MATH 2025 Name: Solutions Fall 2009

Test 3, Thursday, Nov. 19, 2009

For partial credit, show all your work!

1[15P]) Consider the inner product $((x,y,z),(u,v,w)) = 2xu + 4yv + zw$ on $\mathbb{R}^3$:

a) Evaluate $((1,-2,3),(1,1,-3)) = -15$

$$2-8-9=-15$$

b) Evaluate $\|(3,3,1)\| = \sqrt{55}$

$$2\cdot 9+4\cdot 9+1=18+36+1=55$$

b) Are the vectors $(1,1,-2)$ and $(1,1,1)$ orthogonal with respect to this inner product? NO

$$2\cdot 1+4\cdot 1-2=4$$

2[15]) Use the inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$ on $C([0,1])$ to evaluate the following:

a) $(x^3+x^2-3x, x^2) = -23$

$$\int_0^1 x^5+x^4-3x^3\,dx = \frac{1}{6}+\frac{1}{5}-\frac{3}{4} = \frac{10+12-45}{60}$$

b) $(\cos(\pi x), \sin(\pi x)) =$ 0

c) $\|x^2-1\| = \sqrt{8/15}$

$$\int_0^1 (x^2-1)^2\,dx = \int_0^1 x^4-2x^2+1\,dx = \frac{1}{5}-\frac{2}{3}+1$$

$$= \frac{3-10+15}{15} = \frac{8}{15}$$

3[5P]) Consider the inner product $((x,y,z),(u,v,w)) = x\bar{u} + y\bar{v} + z\bar{w}$ on $\mathbb{C}^3$. Evaluate $((1+2i, 2-i, 3+i),(1+2i, 1+3i, 1-i)) =$ $6-3i$

$(1+2i)(1-2i) + (2-i)\cdot(1-3i) + (3+i)(1+i)$

$= \cancel{1} + 4 + 2 - 6i - i \cancel{= 3 + 3} \cancel{1} + 3i + i$

$= 6 - 3i$

4[5P]) Use the definition of linearly independent directly to show that the vectors $(1,1,1), (1,0,-1)$, and $(1,2,1)$ are linearly independent, i.e., show that $c_1(1,1,1) + c_2(1,0,-1) + c_3(1,2,1) = (0,0,0)$ implies that $c_1 = c_2 = c_3 = 0$.

$c_1 + c_2 + c_3 = 0$

$c_1 + \quad 2c_3 = 0$

$c_1 - c_2 + c_3 = 0$

$2c_1 + 2c_3 = 0$

using (2) $\Rightarrow c_1 = 0 \Rightarrow c_3 = 0$

$\Rightarrow c_2 = 0$

5[16P]) Which of the following sets of vectors is linearly independent and which are not? Explane why.

a) $V = \mathbb{R}^2$, $(1,-3)$, $(-6,18)$. $(-6,18) = -6(1,-3)$ No

b) $V = \mathbb{R}^3$, $(1,1,0)$, $(1,-1,1)$, $(-1,1,2)$. pairwise orthogonal

$\Rightarrow$ linearly independent

No c) $V = \mathbb{R}^3$, $(1,0,1)$, $(2,0,-1)$, $(0,1,1)$, $(1,3,1)$. Too many

6

e) $V = C([0,1])$, $\cos(\pi x)$, $\sin(\pi x)$. orthogonal.

Linearly independent.

6[5P]) Write the vector $(3,3,6)$ as a linear combination of $(1,-2,0)$, $(2,1,1)$, and $(2,1,-5)$.

$$(3,3,6) = \frac{-3}{5}(1,-2,0) + \frac{15}{6}(2,1,1) + \frac{-21}{30}(2,1,-5)$$
$$= -\frac{3}{5}(1,-2,0) + \frac{5}{2}(2,1,1) - \frac{7}{10}(2,1,-5)$$

7[15P] Which of the following sets of vectors is a basis for the given vector space? Correct reasoning counts for half of the points!

a) $V = \mathbb{R}^2$, $(1,4)$, $(-1,-8)$, $(3,1)$.

NO, too many, linearly dependent

b) $V = \mathbb{R}^3$, $(1,1,-2)$, $(1,1,1)$, $(1,-1,0)$.

Yes: Orthogonal $\Rightarrow$ linearly indepent.

c) $V = \mathbb{R}^3$, $(1,3,1)$, $(1,-3,2)$. Need exactly three

No,

8) Let $W \subset \mathbb{R}^3$ be the plane spanned by the vectors $(1,1,1)$ and $(2,0,1)$.
a[5P]) Write the formula for the orthogonal projection $P : \mathbb{R}^3 \to W$.

$P((x,y,z)) = \frac{1}{6}(5x-y+2z, -x+5y+2z, 2x+2y+2z)$

$W_1 = (1,1,1)$

$W_2 = (2,0,1) - \frac{3}{3}(1,1,1) = (1,-1,0)$

$$P(x,y,z) = \frac{x+y+z}{3}(1,1,1) + \frac{x-y}{2}(1,-1,0)$$
$$= \frac{2x+2y+2z}{6}(1,1,1) + \frac{3x-3y}{6}(1,-1,0)$$
$$= \frac{1}{6}(5x-y+2z, -x+5y+2z, 2x+2y+2z)$$

b[4P]) Find the point in W closest to the vector $(3, -1, 2)$. $\frac{1}{3}(10,-2,4)$

$$P(3,-1,2) = \frac{1}{6}(15+1+4, -3-5+4, 6-2+4) = \frac{1}{6}(20,-4,8)$$
$$= \frac{1}{3}(10,-2,4)$$

9[10P]) Apply Gram-Schmidt to the set of vectors

a) $(1, 1, 0, 0)$, $(2, 0, 1, 0)$, and $(1, 0, -1, 1)$.

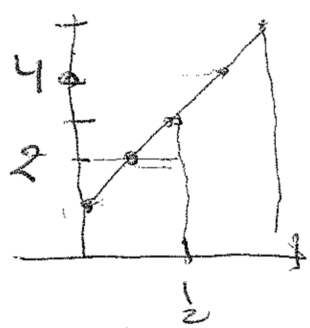
$$w_1 = (1,1,0,0)$$
$$w_2 = (2,0,1,0) - \frac{2}{2}(1,1,0,0) = (1,-1,1,0)$$
$$w_3 = (1,0,-1,1) - \frac{1}{2}(1,1,0,0) - 0$$
$$= \frac{1}{2}(1,-1,-2,2).$$

b) 1 and x^2 (using the inner product $(f, g) = \int_0^1 f(x)g(x)\,dx$).

$$w_1 = 1$$
$$w_2 = x^2 - \frac{(1,x^2)}{1} 1 = \underline{\underline{x^2 - \frac{1}{3}}}$$

10[5P]) Let $W^{(1)}$ be the wavelet space spanned by the functions $\varphi_0^{(1)}$ and $\varphi_1^{(1)}$. Find the function in $W^{(1)}$ closest to the polynomial $1 + 4x$.

$$c_1 = 2\int_0^{1/2} 1+4x\,dx = 2(x+2x^2)\Big]_0^{1/2}$$
$$= 1+1 = 2$$
$$2(x+2x^2)\Big]_{\frac{1}{2}}^{1} = 6-2 = 4$$



$$p(x) = 2\varphi_0^{(1)}(x) + 4\varphi_1^{(1)}(x)$$