Homework

9.24} If \( f \) is continuous, \( f : D \to \mathbb{R} \) and if \( U \subseteq \mathbb{R} \) is open, then \( f^{-1} (U) \subseteq D \) is open in \( D \) but does not have to be open in \( \mathbb{R}^n \) (if \( D = \mathbb{R}^m \)). So, let \( f \in C([0,1], \mathbb{R}) \), \( f(x) = x \). Then \( f^{-1} (\mathbb{R}) = [0,1] \) which is not open in \( \mathbb{R} \).

9.33} Let \( D = \{ x \mid \ln x \in \mathbb{N} \} \).

a) \( D \) is compact: \( D \) is bounded and also closed. The only cluster point is \( 0 \), so and \( 0 \in D \).

b) Let \( f \in C(D, \mathbb{R}^m) \). Then \( x \mapsto \| f(x) \| \) is continuous. As \( D \) is compact, it follows from Theorem 9.3.2 that there exist \( a, b \in D \) such that \( \| f(a) \| = \min_{x \in D} \| f(x) \| \leq \| f(b) \| = \max_{x \in D} \| f(x) \| .

9.35} \( q(t) = (\cos(\pi t), \sin(\pi t)) \), \( \theta_1 \) is a limit.

(Note that the image of \( q \) in the circle \( S(\sqrt{13}) \times \mathbb{R}^2 = \mathbb{C} \) which is compact, but \( [0,1] \) is not compact.)

a) \( (x,y) = (\sqrt{13}, \sqrt{13}) \). Then there exist an unique \( \Theta \in [0,2\pi) \) such that \( x = \cos \Theta, y = \sin \Theta \). Let \( t = \frac{\Theta}{\pi} \), then \( f(x,y) = (x,y) \).

b) \( f \) is continuous except at \( (1,0) = f(0) \). Let \( 0 < t_n \to 1 \) be an increasing sequence converging to 1. (Note: Then \( (\cos(\pi t_n), \sin(\pi t_n)) \to (1,0) \). But \( f^{-1} (1,0) = 0 = \lim_{n \to \infty} f^{-1} (\cos(\pi t_n), \sin(\pi t_n)) = 1 \), on the other hand, \( y \to 1 > \sin(\pi t_n) \), then \( (\cos(\pi t_n), \sin(\pi t_n)) \to (1,0) \) and \( f^{-1} (\cos(\pi t_n), \sin(\pi t_n)) \to 0 \) so \( \lim (xy) = (1,0) \) does not exists. As \( [0,1] \) is not compact, this does not contradict Thm. 9.3.3.

c) No, how to define \( f^{-1} (1,0) ? \)
9.37. a) $S \subseteq \mathbb{E}^m$, $K \subseteq \mathbb{E}^m$, $K$ compact, $f : S \rightarrow K$ 1-1 and onto. Assume that $f^{-1}$ is continuous. Then $S$ is compact because

$$S = f^{-1}(K), K \text{ compact, } f^{-1} \text{ continuous, and}$$

Thm. 9.3.1.

b) $S = (0,1), K = [0,1]$; $f(x) = x$. A "better" example is $S = [0,\infty), K = [0,1], f(x) = \frac{1}{1 + x^2}$. Then $f(S) = [0,1]$ and $f(y) = \sqrt{\frac{1}{1-y^2}}$ which is continuous on $(0,1)$.

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9.4. (1) Suppose that $D \subseteq \mathbb{E}^n$ is compact and $f \in C(D, \mathbb{E}^m)$. Then $f$ is uniformly continuous.

Proof: Let $\varepsilon > 0$. For $x \in D$ there exist $\delta(x) > 0$ s.t. if $y \in B_\delta(x)(x) \cap D$ then

$$\|f(y) - f(x)\| < \varepsilon/2.$$ 

We have

$$D \subseteq \bigcup_{x \in D} B_\delta(x)(x).$$

As $D$ is compact, there exist a finite subcover. Thus, there are $x_1, \ldots, x_k \in D$ such that

$$D \subseteq \bigcup_{j=1}^k B_\delta(x_j)(x_j).$$

Let $\delta = \min_{j=1,\ldots,k} \delta(x_j)/2$. Let $x, y \in D$ be such that

$$\|x - y\| < \delta.$$
There exist \( j \in \{ k, \ldots, k\} \) such that \( x \in B_{\delta_0}(x_j) \).

In particular
\[
\|f(x) - f(x_j)\| \leq \varepsilon/2.
\]

We have
\[
\|y - x_j\| = \|y - x + x - x_j\| \\
\leq \|y - x\| + \|x - x_j\| \\
< \delta + \delta(x_j)/2 \leq \delta
\]

It follows that
\[
\|f(y) - f(x_j)\| < \varepsilon/2.
\]

Hence
\[
\|f(y) - f(x)\| \leq \|f(y) - f(x_j)\| + \|f(x_j) - f(x)\| \\
< \varepsilon/2 + \varepsilon/2 \\
= \varepsilon.
\]

9.49] \( D \subset \mathbb{R}^n \) compact, \( f \in \mathcal{C}(D, \mathbb{R}^m) \) 1-1. Then, if \( f(D) \) is connected so is \( D \).

**Proof.** \( f(D) \subseteq \mathbb{R}^m \) is compact, by Thm. 9.3.3 we know that \( f^{-1}(f(D)) \Rightarrow D \subseteq \mathbb{R}^n \) is continuous.

As \( D = f^{-1}(f(D)) \) it follows by Thm. 9.4.1 that \( D \) is connected.

10.5] \( [AB] = (0 \ 0), \Sigma B3 = (0 \ 0) \).

* \( \|A\| = 1 = \|B\| = \|A + B\|

We first note that
\[
A(x, y) = (x)
\]

Hence
\[
\|A(x, y)\| = |x| \leq \|A(x, y)\| = \sqrt{x^2 + y^2}.
\]
This implies that \( \|A\| \leq 1 \). On the other hand
\[
\|A(\cdot)\| = \|A(\cdot)\| = 1,
\]
Hence \( \|A\| \geq 1 \). It follows that
\( \|A\| = 1 \). The proof for \( B \) is the same by interchanging
the role of \( x \) and \( y \).

For \( A + B \) we note that
\[
[A + B]^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Hence \( (A + B)^2 = \sum \) for all \( \nu \in \mathbb{F}^2 \). Hence
\[
\|A + B\| = \sup_{\|\nu\| = 1} \|(A + B)\nu\| = \sup_{\|\nu\| = 1} \|\nu\| = 1.
\]

10.13 Define \( T_{ij} e_i = f_j \), \( T_{ij} e_y = 0 \) if \( y \neq i \) and
then by linear extension
\[
T_{ij} (\sum_i y_i e_i) = \sum_i y_i T_{ij} (e_i) = x_i f_j.
\]
Now, let \( T \in L(\mathbb{F}^n, \mathbb{F}^m) \). Define for \( 1 \leq i \leq n \), \( 1 \leq j \leq m \)
\[
t_{ij} = (Te_i, f_j).
\]
Then in particular
\[
Te_i = \sum_{j=1}^m t_{ij} f_j.
\]

We claim that
\[
T = \sum_{i,j} t_{ij} T_{ij}.
\]

**Proof:** Let \( \nu = y_1 \ldots \ldots \ldots y_n \). Then
\[
Te_\nu = \sum_i t_{\nu i} f_i.
\]
And
\[
(\sum_{i} t_{\nu i} T_{ij})(e_\nu) = \sum_{i,j} t_{\nu i} T_{ij}(e_\nu) = \sum_{i,j} t_{\nu i} T_{ij}(e_\nu)
\]
\[
= \sum_{j} t_{\nu j} f_j.
\]
Hence \( T(e_i) = (\sum b_{ij} T_{ij})(e_i) \) for all of the basis vectors. As both maps are linear, it follows that they agree everywhere, hence \( \mathbf{T}_{ij} \) is a generating set.

Linearly independent. Let \( c_{ij} \in \mathbb{R} \) be so that

\[
\sum c_{ij} T_{ij} = 0
\]

Let \( v = e_1, \ldots, e_n \). Then

\[
0 = (\sum c_{ij} T_{ij})(e_v) = \sum_{y=1}^{m} c_{y} f_{ij}.
\]

As the vectors \( f_1, \ldots, f_m \) are linearly independent, it follows that \( c_1 = \cdots = c_m = 0 \). As \( v \) was taking arbitrary in \( e_1, \ldots, e_n \), it follows that this holds for all \( v = 1, \ldots, n \). Hence \( c_{ij} = 0 \).

It follows that

\[
\dim L(\mathbb{E}^n, \mathbb{E}^m) = nm.
\]

10.11. Let \( x \in L(\mathbb{E}^n) \), \( \|x\| < 1 \implies T_k = \sum_{k=0}^{K} x^k \) is Cauchy.

Proof. We have (with \( \forall v \not\in e_0 \))

\[
\|x^0 + x^1\| = \|I + x\| \leq \|I\| + \|x\|
\]

\[
= 1 + \|x\|
\]

Induction then shows that

\[
\|T_k\| \leq \sum_{k=0}^{K} \|x\|^k.
\]

Similarly for \( k < n \):

\[
\|T_k - T_n\| = \|\sum_{k=n+1}^{K} x^k\| \leq \sum_{k=n+1}^{K} \|x\|^k.
\]
Let $\varepsilon > 0$.
As the geometric series \( \sum_{k=0}^{\infty} \|x\|^k \) converges, there exist $N > 0$ so that for all $N \leq k < L$ we have
\[
\sum_{k=N}^{L} \|x\|^k < \varepsilon.
\]
But then $\|T_k - T_L\| < \varepsilon$ for $k, L > N$.

b) Let $K \in \mathbb{N}$. Then
\[
(I - X)T_k = \sum_{k=0}^{K} (x^k - x^{k+1})
= I - x^{k+1}
\]
Hence
\[
\|(I - X)T_k - I\| = \|x^{k+1}\| \leq \|x\|^{k+1} \to 0.
\]

C) Let $T = \lim_{k \to \infty} T_k$. Then (b) shows that
\[
(I - X)T = I
\]
or
\[
(I - X)(T(v)) = v \quad \text{for all } v \in \mathbb{R}^n.
\]
Thus $(I - X)$ is surjective and hence also injective.
Hence invertible with
\[
(I - X)^{-1} = T.
\]