace in $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{k}$. Let $U_{\mathfrak{s}} \subset \mathfrak{s}$, $U_{\mathfrak{k}} \subset \mathfrak{k}$, and $U_G \subset G$ be as in Theorem 4.3 and such that $U_G \cap K = \exp(U_{\mathfrak{h}})$. In particular, $\exp(U_{\mathfrak{s}}) \cap K = \{I\}$. Let $U \subset \mathfrak{s}$ be a relatively compact neighborhood of zero in $\mathfrak{s}$ such that $U = -U \subset U_{\mathfrak{s}}$ and $\exp(\overline{U})^2 \subset U_G$. Let $U_{G/K} = \kappa(\exp(U))$. As $U_{G/K} = \kappa(\exp(U)\exp(U_{\mathfrak{k}}))$ and κ is an open map, it follows that $U_{G/K}$ is open in G/K. It is easily seen that the map $\mathrm{Exp} : \mathfrak{s} \to G/K$, $X \mapsto \exp(X)K$, restricted to U is a homeomorphism. For $g \in G$ define $\psi_g : gU_{G/K} \to U$ by $g\mathrm{Exp}(X) \mapsto X$. Then the collection $\{(\psi_g, gU_{G/K})\}_{g\in G}$ defines an atlas for G/K which makes G/K into a analytic manifold such that the action of G on G/K is analytic. One can show (see [**?**] Theorem 4.2., p. 123, that this is the *unique* analytic structure on G/K which makes the action smooth.

THEOREM 9.1. *Assume that G acts transitively on $\mathcal{M}$. Let $x_o \in \mathcal{M}$ and $G^{x_o} = \{g \in G \mid g \cdot x_o = x_o\}$. Then the map*

$$\widetilde{\varphi} : G/G^{x_o} \to \mathcal{M}$$

is a G-isomorphism.

PROOF. Let $K = G^{x_o}$. $\widetilde{\varphi}$ is by Theorem **??** a homeomorphism and also clearly a G-map. Let $\mathfrak{s}$, $U \subset U_{\mathfrak{s}} \subset \mathfrak{s}$, and $U_{G/K} = \mathrm{Exp}(U)$ be as above. Let $g \in G$. Then the map

$$\Psi_g : gU_{G/K} \to \exp(U_{\mathfrak{s}}) \cdot x_o$$
$$g\mathrm{Exp}(X) \mapsto X \mapsto g\exp(X) \mapsto g\exp(X) \cdot x_o$$

is an analytic diffeomorphism of the open subset–and hence submanifold–$U_{G/K}$ of G/K onto the open submanifold $g\exp(U_{\mathfrak{s}})\cdot x_0 = g\cdot\widetilde{\varphi}(U_{G/K})$. The claim now follows. $\square$

Here

10. Integration

Let $\mathcal{M}$ be a locally compact Hausdorff topological space and assume that the topological group G acts on $\mathcal{M}$. A Radon measure μ on $\mathcal{M}$ is *G-invariant* if for all $f \in C_c(\mathcal{M})$ and all $a \in G$ we have

$$\int_{\mathcal{M}} f(a\cdot x)\,d\mu(x) = \int_{\mathcal{M}} f(x)\,d\mu(x)\,.$$

As $L^1(\mathcal{M})$ is the closure of $C_c(\mathcal{M})$ in the L^1-norm

$$\|f\|_1 := \int_{\mathcal{M}} |f(x)|\,d\mu(x)$$

it follows that $\int_{\mathcal{M}} f(a\cdot x)\,d\mu(x) = \int_{\mathcal{M}} f(x)\,d\mu(x)$ for all $f \in L^2(\mathcal{M})$. Inserting for f an indicator function χ_E and noting that $\chi_E(g\cdot x) = \chi_{g^{-1}\cdot E}$ it follows that μ is G-invariant if and only if $\mu(E) = \mu(a\cdot E)$ for all measurable sets E and all $a \in G$.

As G is a topological group then there exists a Radon measure μ_G invariant under left multiplication, see [**?**], p. 37 or [**?**]. This measure is called (left) *Haar measure* on G. It is unique up to multiplication by positive scalars. Note that all continuous compactly supported functions are integrable and that compact sets are measurable with finite measure. In particular, if G is compact, then $\mu_G(G) < \infty$ and we will–without further comments–always normalize it so that $\mu_G(G) = 1$.

Assume that the topological group acts continuously on the topological space $\mathcal{M}$. Assume that μ is an non-zero invariant measure on $\mathcal{M}$. Define $\lambda(a)f(x) = \lambda_{\mathcal{M}}(a)f(x) = f(a^{-1}x)$.

Lemma 10.1. *Let $a \in G$. Then $\lambda(a) : L^p(\mathcal{M}) \to L^p(\mathcal{M})$, $1 \leq p \leq \infty$ is an isometry. Furthermore, $\lambda : G \to \mathrm{B}(L^p(\mathcal{M}))$ is a homomorphism. If $1 \leq p < \infty$, then the map $G \mapsto L^p(\mathcal{M})$, $a \mapsto \lambda(a)f$ is continuous for all $f \in L^2(\mathcal{M})$, i.e., $\lambda : G \to \mathrm{B}(L^p(\mathcal{M}))$ is strongly continuous.*

Proof. The first statement follows from the invariance of the measure. Let $a, b \in G$ and $f \in L^p(G)$. Then

$$\begin{aligned}\lambda(ab)f(x) &= f(b^{-1}a^{-1}x)\\ &= [\lambda(b)f](a^{-1}x)\\ &= (\lambda(a)[\lambda(b)f])(x)\,.\end{aligned}$$

As λ is a homomorphism and an isometry, then it is enough to show that $a \mapsto \lambda(a)f$ is continuous at $a = e$. Assume first that $f \in C_c(\mathcal{M})$. Let $\epsilon > 0$. As f is continuous with compact support it follows that f is uniformly continuous. A set W in G is *symmetric* if $W^{-1} := \{a^{-1} \mid a \in G\} = W$. Let W be a compact symmetric neighborhood of e. Then $K := W \cdot \mathrm{Supp}(f)$ is compact. In particular $\mu(K) < \infty$. Let V be an open neighborhood of e such that $V \subseteq W$ and

$$|f(a^{-1}b) - f(b)| < \frac{\epsilon}{\mu(K)^{1/p}} \quad \text{for all } a \in V \text{ and} b \in G\,.$$

Then

$$\|\lambda(a)f - f\|_p \leq \frac{\epsilon}{\mu_G(K)^{1/p}} \left(\int_G \chi_K \, d\mu_G\right)^{1/p} = \epsilon\,.$$

For $f \in L^p(\mathcal{M})$ let $g \in C_c(\mathcal{M})$ be such that $\|f - g\|_p < \epsilon/3$. Let V be a neighborhood of e such that $\|\lambda(a)g - g\|_p < \epsilon/3$ for all $a \in V$. Then

$$\|\lambda(a)f - f\|_p \leq \|\lambda(a)f - \lambda(a)g\|_p + \|\lambda(a)g - g\|_p + \|g - f\|_p < \epsilon$$

and it follows that $a \mapsto \lambda(a)f$ is continuous. □

Remark 10.2. Note that the last statement does not hold for $p = \infty$. For that let $\mathcal{M} = G = \mathbb{R}$. Let

$$A = [-1,1] \setminus \bigcup_{n=1}^{\infty} [(-2^{-2n}, 2^{-2n-1}) \cup (2^{-2n-1}, 2^{-2n})]\,.$$

Then A is measurable with positive measure. Let $f = \chi_A$. Then there exists sequence $\{a_n\}$ and $\{b_n\}$ such that $\lim a_n = \lim b_n = 0$ and

$$\|\lambda(a_n)f - f\|_\infty = 1 \quad \textit{and} \quad \|\lambda(b_n)f - f\|_\infty = 0\,.$$

For $p = 2$, $\lambda_{\mathcal{M}}$ is called *the regular representation* of G on $L^2(\mathcal{M})$. If $\mathcal{M} = G$ we sometimes say *left regular representation* to distinguish it from the *right regular representation* ρ which we now define. For $a \in G$ and $f : G \to \mathbb{C}$ measurable let $\rho(a)f(b) = f(ab)$. It is then easy to show that $\rho(ab) = \rho(a)\rho(b)$. If $f \in G$ then $C_c(G) \to \mathbb{C}$, $f \mapsto \int_G \rho(a)f \, d\mu_G$, is a left-invariant Radon measure on G. Hence, there exists a positive number $\Delta_G(a)$ such that

$$\Delta_G(a) \int_G f(xa) \, d\mu_G(x) = \int_G f(x) \, d\mu_G(x)\,.$$

The function $\Delta_G : G \to \mathbb{R}^+$ is called *the modular function*.

Lemma 10.3. *Δ_G is a continuous homomorphism $G \to \mathbb{R}^+$.*

Proof. Let $f \in C_c(G)$, $f \geq 0$, $\int_G f d\mu_G = 1$. Then

$$\Delta_G(b) = \int_G \rho(b^{-1}) f \, d\mu_G\,.$$

Thus

$$\begin{aligned}
\Delta_G(ab) &= \int_G \rho((ab)^{-1})f\,d\mu \\
&= \int_G \rho(b^{-1})[\rho(a^{-1})f]\,d\mu_G \\
&= \Delta(b)\int_G \rho(a^{-1})f\,d\mu_G \\
&= \Delta(b)\Delta(a)\,.
\end{aligned}$$

As Δ_G is a homomorphism, we only have to show that Δ_G is continuous at e. Let $f \in C_c(G)$ be as above. f is uniformly continuous as f is compactly supported. Let W be a compact symmetric neighborhood of $e \in G$ such that $|f(ab^{-1}) - f(a)| < \epsilon/\mu(K)$ where $K := \mathrm{Supp}(f)W$ is compact. Then

$$\begin{aligned}
|\Delta_G(b) - 1| &= |\int_G f(ab^{-1})\,d\mu_G(a) - \int_G f(a)\,d\mu_G(a)| \\
&= \int_G |f(ab^{-1}) - f(a)|\,d\mu_G(a) \\
&< \epsilon\,.
\end{aligned}$$

Hence Δ_G is continuous. ☐

LEMMA 10.4. *Assume that $1 \le p < \infty$. Then $\rho : G \to \mathrm{B}(L^p(G))$ is a strongly continuous homomorphism.*

PROOF. The proof is similar to the proof of Lemma 10.1 and is left to the reader. ☐

DEFINITION 10.5. The topological group G is *unimodular* if $\Delta_G = 1$.

Thus G is unimodular if and only if the left invariant measure μ_G is also right invariant. In particular, G is abelian, then G is unimodular. If G is compact, then $\Delta_G(G)$ is a compact (multiplicative) subgroup of $\mathbb{R}^+$ and hence $\Delta_G(G) = \{1\}$. Thus compact groups are unimodular. More generally, let $[G,G] = \overline{\{aba^{-1}b^{-1} \mid a, b \in G\}}$. Then $[G,G]$ is a closed subgroup of G. Clearly $\Delta_G|_{[G,G]} = 1$. Hence Δ_G defines a continuous homomorphism on $G/[G,G]$. Thus, if $G/[G,G]$ is compact, it follows that $\Delta_G = 1$.

For $f : G \to \mathbb{C}$ measurable let $f^\vee(a) = f(a^{-1})$. Then $f^\vee$ is measurable. $f^\vee$ is continuous if f is continuous and has compact support if f has.

LEMMA 10.6. *Let $f \in C_c(G)$. Then*

$$\int_G \frac{f^\vee(a)}{\Delta_G(a)}\,d\mu_G(a) = \int_G f(a)\,d\mu_G(a)\,.$$

PROOF. Define a Radon measure ν on G by

$$f \mapsto \int_G f(x)\, d\nu(x) := \int_G \frac{f^\vee(x)}{\Delta_G(x)}\, d\mu_G(x)\,.$$

Then a simple calculation shows that ν is left-invariant. Hence

$$\nu = c\mu_G \text{ for some constant } c > 0\,. \tag{10.1}$$

Let W be a compact symmetric neighborhood of e. Let $f \in C_c(G)$ be such that $f(e) = 1$, $f \geq 0$, and $\mathrm{Supp}(f) \subseteq W$. Define $g(a) := f(a)f(a^{-1})$. Then $g^\vee = g$ and $\mathrm{Supp}(g) \subseteq W$. Finally, let $h(a) := g(a)/\sqrt{\Delta_G(a)}$. Then $h \geq$ and $h(e) > 0$. Hence $\int_G h\, d\mu_G > 0$. We have

$$h^\vee(a)/\Delta(a) = \sqrt{\Delta_G(a)}g(a^{-1})/\Delta_G(a) = h(a)\,.$$

Hence by (10.1) and the definition of ν we get

$$c\int_G h\, d\mu_G = \int_G h\, d\nu = \int_G h\, d\mu_G > 0\,.$$

Hence $c = 1$ and the claim follows. □

COROLLARY 10.7. *If G is unimodular, then*

$$\int_G f(x^{-1})\, d\mu_G(x) = \int_G f(x)\, d\mu_G(x)$$

for all $f \in C_c(G)$.

PROOF. This follows from Lemma 10.6 because $\Delta_G = 1$. □

If $f, g : G \to \mathbb{C}$ are measurable, then the convolution $f * g$ is defined by

$$f * g(b) := \int_G f(a)g(a^{-1}b)\, d\mu_G(a)$$

whenever the integral exists. By Hölders inequality, the convolution is always defined for $f \in L^p(G)$ and $g \in L^{p'}(G)$. Furthermore, $f*g$ is continuous and bounded with

$$\|f * g(b)\|_\infty \leq \|f\|_p \|g\|_q\,.$$

It is harder to show that if $f \in L^1(G)$ and $g \in L^p(G)$, $1 \leq p \leq \infty$, then $f * g$ is defined and $f * g \in L^p(G)$ with

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p\,,$$

see [**?**], p. 52. If G is not commutative, then in general $f * g \neq g * f$.

For $f \in L^1(G)$ let $f^*(a) := \overline{f^\vee(a)}/\Delta_G(a)$. Then $\|f^*\|_1 = \|f\|_1$ and $(f^*)^* = f$. The following is now clear.

THEOREM 10.8. *$L^1(G)$ with convolution as a product and involution $f \mapsto f^*$ is a Banach $*$-algebra.*

Assume now that H is a closed subgroup of G on $\mathcal{M} = G/H$. It is not always true that there exists an invariant Radon measure on G/H. The next example gives a simple counter example.

THEOREM 10.9. *There exists an invariant Radon measure $\mu_{\mathcal{M}}$ on $\mathcal{M}$ if and only if $\Delta_G|_H = \Delta_H$. In this case $\mu_{\mathcal{M}}$ is unique up to multiplication by positive scalars.*

PROOF. See [?] p. 57. □

In particular, if H is compact then there is always a G-invariant Radon measure on G/H. This measure is simply given by

$$C_c(G/H) \to \mathbb{C}\,, \quad f \mapsto \int_G f \circ \kappa \, d\mu_G\,.$$

For more general considerations one needs *quasi-invariant* measures.

DEFINITION 10.10. Assume that G acts continuously on the topological space $\mathcal{M}$. A Radon measure μ is *quasi-invariant* (or G quasi-invariant) if there exists a measurable function $j : G \times \mathcal{M} \to \mathbb{R}^+$ such that for all $a \in G$ and all $f \in C_c(\mathcal{M})$

$$\int_{\mathcal{M}} f(a \cdot x) j(a,x)\, d\mu(x) = \int_{\mathcal{M}} f\, d\mu\,. \tag{10.2}$$

Replacing f by $\lambda(a)f$ in (10.2) shows that (10.2) is equivalent to

$$\int_{\mathcal{M}} f(x) j(a,x)\, d\mu(x) = \int_{\mathcal{M}} f(a^{-1} \cdot x)\, d\mu(x) \tag{10.3}$$

for all $f \in L^1(\mathcal{M})$.

For $a \in G$ define a measure μ_a on $\mathcal{M}$ by $\mu_a(E) = \mu(a \cdot E)$ then (10.3) shows that μ is quasi-invariant if and only if μ_a are equivalent, i.e., μ and μ_a have the same zero-sets for all $a \in G$, and in that case $j(a, \cdot)$ is the Radon-Nikodym derivative

$$j(a,x) = \frac{d\mu_a}{d\mu}(x)\,.$$

THEOREM 10.11. *Let $H \subseteq G$ be a closed subgroup and $\mathcal{M} = G/H$. There exists a non-trivial quasi invariant measure on $\mathcal{M}$ and every two such are equivalent.*

PROOF. See [?], Proposition 2.54, p. 59 and Theorem 2.59, p. 61, or [?], Chapter 6. □

EXAMPLE 10.12 (The $ax+b$-group). As a set the $ax+b$-group G is $\mathbb{R}^+ \times \mathbb{R}$. It can be viewed as the group of affine linear transformation on the line $\mathbb{R}$ where $\varphi_{(a,b)}(x) = ax + b$, hence the name of the group. The multiplication is given by composition of maps

$$\varphi_{(a,b)(c,d)} = \varphi_{(a,b)} \circ \varphi_{(c,d)} .$$

A simple calculation then shows that

$$(a, b)(c, d) = (ab, ad + b) .$$

In particular the identity element is $e = (1, 0)$ and $(a, b)^{-1} = (a^{-1}, -a^{-1}b)$. Define a Radon measure on G by

$$\int_G f(x)\, d\mu_G(x) := \int_0^\infty \int_{-\infty}^\infty f(a, b)\, \frac{dadb}{a^2} .$$

Then μ_G is left-invariant. But a simple change of variables shows that

$$\int_0^\infty \int_{-\infty}^\infty f((a, b)(c, d))\, \frac{dadb}{a^2} = c \int_0^\infty \int_{-\infty}^\infty f(a, b)\, \frac{dadb}{a^2} .$$

Hence $\Delta_G(c, d) = c^{-1}$. In particular, the $ax + b$-group is not unimodular.

Let $H = \{(a, 0) \mid a \in \mathbb{R}^+\}$. Then H is a closed subgroup, $H = G^0$ and $G/H \simeq \mathbb{R}$. An invariant measure on G/H has to be invariant under translations and dilations. The invariance under translation implies that the measure is–up to a constant–the Lebesgue measure. But the Lebesgue measure is not invariant under dilations. Hence, there is no invariant measure on G/H. But clearly the Lebesgue measure is quasi-invariant. This follows also from Theorem 10.9 because H is abelian and hence unimodular, but $\Delta_G(a, 0) = a^{-1}$ so $\Delta_G|_H \neq \Delta_G = 1$. $\diamondsuit$

EXAMPLE 10.13 (Symmetric Matrices). Recall that a matrix $X \in M_n(\mathbb{R})$ is *positive definite* if $(Xu, u) > 0$ for all $u \neq 0$. Denote the set of positive definite matrices by $\mathrm{Sym}_+(n, \mathbb{R})$. Then $\mathrm{Sym}_+(n, \mathbb{R})$ is open. We note that X is positive definite if and only if $g \cdot X$ is positive definite. Consider the action of $GL(n, \mathbb{R})$ on the space of symmetric matrices give by

$$g \cdot X := gXg^T .$$

Let $X \in \mathrm{Sym}_+(n, \mathbb{R})$. Then X can be diagonalized, in fact, there exists an orthonormal basis basis consisting of eigenvectors for X. We can reformulate this fact as: there exists $g \in \mathrm{SO}(n)$ such that

$$g \cdot X = d(\lambda_1, \ldots, \lambda_n) := \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} .$$

As X is positive definite $\lambda_j > 0$, $j = 1, \ldots, n$. Let a be the diagonal matrix $a = d(\lambda_1^{-1/2}, \ldots, \lambda_n^{-1/2})$. Then

$$a \cdot (g \cdot X) = I_n .$$

It follows that $\mathrm{GL}(n, \mathbb{R}) \cdot I_n = \mathrm{Sym}_+(n, \mathbb{R})$. Note that we can replace the group $\mathrm{GL}(n, \mathbb{R})$ by the connected component $\mathrm{GL}_+(n, \mathbb{R}) := \{a \in \mathrm{GL}(n, \mathbb{R}) \mid$

$\det(a) > 0\}$ Finally $a \cdot I_n$ if and only if $aa^T = I_n$. But that is if and only if $a \in \mathrm{O}(n)$, where $\mathrm{O}(n)$ denotes the group of $n \times n$ orthogonal matrices. Thus

$$\mathrm{Sym}_+(n, \mathbb{R}) \simeq \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n) \simeq \mathrm{GL}_+(n, \mathbb{R})/\mathrm{SO}(n)\,.$$

As $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact as we will prove in a moment, it follows that there exists a $\mathrm{GL}(n, \mathbb{R})$-invariant measure on $\mathrm{Sym}_+(n, \mathbb{R})$.

Let

$$A^+ = \{d(\lambda_1, \ldots, \lambda_n) \mid \lambda_1 > \lambda_2 > \ldots > \lambda_n > 0\}\,.$$

Then $\mathrm{Sym}_+^{\mathrm{reg}}(n, \mathbb{R}) := \mathrm{SO}(n, \mathbb{R})A^+ \cdot I_n$ is open and dense in $\mathrm{Sym}_+(n, \mathbb{R})$. If $a \in A^+$ and $k \in \mathrm{SO}(n, \mathbb{R})$ are such that $k \cdot a = a$ then $k = d(\pm 1, \ldots, \pm 1)$. Let $K := \mathrm{SO}(n, \mathbb{R})$ and

$$M := \{d(m_1, \ldots, m_n) \mid m_j = \pm 1 \text{ and } m_1 \cdots m_n = 1\}\,.$$

Then

$$K/M \times A^+ \to \mathrm{Sym}_+^{\mathrm{reg}}\,, \quad (kM, a) \mapsto ka \cdot I_n$$

is a diffeomorphism. ♢

11. The Tangent Space and Homogeneous Vector Bundles

In this section we introduce homogeneous vector bundles and then discuss the connection to the tangent bundle of a homogeneous manifold. We refer to [**?**] for more detailed discussion. In this section G is always a Lie group and $\mathcal{M} = G/K$ is a homogeneous manifold, $x_o \in \mathcal{M}$ is a fixed base point and $K = G^{x_o}$.

Definition 11.1. A (real) *vector bundle* over $\mathcal{M}$ is a manifold $\mathcal{V}$ together with a smooth map $\pi : \mathcal{V} \to \mathcal{M}$ such that

(1) For each $m \in \mathcal{M}$ the set $\pi^{-1}(m) =: \mathcal{V}_m$ is a vector space.
(2) For each $m \in \mathcal{M}$ there exists an open neighborhood U of m, a $k \in \mathbb{N}$, and a diffeomorphism

$$\Psi : \pi^{-1}(U) =: \mathcal{V}_U \to U \times \mathbb{R}^k$$

such that for each $m \in U$, $\Psi(m) \in \{x\} \times \mathbb{R}^k$ and the map

$$\Psi_m := \mathrm{pr}_2 \circ \Psi|_| \mathcal{V}_m : \mathcal{V}_m \to \mathbb{R}^k,$$

where pr_2 is the projection onto the second factor, is linear.

If we in (2) replace $\mathbb{R}^k$ by $\mathbb{C}^k$, then we speak of *complex vector bundle*.

If nothing else is said, the assumption is that $\mathcal{V}$ is real vector bundle.

An isomorphism of vector bundles is a diffeomorphism $\Phi : \mathcal{V} \to \mathcal{W}$ such that the diagram

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "\Phi"] \arrow[dr, "\pi_{\mathcal{V}}"'] & & \mathcal{W} \arrow[dl, "\pi_{\mathcal{W}}"] \\ & \mathcal{M} & \end{tikzcd} .$$

commutes and $\Phi|_{\mathcal{V}_m} : \mathcal{V}_m \to \mathcal{W}_m$ is a linear isomorphism for all $m \in M$.

Remark 11.2. One can also define vector bundles where the fibers are a fixed Banach spaces or Hilbert spaces $\mathcal{H}$. Then $\mathcal{V}$ would be a manifold modeled on $\mathbb{R}^{\dim \mathcal{M}} \times \mathcal{H}$ and $\mathcal{V}$ is locally isomorphic to $U \times \mathcal{H}$.

Let V be a vector space. The vector bundle

$$\mathcal{M} \times V \stackrel{\mathrm{pr}_1}{\to} \mathcal{M}$$

is called a *trivial vector bundle*. The second part of the definition of a vector bundle then says, that $\mathcal{V}$ is locally trivial.

Operations on vector bundles are defined fiberwise. As an example

$$\mathcal{V} \times \mathcal{W} := \bigcup_{m \in \mathcal{M}} \mathcal{V}_m \times \mathcal{W}_m \,,$$

$$\mathcal{V}^* := \bigcup_{m \in \mathcal{M}} \mathcal{V}_m^* \,,$$

$$\mathcal{V} \otimes \mathcal{W} := \bigcup_{m \in \mathcal{M}} \mathcal{V}_m \otimes \mathcal{W}_m \,,$$

and

$$\mathrm{Hom}(\mathcal{V}, \mathcal{W}) := \bigcup_{m \in \mathcal{M}} \mathrm{Hom}(\mathcal{V}_m, \mathcal{W}_m) = \mathcal{W}_m \otimes \mathcal{V}_m^* \,.$$

If V is a real vector space, then $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ denotes its complexification. For a vectorbundle $\mathcal{V}$ we define the *complexification* of $\mathcal{V}$ to be the complex vector bundle

$$\mathcal{V}_{\mathbb{C}} := \bigcup_{m \in \mathcal{M}} \mathcal{V}_{m\mathbb{C}} \,. \tag{11.1}$$

A vector bundle is *trivial*

Definition 11.3. A vector bundle $\mathcal{V} \to \mathcal{M}$ over $\mathcal{M}$ is a homogeneous vector bundle if G acts on $\mathcal{V}$ (from the left) in such a way that

(1) If $a \in G$ and $m \in \mathcal{M}$ then $a \cdot \mathcal{V}_m = \mathcal{V}_{a \cdot m}$.
(2) The map $\mathcal{V}_m \to \mathcal{V}_{a \cdot m}$, $v \mapsto a \cdot v$, is linear.

Two G bundles $\mathcal{V}$ and $\mathcal{W}$ are G-isomorphic if there exists a vector bundle isomorphism $\Psi : \mathcal{V} \to \mathcal{W}$ that commutes with the G-action, $\Psi(a \cdot v) = a \cdot \Psi(v)$. We then write $\mathcal{V} \simeq_G \mathcal{W}$. The bundle $\mathcal{V}$ is trivial if there exists a G-isomorphism $\mathcal{V} \to \mathcal{M} \times V$, where V is a finite dimensional vector space, and the action on $\mathcal{M} \times V$ is defined by $a \cdot (m, v) = (a \cdot m, v)$.

Let $\mathcal{H}$ real or complex Hilbert space. A *representation* of G in $\mathcal{H}$ is a homomorphism π of G into the group of invertible elements of $\mathrm{B}(\mathcal{H})$ such that

$$G \to \mathcal{H}\,, \quad a \mapsto \pi(a)v$$

is continuous for all $v \in \mathcal{H}$. The representation π is *unitary* if $\pi(a)$ is unitary for all $a \in G$. If $n = \dim \mathcal{H} < \infty$ then $\pi : G \to \mathrm{GL}(n, \mathbb{C})$ is a continuous homomorphism and hence analytic, see [**?**], Theorem 2.6, p. 117. We will show later how to prove this for linear Lie groups.

Let (π, W) be a real or complex representation of K. Let K act on $G \times W$ from the right by

$$(a, w) \cdot k = (ak, \pi(k)^{-1}(w))\,.$$

(a_1, w_1) and (a_2, w_2) are said to be equivalent, $(a_1, w_1) \sim (a_2, w_2)$, if they are in the same K-orbit, i.e., there exists a $k \in K$ such that

$$a_1 k = a_2 \quad \text{and} \quad \pi(k)^{-1}(w_1) = w_2\,.$$

We denote the equivalence class of (a, w) by $[a, w]$. The quotient map $(a, w) \mapsto [a, w]$ is denoted by κ_W. Let

$$G \times_\pi W := (G \times W)/K = \{[a, w] \mid a \in G\,,\ w \in W\}\,. \tag{11.2}$$

We will sometimes write $G \times_K W$ for $G \times_\pi W$ to indicate the role of K. In the following we will simply write $\mathcal{W}$ for $G \times_\pi W$.

It is clear that the left G-action on $G \times W$, $a \cdot (g, w) = (ag, w)$ and the right K action commutes. G therefore acts smoothly on $\mathcal{W}$ by

$$a \cdot [b, w] := [ab, w]\,.$$

This action can be visualized by the commutative diagram:

$$\begin{tikzcd} G \times W \arrow[r, "(b,w)\mapsto(ab,w)"] \arrow[d, "(b,w)\mapsto[b,w]"'] & G \times W \arrow[d, "(ab,w)\mapsto[ab,w]"] \\ \mathcal{W} \arrow[r, "{[b,w]\mapsto a\cdot[b,w]}"'] & \mathcal{W} \end{tikzcd} \tag{11.3}$$

It is also clear, that the map $G \times W \to G/K$, $\pi((a, w)) = aK$ factors through K and defines a G-map $\mathcal{W} \to \mathcal{M}$. Consider $G \times W$ as the trivial vector bundle over G by $\lambda(a, v) + (a, w) := (a, \lambda v + w)$. This commutes with the K action on the right, hence $\lambda[a, v] + [a, w] := [a, \lambda v + w]$ is well defined.

Suppose that $\mathcal{V} \to \mathcal{M}$ is a homogeneous vector bundle. Let $m_o = eK$ and $V = \mathcal{V}_{m_o}$. As K fixes m_o it follows by (1) and (2) above that the action on $\mathcal{V}$ defines a finite dimensional representation π of K on V. The following theorem explains how to recover the vector bundle $\mathcal{V}$ from π_V, and in fact shows how to construct *all* homogeneous vector bundles over $\mathcal{M}$.

We refer to Chapter 5 in [**?**] for the proof that $G \times_\pi W$ is locally trivial and a smooth manifold.

THEOREM 11.4. *Let (π, W) be a finite dimensional representation of of K then $G \times_\pi W$ is a homogeneous vector bundle. If $\mathcal{V} \to \mathcal{M}$ is a homogeneous vector bundle and (π, V) is the representation of K on $V = \mathcal{V}_{m_o}$ defined by the action of G on the $\mathcal{V}$, then the map*

$$G \times_\pi V \to \mathcal{V}\,, \quad [a, v] = a \cdot v \tag{11.4}$$ *eq-IsoVectBund*

is a G-isomorphism.

PROOF. Denote the map in (11.4) by Ψ. We start by showing that Ψ is well defined. For that we note first, that $(a_1, v_1) \sim (a_2, v_1)$ if and only if there exist a $k \in K$ such that

$$a_1 k = a_2 \text{ and } \pi(k)^{-1} v_1 = v_2\,.$$

But then, as π is defined as the action of K on $\mathcal{V}_{m_o}$, we get

$$a_2 \cdot v_2 = (a_1 k) \cdot (k^{-1} \cdot v_1) = a_1 \cdot v_1\,.$$

Next we show that the map is injective. Assume that $a_1 \cdot v_1 = a_2 \cdot v_2$. As $v_1, v_2 \in \mathcal{V}_{m_o}$ it follows by (1) in the definition of a homogeneous vector bundle that $k := a_1^{-1} a_2$ fixes m_o. Thus $k \in K$. By definition it follows that $\pi(k^{-1})(v_1) = v_2$ and $a_1 k = a_2$. Hence $[a_1, v_1] = [a_2, v_2]$.

Next we show that Ψ is surjective. Let $w \in \mathcal{V}$ and let $x \in \mathcal{M}$ be such that $w \in \mathcal{V}_x$. As the G action on $\mathcal{M}$ is transitive there exists $a \in G$ such that $a^{-1} \cdot x = m_o$. Thus $v := a^{-1} \cdot w \in V$ and clearly $\Psi([a, v]) = w$.

Consider the commutative diagram

$$\begin{tikzcd} G \times V \arrow[r] \arrow[d, "(a,v)\mapsto[a,v]"'] & \mathcal{V} \\ G \times_\pi V \arrow[ur, "\Psi"'] & \end{tikzcd} \qquad (g, v) \mapsto a \cdot v$$

As the horizontal map on the top is differentiable, it follows that Ψ is differentiable.

A local trivialization of $\mathcal{V}$ shows that the inverse of Ψ is smooth proving the theorem. $\square$

Let $\mathcal{V} \xrightarrow{\pi} \mathcal{M}$ be a vector bundle over $\mathcal{M}$. A *section* $s : \mathcal{M} \to \mathcal{V}$ is a smooth map such that $\pi \circ s = \mathrm{id}_{\mathcal{M}}$ or equivalently, $s(m) \in \mathcal{V}_m$ for all

$m \in \mathcal{M}$. The space of smooth sections is denoted by $\Gamma^\infty(\mathcal{V})$. The space of smooth, compactly supported sections is $\Gamma^\infty_c(\mathcal{V})$. Note that $\Gamma^\infty(\mathcal{V})$ and $\Gamma^\infty_c(\mathcal{V})$ are vector spaces.

Assume that $\mathcal{V}$ is homogeneous. Then G acts linearly on $\Gamma^\infty(\mathcal{V})$ by

$$(a \cdot s)(m) := a \cdot (s(a^{-1} \cdot m)) .$$

for all $a \in G$ and $m \in \mathcal{M}$. Let us do the calculations. It is clear that $e \cdot s = s$ for all s. Let $a, b \in G$. Then

$$\begin{aligned} (ab) \cdot s(m) &= (ab) \cdot (s(b^{-1}a^{-1} \cdot m)) \\ &= a \cdot [b \cdot (s(b^{-1} \cdot (a^{-1} \cdot m)))] \\ &= a \cdot [(b \cdot s)(a^{-1} \cdot m)] \\ &= [a \cdot (b \cdot s)](m) \end{aligned}$$

or $ab \cdot s = a \cdot (b \cdot s)$.

This action leaves $\Gamma^\infty_c(\mathcal{V})$ invariant. The section X is invariant if $a \cdot X = X$ for all $a \in G$. Thus s is invariant if and only if

$$s(a \cdot m) = a \cdot s(m)$$

The space of invariant sections is denoted by $\Gamma^\infty(\mathcal{V})^G$.

Let $G \times_\pi V$ be a homogeneous vector bundle over $\mathcal{M} = G/K$. Assume that $F : G \to V$ is a smooth function such that for all $a \in G$ and $k \in K$ the relation

$$F(ak) = \pi(k)^{-1} F(a) \tag{11.5}$$

holds. Consider the map $G \to G \times V$ defined by $a \mapsto (a, F(a))$. Let $k \in K$. Then

$$(ak, F(ak)) = (ak, \pi(k)^{-1} F(a)) \simeq (a, F(a)) .$$

Hence the map

$$s_F(aK) := [a, F(a)] = a \cdot [e, F(a)] \tag{11.6}$$ de-SF

is well defined and smooth. Clearly s_F is a smooth section.

THEOREM 11.5. *Denote by $C^\infty(G;\pi)$ the space of smooth functions such that (11.5) hold. Then the map* *th-Sections*

$$C^\infty(G;\pi) \to \Gamma^\infty(G \times_\pi V) , \quad F \mapsto s_F$$

is a linear bijection. Furthermore

$$s_{\lambda(a)F} = a \cdot F . \tag{11.7}$$ *stosF*

PROOF. We only have to show that the map is surjective. Let $s \in \Gamma^\infty(G \times_\pi V)$. Identify the fiber over eK by V. Define $F : G \to V$ by

$$F(a) := a^{-1} \cdot s(aK) .$$

Then F is smooth as s and smooth and the action of G on $G \times_\pi V$ is smooth.

Let $k \in K$. Then

$$\begin{aligned} F(ak) &= (ak)^{-1} \cdot s(aK) \\ &= k^{-1} \cdot (a^{-1} \cdot s(aK)) \\ &= \pi(k)^{-1} F(a) \,. \end{aligned}$$

Thus $F \in C^\infty(G; \pi)$

We also have

$$s(aK) = a \cdot (a^{-1} \cdot s(aK)) = a \cdot [e, F(a)] \,.$$

Hence $s = s_F$ by (11.6).

For (11.7) we simply calculate

$$\begin{aligned} s_{\lambda(a)F}(b \cdot m_o) &= [b, \lambda(a)F(b)] \\ &= [b, F(a^{-1}b)] \\ &= a \cdot [a^{-1}b, F(a^{-1}b)] \\ &= a \cdot s_F(b \cdot m_o) \,. \end{aligned}$$

□

Definition 11.6. Let $\mathcal{V} \to \mathcal{M}$ be a vector bundle. Let $\mathbb{F}$ is the field of real or complex numbers. Assume that each fiber $\mathcal{V}_m$ is a $\mathbb{F}$ vector space. A *Hermitian form* on $\mathcal{V}$ is a smooth section $g \in \Gamma^\infty(\mathcal{V}^* \otimes \mathcal{V}^*)$ such that $g_m : \mathcal{V}_m \times \mathcal{V}_m \to \mathbb{F}$ is an inner product on $\mathcal{V}_m$ for all $m \in \mathcal{M}$. If G is a Lie group and $\mathcal{V}$ is G-homogeneous, then g is G-invariant if for all $a \in G$, all $m \in \mathcal{M}$ and all $v, w \in \mathcal{V}_m$ we have

$$g_m(u, v) = g_{a \cdot m}(a \cdot v, a \cdot w) \,.$$

Lemma 11.7. *Let $\mathcal{V} = G \times_\pi V$ be a homogeneous vector bundle over $\mathcal{M} = G/K$. Then a G-invariant Hermitian form on $\mathcal{V}$ exists if and only if there exists a K-invariant Hermitian form β on V.*

Proof. Let $x_o = eK \in \mathcal{M}$. Assume that g is a G-invariant form on $\mathcal{V}$. Then clearly $g|_{V \times V}$ is a K-invariant Hermitian form on $V = \mathcal{V}_{x_o}$.

Assume that β is a K-invariant Hermitian form on V. Define

$$g_{a \cdot x_o}([a, v], [a, w]) := \beta(v, w) \,.$$

To show that g is well defined assume that $a \cdot x_o = b \cdot x_o$, $[a, v_1] = [b, v_2]$ and $[a, w_1] = [b, w_2]$. Then there exists $k \in K$ such that such that $a = bk$, $\pi(k)^{-1} v_2 = v_1$, and $\pi(k)^{-1} w_2 = w_1$. But then $\beta(v_1, w_1) = \beta(v_2, w_2)$ because of the K-invariance of β. That g is smooth is clear. □

Lemma 11.8. *If K is compact and $G \times_\pi V = \mathcal{V} \to G/K$ is a homogeneous vector bundle. Then a G-invariant Hermitian form exists.*

PROOF. Let γ be an inner product on $\mathcal{V}_{x_o}$. Let μ_K be a normalized Haar measure on K. Define β by

$$\beta(u,v) := \int_K \gamma(\pi(k)u, \pi(k)v)\, dk$$

Then β is clearly K-invariant and Hermitian. Furthermore, if $u \neq 0$, then

$$\beta(u,u) = \int_K \|\pi(k)u\|_\gamma^2 \, d\mu_K(k) > 0\,.$$

Hence β is a non-zero K-invariant form on V and the claim follows. $\square$

Let now $\mathcal{M} = G/K$ and take the vector bundle to be $T(\mathcal{M})$. Then $T(\mathcal{M})$ is homogeneous with respect to the action $g \cdot v = (d\ell_g)_x(v)$ if $v \in T_x(\mathcal{M})$. Let $x_o = eK$. Then $\ell_k(x_o) = x_o$ which implies that $\pi(k) := (d\ell_k)_{x_o} : T_{x_o}(\mathcal{M}) \to T_{x_o}(\mathcal{M})$ define a representation of K.

LEMMA 11.9. *$T(\mathcal{M}) \simeq G \times_\pi T_{x_o}(\mathcal{M})$.*

PROOF. This follows from Theorem 11.4. $\square$

This shows that G/K has a G-invariant Riemannian structure if and only if there exists a ℓ_K-invariant inner product on $T_{x_o}(\mathcal{M})$. In particular that is the case if K is compact.

12. Invariant Differential Operators

In this section we introduce the notation of invariant differential operators on G-spaces, where G is a Lie group. We determine the space of invariant differential operators on homogeneous manifolds $M = G/K$ in the case where K is compact. We use this characterization to introduce the heat equation on those manifolds.

Let M be a manifold. A linear map $D : C^\infty(M) \to C^\infty(M)$ is a *differential operator* if in local coordinates $\phi : U \to V$, U open in M and V open in $\mathbb{R}^n$

$$(Df) \circ \phi^{-1} = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha (f \circ \phi^{-1}) \tag{12.1}$$

with $a_\alpha \in C^\infty(V)$. Denote by $\mathrm{Diff}(M)$ the algebra of differential operators on M. Assume now that the group G acts on M by diffeomorphisms ℓ_a, $a \in G$. For $D \in \mathrm{Diff}(M)$ and $a \in G$ define $a \cdot D \in \mathrm{Diff}(M)$ by

$$a \cdot D(f) := [D(f \circ \ell_a) \circ \ell_{a^{-1}} = \lambda(a)[D(\lambda(a^{-1})f) \text{ for } f \in C^\infty(M)\,. \tag{12.2}$$

LEMMA 12.1. *Let $a, b \in U$ and $D, E \in \mathit{Diff}(M)$. Then $(ab) \cdot D = a \cdot (b \cdot D)$ and $a \cdot (DE) = (a \cdot D) \circ (a \cdot E)$. Thus G acts on $\mathit{Diff}(M)$ by algebra homomorphisms.*

PROOF. Let $f \in C^\infty(M)$. Then

$$\begin{aligned}(ab)\cdot D(f) &= D(f\circ \ell_{ab})\circ \ell_{(ab)^{-1}}\\ &= [[D((f\circ\ell_a)\circ\ell_b)]\circ\ell_{b^{-1}}]\circ\ell_{a^{-1}}\\ &= [b\cdot D(f\circ\ell_a)]\circ\ell_{a^{-1}}\\ &= a\cdot[b\cdot D](f)\,.\end{aligned}$$

For the second statement

$$\begin{aligned}a\cdot(DE)(f) &= [DE(f\circ\ell_a)]\circ\ell_{a^{-1}}\\ &= [D([E(f\circ\ell_a)]\circ\ell_{a^{-1}})\circ\ell_a]\circ\ell_{a^{-1}}\\ &= (a\cdot D)\circ(a\cdot E)(f)\,.\end{aligned}$$

□

A differential operator D is called G-*invariant* or simply *invariant* if $a\cdot D = D$ for all $a\in G$. Denote by $\mathrm{Diff}(M)^G$ the space of G-invariant differential operator. Then Lemma 12.1 implies that $\mathrm{Diff}(M)^G$ is a subalgebra of $\mathrm{Diff}(M)$. We note that D is G invariant if and only if for all $a\in G$ and $f\in C^\infty(M)$:

$$D(f\circ\ell_a) = D(f)\circ\ell_a\,. \tag{12.3}$$

Let V be a Euclidean vector space and $p: V\to\mathbb{C}$ a polynomial function. Let $a\in \mathrm{GL}(V)$. then $a\cdot pIv) = p(a^{-1}v)$ is again a polynomial function. Denote by θ the homomorphism $\mathrm{GL}(V)\to\mathrm{GL}(V)$, $a\mapsto(a^{-1})^t$.

LEMMA 12.2. *Suppose $\emptyset\neq\Omega\subseteq V$ is open. Assume that $a\in\mathrm{GL}(V)$ satisfies $a(\Omega)=\Omega$. Let $p\in P(V)$. Then*

$$a\cdot(p(\partial)) = (\theta(a)\cdot p)(\partial)\,.$$

PROOF. As every polynomial is a linear combination of products of degree one polynomials $w\mapsto p_v(w) := (v,w)$ we can assume that $p=p_v$ for some $v\in V$. Then $p_v(\partial)f(w) = \partial_v(f)(w) = D(f)(w)v$. Hence by the chain rule:

$$\begin{aligned}a\cdot p_v(\partial)f(w) &= p_v(\partial)(f\circ\ell_a)(a^{-1}w)\\ &= D(f\circ\ell_a)(a^{-1}w)v\\ &= D(f)(w)D\ell_a(v)\\ &= D(f)(w)a(v)\\ &= p_{a(v)}(f)(w)\,.\end{aligned}$$

We have $p_{a(v)}(w) = (a(v),w) = (v,a^t(w)) = p_v(\theta(a)^{-1}w) = \theta(a)\cdot p_v(w)$ and the Lemma follows. □

EXAMPLE 12.3 (Invariant Differential Operators on V). Consider V as an abelian Lie group acting on V by translaton. Let $u, w \in V$. It follows immediately from the definition.

Let $D = \sum_\alpha a_\alpha \partial^\alpha \in \mathrm{Diff}(V)^V$. Then

$$D(f \circ \ell_v)(w) = \sum_\alpha a_\alpha(w) \partial^\alpha(f)(w+v)$$

as each ∂^α is V-invariant. On the other hand

$$D(f)(w+v) = \sum_\alpha a_\alpha(w+v) \partial^\alpha(f)(w+v)\,.$$

Thus, by (12.3) D is invariant if and only if $a_\alpha(w) = a_\alpha(w+v)$ for all $w, v \in V$. Thus D is invariant if and only if a_α is constant. As $pq(\partial) = p(\partial)q(\partial)$ it follows that

LEMMA 12.4. *$\mathit{Diff}(V)^V \simeq P(V)$ as an algebra.*

Replace now V by the (connected) *Euclidean motion group*

$$E(V) = \mathrm{SO}(V) \ltimes V \tag{12.4}$$

where as usually $\mathrm{SO}(V) = \{a \in \mathrm{GL}(V) \mid (av, aw) = (v, w) \text{ and } \det a = 1\}$. $E(V)$ acts on V by

$$(a, v) \cdot w = a(w) + v\,.$$

The group multiplication in $E(V)$ is the one that comes from composition of maps

$$\begin{aligned} [(a, v)(b, w) \cdot u &= (a, v) \cdot (b(u) + w) \\ &= ab(u) + a(w) + v \\ &= (ab, a(w) + v) \cdot u \end{aligned}$$

or

$$(a, v)(b, w) = (ab, a(w) + v)\,.$$

In particular

$$(a, v)^{-1} = (a^{-1}, -a^{-1}(v))\,.$$

LEMMA 12.5. *Let $p \in P(V)$ be invariant under rotations. Then there exists a polynomial in one variable such that $p(w) = q(\|w\|^2)$ for all $w \in V$.*

PROOF. Fix $v \in \mathrm{S}$ and define $\hat{q}(t) := p(tv)$. Let $w \in V$. Then there exists $a \in U$ such that $a^{-1}(w) = \|w\|v$. Hence

$$p(w) = p(a^{-1}(w)) = p(\|w\|v) = \hat{q}(\|w\|)\,.$$

As $p(w)$ is a polynomial it follows that $\hat{q}(t)$ can only contain even powers of t. Hence $q(t^2) := \hat{q}(t)$ is a well defined polynomial and $p(w) = q(\|w\|^2)$. $\square$

LEMMA 12.6. *$\mathit{Diff}(V)^{E(V)} = \mathbb{C}[\Delta]$.*

Proof. Let $q(\Delta) \in \mathbb{C}[\Delta]$ and let $p(w) := q(\|w\|^2)$. Then $q(\Delta) = p(\partial)$. It follows from Lemma 12.2, Lemma 12.4, and the fact that $p(w) = q(\|w\|^2)$ is rotational invariant, that $\mathbb{C}[\Delta] \subset \mathrm{Diff}(V)^{E(V)}$.

Let $D \in \mathrm{Diff}(V)^{E(V)}$. Then D is translation invariant. By Lemma 12.4 there exists a polynomial $p \in P(V)$ such that $D = p(\partial)$. As D is $\mathrm{SO}(V)$-invariant it follows from Lemma 12.2 that p is $\mathrm{SO}(V)$-invariant. Hence there exists a polynomial of one variable such that $p(w) = q(t^t)$. But then $p(\partial) = q(\Delta)$ and the claim follows. $\square$

Let $M = G/K$ be a homogeneous manifold with K compact. Then there exists a K-stable complement $\mathfrak{q}$ of $\mathfrak{k}$ in $\mathfrak{g}$ and we have local coordinates given by $\mathrm{Exp}_g : U \subset gV \subset M$, $X \mapsto g\exp(X) \cdot m_o$, where $U \subset \mathfrak{q}$ is an open neighborhood of zero and V is an open neighborhood of $m_o = eK$. Let $(\,\cdot\,,\,\cdot\,)$ be a $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{q}$. Then $(\mathrm{Ad}(k)^{-1})^t = \mathrm{Ad}(k^{-1})^t$ is well defined and $\mathrm{Ad}(k^{-1})^t = \mathrm{Ad}(k)$ for all $k \in K$.

For a polynomial $p \in P(\mathfrak{q})$ define

$$[Df](a \cdot m_o) := p(\partial)(f \circ \mathrm{Exp}_a)(0)\,. \tag{12.5}$$

This is a well defined differential operator on M if and only if

$$D_p(f \circ \mathrm{Exp}_a)(0) = p(\partial)(f \circ \mathrm{Exp}_{ak})(0) \tag{12.6}$$

for all $k \in K$ and $a \in G$. As K acts on $\mathfrak{q}$ by the adjoint action $\mathrm{Ad}(k)|_{\mathfrak{q}}$ it follows by Lemma 12.2 that (12.6) holds for all a and k if and only if p is $\mathrm{Ad}(K)$-invariant. Denote the algebra of invariant differential operators on M by $\mathbb{D}(M)$. This notion is not quite correct as M might have several Lie groups acting transitively giving rise to different algebras of invariant differential operators as the case of the V above shows. But it will always be clear what G is, so this should not hurt.

Theorem 12.7. *The map $P(\mathfrak{q})^K \to \mathit{Diff}(M)^G$, $p \mapsto D_p$ is a linear isomorphism.*

Proof. We have already seen that $p \mapsto D_p \in \mathrm{Diff}(M)$ is well defined. The definition (12.5) is made so that

$$D_p(f)(a \cdot m_o) = D_p(f \circ \ell_a)(m_o)\,.$$

Hence, if $m = b \cdot m_o$,

$$\begin{aligned} D_p(f)(a \cdot m) &= D_p(f \circ \ell_{ab})(m_o) \\ &= D_p((f \circ \ell_a) \circ \ell_b)(m_o) \\ &= D_p(f \circ \ell_a)(b \cdot m_o) \\ &= D_p(f \circ \ell_a)(m)\,. \end{aligned}$$

Hence D_p is G-invariant.

Let $D \in \mathrm{Diff}(M)^G$. Let $m = g \cdot m_o \in M$ and consider the local coordinates Exp_g above. Then there exists smooth functions $a_\alpha \in C^\infty(U)$ such that

$$Df(\mathrm{Exp}_g(X) \cdot m_o) = \sum_\alpha a_\alpha(X)\partial^\alpha(f \circ \mathrm{Exp}_g)(X)\,.$$

Define a polynomial $p \in P(\mathfrak{q})$ by $p(X) := \sum_\alpha a_\alpha(0)X^\alpha$. As D is invariant, it follows that $Df(a \cdot m_o) = D(f \circ \ell_a)(m_o)$. Thus

$$\begin{aligned} Df(a \cdot m_o) &= D(f \circ \ell_a)(m_0) \\ &= \left.\sum_\alpha a_\alpha(X)\partial^\alpha((f \circ \ell_a) \circ \mathrm{Exp})(X)\right|_{X=0} \\ &= p(\partial)((f \circ \ell_a) \circ \mathrm{Exp})(0) \\ &= D_p(f)(a \cdot m_o)\,. \end{aligned}$$

Assume that $D_p = 0$. Let $\alpha \in (\mathbb{N}_0)^{\dim \mathfrak{q}}$. Let $W \subset U$ be a compact neighborhood of O and let $\varphi \in C_c^\infty(U)$ such that $\varphi|_W = 1$. Then we can define $f \in C_c^\infty(M)$ by

$$\begin{aligned} f(\mathrm{Exp}(X)) &= X^\alpha\varphi(X)\,, \quad X \in U \\ f(m) &= 0\,, \quad m \in M \setminus V\,. \end{aligned}$$

Then, if $p = \sum_\alpha a_\alpha X^\alpha$,

$$D_p f(m_o) = \alpha! a_\alpha = 0\,.$$

Hence $p = 0$ and $p \mapsto D_p$ is injective. $\square$

Let $\Delta_\mathfrak{q}$ be the Laplace operator on the Euclidean vector space $\mathfrak{q}$. Then $\Delta_\mathfrak{q} = Q(\partial)$ where $Q(X) = \|X\|^2$. As Q is K-invariant it follows that $\Delta_\mathfrak{q}$ defines a G-invariant differential operator Δ_M on M. Δ_M is called the *Laplace operator on* M. The *heat equation* on M is then

$$\Delta_M u(m,t) = \partial_t u(m,t) \text{ and } u(m,0) = f(m)$$

for $f \in C^\infty(M)$. For M the sphere in an Euclidean vector space V we discuss this in the next chapter and give a brief overview over the heat equation on some other homogeneous manifolds later.

13. Orthogonal Groups

Let $\mathrm{Bil}_s(n,\mathbb{R})$ denote the space of symmetric bilinear forms on $\mathbb{R}^n$. It is isomorphic to the space $\mathrm{Sym}(n,\mathbb{R})$ of symmetric $n \times n$-matrices via $A \mapsto \beta_A$, $\beta_A(u,v) := (A(u),v)$. The group $\mathrm{GL}(n,\mathbb{R})$ acts on $\mathrm{Bil}_s(n,\mathbb{R})$ by

$$(a \cdot \beta)(u,v) := \beta(a^{-1}u, a^{-1}v)\,.$$

This corresponds to the action $a \cdots X := (a^{-1})^T X a^{-1}$ on $\mathrm{Sym}(n,\mathbb{R})$. The form β is non-degenerate, ie., if $u \neq 0$, then there exists $v \in \mathbb{R}^n$ such that $\beta(u,v) \neq 0$, if and only if $a \cdot \beta$ is non-degenerate. Hence $\mathrm{GL}(n,\mathbb{R})$ acts on the subset $\mathrm{Bil}_s^{nd}(n,\mathbb{R})$ of non-degenerate bilinear form. Note that β_A is non-degenerate if and only if $A \in \mathrm{GL}(n,\mathbb{R})$.

For $\beta \in \mathrm{Bil}_s^{nd}(n,\mathbb{R})$ let

$$\mathrm{O}(\beta) := \{a \in \mathrm{GL}(n,\mathbb{R}) \mid a \cdot \beta = \beta\}$$

be the stabilizer of β_A in $\mathrm{GL}(n,\mathbb{R})$. Then $O(\beta_A)$ is is a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ and hence a Lie group. We have

Lemma 13.1. $\mathrm{O}(\beta_A) = \{a \in \mathrm{GL}(n,\mathbb{R}) \mid a^T A a = A\}$.

Proof. As $\mathrm{O}(\beta_A)$ is a group it follows that $a \in \mathrm{O}(\beta_A)$ if and only if $a^{-1} \in \mathrm{O}(\beta_A)$. That is the case if and only if

$$\begin{aligned} a^{-1} \cdot \beta_A(u,v) &= (Aa(u), a(v)) \\ &= (a^T A a(u), v) \\ &= (A(u), v) \\ &= \beta_A(u,v)\,. \end{aligned}$$

As v is arbitrary it follows that $A(u) = a^T A a(u)$ for all u. Hence $a^T A a = A$. $\square$

There are only finitely many orbits in $\mathrm{Bil}_s^{nd}(n,\mathbb{R})$:

1) *The orthogonal group* $\mathrm{O}(n)$. This is the group that corresponds to $A = \mathrm{I}_n$. Thus

$$\begin{aligned} \mathrm{O}(n) &= \{a \in \mathrm{GL}(n,\mathbb{R}) \mid (\forall u, v \in \mathbb{R}^n)\ (a(u), a(v)) = (u,v)\} \\ &= \{a \in \mathrm{GL}(n,\mathbb{R}) \mid a^T a = \mathrm{I}_n\}\,. \end{aligned}$$

We let $\mathrm{SO}(n) = \{a \in \mathrm{O}(n) \mid \det(a) = 1\}$. Then $\mathrm{SO}(n)$ is a closed subgroup in $\mathrm{O}(n)$. The group $\mathrm{O}(n)$ is not connected as elements with $\det(a) = -1$ can not be connected to any element in $\mathrm{SO}(n)$. We denote by $\mathrm{SO}_o(n) = \mathrm{O}_o(n)$ the connected component containing the identity element. Similar notation will be used for other groups.

Lemma 13.2. *The group* $\mathrm{O}(n)$ *is compact.*

Proof. For $a \in \mathrm{GL}(n,\mathbb{R})$ write $a = [a_1, \ldots, n_n]$ where a_j is a column vector. Then

$$a^T a = (a_i \cdot a_j)\,.$$

Hence $a \in \mathrm{O}(n)$ if and only if the columns are orthonormal. Thus $\mathrm{O}(n)$ is homeomorphic to a closed subset of $\underbrace{S^{n-1} \times \ldots S^{n-1}}_{n-\text{times}}$ and hence compact. $\square$

For $a \in \mathrm{O}(n)$ and $u \in S^{n-1}$ clearly $a(u) \in S^{n-1}$ hence $\mathrm{O}(n)$ acts on S^{n-1}. Let $e_1 = (1, 0, \ldots, 0)^T$. Then

$$[a_1, \ldots, a_n](e_1) = a_1\,.$$

If $v \in S^{n-1}$ then we can extend v to an orthonormal basis $v_1 = v, v_2, \ldots, v_n$ on S^{n-1}. Then $a = [v_1, \ldots, v_n] \in \mathrm{O}(n)$ and $a(e_1) = v$. It follows that the action is transitive. By permuting the vectors we can assume that $\det(a) = 1$ and hence $\mathrm{SO}(n)$ acts transitively.

We see that $a \in \mathrm{O}(n)^{e_1}$ if and only if $a_1 = e_1$. But that happens if and only if

$$a \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathrm{O}(n-1) \right\} \simeq \mathrm{O}(n-1)\,.$$

Thus $\mathrm{O}(n)^{e_1} \simeq \mathrm{O}(n-1)$ and similarly $\mathrm{SO}(n)^{e_1} \simeq \mathrm{SO}(n-1)$. It follows that

$$S^{n-1} \simeq \mathrm{O}(n)/\mathrm{O}(n-1) = \mathrm{SO}(n)/\mathrm{SO}(n-1)\,.$$

2) $\mathrm{O}(p,q)$: For $1 \leq p, q \leq n$ such that $n = p + q$ let

$$I_{p,q} = \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & -I_q \end{pmatrix}\,.$$

The form $\beta_{p,q} = \beta_{I_{p,q}}$ has signature (p,q) and

$$\beta_{p,q}(u,v) = u_1 v_1 + \ldots + u_p v_p - u_{p+1} v_{p+1} - \ldots - u_n v_n\,.$$

We will always assume that $1 \leq p \leq q$. We set

$$\mathrm{O}(p,q) := \mathrm{O}(\beta_{p,q}) \text{ and } \mathrm{SO}(p,q) := \{a \in \mathrm{O}(p,q) \mid \det(a) = 1\}\,.$$

For $r \in \mathbb{R}$ let

$$H(p,q;r) := \{u \in \mathbb{R}^n \mid \beta_{p,q}(u,u) = r\}$$

except in the case $p = 1$ and $r > 0$. In that case we set

$$H(1, n-1; r) := \{x \in \mathbb{R}^n \mid \beta_{1,n-1}(x,x) = r \text{ and } x_1 > 0\}\,.$$

According to Theorem 5.10 $\mathcal{M}(r)$ is a closed submanifold of $\mathbb{R}^n$ of dimension $n-1$ if $r \neq 0$, which we will assume from now in. The groups $\mathrm{O}(p,q)$ and $\mathrm{SO}(p,q)$ act on each one of those manifolds. Furthermore, if $s = \sqrt{|r|}$ then multiplication by s is diffeomorphism $H(p,q;\mathrm{sign}(r)) \simeq H(p,q;r)$ commuting with the action of $\mathrm{O}(p,q)$. We can therefore assume that $r = 1$ or $r = -1$.

Let us prove the following.

THEOREM 13.3. *Let $p = 1$.*

(1) $\mathrm{SO}(1, n-1)$ *acts transitively on* $H(1, n-1; 1)$ *and the stabilizer of* e_1 *is*

$$\mathrm{SO}(1, n-1)^{e_1} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathrm{SO}(n-1) \right\} \simeq \mathrm{SO}(n-1)\,.$$

(2) *Suppose* $n \geq 3$. $\mathrm{SO}(1, n-1)$ *acts transitively on* $H(1, n-1; -1)$ *and the stabilizer of* $e_n = (0, \dots, 0, 1)^T$ *is*

$$\mathrm{SO}(1, n-1)^{e_n} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \middle| \; A \in \mathrm{SO}(1, n-2) \right\} \simeq \mathrm{SO}(1, n-2)\,.$$

PROOF. Let us start by defining two subgroups of $\mathrm{SO}(1, n-1)$.

$$\begin{aligned} K &:= \left\{ k_A := \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \middle| \; A \in \mathrm{SO}(n-1) \right\} \simeq \mathrm{SO}(n-1) \\ A &:= \left\{ a_t := \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix} \middle| \; t \in \mathbb{R} \right\}. \end{aligned}$$

We leave the simple proof that those are indeed subgroups of $\mathrm{SO}(1, n-1)$ to the reader.

(1) Let $v = (t, w) \in H(1, n-1; 1)$, $t > 0$, $w \in \mathbb{R}^{n-1}$. Then there exists $A \in \mathrm{SO}(n-1)$ such that $A(w) = \|w\| e_n$. Thus $k_A v = (t, 0, \dots, \|w\|)^T$. As $v \in H(1, n-1; 1)$ it follows that

$$t^2 - \|w\|^2 = 1\,.$$

Hence there exist $s \in \mathbb{R}$, $s \geq 0$, such that

$$t = \cosh(s) \text{ and } \|w\| = \sinh(s)\,.$$

It is then clear that

$$a_s(e_1) = (t, 0, \dots, \|w\|)^T\,.$$

Thus

$$k_A^{-1} a_s(e_1) = v\,.$$

It is clear that a is in the stabilizer of e_1 if and only if a has the form

$$a = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Inserting this into $\beta_{p,q}$ it follows that a leaves $\beta_{p,q}$ invariant if and only if $A \in \mathrm{O}(n-1)$. If $a \in \mathrm{SO}(1, n-1)$ then $A \in \mathrm{SO}(n-1)$ and the claim follows.

(2) Let $v = (t, w)^T \in H(1, n-1; -1)$. Let $A \in \mathrm{SO}(n-1)$ be so that $A(w) = \|w\| e_n$. Then $\|w\|^2 - t^2 = 1$. Hence, there exists $s \in \mathbb{R}$ such that $t = \sinh(s)$ and $\|w\| = \cosh(s)$. Hence $a_s(e_n) = (t, 0, \dots, 0, \|w\| e_n)^T$. It follows that $k_A^{-1} a_s(e_n) = v$.

A matrix a fixes e_n if and only if a is of the form

$$a = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

for some $(n-1) \times (n-1)$-matrix. Inserting into the bilinear form we see that $a \in \mathrm{O}(1, n-1)$ if and only if $B \in \mathrm{O}(1, n-2)$, respectively $a \in \mathrm{SO}(1, n-1)$ if and only if $B \in \mathrm{SO}(1, n-2)$. $\square$

We note that there is a fundamental difference between $H(1, n-1; 1)$ and $H(1, n-1, -1)$. In the first case the stabilizer is compact, in the second case it is non-compact. Geometrically $H(1, n-1; 1)$ a Riemannian manifold, but $H(1, n-1; -1)$ is a pseudo-Riemannian manifold.

The proof of Theorem 13.3 actually show the following:

THEOREM 13.4 (Polar-coordinates for $H(1, n-1; 1)$). *The map*

$$S^{n-2} \times \mathbb{R}^+ \to H(1, n-1; 1) \setminus \{(1, 0)^T\} \quad (t, w) \mapsto (\cosh(t), \sinh(t)w)^T$$

is a diffeomorphism.

We remark that each of the manifolds $H(p, q; r)$ has an invariant Radon measure.

EXAMPLE 13.5. Let $\beta : \mathbb{R}^n \to \mathbb{R}$ be a non-degenerated symmetric bilinear form and $\mathcal{M}$ a connected component of $\{u \in \mathbb{R}^n \mid \beta(u, u) = r\}$. Assume that $G = \mathrm{SO}_o(\beta)$ acts transitively on $\mathcal{M}$. Then $\mathcal{M} = G/K$ where $K = G^{x_o}$ for some $x_o \in \mathcal{M}$. Let

$$\mathcal{V} := \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid u \in \mathcal{M} \text{ and } \beta(u, v) = 0\}.$$

Then $\mathcal{V}$ is a closed submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. $\mathcal{V}$ is clearly a vector bundle over $\mathcal{M}$ with projection π simply the projection onto the first factor. If $(u, v) \in \mathcal{V}$ and $g \in G$, then $(g(u), g(v)) \in \mathcal{V}$ as β is G-invariant. It follows that $\mathcal{V}$ is a homogeneous vector bundle. Now we claim that $T_x(\mathcal{M}) \simeq \{v \in \mathbb{R}^n \mid \beta(x, v) = 0\}$. For that recall, that if $\gamma : (-\epsilon, \epsilon) \to \mathcal{M}$ is a smooth curve with $\gamma(0) = x$ then γ defines a tangent vector $\dot{\gamma}(0)$ by

$$\dot{\gamma}(0)f = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t))$$

and every tangent vector is obtained in this way. But if γ is such a curve then $t \mapsto \beta(\gamma(t), \gamma(t))$ is constant and hence

$$0 = \left.\frac{d}{dt}\right|_{t=0} \beta(\gamma(t), \gamma(t)) = 2\beta(\gamma(0), \left.\frac{d}{dt}\right|_{t=0} \gamma(t)).$$

Hence $\beta(\gamma(0), \left.\frac{d}{dt}\right|_{t=0} \gamma(t)) = 0$. It follows that the map

$$(\gamma(0), \dot{\gamma}(0)) \mapsto (\gamma(0), \left.\frac{d}{dt}\right|_{t=0} \gamma(t))$$

defines a map $T(\mathcal{M}) \to \mathcal{V}$. This map is clearly injective and linear on the fibers. Comparing the dimensions we see that it is a diffeomorphism. We need to see that this map is a G-map. Let γ be as above and $a \in G$. Then

$$\begin{aligned}(d\ell_a)_x(\dot\gamma(0))f &= \dot\gamma(0)(f \circ \ell_a)\\ &= \left.\frac{d}{dt}\right|_{t=0} f \circ \ell_a(\gamma(t))\\ &= \left.\frac{d}{dt}\right|_{t=0} f(a\gamma(t))\,.\end{aligned}$$

But $\gamma_a(t) = a\gamma(t)$ is a smooth curve in $\mathcal{M}$ with $\gamma_a(0) = a \cdot x$ and

$$\left.\frac{d}{dt}\right|_{t=0})\gamma_a(t) = a\left.\frac{d}{dt}\right|_{t=0})\gamma(t)\gamma(0)\,.$$

The claim now follows.

A special case of this is that $T(S^{n-1}) = \mathrm{SO}(n) \times_\pi \mathbb{R}^{n-1}$ where π is the natural representation of $\mathrm{SO}(n-1)$, $\pi(k)v = k(v)$. Similar remark holds for the manifolds $H(p, q; r)$. We note, that

$$H(1, n-1, 1) \simeq \mathrm{SO}(1, n-1)/\mathrm{SO}(n-1)$$

has a $\mathrm{SO}(1, n-1)$-invariant Riemannian structure as $\mathrm{SO}(n-1)$ is compact.

On the other hand for $n \geq 3$

$$H(1, n-1; -1) = \mathrm{SO}(1, n-1) \simeq \mathrm{SO}(1, n-1)/\mathrm{SO}(1, n-2)\,.$$

There is no positive definite $\mathrm{SO}(1, n-2)$-invariant form on $\mathbb{R}^{n-1}$ because such a form would give a homomorphism $\mathrm{SO}(1, n-2) \to \mathrm{O}(n-1)$ and there is no such non-trivial homomorphism. But the form $\beta_{1,n-2}$ is invariant. Thus $H(1, n-1; -1)$ has an invariant form of signature $(1, n-1)$. It is an Lorentzian manifold.

Let $\mathfrak{s} \subset \mathfrak{g}$ be a complementary subspace to $\mathfrak{k}$. Then $\mathfrak{s}$ can be identified with the tangent space $T_{x_o}(\mathcal{M})$ by

$$X_{x_o}(f) = \left.\frac{d}{dt}\right|_{t=0} f(e^{tX} \cdot x_o) = \partial_X(f \circ \mathrm{Exp})(0)\,. \tag{13.1}$$

The map is a linear isomorphism because Exp defines local coordinates around x_o.

Assume that we can chose $\mathfrak{s}$ $\mathrm{Ad}(K)$-invariant, i.e., if $k \in K$ and $X \in \mathfrak{s}$ then $\mathrm{Ad}(k)X \in \mathfrak{s}$. Thus $\mathrm{Ad}|_K$ defines a representation of K in $\mathfrak{s}$ which we denote by $\mathrm{Ad}_\mathfrak{s}$ and we have the homogeneous vector bundle $G \times_{\mathrm{Ad}_\mathfrak{s}} \mathfrak{s}$.

LEMMA 13.6. *If $\mathfrak{s}$ is a K-invariant complementary subspace to $\mathfrak{k}$ in $\mathfrak{g}$, then $T(\mathcal{M}) \simeq_G G \times_{\mathrm{Ad}_\mathfrak{s}} \mathfrak{s}$.*

PROOF. We identify $\mathfrak{s}$ with $T_{x_o}(\mathcal{M})$ as before. If $k \in K$ and $X \in \mathfrak{s}$ then

$$\begin{aligned} T_{x_o}(\ell_k)(X)f &= \frac{d}{dt}\bigg|_{t=0} f \circ \ell_k(e^{tX} \cdot x_o) \\ &= \frac{d}{dt}\bigg|_{t=0} f(ke^{tX} \cdot x_o) \\ &= \frac{d}{dt}\bigg|_{t=0} f(e^{t\mathrm{Ad}(k)X} \cdot x_o) \\ &= (\mathrm{Ad}(k)X)_{x_o} f \, . \end{aligned}$$

Thus $T_{x_o}\ell_k$ is nothing but $\mathrm{Ad}_{\mathfrak{s}}(k)$. The claim now follows from Theorem 11.4. □

The following lemma gives a simple criteria for the existence of a K-invariant complementary subspace $\mathfrak{s}$ in $\mathfrak{g}$.

LEMMA 13.7. *Denote by $Z(G)$ the center of the group G. If $g \in Z(G)$, then* $\mathrm{Ad}(g) = \mathrm{id}$.

PROOF. Let $U_{\mathfrak{g}} \subset \mathfrak{g}$ be an open ball with center zero such that $\exp : U_{\mathfrak{g}} \to U_G$, $U_G = \exp U_{\mathfrak{g}}$ open, is a diffeomorphism. Let $X \in \mathfrak{g}$. Let $\epsilon > 0$ be such that $tX, t\mathrm{Ad}(g)X \in U_{\mathfrak{g}}$ for all $|t| < \epsilon$. As g commutes with all element in G we get

$$e^{tX} = ge^{tX}g^{-1} = e^{t\mathrm{Ad}(g)X} \, .$$

But then $tX = t\mathrm{Ad}(g)X$ for all $|t| < \epsilon$, which implies that $X = \mathrm{Ad}(g)X$. □

Let $\theta : \mathrm{GL}(n, \mathbb{F}) \to \mathrm{GL}(n, \mathbb{F})$ be defined by

$$\theta(g) = (g^*)^{-1} \, . \tag{13.2}$$

Then θ is a continuous homomorphism and hence analytic. We have

$$\dot{\theta}(X) = -X^*$$

by Theorem 13.10.

LEMMA 13.8. *Assume that $\theta(G) = G$ and $\theta(K) = K$. Then the following holds:*

(1) *If $X \in \mathfrak{g}$ then $X^* \in \mathfrak{g}$ and if $X \in \mathfrak{k}$ then $X^* \in \mathfrak{k}$.*
(2) *The bilinear form*

$$\beta(X, Y) = \mathrm{Tr}(XY)$$

is non-degenerate on $\mathfrak{g}$ and $\mathfrak{k}$ and $\beta(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = \beta(X, Y)$ for all $X, Y \in \mathfrak{g}$ and $g \in G$, i.e., β is G-invariant.
(3) *Let $\mathfrak{s} = \{Y \in \mathfrak{g} \mid (\forall X \in \mathfrak{k})\ \beta(X, Y) = 0\}$. Then*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$$

PROOF. (1) Let $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. Then

$$e^{tX^*} = (e^{tX})^* \in G\,.$$

By Theorem 4.3 it follows that $X^* \in \mathfrak{g}$. The same argument shows that $\mathfrak{k}^* = \mathfrak{k}$.

(2) Let $X \in \mathfrak{g}$, $X \neq 0$. Then $X^* \in \mathfrak{g}$ and $\beta(X, X^*) = \mathrm{Tr}(XX^*) > 0$. If $X \in \mathfrak{k}$ then $X^* \in \mathfrak{k}$ showing that β is non-degenerate on $\mathfrak{g}$ and $\mathfrak{k}$. If $g \in G$, then

$$\begin{aligned} \beta(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) &= \mathrm{Tr}((gXg^{-1})(gYg^{-1})) \\ &= \mathrm{Tr}(gXYg^{-1}) \\ &= \mathrm{Tr}(XY) \\ &= \beta(X, Y)\,. \end{aligned}$$

(3) The bilinear form $(X, Y) = \mathrm{Tr}(XY^*)$ is an inner product on $\mathfrak{g}$. As $\mathfrak{k}^* = \mathfrak{k}$ it follows that $\mathfrak{s}$ is the orthogonal complement to $\mathfrak{k}$ with respect to this inner product. Hence $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. □

LEMMA 13.9. *Let G be a linear Lie group with Lie algebra $\mathfrak{g}$ and let K is a closed subgroup with Lie algebra $\mathfrak{k}$. Denote by $Z(G)$ the center of G. If $K/(Z(G) \cap K)$ is compact or if G and K are invariant under the involution $\theta(x) = (x^*)^{-1}$, then a K-invariant complementary subspace $\mathfrak{s}$ exists.*

PROOF. Assume that $K/Z(G) \cap K$ is compact. Let $(\cdot, \cdot)$ be an inner product on $\mathfrak{g}$. As $K_1 := K/Z(G) \cap K$ is a compact topological group, we can normalize the Haar measure on K_1 such that $\int_{K_1} dk = 1$. As $\mathrm{Ad}(z) = \mathrm{id}$ for all $z \in Z(G)$ it follows that Ad defines a homomorphism of K_1 into $\mathrm{GL}(\mathfrak{g})$. Define a new inner product on $\mathfrak{g}$ by

$$\langle X, Y \rangle := \int_{K_1} (\mathrm{Ad}(k)X, \mathrm{Ad}(k)Y)\, dk\,.$$

Then $\langle \cdot, \cdot \rangle$ is K-invariant, $\langle \mathrm{Ad}(k)X, \mathrm{Ad}(k)Y \rangle = \langle X, Y \rangle$. Let $\mathfrak{s} = \mathfrak{k}^\perp$ be the orthogonal complement of $\mathfrak{k}$ with respect to $\langle \cdot, \cdot \rangle$. Then $\mathfrak{s}$ is K-invariant.

Assume now that $\theta(G) = G$ and $\theta(K) = K$. Let β and $\mathfrak{s}$ be as in Lemma 13.8. As β is K-invariant and $\mathrm{Ad}(K)\mathfrak{k} = \mathfrak{k}$ it follows that $\mathrm{Ad}(K)\mathfrak{s} = \mathfrak{s}$ and the claim follows. □

We remark that all the Lie groups and Lie algebras in Example 6.2 are invariant under θ respectively $\dot{\theta}$.

A *vector bundle* over $\mathcal{M}$ is a smooth manifold $\mathcal{V}$ together with a surjective, smooth map $\pi : \mathcal{V} \to \mathcal{M}$ such that the following holds:

(1) $\mathcal{V}_m := \pi^{-1}(m)$ is a vector space for all $m \in \mathcal{M}$. $\mathcal{V}_m$ is called the *fiber* over the point m.

(2) For all $m \in \mathcal{M}$ there exists an open neighborhood $m \in U$, a vector space $V = V_U$, and a diffeomorphism
$$\Phi_U : \pi^{-1}(U) \to U \times V$$
such that $\Phi_U(v) = (\pi(v), F(v))$, where $F : \mathcal{V}_x \to V$ is a linear.

An isomorphism of vector bundles is a diffeomorphism $\Phi : \mathcal{V} \to \mathcal{W}$ such that the diagram

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "\Phi"] \arrow[dr, "\pi_{\mathcal{V}}"'] & & \mathcal{W} \arrow[dl, "\pi_{\mathcal{W}}"] \\ & \mathcal{M} & \end{tikzcd} .$$

commutes and $\Phi|_{\mathcal{V}_m} : \mathcal{V}_m \to \mathcal{W}_m$ is a linear isomorphism for all $m \in M$. The vector bundle $\mathcal{M} \times V$, V a vector space, with projection map $(m, v) \mapsto m$, is called a *trivial vector bundle*. The second part of the definition of a vector bundle then says, that $\mathcal{V}$ is locally trivial.

Operations on vector bundles are defined fiberwise. Thus, let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles. Define
$$\mathcal{V} \otimes \mathcal{W} := \bigcup_{m \in \mathcal{M}} \mathcal{V}_m \otimes \mathcal{W}_m \,,$$
$$\mathcal{V} \times \mathcal{W} := \bigcup_{m \in \mathcal{M}} \mathcal{V}_m \times \mathcal{W}_m \,.$$
and
$$\mathcal{V}^* := \bigcup_{m \in \mathcal{M}} \mathcal{V}_m^* \,.$$
Then $\mathcal{V} \otimes \mathcal{W}$, $\mathcal{V} \times \mathcal{W}$, and $\mathcal{V}^*$ are vector bundles in a natural way.

If V is a real vector space, then $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ denotes its complexification. For a vectorbundle $\mathcal{V}$ we define the $i$complexification of $\mathcal{V}$ to be the complex vector bundle
$$\mathcal{V}_{\mathbb{C}} := \bigcup_{m \in \mathcal{M}} \mathcal{V}_{m\mathbb{C}} \,. \tag{13.3}$$

If $\mathcal{V} \stackrel{\pi}{\to} \mathcal{M}$ is a vector bundle then a section $X : \mathcal{M} \to \mathcal{V}$ is a smooth map such that $\pi \circ X \mathrm{id}_{\mathcal{M}}$ or equivalently, $X(m) \in \mathcal{V}_m = \pi^{-1}(m)$ for all $m \in \mathcal{M}$. The space of smooth sections is denoted by $\Gamma^\infty(\mathcal{V})$. The space of smooth, compactly supported sections is $\Gamma_c^\infty(\mathcal{V})$. If G acts on $\mathcal{M}$, then G also acts on $\Gamma^\infty(\mathcal{V})$. The action is given by
$$(g \cdot X)(x) = g \cdot X(g^{-1} \cdot x) \,.$$

The standard example of a vector bundle is the *tangent bundle*
$$T(\mathcal{M}) = \bigcup_{x \in \mathcal{M}} T_x(\mathcal{M})$$

with projection map

$$T(\mathcal{M}) \to \mathcal{M}\,, \quad (x,v) \mapsto x\,.$$

A real *vector field* is an element of $\Gamma^\infty(T(\mathcal{M}))$, whereas a (complex) vector field is a section $X \in \Gamma^\infty(T(\mathcal{M})_{\mathbb{C}})$. Denote the space real respectively complex vector fields by $\Gamma(\mathcal{M})$ respectively $\Gamma_{\mathbb{C}}(\mathcal{M})$.

Let $\mathbb{F}$ denote the field of real or complex numbers. The exponential map $\exp : \mathrm{M}(n,\mathbb{F}) \to \mathrm{M}(n,\mathbb{F})$ is given by the power series

$$\exp(X) = \sum_{j=0}^{\infty} \frac{X^n}{n!}$$

which converges uniformly on every closed ball $B_R(0) \subset \mathrm{M}(n,\mathbb{F})$ with center 0 and radius $R > 0$. We denote $\exp(X)$ also by e^X. The exponential function satisfies the differential equation

$$\frac{d}{dt} e^{tX} = X e^{tX} = e^{tX} X\,.$$

For $x_1, \ldots, x_n \in \mathbb{F}$, where $\mathbb{F}$ is a field, denote by $\mathrm{diag}(x_1, \ldots, x_n)$ the diagonal matric with diagonal elements $x_1, \ldots, x_n$, i.e.,

$$\mathrm{diag}(x_1, \ldots, x_n) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_n \end{pmatrix}.$$

We note the following simple facts, and leave most of the proof for the reader as an exercise:

Lemma 13.10. *Let $X, Y \in \mathrm{M}(n,\mathbb{F})$ and $g \in \mathrm{GL}(n,\mathbb{F})$. Then the following holds:*

(1) $\|e^X\| \leq e^{\|X\|}$.
(2) *If X and Y commutes, then*
$$e^{X+Y} = e^X e^Y\,.$$
(3) *If $g \in \mathrm{GL}(n,\mathbb{F})$ then*
$$g e^X g^{-1} = \exp(g X g^{-1})\,.$$
(4) *If $X = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ then*
$$e^X = \mathrm{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})\,.$$
More generally, if X is upper triangular
$$X = \begin{pmatrix} \lambda_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_n \end{pmatrix}$$

then e^X is the upper triangular matric

$$e^X = \begin{pmatrix} e^{\lambda_1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & e^{\lambda_n} \end{pmatrix}$$

(5) $(e^X)^t = e^{X^t}$ *and* $(e^X)^* = e^{X^*}$.
(6) *If X is a lower triangular matric, then so is e^X.*
(7) $\det(e^X) = e^{\mathrm{Tr}(X)}$. *In particular* $e^X \in \mathrm{GL}(n, \mathbb{F})$.

We recall the the following. Assume that G is a Lie group and $H \subset G$ a closed subgroup. Let $\kappa G \to \mathcal{M} := G/H$ be the canonical quotient map. The space $\mathcal{M}$ is a locally compact Hausdorff topological space in the quotient topology: $U \subset \mathcal{M}$ is open if and only if $\pi^{-1}(U) \subseteq G$ is open.

In some cases $\mathrm{Iso}(\mathcal{M})$ is in fact a Lie group acting smoothly on $\mathcal{M}$. It follows that

$$G/K \to \mathcal{M}\,, \quad gK \mapsto g(x_o)$$

is a diffeomorphism. For a detailed discussion of this see [**?**], pp. 201–208.

A measure μ on $\mathcal{M}$ is *G-invariant* if for all $f \in C_c^\infty(\mathcal{M})$ and all $g \in G$ we have

$$\int_{\mathcal{M}} f(g(x))\, d\mu(x) = \int_{\mathcal{M}} f(x)\, d\mu(x)\,.$$

Let dg be a left invariant measure (Haar measure) on G. If K is compact, it follows that

$$\int_{\mathcal{M}} f(x)\, d\mu(x) := \int_G f(g(x_o))\, dg \tag{13.4}$$

defines a G-invariant measure on $\mathcal{M}$. The invariant measure is unique up to a multiple with positive constant. Thus, if K is compact, an invariant measure is always of the form (13.4) up to a positive constant.

The volume form is always invariant under the group of isometries. Hence, if G is a finite dimensional Lie group acting smoothly, isometrically, and transitively on $\mathcal{M}$ such that the stabilizer of a point is $\mathcal{M}$ is compact, then the volume element is given by a left invariant measure on the group G as in (13.4).

14. Representations

In this section we introduce the basic ideas from representations theory. In this section G is always a Lie group, even if most of the definitions work just as well for topological groups.

14.1. Representations. A *representation* of G in a vector space $\mathcal{H}$ is a homomorphism $\pi : G \to \mathrm{GL}(\mathcal{H})$, the group of invertible elements in $\mathcal{H}$. If $\mathcal{H}$ is a topological vector space, in particular a Hilbert space, then we also assume that for all $u \in \mathcal{H}$ the map

$$G \to \mathcal{H}\,, \quad g \mapsto \pi(g)u$$

is continuous. We write $(\pi, \mathcal{H})$ or simple π. Then $\mathcal{H}_\pi$ or V_π denote the vector space on which π acts.

From now on $\mathcal{H}_\pi$ will be a Hilbert space if nothing else is said. The representation π is *unitary* if $\pi(G) \subset \mathrm{U}(\mathcal{H})$, ie., each operator $\pi(g)$ is unitary. Note, if π is an unitary representation, then

$$\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1})\,.$$

A subspace $\mathcal{K} \subseteq \mathcal{H}$ is *invariant* if $\pi(G)\mathcal{K} \subseteq \mathcal{K}$. The representation π is *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only closed invariant subspaces of $\mathcal{H}$.

14.2. Intertwining Operators. If π and σ are two representations, then an operator, not necessarily bounded, $T : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ is an *intertwining operator* if for all $g \in G$ we have $T \circ \pi(g) = \sigma(g) \circ T$. Write $\mathrm{B}_G(\mathcal{H}_\pi, \mathcal{H}_\sigma)$ or simply $\mathrm{B}_G(\pi, \sigma)$ for the set of bounded intertwining operators. Similarly, write $\mathrm{U}_G(\mathcal{H}_\pi, \mathcal{H}_\sigma)$ for the set of unitary intertwining operators. If $\pi = \sigma$, then we simply write $\mathrm{B}_G(\mathcal{H}_\pi)$, $\mathrm{B}_G(\pi)$, and $\mathrm{U}_G(\pi)$. It is clear that $\mathrm{B}_G(\pi, \rho)$ is closed in $\mathrm{B}(\mathcal{H}_\pi, \mathcal{H}_\sigma)$. We say that π and σ are *equivalent* if there exists $T \in \mathrm{B}_G(\pi, \sigma)$ an isomorphism. π and σ are *unitary equivalent* if $\mathrm{U}(\pi, \sigma) \neq \emptyset$.

LEMMA 14.1. *Let π, σ, and τ be unitary representations of G.*

1. *If $T \in \mathrm{B}_G(\pi, \sigma)$ and $S \in \mathrm{B}_G(\sigma, \tau)$ then $S \circ T \in \mathrm{B}_G(\pi, \tau)$.*
2. *If $T \in \mathrm{B}_G(\pi, \sigma)$ then $T^* \in \mathrm{B}_G(\sigma, \pi)$.*
3. *$\mathrm{B}_G(\pi, \sigma)$ is closed in $\mathrm{B}(\pi, \sigma)$ in the norm topology and the strong topology.*
4. *$\mathrm{B}_G(\pi)$ is a B^* algebra.*
5. *If $T \in \mathrm{B}(\pi, \sigma)$ then $\overline{\mathrm{Im}(T)}$ is σ-invariant.*
6. *If $\mathcal{K} \subset \mathcal{H}_\pi$ is a closed subspace invariant under π, then $\mathcal{K}^\perp$ is invariant and the orthogonal projection $P_\mathcal{K}$ onto $\mathcal{K}$ is in $\mathrm{B}(\pi)$.*

PROOF. In this proof a will always stand for an arbitrary element of G.

(1) Let $a \in G$, then $S(T(\pi(a))) = S(\sigma(a)T) = \tau(a) \circ (S \circ T)$.

(2) Let $a \in G$. Let

$$\begin{aligned} T^*\sigma(a) &= (\sigma(a)^*T)^* \\ &= (\sigma(a^{-1})T)^* \\ &= (T\pi(a^{-1}))^* \\ &= \pi(a)T^* \end{aligned}$$

as $\pi(a^{-1})^* = \pi(a)$.

(3) If $T_j \to T$ in the norm topology, then $T_j \to T$ in the strong topology. Therefore we only have to show that $\mathrm{B}_G(\pi, \sigma)$ is closed in the norm topology. Assume that $T_j \in \mathrm{B}_G(\pi, \sigma)$ and that $T_j \to T \in \mathrm{B}(\pi, \sigma)$. Then

$$\begin{aligned} \|T\pi(a) - \sigma(a)T\| &= \|T\pi(a) - T_j\pi(a) + \sigma(a)T_j - \sigma(a)T\| \\ &\leq \|(T - T_j)\pi(a)\| + \|\sigma(a)(T_j - T)\| \\ &= 2\|T - T_j\| \\ &\to 0 \end{aligned}$$

Hence $\|T\pi(a) - \sigma(a)T\| = 0$ or $T\pi(a) = \sigma(a)T$ showing that $T \in B_G(\pi, \sigma)$.

(4) This follows from (2) and (3).

(5) Let $a \in G$ and $v \in \mathcal{H}_\pi$. Then $\sigma(a)T(v) = T(\pi(a)v)$. Hence $\mathrm{Im}(T)$ is σ-invariant. The claim follows as π is unitary.

(6) Let $u \in \mathcal{K}^\perp$. If $a \in G$ and $v \in \mathcal{K}$ then

$$(v, \pi(a)u) = (\pi(a^{-1})v, u) = 0$$

as $\pi(a^{-1})v \in \mathcal{K}$. It follows that $\pi(a)u \in \mathcal{K}^\perp$.

(7) Write $P = P_\mathcal{K}$. Let $u \in \mathcal{K}$. Then $\pi(a)u \in \mathcal{K}$ and hence $P(\pi(a)u) = \pi(a)u = \pi(a)P(u)$. If $u \in \mathcal{K}^\perp$ then $\pi(a)u \in \mathcal{K}^\perp$ and hence $P(\pi(a)u) = 0 = \pi(a)P(u)$. If $u \in \mathcal{H}$ is arbitrary then write $u = v + w$ with $v \in \mathcal{K}$ and $w \in \mathcal{K}^\perp$. Then

$$\begin{aligned} P(\pi(a)u) &= P(\pi(a)v) + P(\pi(a)w) \\ &= \pi(a)P(v) \\ &= \pi(a)P(u)\,. \end{aligned}$$

□

LEMMA 14.2. *Suppose π and σ are unitary representations of G. If $T \in B_G(\pi, \sigma)$ then $T^* \in B_G(\sigma, \pi)$.*

PROOF. Let $u \in \mathcal{H}_\pi$ and $v \in \mathcal{H}_\sigma$. Then

$$\begin{aligned} (u, T^*(\sigma(g))v) &= (T(u), \sigma(g)v) \\ &= (\sigma(g^{-1})T(v), u) \\ &= (T(\pi(g^{-1})v), u) \\ &= (\pi(g^{-1})v, T^*(u)) \\ &= (v, \pi(g)T^*(u))\,. \end{aligned}$$

As u and v are arbitrary it follows that t^* intertwines σ and π. □

THEOREM 14.3. *Let π and σ be two unitary representations. Assume that $T : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ is a densely defined, closed intertwining operator. Suppose that T is injective with dense image. Then π and σ are unitary equivalent.*

PROOF. We can, according to Theorem **??**, write

$$T = U\sqrt{T^*T}$$

where $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ is a partial isometry that intertwines π and σ. $\mathrm{Im}(U)$ is closed as U is a partial isometry. But $\mathrm{Im}(T) \subseteq \mathrm{Im}(U)$ and $\mathrm{Im}(T)$ is dense in $\mathcal{H}_\sigma$, hence $\mathrm{Im}(U) = \mathcal{H}_\sigma$. It follows that $U \in \mathrm{U}_T(\pi, \sigma)$ and the claim follows. □

THEOREM 14.4 (Schur's Lemma). *Let $(\pi, \mathcal{H})$ be an unitary representation. Then π is irreducible if and only if* $\mathrm{B}(\pi) = \mathbb{C}\mathrm{id}$.

PROOF. Suppose that π is irreducible. Let $T \in \mathrm{B}(\pi)$. Then $T_1 := \frac{1}{2}(T + T^*) \in \mathrm{B}(\pi)$ is self adjoint. The spectral decomposition of T_1 is $T_1 = \int_{\sigma(T_1)} \lambda dE$, Theorem 6.3, and each $E(A)$ is an intertwining operator. If $T_1 \neq \lambda\mathrm{id}$, then there exists a spectral projection $0 \neq E(A) \neq \mathrm{id}$. But then $E(A)\mathcal{H}$ is a closed invariant subspace, $\{0\} \neq E(A)\mathcal{H} \neq \mathcal{H}$, contradicting the irreducibility of π. Hence, there exists a real number λ such that $T_1 = \lambda\mathrm{id}$. Similarly, there exists a real number μ such that $T_2 = \frac{1}{2i}(T - T^*) = \mu\mathrm{id}$. But then

$$T = T_1 + iT_2 = (\lambda + i\mu)\mathrm{id}\,.$$

Assume that $\mathcal{B}(\pi) = \mathbb{C}\mathrm{id}$. If π is not irreducible, then there exists an invariant subspace $\mathcal{K} \subset \mathcal{H}$ such that $0 \neq \mathcal{K} \neq \mathcal{H}$. But then the orthogonal projection $P_\mathcal{K}$ is in $\mathrm{B}(\c{P})$ but is not a multiple of the identity. □

COROLLARY 14.5. *Assume that G is abelian. If π is an irreducible representation of G then* $\dim \mathcal{H}_\pi = 1$.

14.3. Orthogonal Direct Sums. Let I be a finite or countably infinite set. Suppose that for each $i \in I$ there is given an unitary representation $(\pi_i, \mathcal{H}_i)$. Let

$$\mathcal{H} := \bigoplus \mathcal{H}_i = \{(u_i)_{i\in I} \mid \sum_{i\in I} \|u_i\|^2 < 0\}\,.$$

If $(u_i), (v_i) \in \mathcal{H}_i$ then

$$((u_i), (v_i)) := \sum_{i\in I} (u_i, v_i)_{\mathcal{H}_i}$$

converges absolutely and defines an inner product on $\mathcal{H}$, making $\mathcal{H}$ into a Hilbert space. For $a \in G$ define an operator $\pi(a) = \oplus\pi_i(a) : \mathcal{H} \to \mathcal{H}$ by

$$\pi(a)(u) := (\pi_i(a)u_i)_i$$

where $u = (u_i)_i$. As $\sum_{i\in I} \|\pi_i(a)u_i\|^2 = \sum_{i\in I} \|u_i\|^2$ it follows that $\pi(a)u \in \mathcal{H}$.

LEMMA 14.6. $\pi = \oplus_{i\in I}\pi_i$ *is an unitary representation of G.*

Proof. It is an easy exercise to show that $\pi(a)$ is unitary and $\pi(ab) = \pi(a)\pi(b)$. Let us now show that π is continuous in the strong operator topology. For that it is enough to show that π is continuous at e. Let $\epsilon > 0$. As $\sum_i \|u\|^2 < \infty$ there exists a finite subset $\emptyset \neq J \subset I$ such that

$$\sum_{i \in I \setminus J} \|u_i\|^2 < \frac{\epsilon}{2\sqrt{2}}\,.$$

For $j \in J$ let V_j be an open neighborhood of e such that for all $a \in V_j$

$$\|\pi_j(a)u_j - u_j\| \leq \frac{\epsilon}{\sqrt{2\#J}}\,.$$

Let

$$V := \bigcap_{j \in J} V_j\,.$$

Then V is an open neighborhood of e as J is finite. Using that $\|\pi(a)u_i - u_i\| \leq 2\|u_i\|$ as $\pi_i(a)$ is unitary, we get for $a \in V$:

$$\begin{aligned} \|\pi(a)u - u\|^2 &= \sum_{j \in J} \|\pi_i(a)u_i - u_i\|^2 + + \sum_{j \notin J} \|\pi_i(a)u_i - u_i\|^2 \\ &< \#J \frac{\epsilon^2}{2\#J} + 4 \sum_{i \notin J} \|u_i\|^2 \\ &< \epsilon^2\,. \end{aligned}$$

Hence $a \mapsto \pi(a)u$ is continuous. □

14.4. Cyclic Representations. Let $(\pi, \mathcal{H})$ be an unitary representation. Suppose that π is not irreducible. Then there exists an invariant subspace $\{0\} \neq \mathcal{K} \neq \mathcal{H}$ and $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$ is a decomposition of $\mathcal{H}$ into two orthogonal subspaces. We can then apply the same argument to $\mathcal{K}$ and $\mathcal{K}^\perp$. Either, this process stops in finite steps, or it goes on without an end. In the first case we have that $\mathcal{H} = \bigoplus_{j=1}^k \mathcal{H}_j$ is a decomposition of $\mathcal{H}$ into finitely many irreducible invariant subspaces. In the second case it can happen that there are countably many irreducible subspaces $\mathcal{H}_j \subset \mathcal{H}$ such that $\mathcal{H} = \bigoplus_{j=1}^\infty \mathcal{H}_j$ or no so decomposition exists (see next section). Irreducible subspaces are therefore not always the best tool to work with.

Definition 14.7. Let $(\pi, \mathcal{H})$ be an unitary representation. A vector u is *cyclic* if the space $\{\sum_{\text{finite}} c_j \pi(a_j)u \mid c_j \in \mathbb{C}\,,\ a_j \in G\}$ is dense in $\mathcal{H}$. If there exists a cyclic vector, then we say that π is a *cyclic representation*.

Theorem 14.8. *If π is an unitary representation. Then π is a orthogonal direct sum of cyclic representations.* *th-cyclic*

Proof. Let Λ be the collection of families of pairwise orthogonal invariant and cyclic subspaces of $\mathcal{H}$. Thus $\{\mathcal{H}_j\} \in \Lambda$ if and only if each $\mathcal{H}_j$ is invariant under π, cyclic under the representation $\pi_j(a) := \pi(a)|_{\mathcal{H}_j}$, and

$\mathcal{H}_i \perp \mathcal{H}_j$ if $i \neq j$. We order Λ by inclusion. If A_j is a linearly ordered subset in Λ, then $A := \bigcup A_j \in \Lambda$ and $A_j \leq A$. Thus every linearly ordered subset of Λ has an upper bound. By Zorn's Lemma Λ has a maximal element $\{\mathcal{H}_j\}_{j\in J}$. Let $\mathcal{K} := \bigoplus_{j\in J} \mathcal{H}_j$. Then $\mathcal{K}$ is a orthogonal direct sum of cyclic subspaces. We claim that $\mathcal{K} = \mathcal{H}$. Otherwise $\mathcal{K}^\perp \neq \{0\}$. Take $u \in \mathcal{K}^\perp$, $u \neq 0$. Then

$$\mathcal{L} := \{\sum_{\text{finite}} c_j \pi(a_j) u \mid c_j \in \mathbb{C}\,,\ a_j \in G\}$$

is a non-zero G-invariant subspace of $\mathcal{H}$ and $\mathcal{L}$ is orthogonal to all $\mathcal{H}_j$. Hence $\{\mathcal{H}_j\} < \{\mathcal{H}_j\} \cup \{\mathcal{L}\} \in \Lambda$ contradicting the maximality of $\{\mathcal{H}_j\}$. Thus $\mathcal{K} = \mathcal{H}$. □

Remark 14.9. We will see that the decomposition in Theorem 14.8 is far from being unique. ♢

15. Examples of Representations

In this section we give three important examples of representations. Some of those will be important for later discussion. All sets will be measurable. In the following we will always view two sets A and B as equal if the set theoretical difference $(A \setminus B) \cup (B \setminus A)$ has measure zero.

Take now $G = V$ and $H = \{0\}$. Then $L^2(\mathcal{M}) = L^2(V)$ and the representation λ is given by

$$[\lambda(v)f](x) = f(x-y)\,.$$

Define a representation $\hat{\lambda}$ on $L^2(V*)$ by

$$[\hat{\lambda}(v)\varphi](\lambda) = e^{-i\lambda(v)}\varphi(\lambda)\,.$$

Both representations are unitary. Theorem **??** implies that the Fourier transform is an unitary intertwining operator, $\mathcal{F} \in \mathrm{U}(\lambda, \widehat{\lambda})$. In particular, λ and $\widehat{\lambda}$ are equivalent.

For a measurable set $A \subseteq V$ let

$$L^2_A(V) = \{f \in L^2(V) \mid \text{ for almost all} \lambda \notin A \,:\, \widehat{f}(\lambda) = 0\} \simeq L^2(A)\,.$$

Then $L^2_A(V) \neq \{0\}$ if and only if A has a positive measure. $L^2(A)$ is invariant under the representation $\hat{\lambda}$. Hence $L^2_A(V)$ is invariant under λ.

Lemma 15.1. *A subspace $\{0\} \neq \mathcal{K} \subseteq L^2(V)$ is invariant if and only if there exists a measurable subset A, $|A| > 0$, such that $\mathcal{K} = L^2_A(V)$.*

Proof. See [**?**], Corollary 15.1. □

15.1. Representations on Spaces of Holomorphic Functions.

DEFINITION 15.2. A separately continuous map $\alpha : G\times X \to \mathbb{C}^*$ is called a cocycle if for all $a, b \in G$ and all $x \in X$ we have

$$\alpha(ab, x) = \alpha(a, b \cdot x)\alpha(b, x)\,.$$

LEMMA 15.3. *Let α be a cocyle. Then $\alpha(e, x) = 1$ for all $x \in X$ and*

$$\alpha(b, x)^{-1} = \alpha(b^{-1}, b \cdot x)\,.$$

PROOF. Take $a = b = e$ in the definition of a cocyle. Then $\alpha(e, x) = \alpha(e, x)^2$. As $\alpha(e, x) \neq 0$ it follows that $\alpha(e, x) = 1$ for all $x \in X$. Taking $a = b^{-1}$ then implies that

$$1 = \alpha(b^{-1}, b \cdot x)\alpha(b, x)\,.$$

$\square$

If α is a cocycle, then we can define a linear action of G on $C(X)$ by

$$[\pi_\alpha(g)f](x) = \alpha(g^{-1}, x)f(g^{-1}x)\,. \tag{15.1}$$

That this defines an action follows by:

$$\begin{aligned}
[\pi_\alpha(ab)f](x) &= \alpha(b^{-1}a^{-1}, x)f((b^{-1}a^{-1}) \cdot x)\\
&= \alpha(b^{-1}, a^{-1} \cdot x)\alpha(a^{-1}, x)f(b^{-1} \cdot (a^{-1} \cdot x))\\
&= \alpha(a^{-1}, x)[\pi_\alpha(b)f](a^{-1} \cdot x)\\
&= \pi_\alpha(a)[\pi_\alpha(b)f](x)\,.
\end{aligned}$$

REMARK 15.4. If one is working with vector valued functions instead of scalar valued, one has to define the representation by $\pi_\alpha(g)f](x) = \alpha(g^{-1}, x)^{-1}f(g^{-1}x)$ as can be seen by the step for second display to the third one.

Note, that if we were considering vector valued functions and operator valued cocycles, then the second last step in the above explains why we use the inverse.

LEMMA 15.5. *Suppose that π is an unitary representation of G given by the cocycle α on a reproducing kernel Hilbert space $\mathcal{H} \subset C(X)$. Then*

$$\alpha(a, x)K(a \cdot x, y) = \overline{\alpha(a^{-1}, y)}K(x, a^{-1} \cdot y)$$

and

$$\alpha(a, x)\overline{\alpha(a, y)}K(a \cdot x, a \cdot y) = K(x, y)\,.$$

In particular, if $\alpha \equiv 1$, then

$$(\forall g \in G,\ x, y \in X) \qquad K(a \cdot x, a \cdot y) = K(x, y)\,. \tag{15.2}$$

PROOF. Let $f \in \mathcal{H}$. Then

$$\begin{aligned}(f, \overline{\alpha(a^{-1}, y)} K_{a^{-1}\cdot y}) &= \alpha(a^{-1}, y) f(a^{-1} \cdot y) \\ &= [\pi(a) f](y) \\ &= (\pi(a) f, K_y) \\ &= (f, \pi(a^{-1}) K_y)\,.\end{aligned}$$

As f was arbitrary it follows that

$$\alpha(a, x) K(a \cdot x, z) = \overline{\alpha(a^{-1}, z)} K(x, a^{-1} \cdot z)$$

which is the first claim. Then replace y by $a \cdot y$ and use that $\alpha(a, y)^{-1} = \alpha(a^{-1}, a \cdot y)$ to derive the second statement from the first. □

We say that the kernel K is *invariant* if it satisfies (15.2). If G acts transitively on X, i.e., for all $x \in G$ we have $X = G \cdot x$, fix $x_o \in X$ and let

$$H = G^{x_o} := \{a \in G \mid a \cdot x_o = x_o$$

be the stabilizer of x_o. Then the map $X \simeq G/H$. Define a continuous function $k : G \to \mathbb{C}$ by

$$k(a) = K(a \cdot x_o, x_o)\,.$$

Then

$$\begin{aligned}k(h_1 a h_2) &= K(h_1 a h_2 \cdot x_o, x_o) \\ &= K(a \cdot x_0, h_2^{-1} \cdot x_o) \\ &= K(a \cdot x_o, x_o) \\ &= k(a)\end{aligned}$$

Thus k is H-biinvariant. We can also view k as H-invariant function on X by $k(x) = K(x, x_o)$. Finally, the kernel K is given by

$$K(a \cdot x_o, b \cdot x_o) = k(b^{-1} a)\,.$$

The condition (3) in Lemma **??** is now equivalent to saying that the function k is positive definite, i.e., for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in G$, the matrix $(k(a_i^{-1} a_j))_{i,j=1}^n$ is positive semidefinite.

LEMMA 15.6. *Let $f \in L^1(\mathbb{R}^n)$ be positive. Then*

$$g(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\lambda) e^{i\lambda \cdot x}\, d\lambda$$

is positive definite. In particular, the heat kernel h_t is positive definite.

PROOF. Let $\alpha_\nu \in \mathbb{C}$ and $x_\nu \in \mathbb{R}^n$, $\nu = 1, \ldots, m$. Then

$$\begin{aligned}
\sum_{\nu,\mu=1}^{m} \alpha_\nu \overline{\alpha_\mu} g(x_\mu - x_\nu) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\lambda) \sum_{\nu,\mu=1}^{m} f(\lambda) \alpha_\nu \overline{\alpha_\mu} e^{i\lambda\cdot(x_\mu - x_\nu)}\, d\lambda \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\lambda) \sum_{\nu,\mu=1}^{m} \alpha_\nu e^{i\lambda\cdot x_\mu} \overline{\alpha_\mu e^{i\lambda\cdot x_\nu}}\, d\lambda \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\lambda) | \sum_{\nu,\mu=1}^{m} \alpha_\nu e^{i\lambda\cdot x_\mu} |^2\, d\lambda \\
&\geq 0\,.
\end{aligned}$$

□

REMARK 15.7 (Bochner's Theorem). The above is a part a well known theorem of Bochner: A function g on $\mathbb{R}^n$ is positive definite if and only if

$$g(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\lambda\cdot x}\, d\mu(\lambda)$$

for some finite (positive) measure μ on $\mathbb{R}^n$.

16. The Operator Valued Fourier Transform

Note, if $\dim \mathcal{H}_\pi = 1$, i.e., $\mathcal{H}_\pi = \mathbb{C}$, then the linear maps on $\mathcal{H}_\pi$ are the multiplication operators $z \mapsto T_a(z) = az$, $a \in \mathbb{C}$. The map T_a is regular if and only if $a \in \mathbb{C}^*$ and unitary if and only if $|a| = 1$. This says, that the unitary irreducible representations of an abelian group G are the continuous homomorphisms $\chi : G \to \mathbb{T}$. Finally, π and τ are equivalent if and only if $\pi(a) = \tau(a)$ for all $a \in G$. Denote by $\widehat{G}$ the set of all continuous homomorphism $\pi : G \to \mathbb{T}$. If $G = \mathbb{T}^n$, then $\widehat{G} = \mathbb{Z}^n$, if $G = \mathbb{R}^n$, then $\widehat{G} = \mathbb{R}^n$, and if $G = \mathbb{Z}^n$, then $\widehat{G} = \mathbb{T}^n$. In those classical cases, we have $\widehat{\widehat{G}} \simeq G$. Also, the Fourier transform of a function on G is a function $\widehat{f} : \widehat{G} \to \mathbb{C}$. The Plancherel Theorem states that there exists a measure $\widehat{\mu}$ on $\widehat{G}$ such that the Fourier transform extends to an unitary isomorphism $L^2(G) \to L^2(\widehat{G})$ such that the inverse is given by

$$\widehat{\mathcal{F}}(F)(x) = \int F(\chi)\chi(x)\, d\widehat{\mu}(\chi)\,, \quad F \in L^1(\widehat{G}) \cap L^2(\widehat{G})\,.$$

All of this goes through for abelian locally compact Hausdorff topological groups. Then $\widehat{G}$ is a group, where the multiplication $(\pi \cdot \chi)(a) = \pi(a)\chi(a)$. For $\epsilon > 0$ and $K \subseteq G$ compact neighborhood of e let

$$W(K, \epsilon) := \{\chi \in \widehat{G} \mid \sup_{a\in K} |\chi(a) - 1| < \epsilon\}\,.$$

If $\chi \in \widehat{G}$ let $W(\chi; K, \epsilon) := W(K, \epsilon)\chi$. Then the collection $\{W(\chi; K, \epsilon)\}$ forms a basis of a topology that makes $\widehat{G}$ into a locally compact Hausdorff topological group.

Let $(\pi, \mathcal{H})$ be an unitary representation of G. For $f \in L^1(G)$ and $u, w \in \mathcal{H}$ we have

$$\begin{aligned} |\int_G f(a)(\pi(x)u, w)\, d\mu_G(a)| &\leq \int_G |f(a)(\pi(x)u, w)|\, d\mu_G(a) \\ &\leq \int_G |f(a)|\, d\mu_G(a) \|u\| \|w\| \\ &\leq \|f\|_1 \|u\| \|w\| \,. \end{aligned}$$

Hence there exists a bounded operator $\pi(f) \in \mathcal{B}(\mathcal{H})$ with $\|\pi(f)\| \leq \|f\|_1$ such that for all $u, w \in \mathcal{H}$ we have

$$(\pi(f)u, w) = \int_G f(a)(\pi(a)u, w)\, d\mu_G(a) \,.$$

Thus $\pi(f)u$ is the weak integral $\int_G f(a)\pi(a)u\, d\mu_G(a)$. Assume that $\mathcal{H} \subseteq L^2(G)$ and that π is given by $\pi(a) = \lambda(a)|_{\mathcal{H}}$. Then

$$(\pi(f)u, w) = \int_G f(a) \left(\int_G u(a^{-1}b)\overline{w(b)}\, d\mu_G(b) \right) d\mu_G(a)$$

The function

$$(a, b) \mapsto f(a)u(a^{-1}b)\overline{w(b)}$$

is integrable on $G \times G$. Hence we can interchange the order of integration to get

$$\begin{aligned} (\pi(f)u, v) &= \int_G \left(\int_G f(a)u(a^{-1}b)\, d\mu_G(a) \right) \overline{w(b)}\, d\mu_G(b) \\ &= (f * u, w) \,. \end{aligned}$$

Hence $\pi(f)u = f * u$.

Proof the following

- $\pi(f * g) = \pi(f)\pi(g)$.
- $\pi(f^*) = \pi(f)^*$.
- If f is central, then $\pi(f)$ is an intertwining operator. In particular, if π is irreducible then $\pi(f) = \lambda(f)\mathrm{id}$.
- If G is compact, then $\pi(f)$ is compact.

Assume now that G is compact and that $f \in C(G)$. As G is unimodular, we can also write $f * u$ as

$$\begin{aligned} f * u(b) &= \int_G f(a)u(a^{-1}b)\, d\mu_G(a) \\ &= \int_G f(bc^{-1})u(c)\, d\mu_G(c) \,. \end{aligned}$$

THEOREM 16.1. *Suppose that G is compact. If π is an unitary representation of G, then π is an orthogonal direct sum of irreducible representations.*

PROOF. We can assume that π is cyclic by Theorem 14.8. Let u be a cyclic vector of norm 1. For $w \in \mathcal{H}$ let $T(w) \in C(G)$ be defined by

$$T(w)(a) := (w, \pi(g)u)\,.$$

Then

$$\begin{aligned} T(\pi(b)w)(a) &= (\pi(b)w, \pi(a)u) \\ &= (w, \pi(b^{-1}a)u) \\ &= [\lambda(b)T(w)]\,. \end{aligned}$$

As $T(w)$ is continuous and G compact, it follows that $T(w) \in L^2(G)$. Furthermore

$$\|T(w)\|^2 = \int_G |(w, \pi(a)u)|^2 \, d\mu_G(a) \leq \|w\|^2\,.$$

Hence T is continuous with norm ≤ 1. If $T(w) = 0$, then $(w, \pi(a)u) = 0$ for all $a \in G$, and hence $0 = (w, \sum_{\text{finite}} c_j \pi(a_j)u) = 0$. As u is cyclic $w \perp \mathcal{H}$ and hence $w = 0$. By Theorem 14.3 there exists an isometric intertwining operator $U : \mathcal{H} \to L^2(G)$. We can therefore assume that $\mathcal{H} \subseteq L^2(G)$.

□

add: Unitary reps of compact groups are unitary sums of finite dimensional reps. Corollary: G compact then irreducible reps are finite dimensional. Vector bundles and induced reps. Decomposition of L^2-sections of G-bundles.

17. Harmonic analysis on $\mathbb{R}^n$ and the Euclidean motion group

In this section we give a short description of some of the ideas and results of this article, when applied to $X = \mathbb{R}^n$ and G the group of orientation preserving Euclidean motions. As a set $G = \mathrm{SO}(n) \times \mathbb{R}^n$. It acts transitively on $\mathbb{R}^n$ by

$$(A, x) \cdot y = A(y) + x\,.$$

The product in G is determined by composition of automorphisms of $\mathbb{R}^n$ and it is given by $(A, x)(B, y) = (AB, A(y) + x)$. In this way G is isomorphic to the semidirect product $\mathbb{R}^n \rtimes \mathrm{SO}(n)$. Denote by $I_n \in \mathrm{SO}(n)$ the identity matrix. To adapt to the notation in later sections we define the following subgroups of G:

$$K = \{(A, 0) \mid A \in \mathrm{SO}(n)\} \simeq \mathrm{SO}(n),$$
$$N = \{(I_n, (x, 0)) \mid x \in \mathbb{R}^{n-1}\} \simeq \mathbb{R}^{n-1},$$

and

$$A = \{(I_n, (0, p)) \mid p \in \mathbb{R}\} \simeq \mathbb{R}.$$

Then $G = KAN = NAK$ where $K \times A \times N \ni (k, a, n) \mapsto kan \in G$ is a diffeomorphism. The stabilizer of $0 \in \mathbb{R}^n$ is the group K. Hence $\mathbb{R}^n \simeq G/K$.

17.1. Fourier analysis on X. The regular action of G on $L^2(X)$ is given by

$$L_g f(y) = f(g^{-1} \cdot y) = f(A^{-1}(y - x)), \quad g = (A, x) \in G.$$

The Fourier transform

$$\mathcal{F}_{\mathbb{R}^n}(f)(\lambda) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\lambda}\, dx$$

decomposes $L^2(X)$ into irreducible representations of the abelian group $\mathbb{R}^n$. However, in order to decompose in representations of the larger group G we need a reinterpretation, which we shall now discuss.

Recall that the irreducible unitary representations of G are constructed by the method of Mackey [**?**]. We consider the following representations of G, called the *principal series* of representations. Let $d\omega$ be the surface measure on $\mathrm{S^n}$ and put $L^2(\mathrm{S^n}) = L^2(\mathrm{S^n}, d\omega)$. For $r \in \mathbb{R}$ a unitary representation π_r of G on $L^2(\mathrm{S^n})$ is defined by

$$\pi_r(A, x)\varphi(\omega) := e^{-irx\cdot\omega}\varphi(A^{-1}(\omega)). \tag{17.1}$$

Then π_r is irreducible if $r \neq 0$, and $\pi_r \simeq \pi_{r'}$ if and only if $r = \pm r'$. Together with the characters of K, trivially extended to G, the representations π_r for all $r > 0$ exhaust the irreducible representations of G up to equivalence (see for example [**?**], Chapter IV, where $n = 2$). Notice that the constant function $p_r(\omega) := 1$ on S^{n-1} is a K-fixed vector for π_r.

The representation π_r can easily be realized as an induced representation. Let $M' \subset K$ denote the subgroup

$$M' = \left\{ \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix} \mid A \in \mathrm{O}(n-1) \right\} \simeq \mathrm{O}(n-1),$$

and put $M = M'_o \simeq \mathrm{SO}(n-1)$ and $P = MAN \subset G$. Define a character χ_r of $\mathbb{R}^n$ by $\chi_r(x) = e^{irx_n} = e^{irx\cdot e_n}$, and note that M stabilizes χ_r. Hence χ_r extends to a character on P. Next we note that $G/P = \mathrm{SO}(n)/\mathrm{SO}(n-1) \simeq \mathrm{S^n}$. Viewing functions on $\mathrm{S^n}$ as right M-invariant functions on K, the restriction $f \mapsto f|_K$ defines a unitary isomorphism from the Hilbert space of $\mathrm{ind}_P^G \chi_r$ onto $L^2(S^{n-1})$, intertwining the action of $\mathrm{ind}_P^G \chi_r$ and π_r. Thus $\pi_r \simeq \mathrm{ind}_P^G \chi_r$.

For $f \in L^2(X)$ and $r > 0$ define $\hat{f}_r \in L^2(S^{n-1})$ by

$$\widehat{f_r}(\omega) = \mathcal{F}_{\mathbb{R}^n} f(r\omega) \tag{17.2}$$

and call

$$L^2(\mathbb{R}^n) \ni f \mapsto \hat{f} \in L^2(\mathbb{R}^+, L^2(S^{n-1}))$$

the *Fourier transform* on X (with respect to G). Notice that if f is viewed as a right K-invariant function on G, then

$$[\pi_r(f)p_r](\omega) = \int_{\mathbb{R}^n} f(x)e^{-irx\cdot\omega}\,dx = (2\pi)^{n/2}\,\hat{f}_r(\omega)$$

(if f is integrable).

Interpreted in this fashion, the Fourier transform is an intertwining operator, $\widehat{(L_g f)}_r = \pi_r(g)\widehat{f}_r$ and $f \mapsto \widehat{f}_r$ sets up an unitary G-isomorphism

$$(L^2(X), L) \simeq \int_{\mathbb{R}^+}^{\oplus} (L^2(S^{n-1}), \pi_r)\, r^{n-1} dr, \tag{17.3}$$

which gives the decomposition of $L^2(X)$ into irreducible representations of G.

17.2. The Radon transform. The representations π_r are naturally associated to the homogeneous space $\widetilde{\Xi} := G/MN$. Define

$$G \ni (A, x) \mapsto (Ae_n, (Ae_n)\cdot x) \in S^{n-1} \times \mathbb{R},$$

where the dot denotes the standard inner product on $\mathbb{R}^n$. It is easy to see that this map is right MN-invariant and factors to a map $G/MN \to S^{n-1} \times \mathbb{R}$ with inverse

$$(\omega, p) \mapsto \{(A, x) \in G \mid Ae_n = \omega\,,\ \omega \cdot x = p\}.$$

Hence $G/MN \simeq S^{n-1} \times \mathbb{R}$. In these coordinates, the action of G is given by

$$(A, x)\cdot(\omega, p) = (A\omega, p + x\cdot A\omega)\,. \tag{17.4}$$

The corresponding action on functions on $\widetilde{\Xi}$ is then

$$L_{(A,x)}\varphi(\omega, p) = \varphi(A^{-1}\omega, p - x\cdot\omega)\,. \tag{17.5}$$

Hence $d\omega\, dp$ is a left invariant measure on $\widetilde{\Xi}$ with respect to which

$$L^2(\widetilde{\Xi}) \simeq L^2(\mathrm{S^n} \times \mathbb{R}) \simeq L^2(\mathrm{S^n})\bar{\otimes}L^2(\mathbb{R})\,. \tag{17.6}$$

Note that MN acts trivially on $P/MN \simeq A = \mathbb{R}$. Thus, the spectral decomposition of $L^2(P/MN)$ as a representation of P is $\int_{\mathbb{R}}^{\oplus} \chi_r\, dr$. Denote by ϵ the trivial representation of MN. Using the fact (see [**?**], p. 284 and

287) that induction and direct integral commute, induction in stages gives

$$
\begin{aligned}
L_{\widetilde{\Xi}} &\simeq \operatorname{ind}_{MN}^{G}\epsilon \\
&\simeq \operatorname{ind}_{P}^{G}\operatorname{ind}_{MN}^{P}\epsilon \\
&\simeq \operatorname{ind}_{P}^{G}\int_{\mathbb{R}}^{\oplus}\chi_r\,dr \\
&\simeq \int_{\mathbb{R}}^{\oplus}\operatorname{ind}_{P}^{G}\chi_r\,dr \\
&\simeq \int_{\mathbb{R}}^{\oplus}\pi_r\,dr \\
&\simeq 2\int_{\mathbb{R}^+}^{\oplus}\pi_r\,dr\,.
\end{aligned}
\tag{17.7}
$$

Denote by Ξ the set of hyperplanes $\xi \subset X$. Then G acts transitively on Ξ and the stabilizer of the hyperplane $\xi_0 = \{(x,0) \mid x \in \mathbb{R}^{n-1}\}$ is $M'N$. Hence $\Xi = G/M'N$. Each hyperplane is determined by it normal vector and the signed distance from the origin

$$
\xi = \xi(\omega,p) = \{x \in \mathbb{R}^n \mid x\cdot\omega = p\}\,, \quad \omega \in S^{n-1},\ p \in \mathbb{R}\,. \tag{17.8}
$$

As $\xi(\omega,p) = \xi(-\omega,-p)$ it follows that

$$
\Xi \simeq S^{n-1}\times_{\mathbb{Z}_2}\mathbb{R}\,. \tag{17.9}
$$

Note that $\mathbb{Z}_2 = M'/M = W$ acts on A by $s(0,\ldots,0,x) = (0,\ldots,0,-x)$ where s is the non-trivial element in W and that $\widetilde{\Xi}$ is a double covering of Ξ. Furthermore, functions on Ξ can be realized as even functions on $\widetilde{\Xi}$.

The *Radon transform* of a function $f \in C_c^\infty(\mathbb{R}^n)$ is the function $\mathcal{R}f \in C^\infty(\Xi)$ given by

$$
\mathcal{R}f(\xi) = \int_\xi f(x)\,dx\,. \tag{17.10}
$$

For this definition we need to fix the measure on the hyperplanes in a coherent way. Let $\xi \in \Xi$ and take $g \in G$ such that $\xi = g\cdot\xi_o$. Define

$$
\int_\xi f(x)\,dx := \int_{\xi_o} f(g\cdot(x_1,\ldots,x_{n-1},0))\,dx_1\ldots dx_{n-1}\,.
$$

It is easy to see that this integral is well defined, and that by Definition (17.10) we then obtain $\mathcal{R}f \in C^\infty(\Xi)$. Furthermore, for $h \in G$ it follows that $\mathcal{R}(L_h f) = L_h(\mathcal{R}f)$, i.e., $\mathcal{R}$ is a G-intertwining operator. For $\omega \in S^{n-1}$ let $\omega^\perp = \{x \in \mathbb{R}^n \mid x\cdot\omega = 0\}$. Then (17.10) can be written as

$$
\mathcal{R}f(\omega,p) = \int_{\omega^\perp} f(x+p\omega)\,dx\,. \tag{17.11}
$$

Denote by ∂_j the partial derivative $\partial/\partial x_j$, and $L_X = \sum_{j=1}^n \partial_j^2$ the *Laplace operator* on $\mathbb{R}^n$. Note that $\mathbb{D}(X)$, the algebra of G-invariant differential operators on X, is $\mathbb{C}[L_X] = \{p(L) \mid p \in \mathbb{C}[x]\}$. Denote by $\Box$ the differential operator on $\widetilde{\Xi}$ given by

$$\Box\varphi(\omega, p) = \left(\frac{\partial}{\partial p}\right)^2 \varphi(\omega, p)\,. \tag{17.12}$$

It follows from (17.4) that $\Box$ commutes with the G-action on Ξ. In fact the algebra of G-invariant differential operators on Ξ is $\mathbb{D}(\Xi) = \mathbb{C}[\Box]$.

Lemma 17.1. *Let $f \in C_c^\infty(X)$. Then*

$$\widehat{f}_r(\omega) = (2\pi)^{-\frac{n-1}{2}} \mathcal{F}_{\mathbb{R}}(\mathcal{R}f(\omega, \cdot))(r) \tag{17.13}$$

for all $r \in \mathbb{R}^+$ and $\omega \in S^{n-1}$. Furthermore, we have

$$\mathcal{R}(L_X f) = \Box(\mathcal{R}f)\,. \tag{17.14}$$

Proof. This is well known. As the proof of (17.13) is very simple, we give it here. We simply calculate:

$$\begin{aligned}
\mathcal{F}_{\mathbb{R}^n} f(p\omega) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ipx\cdot\omega}\, dx \\
&= (2\pi)^{-n/2} \int_{-\infty}^{\infty} \int_{\omega^\perp} f(y + r\omega) e^{-ipr}\, dy dr \\
&= (2\pi)^{-\frac{n-1}{2}} \mathcal{F}_{\mathbb{R}}(\mathcal{R}f)(\omega, p).
\end{aligned}$$

For (17.14), see [**?**] Lemma 2.1. $\Box$

Observe that it follows from the Plancherel formula (17.3) that

$$\|f\|^2 = \int_0^\infty \|\hat{f}_r\|^2_{L^2(S^{n-1})}\, r^{n-1}\, dr = \int_{S^{n-1}} \int_0^\infty |\widehat{f}_r(\omega)|^2 r^{n-1}\, dr\, d\omega \tag{17.15}$$

for $f \in C_c^\infty(X)$, hence in particular $\int_0^\infty |\widehat{f}_r(\omega)|^2 r^{n-1}\, dr < \infty$ for almost all ω.

Define an operator Λ from $C_c^\infty(X)$ to functions on Ξ by

$$\mathcal{F}_{\mathbb{R}}(\Lambda f(\omega, \cdot))(r) = r^{\frac{n-1}{2}} \widehat{f}_r(\omega)$$

for $\omega \in S^{n-1}$ and $r > 0$. It follows from the preceding observation that $\Lambda f(\omega, \cdot)$ is well defined for almost all $\omega \in S^{n-1}$. Notice that it follows from (17.12) and (17.13) that

$$\Lambda = (2\pi)^{-\frac{n-1}{2}} \Box^{\frac{n-1}{4}} \circ \mathcal{R},$$

if the power of $\Box$ is properly defined.

By [**?**], Theorem 1.4, see also [**?**], Theorem 4.1, we have

THEOREM 17.2. *The operator* Λ *extends to an unitary intertwining operator*

$$\Lambda : (L^2(X), L_X) \to (L^2(\Xi), L_\Xi) \simeq \int_{\mathbb{R}^+}^{\oplus} (L^2(S^{n-1}), \pi_r)\, dr\,.$$

PROOF. We will only show that the map is an isometry. In fact, it follows immediately from (17.15) and the Plancherel formula for $\mathbb{R}$ that

$$\|f\|^2 = \int_0^\infty \|\Lambda(f)(\cdot, p)\|^2_{L^2(S^{n-1})}\, dp. \tag{17.16}$$

□

17.3. The heat equation. The *heat equation* on $\mathbb{R}^n$ is the Cauchy problem

$$Lu(x,t) = \partial_t u(x,t)\,, \qquad u(\cdot, 0) = f \in L^2(\mathbb{R}^n)\,. \tag{17.17}$$

We write $u(x,t) = H_t f(x)$. One can represent the solution in two different ways. First, let $h_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ be the *heat kernel* on $\mathbb{R}^n$. Then, with $w^2 = w_1^2 + \ldots + w_n^2$, $w \in \mathbb{C}^n$,

$$u(x,t) = f * h_t(x) = (4\pi t)^{-n/2} \int_X f(x) e^{-(x-y)^2/4t}\, dy. \tag{17.18}$$

Secondly, by means of the Fourier transform,

$$\begin{aligned} u(x,t) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-t\lambda^2} \mathcal{F}_{\mathbb{R}^n}(f)(\lambda) e^{ix\cdot\lambda}\, d\lambda && (17.19)\\ &= (2\pi)^{-n/2} \int_{S^{n-1}\times\mathbb{R}^+} e^{-tr^2} \widehat{f}_r(\omega) e^{irx\cdot\omega}\, r^{n-1} d\omega\, dr\,. && (17.20) \end{aligned}$$

Note, that $H_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an intertwining operator, i.e., $H_t(L_g f) = L_g(H_t f)$. The fact that $\mathcal{F}_{\mathbb{R}^n}(H_t f)(\lambda) = e^{-t|\lambda|^2} \mathcal{F}_{\mathbb{R}^n}(f)(\lambda)$ shows that H_t is contractive. To make it an isometry we define the measure $d\mu_t(r) = e^{2tr^2} r^{n-1} dr$ on $\mathbb{R}^+$, and define

$$\|f\|_t^2 = \int_0^\infty \|\widehat{f}_r\|^2_{L^2(S^{n-1})}\, d\mu_t(r) \in [0, \infty]$$

for $f \in L^2(\mathbb{R}^n)$. Using that

$$\|\widehat{H_t f}_r\|^2_{L^2(S^{n-1})} = e^{-2tr^2} \|\widehat{f}_r\|^2_{L^2(S^{n-1})}$$

it follows that H_t is a unitary isomorphism of $L^2(\mathbb{R}^n)$ onto

$$\{f \in L^2(\mathbb{R}^n) \mid \|f\|_t < \infty\},$$

equipped with the norm $\|\cdot\|_t$. Hence

$$(\mathrm{Im}(H_t), L) \simeq \int_{\mathbb{R}^+} (L^2(S^{n-1}), \pi_r)\, d\mu_t(r)\,. \tag{17.21}$$

Replace x in the above integral (17.19) by $z = x+iy \in \mathbb{C}^n$ and note that the integral converges uniformly for z in a compact subset of $\mathbb{C}^n$. Hence, $X \ni x \mapsto u(x,t) \in \mathbb{C}$ extends to a holomorphic function on $\mathbb{C}^n$. The linear map $H_t : L^2(X) \to \mathcal{O}(\mathbb{C}^n)$ is the *Segal-Bargmann* transform. To describe the image of the Segal-Bargmann transform for $t > 0$ define

(17.22)

$$\mathcal{F}_t(\mathbb{C}^n) := \{F \in \mathcal{O}(\mathbb{C}^n) \mid \|F\|^2 := \int_{\mathbb{C}^n} |F(x+iy)|^2 \, h_{t/2}(y) \, dxdy < \infty\} .$$

It is easy to see that the left translation $L_g F(z) = F(g^{-1} \cdot z)$ defines a unitary representation of the motion group G on $\mathcal{F}_t(\mathbb{C}^n)$.

THEOREM 17.3. (Segal, Bargmann) *The following holds:*

(1) $\mathcal{F}_t(\mathbb{C}^n)$ *is a Hilbert space and* $H_t : L^2(X) \to \mathcal{F}_t(\mathbb{C}^n)$ *is an unitary* G*-isomorphism.*
(2) *The point-evaluations*

$$\mathcal{F}_t(\mathbb{C}^n) \ni F \mapsto F(z) \in \mathbb{C} , \qquad z \in \mathbb{C}^n$$

are continuous.
(3) *The reproducing kernel is given by*

$$K(z,w) = h_{2t}(z - \bar{w}) = (8\pi t)^{-n/2} e^{-(z-\bar{w})^2)/8t} .$$

(4) *If* $f \in C_c^\infty(X)$ *then*

$$f(x) = \int_X H_t f(x+iy) h_t(y) \, dydx .$$

PROOF. We will sketch the proof of (1) and (4) (the proof of (2) and (3) is completely similar to that of Theorem **??** below). We refer to [**?**] for more detailed discussion and further references. Let $f \in L^2(\mathbb{R}^n)$. Then, with $c = (2\pi t)^{-n/2} = (\int_X e^{-y^2/2t} \, dy)^{-1}$ and using that $\mathcal{F}_{\mathbb{R}^n}(H_t f)(\lambda) = e^{-t\lambda^2} \mathcal{F}_{\mathbb{R}^n} f(\lambda)$, we obtain

$$\begin{aligned}
\|H_t f\|^2 &= c \iint |H_t f(x+iy)|^2 dx e^{-y^2/2t} \, dy \\
&= c \iint |\mathcal{F}_{\mathbb{R}^n}(H_t f)(\lambda)|^2 e^{-2y\cdot\lambda} e^{-y^2/2t} \, d\lambda dy \\
&= \int |\mathcal{F}_{\mathbb{R}^n} f(\lambda)|^2 \left(c \int e^{-(y+2t\lambda)^2/2t} \, dy \right) d\lambda \\
&= \int |\mathcal{F}_{\mathbb{R}^n} f(\lambda)|^2 \, d\lambda \\
&= \|f\|_2^2 .
\end{aligned}$$

The proof of the inversion formula is similarly based on the fact that the holomorphic extension of $H_t f$ is given by the Fourier transform (17.19).

Let $f \in C_c^\infty(X)$. With $c = (2\pi)^{-n/2}$ and $\varphi = \mathcal{F}_{\mathbb{R}^n}(f)$ we have:

$$\begin{aligned}
\int H_t f(x+iy) h_t(y)\, dy &= c \int \left(\int \varphi(\lambda) e^{-t\lambda^2} e^{i(x+iy)\cdot\lambda}\, d\lambda \right) h_t(y)\, dy \\
&= c \int \varphi(\lambda) e^{ix\cdot\lambda} \left(\frac{1}{(4\pi t)^{\frac{n}{2}}} \int e^{-(y+2t\lambda)^2/4t}\, dy \right) d\lambda \\
&= (2\pi)^{-n/2} \int \varphi(\lambda) e^{ix\cdot\lambda}\, d\lambda \\
&= f(x)\,.
\end{aligned}$$

□

The formalism of Theorem 17.3 does not reflect the spectral decomposition of $L^2(X)$ in representations of G. We shall now introduce an alternative description of $\mathcal{F}_t(\mathbb{C}^n) = \mathrm{Im}(H_t)$ which takes this into account. Recall the unitary isomorphism in (17.3). The following definitions are motivated by Theorem 17.3, part (1), for $n = 1$.

Let $\mathcal{F}_{\mathbb{Z}_2,t}(S^{n-1} \times \mathbb{C})$ be the space of holomorphic $L^2(S^{n-1})$-valued functions on $\mathbb{C}$ such that $F(\omega, z) = F(-\omega, -z)$, $(\omega, z) \in S^{n-1} \times \mathbb{C}$ and

$$\|F\|_t^2 := (2\pi t)^{-1/2} \int_{\mathbb{C}} \|F(\cdot, x+iy)\|^2_{L^2(S^{n-1})}\, e^{-y^2/2t}\, dxdy < \infty\,.$$

The G-action (17.5) defines a unitary representation of G on $\mathcal{F}_{\mathbb{Z}_2,t}(S^{n-1} \times \mathbb{C})$.

Our aim is now to use the Radon transform in order to construct a commutative diagram

$$\begin{array}{ccc}
L^2(X) & \xrightarrow{\Lambda} & L^2(\Xi) \simeq L^2_{\mathbb{Z}_2}(\mathbb{R}, L^2(S^{n-1})) \\
{\scriptstyle H_t}\big\downarrow & & \big\downarrow{\scriptstyle H_t^{\mathbb{R}}} \\
\mathcal{F}_t(\mathbb{C}^n) & \xrightarrow[\widetilde{\Lambda}]{} & \mathcal{F}_{\mathbb{Z}_2,t}(S^{n-1} \times \mathbb{C})
\end{array}\,.$$

In order to define the operator $\widetilde{\Lambda}$, we apply the operator Λ on both sides of (17.17), and use (17.14) to derive

(17.23)

$$\Lambda(H_t(f))(\cdot, p) = H_t^{\mathbb{R}}(\Lambda f)(\cdot, p) = (4\pi t)^{-1/2} \int_{\mathbb{R}} \Lambda f(\cdot, u) e^{-(u-p)^2/4t}\, du.$$

Here the middle term is to be understood as the solution at time t of the $L^2(S^{n-1})$-valued heat equation on $\mathbb{R}$, with initial value $p \mapsto \Lambda(f)(\cdot, p) \in L^2(S^{n-1})$. Just as before it follows from (17.23) that $p \mapsto H_t^{\mathbb{R}}(\Lambda f)(\cdot, p)$ extends to a holomorphic $L^2(S^{n-1})$-valued function on $\mathbb{C}$. As in the proof of Theorem 17.3 one shows that this holomorphic extension belongs to $\mathcal{F}_{\mathbb{Z}_2,t}(S^{n-1} \times \mathbb{C})$. For $F \in \mathcal{F}_t(\mathbb{C}^n)$ take $f \in L^2(X)$ such that $H_t(f) = F$ and define $\widetilde{\Lambda}(F)$ to be the holomorphic extension of $H_t^{\mathbb{R}}(\Lambda f)$.

The following is now a consequence of Theorem 17.2 and the diagram above.

THEOREM 17.4. *The map* $\widetilde{\Lambda} : \mathcal{F}_t(\mathbb{C}^n) \to \mathcal{F}_{\mathbb{Z}_2,t}(S^{n-1} \times \mathbb{C})$ *is an unitary* G*-isomorphism.*

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