1[15P] Consider the inner product \(( (x, y, z), (u, v, w)) = 2xu + 4yu + zw \) on \(\mathbb{R}^3\):

a) Evaluate \( (1, -2, 3), (1, 1, -3) = -15 \)

\[
2 - 8 - 9 = -15
\]

b) Evaluate \( ||(3, 3, 1)|| = \sqrt{5^2} \)

\[
2 - 9 + 4 - 9 + 1 = 18 + 36 + 1 = 55
\]

b) Are the vectors \((1, 1, -2)\) and \((1, 1, 1)\) orthogonal with respect to this inner product? \(\text{No}\)

\[
2 \cdot 1 + 4 \cdot 1 - 2 = 4
\]

2[15] Use the inner product \( (f, g) = \int_0^1 f(x)g(x) \, dx \) on \( C([0, 1]) \) to evaluate the following:

a) \( (x^3 + x^2 - 3x, x^2) = -23 \)

\[
\int_0^1 (x^5 + x^4 - 3x^3) \, dx = \frac{1}{6} + \frac{1}{5} - \frac{3}{4} = \frac{10 + 12 - 45}{60}
\]

b) \( (\cos(\pi x), \sin(\pi x)) = 0 \)

c) \( ||x^2 - 1|| = \sqrt{8/15} \)

\[
\int_0^1 (x^2 - 1)^2 \, dx = \int_0^1 x^4 - 2x^2 + 1 \, dx = \frac{1}{5} - \frac{2}{3} + 1
\]

\[
= \frac{3 - 10 + 15}{15} = \frac{8}{15}
\]
Consider the inner product \( ((x, y, z), (u, v, w)) = x \bar{u} + y \bar{v} + z \bar{w} \) on \( \mathbb{C}^3 \). Evaluate
\[
((1 + 2i, 2 - i, 3 + i), (1 + 2i, 1 + 3i, 1 - i)) = \quad 6 - 3i
\]
\[
(1+2i)(1-2i) + (2-i)(1-3i) + (3+i)(1+i)
\]
\[
= 1 + 4 + 2 - 6i + i - 3 + 3 - 1 + 3i + i
\]
\[
= 6 - 3i
\]

Use the definition of linearly independent directly to show that the vectors \((1,1,1), (1,0,-1), \) and \((1,2,1)\) are linearly independent, i.e., show that \(c_1(1,1,1) + c_2(1,0,-1) + c_3(1,2,1) = (0,0,0)\) implies that \(c_1 = c_2 = c_3 = 0\).

\[
\begin{align*}
    c_1 + c_2 + c_3 &= 0 \\
    c_1 - c_2 + c_3 &= 0 \\
    2c_1 + 2c_3 &= 0
\end{align*}
\]

From (2) \(\Rightarrow c_1 = 0 \Rightarrow c_3 = 0 \Rightarrow c_2 = 0\).

Which of the following sets of vectors is linearly independent and which are not? Explain why.

a) \(V = \mathbb{R}^2, (1, -3), (-6, 18)\). \((-6, 18) = -6 \cdot (1, -3)\) \(\text{No}\)

b) \(V = \mathbb{R}^3, (1, 1, 0), (1, -1, 1), (-1, 1, 2)\). \(\text{pairwise orthogonal}\) \(\Rightarrow\) \(\text{linearly independent}\)

\(\text{No}\)

c) \(V = \mathbb{R}^3, (1, 0, 1), (2, 0, -1), (0, 1, 1), (1, 3, 1)\). \(\text{Too many}\)

e) \(V = C([-1, 1]), \cos(\pi x), \sin(\pi x)\). \(\text{orthogonal}\) \(\Rightarrow\) \(\text{linearly independent}\).
6[5P]) Write the vector \((3, 3, 6)\) as a linear combination of \((1, -2, 0)\), \((2, 1, 1)\), and \((2, 1, -5)\).

\[
(3, 3, 6) = -\frac{3}{5}(1, -2, 0) + \frac{15}{6}(2, 1, 1) - \frac{21}{30}(2, 1, -5)
\]

\[
= -\frac{3}{5}(1, -2, 0) + \frac{5}{2}(2, 1, 1) - \frac{7}{10}(2, 1, -5)
\]

7[15P] Which of the following sets of vectors is a basis for the given vector space? Correct reasoning counts for half of the points!

a) \(V = \mathbb{R}^2\), \((1, 4), (1, -8), (3, 1)\).

\(\begin{array}{c}
\text{No, too many, linearly dependent}
\end{array}\)

b) \(V = \mathbb{R}^3\), \((1, 1, -2)\), \((1, 1, 1)\), \((1, -1, 0)\).

\(\begin{array}{c}
\text{Yes: Orthogonal } \implies \text{linearly independent}
\end{array}\)

c) \(V = \mathbb{R}^3\), \((1, 3, 1)\), \((1, -3, 2)\).

\(\begin{array}{c}
\text{Need exactly three}
\end{array}\)

\(\begin{array}{c}
\text{No}
\end{array}\)

8) Let \(W \subset \mathbb{R}^3\) be the plane spanned by the vectors \((1, 1, 1)\) and \((2, 0, 1)\).

a[5P]) Write the formula for the orthogonal projection \(P : \mathbb{R}^3 \to W\).

\[
P((x, y, z)) = \frac{1}{6} \left(5x - y + 2z, -x + 5y + 2z, 2x + 2y + 2z\right)
\]

\[
W_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

\[
W_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}
\]

\[
P(x, y, z) = \frac{1}{6} \left(5x - y + 2z, -x + 5y + 2z, 2x + 2y + 2z\right)
\]

\[
P(x, y, z) = \frac{1}{6} \left(5\left(\frac{x + y + z}{3}\right) - \left(\frac{x - y}{2}\right), -\left(\frac{x + y + z}{3}\right) + \left(\frac{x - y}{2}\right), 2\left(\frac{x + y + z}{3}\right)\right)
\]

\[
= \frac{1}{6} \left(5\left(x + y + z\right), -\left(x + y + z\right), 2\left(x + y + z\right)\right)
\]
b(4P]) Find the point in W closest to the vector \((3, -1, 2)\).

\[ P(3, -1, 2) = \frac{1}{6} (15 + 1 + 4) \cdot 3 - 3 - 5 + 4 \cdot 6 - 2 + 4 = \frac{1}{6}(20 - 4, 8) = \frac{1}{3} (10 - 2, 4) \]

9(10P]) Apply Gram-Schmidt to the set of vectors

a) \((1, 1, 0, 0), (2, 0, 1, 0), \) and \((1, 0, -1, 1)\).

\[ \mathbf{w}_1 = (1, 1, 0, 0) \]
\[ \mathbf{w}_2 = (2, 0, 1, 0) - \frac{2}{2} (1, 1, 0, 0) = (0, -1, 1, 0) \]
\[ \mathbf{w}_3 = (1, 0, -1, 1) - \frac{1}{2} (1, 1, 0, 0) - \mathbf{0} \]
\[ = \frac{1}{2} (1, -1, -2, 2) \]

b) \(1\) and \(x^2\) (using the inner product \((f, g) = \int_0^1 f(x) g(x) \, dx\)).

\[ \mathbf{w}_1 = 1 \]
\[ \mathbf{w}_2 = \sqrt{\int_0^1 \frac{x^2}{1} \, dx} = \frac{1}{3} \]

10(5P]) Let \(W^{(1)}\) be the wavelet space spanned by the functions \(\varphi_0^{(1)}\) and \(\varphi_1^{(1)}\). Find the function in \(W^{(1)}\) closest to the polynomial \(1 + 4x\).

\[ c_1 = 2 \int_0^{1/2} (1 + 4x) \, dx = 2 \left( x + 2x^2 \right) \bigg|_0^{1/2} = 1 + 1 = 2 \]
\[ 2 (x + 2x^2) \bigg|_0^{1/2} = 6 - 2 = 4 \]

\[ p(x) = 2 \varphi_0^{(1)}(x) + 4 \varphi_1^{(1)}(x) \]