# Homework

9.24) If $f$ is continuous, $f: D \Rightarrow \mathbb{R}$ and if $U \subseteq \mathbb{R}$ is open, then $f^{-1}(U) \subseteq D$ is open in $D$ but does not have to be open in $\mathbb{E}^n$ (if $D \subseteq \mathbb{R}^m$). So, let $f \in \mathcal{C}([0,1), \mathbb{R})$, $f(x) = x$. Then $f^{-1}(\mathbb{R}) = [0,1)$ which is not open in $\mathbb{R}$.

9.33 Let $D = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$.

a) $D$ is compact: $D$ is bounded and also closed. The only cluster point is $\{0\}$ and $0 \in D$.

b) Let $f \in \mathcal{C}(D, \mathbb{E}^m)$. Then $x \mapsto \|f(x)\|$ is continuous. As $D$ is compact it follows from Theorem 9.3.2 that there exists $a, b \in D$ s.t. $\|f(a)\| = \min_{x \in D} \|f(x)\| \le \|f(b)\| = \max_{x \in D} \|f(x)\|$.

9.35) $\varphi(t) = (\cos(2\pi t), \sin(2\pi t))$, $\varphi|_{[0,1)} = f$.

(Note that the image of $\varphi$ is the circle $\{(x,y) \mid x^2 + y^2 = 1\}$ which is compact, but $[0,1)$ is not compact.)

a) If $(x,y) \in S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$. Then there exists an unique $\theta \in [0, 2\pi)$ such that $x = \cos\theta$, $y = \sin\theta$. Let $t = \frac{\theta}{2\pi}$, then $f(t) = (x,y)$.

b) $f$ is continuous except at $(1,0) = f(0)$. Let $0 < t_n \nearrow 1$ be an increasing sequence converging to 1. Then $(\cos(2\pi t_n), \sin(2\pi t_n)) \to (1,0)$. But $f^{-1}(1,0) = 0 \neq \lim_{n \to \infty} f^{-1}(\cos(2\pi t_n), \sin(2\pi t_n)) = 1$. On the other hand, if $1 > s_n \searrow 0$, then $(\cos(2\pi s_n), \sin(2\pi s_n)) \to (1,0)$ and $f^{-1}(\cos 2\pi s_n, \sin 2\pi s_n) \to 0$ so $\lim_{(x,y) \to (1,0)} f^{-1}(x,y)$ does not exists. As $[0,1)$ is not compact, this does not contradict Thm. 9.3.3.

c) No, how to define $\varphi^{-1}(1,0)$ ? ✱

9.37 a) $S \subseteq \mathbb{E}^m$, $K \subset \mathbb{E}^m$, $K$ compact. $f: S \to K$ 1-1 and onto. Assume that $f^{-1}$ is continuous. Then $S$ is compact because

$S = f^{-1}(K)$, $K$ compact, $f^{-1}$ continuous and Thm. 9.3.1.

b) $S = (0,1)$, $K = [0,1]$, $f(x) = x$. A "better" example is $S = [0,\infty)$, $k = [0,1]$, $f(x) = \frac{1}{1+x^2}$. Then $f(S) = (0,1]$ and $f(y) = \sqrt{(1-y)/y}$ which is continuous on $(0,1]$.

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9.41 Suppose that $D \subseteq \mathbb{E}^m$ is compact and $f \in C(D, \mathbb{E}^m)$. Then $f$ is uniformly continuous.

Proof: Let $\varepsilon > 0$. For $x \in D$ there exist $\delta(x) > 0$ s.t. if $y \in B_{\delta(x)}(x) \cap D$ then

$$\|f(y) - f(x)\| < \varepsilon/2.$$

We have

$$D \subseteq \bigcup_{x \in D} B_{\delta(x)/2}(x).$$

As $D$ is compact, there exist a finite subcover. Thus, there are $x_1, \ldots, x_k \in D$ such that

$$D \subseteq \bigcup_{j=1}^{k} B_{\delta(x_j)/2}(x_j).$$

Let $\delta = \min_{j=1,\ldots,k} \delta(x_j)/2$. Let $x, y \in D$ be such that

$$\|x - y\| < \delta.$$

These exist $j \in \{1, \dots, k\}$ such that $x \in B_{\delta(x_j)/2}(x_j)$.
In particular
$$\|f(x)-f(x_j)\| < \varepsilon/2 .$$
We have
$$\|y-x_j\| = \|y-x+x_j+x\|$$
$$\leq \|y-x\| + \|x-x_j\|$$
$$< \delta + \delta(x_j)/2 \leq \delta$$
It follows that
$$\|f(y)-f(x_j)\| < \varepsilon/2 .$$
Hence
$$\|f(y)-f(x)\| \leq \|f(y)-f(x_j)\| + \|f(x_j)-f(x)\|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon . \qquad \blacksquare$$

<u>9.49)</u> $D \subseteq \mathbb{E}^n$ compact, $f \in \mathcal{C}(D, \mathbb{E}^m)$ 1-1. Then, if $f(D)$ is connected so is $D$.

<u>Proof</u>. ($f(D) \subseteq \mathbb{E}^m$ is compact). By Thm. 9.3.3 we know that $f^{-1}: f(D) \Rightarrow D \subseteq \mathbb{E}^n$ is continuous.
As $D = f^{-1}(f(D))$ it follows by Thm. 9.4.1 that $D$ is connected.

<u>10.5)</u> $[A] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $[B] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

- $\|A\| = 1 = \|B\| = \|A+B\|$

We first made that
$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$
Hence
$$\left\|A\begin{pmatrix} x \\ y \end{pmatrix}\right\| = |x| \leq \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| = \sqrt{x^2+y^2} .$$

This implies that $\|A\| \leq 1$. On the other hand $\|A\binom{1}{0}\| = \|\binom{1}{0}\| = 1$. Hence $\|A\| \geq 1$. It follows that $\|A\| = 1$. The proof for $B$ is the same interchanging the role of $x$ and $y$.

For $A+B$ we note that

$$[A+B] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $(A+B)\vec{V} = \vec{V}$ for all $V \in \mathbb{E}^2$. Hence

$$\|A+B\| = \sup_{\|\vec{V}\|=1} \|(A+B)V\| = \sup_{\|\vec{V}\|=1} \|V\| = 1.$$

10.13 Define $T_{ij} e_i = f_j$, $T_{ij} e_\nu = 0$ if $\nu \neq i$ and then by linear extension

$$T_{ij}(\Sigma x_\nu e_\nu) = \Sigma x_\nu T_{ij}(e_\nu) = x_i f_j.$$

Now, let $T \in L(\mathbb{E}^n, \mathbb{E}^m)$. Define for $1 \leq i \leq n$, $1 \leq j \leq m$

$$t_{ij} = (Te_i, f_j)$$

Then in particular

$$Te_i = \sum_{j=1}^{m} t_{ij} f_j.$$

We claim that

$$T = \sum_{i,j} t_{ij} T_{ij}$$

proof: Let $\nu = 1, \ldots, n$. Then

$$Te_\nu = \Sigma t_{\nu j} f_j$$

and

$$\left(\sum_{i,j} t_{ij} T_{ij}\right)(e_\nu) = \sum_{i,j} t_{ij} T_{ij}(e_\nu)$$

$$= \sum_j t_{\nu j} T_{\nu j}(e_\nu)$$

$$= \sum_j t_{\nu j} f_j$$

Hence $T(e_\gamma) = (\sum t_{ij} T_{ij})(e_\nu)$ for all of the basis vectors. As both maps are linear it follows that the agree every where. Hence $\{T_{ij}\}$ is a generating set.

- Linear indipendent. Let $c_{ij} \in \mathbb{R}$ be so that

$$\sum c_{ij} T_{ij} = 0$$

Let $\nu \in \{1, .., n\}$. Then

$$0 = (\sum c_{ij} T_{ij})(e_\gamma) = \sum_{j=1}^{m} c_{\gamma j} f_j .$$

As the vectors $f_1, \dots, f_m$ are linearly independent it follows that $c_{\gamma 1} = \dots = c_{\gamma m} = 0$. As $\gamma$ was taking arbitrary in $\{1, .., n\}$ it follows that this holds for all $\gamma = 1, .., n$. Hence $c_{\gamma\mu} = 0$. It follows that

$$\dim L(\mathbb{E}^n, \mathbb{E}^m) = nm .$$

10.11. ] a) $\cancel{X} \in L(\mathbb{E}^n)$, $\|X\| < 1 \Rightarrow T_K = \sum_{k=0}^{K} X^k$ is Cauchy.

<u>proof</u> We have (with $Iv = v$ for all $v$)

$$\|X^0 + X\| = \|I + X\| \leq \|I\| + \|X\|$$
$$= 1 + \|X\|$$

Induction then shows that

$$\|T_K\| \leq \sum_{k=0}^{K} \|X\|^k .$$

Similary for $K < N$:

$$\|T_K - T_N\| = \|\sum_{k=K+1}^{N} X^k\| \leq \sum_{k=K+1}^{N} \|X\|^k .$$

Let $\varepsilon > 0$.
As the geometric series $\sum_{k=0}^{\infty} \|X\|^k$ converges, there exist $N > 0$ so that for all $N \leq K < L$ we have

$$\sum_{k=K+1}^{L} \|X\|^k < \varepsilon$$

But then $\|T_K - T_L\| < \varepsilon$ for $K, L \geq N$.

b) Let $K \in \mathbb{N}$. Then

$$(I - X)T_K = \sum_{k=0}^{K} (X^k - X^{k+1})$$

$$= I - X^{K+1}$$

Hence

$$\|(I - X)T_K - I\| = \|X^{K+1}\| \leq \|X\|^{K+1} \longrightarrow 0.$$

c) Let $T = \lim_{K \to \infty} T_K$. Then (b) shows that

$$(I - X)T = I$$

or $(I - X)(T(v)) = v$ for all $v \in \mathbb{E}^n$.
Thus $(I - X)$ is surjective and hence also injective. Hence invertable with

$$(I - X)^{-1} = T.$$