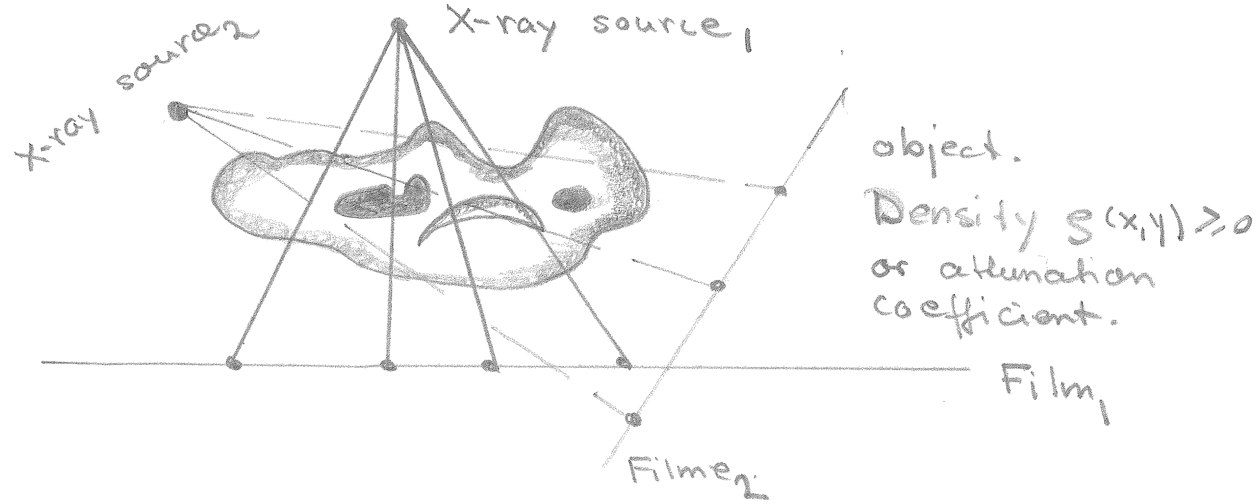
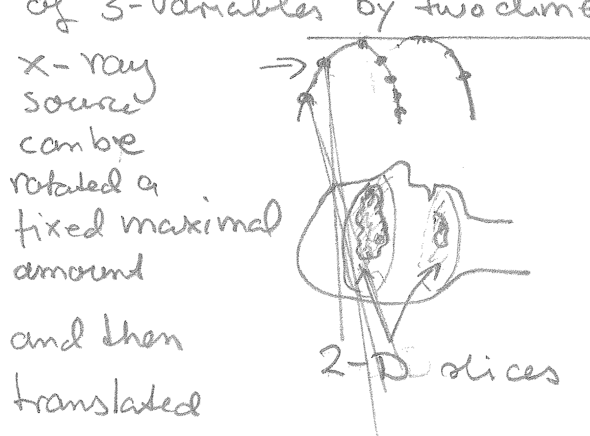
THE RADON TRANSFORM – AND APPLICATIONS

§1 A SIMPLE MODEL

The Radon Transform has become popular because of its application in image reconstruction, in particular x-ray tomography.



The object that we are interested in is described by a function $\rho : \mathbb{R}^3 \to \mathbb{R}$ (or $\mathbb{R}^2 \to \mathbb{R}$), called the attenuation coefficient. It describes the absorbtion (or scattering) of x-rays of given energy. The first assumption is, that $\mu \equiv 0$ outside the body. The aim is to reconstruct μ from the measurments. One – older – way to reconstruct the function $\mu(x,y,z)$ of 3-variables by two dimensional measurment is to fix the z-direction and consider $f_z(x,y) = \mu(x,y,z)$ for several values of z.



The assumptions that are made are:

- X-rays travel along straight lines;
- All the waves are of the same frequency
- The intensity, I, of the x-ray beam satisfies Beer's law

$$(*) \qquad \frac{dI}{ds} = -\rho(x) I$$

where s is the arc-length along the line L given by (1)

From $(*)$ it follows, that

$$\log\left(\frac{I_{in}}{I_{out}}\right) = \int_L \rho(x)\,dx =: \hat{\rho}(L)$$

where I_{in} denotes the intensity of the beam before entering the body and I_{out} is the measured intensity after the beam leaves the body.

The integral transform $\rho \longmapsto \hat{\rho}$ is exactly the Radon transform introduced by Radon in 1917. It transforms "suitable" functions on $\mathbb{R}^d$ to functions on the space of all affine lines (or more generally k-dimensional hypersurfaces in $\mathbb{R}^d$).

It should be noted, that the 3 assumptions that have been made do not quite reflect the true situation, in particular the x-ray is never monochromatic, even if it might be localized around certain spectrum. Denote by $f(E)$ the spectral function. Then a more correct model is given by the

$$\int_0^\infty f(E)\, e^{-\int_L \mu(x,E)\,dx}\, dE$$

which is non-linear in μ!

There are other forms of applications like emission tomography

$$T_\mu f(L) = \int_L f(x)\, e^{-\int_{L_x} \mu(y)\,dy}\, dx$$

around. I refer to the fortcomming volume in "Proceedings of Symposia in Applied Mathematics" for further discussion.

It is also a "fact of life" that we can only use finitely many measurements to reconstruct μ from $\hat{\mu}$, but μ is <u>never</u> determined by finitely many values of $\hat{\mu}$! This brings in topics like approximation and sampling.

If time allows, then we will also discuss 3D-versions like cone-beam tomography.

§2 The Radon Transform

Let $S(\mathbb{R}^n)$ be the space of rapidly decreasing functions on $\mathbb{R}^n$, i.e. for each $N \in \mathbb{N}$

$$\sup_{x \in \mathbb{R}^n} (1+|x|^2)^N |f(x)| \leq C_{N,f} .$$

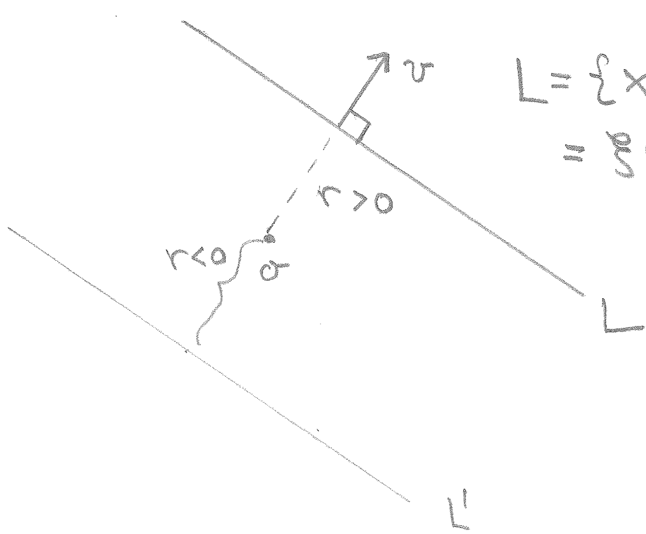
That is $f(x) \to 0$, $x \to \infty$, faster than any $|x|^{-N}$. Let $L^p(\mathbb{R}^n)$ be the space of measurable functions $f: \mathbb{R}^n \to \mathbb{C}$ s.t.

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty .$$

Let Ξ be the manifold of $(d-1)$-dimensional affine hyperplanes in $\mathbb{R}^d$. In 2-dimension this is just the set of lines in $\mathbb{R}^d$. A hyperplane is determined by its <u>normal</u> vector

$$v \in S^{d-1} = \{ x \in \mathbb{R}^d \mid \|x\| = 1 \}$$

and its signed distance from the origin.



$$L = \{ x \in \mathbb{R}^d \mid x \cdot v = r \} = \xi(v, r) .$$

Note that

$$\xi(v,r)=\xi(-v,-r)$$

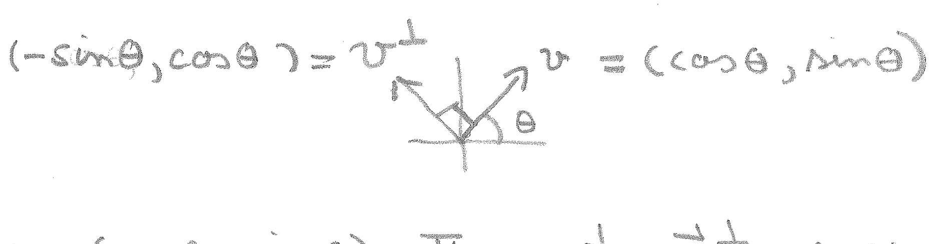
Thus $\Xi = S^{d-1}\times_{\mathbb{Z}_2}\mathbb{R}$ where the subscript $\mathbb{Z}_2$ indicates that (v,r) and $-(v,r)=(-v,-r)$ leads to the same hyperplane. Thus we view functions on Ξ as <u>even</u> functions on $S^{d-1}\times\mathbb{R}$, i.e. $\varphi(v,r)=\varphi(-v,-r)$. The Radon transform $f\mapsto\hat{f}=\mathcal{R}f$ maps functions on $\mathbb{R}^d$ linearly to functions on Ξ:

$$\hat{f}(L)=\hat{f}(\xi(v,r))=$$

$$=\int_L f(x)\,dx$$

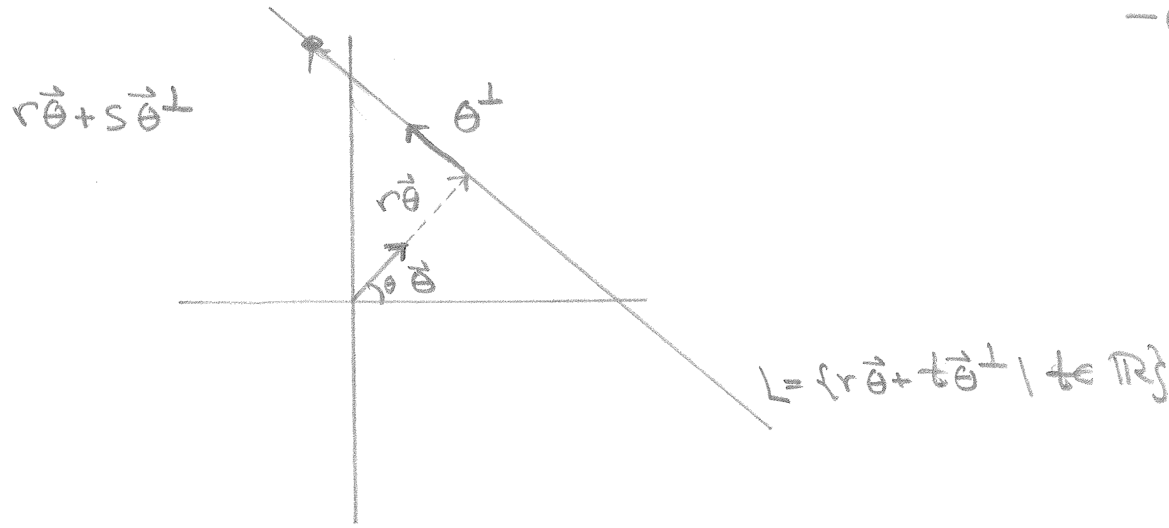
$$=\int_{x\cdot v=r} f(x)\,dx,\qquad f\in S(\mathbb{R}^d).$$

In 2-dimensions we can write this more explicitly as:



$\vec{\theta}=v=(\cos\theta,\sin\theta)$. Then $v^{\perp}=\vec{\theta}^{\perp}=(-\sin\theta,\cos\theta)$ is orthogonal to $\vec{\theta}$ and

$$L=\{r\vec{\theta}+t\vec{\theta}^{\perp}\mid t\in\mathbb{R}\}$$



We have therefore

$$\hat{f}(L)=\hat{f}(\vec{\theta},r)=\int_{-\infty}^{\infty} f(r\vec{\theta}+t\vec{\theta}^{\perp})\,dt.$$

[We have also the d-dim X-Ray transform $Pf(\theta,x)=\int_{-\infty}^{\infty} f(x+t\theta)\,dt$, $x\in\theta^{\perp}$, $\theta\in S^{d-1}$.]

Let $d\sigma(v)$ be the normalized surface measure on S^{d-1}.

In 2D this is just

$$\int_{S^1} g(\xi)\,d\sigma(\xi)=\frac{1}{2\pi}\int_0^{2\pi} g((\cos\theta,\sin\theta))\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi} g(\vec{\theta})\,d\theta.$$

We can then integrate function on Ξ by

$$\int_{\Xi} g(\xi)\,d\xi := \frac{1}{2}\int_{-\infty}^{\infty}\int_{S^{d-1}} g(v,r)\,d\sigma(v)\,dr$$

$$=\int_0^{\infty}\int_{S^{d-1}} g(v,r)\,d\sigma(v)\,dr.$$

The following questions are now natural:

(1) Injectivity of the Radon Transform (on given function spaces)

(2) The Image of $\mathcal{R}$

(3) Inversion formula.

§3 The Central Slice Theorem and the injectivity

Define the Fourier transform of $f \in L^1(\mathbb{R}^d)$ by

$$\mathcal{F}_d f(\lambda) = \mathcal{F} f(\lambda) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-i\lambda \cdot x} dx$$

$$= (f, e_\lambda)$$

where $e_\lambda(x) = e^{i\lambda \cdot x}$.

<u>Theorem 1</u> (The Central Slice Theorem) Let $f \in S(\mathbb{R}^d)$, (also valid for $f \in L^1$) $r \in \mathbb{R}$ and $\omega \in S^{d-1}$. Then

$$\mathcal{F}_d f(r\omega) = \frac{1}{(2\pi)^{d/2}} \int_{-\infty}^{\infty} \mathcal{R}f(\omega, t) e^{-itr} dt$$

$$= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \left[\mathcal{F}_1 \mathcal{R}f(\omega, \cdot)\right](r)$$

<u>Proof</u>. Write $x = t\omega + y$ where $t \in \mathbb{R}$ and $y \perp \omega$. Then $dx = dy\,dt$. Furthermore

$$(r\omega) \cdot x = t$$

as $\|\omega\| = 1$ and $\omega \cdot y = 0$. Thus

$$(2\pi)^{d/2} \mathcal{F}_d f(r\omega) = \int_{-\infty}^{\infty} f(t\omega + y) e^{-irt} dy\,dt$$

$$= \int_{-\infty}^{\infty} \mathcal{R}f(\omega, t) e^{-irt} dt.$$

<u>Corollary</u> The Radon transform is injective on $L^1(\mathbb{R}^d)$

<u>Proof</u>. It follows from the injectivity of the Fourier transform ∎

The central slice theorem implies also the following <u>filtered back-projection (inversion) formula</u>. Note, that if f is rapidly decreasing (i.e. $f \in S(\mathbb{R}^d)$) then all integrals make sense. We will discuss the back-projection in more detail in the next section. For $f \in S(\mathbb{R}^d)$ define

$$\mathcal{G}Rf(\omega,t) = \frac{1}{2}\int_{-\infty}^{\infty} \underbrace{\mathcal{F}_1 Rf(\omega,r)}_{\text{1-D FT in the second variable.}} e^{-itr}\, |r|^{n-1}\, dr$$

Then a simple calculation shows:

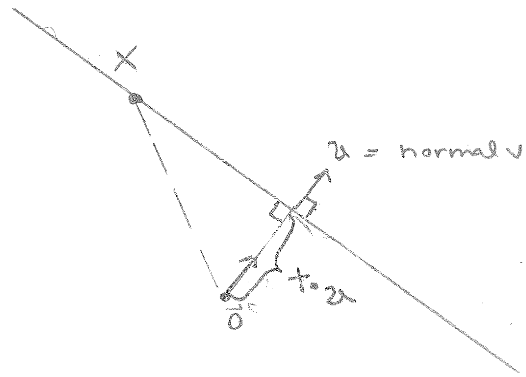
Theorem (Filtered Back-projection formula)
Let $f \in S(\mathbb{R}^d)$. Then $\exists c$

$$f(x) = c\int_{S^{d-1}} \mathcal{G}Rf(\omega, \omega\cdot x)\, d\sigma(\omega).$$

<u>§ 4 The Back-projection</u>

For $x \in \mathbb{R}^d$ let $\check{x}$ denote the set of hyperplanes containing x. Then

$$\check{x} = \{(\omega, \omega\cdot x) \mid \omega \in S^{d-1}\} \simeq S^{d-1}.$$



Hence, if $\varphi : \Xi \to \mathbb{C}$ is continuous, then we can define a function on $\mathbb{R}^d$ by average φ over all hyperplanes containing x

$$\mathcal{R}^\vee \varphi(x) = \check{\varphi}(x) = \int_{S^{d-1}} \varphi(\omega, x\cdot\omega)\, d\sigma(\omega)$$

The integral operator $\mathcal{R}^\vee$ is called the <u>back-projection</u> or dual Radon transform. We have

$$\int_{\mathbb{R}^d} f(x)\check{\varphi}(x)\,dx = \int_{\Xi} \hat{f}(\xi)\varphi(\xi)\,d\xi .$$

Denote by $\Omega_k = 2\dfrac{\pi^{k/2}}{\Gamma(k/2)}$ the area of the unit sphere in $\mathbb{R}^k$.

<u>Lemma</u> $\exists c$: $\displaystyle\int_{S^{d-1}}\int_{\omega^\perp} f(y)\,dy\,d\sigma = c \int |y|^{-1} f(y)\,dy$

<u>Lemma</u> Let $f \in S(\mathbb{R}^d)$. Then $\exists c_n$ such that

$$(\hat{f})^\vee(x) = c_n \int_{\mathbb{R}^d} |x-y|^{-1} f(y)\,dy$$

<u>Proof</u> We have

$$(\hat{f})^\vee(x) = \int_{S^{d-1}} \hat{f}(\omega, \omega\cdot x)\,d\sigma(\omega)$$

all constants are known explicitly and depend only on the normalization of the measure

$$= \int_{S^{d-1}} \int_{\omega^\perp} f((x\cdot\omega)\omega + y)\,dy\,d\omega$$

Next, note that $x - (x\cdot\omega)\omega \in \omega^\perp$ as $\omega\cdot(x - (x\cdot\omega)\omega) =$
$= \omega\cdot x - x\cdot\omega\|\omega\|^2 = 0$. Hence

$$\int_{S^{d-1}}\int_{\omega^\perp} f((x\cdot\omega)\omega + y)\,dy\,d\omega = \int_{S^{d-1}}\int_{\omega^\perp} f(x+y)\,dy\,d\omega$$

$$= c\int_{\mathbb{R}^d} f(x+y)|y|^{-1}\,dy$$

$$= c\int_{\mathbb{R}^d} \frac{f(y)}{|x-y|}\,dy. \qquad \blacksquare$$

<u>Lemma</u> Let $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$P^\vee P f(x) = c\int_{\mathbb{R}^d} f(x+y)|y|^{1-n}\,dy. \qquad \blacksquare$$

Note the occurence of the $y \mapsto |y|^{-\alpha}$ in both cases.

<u>§ 5 Inversion formulas involving power of the Laplacian</u>

For $\operatorname{Re}\alpha < n$ define

$$I^\alpha f(x) = \mathcal{F}^{-1}(|\lambda|^{-\alpha}\hat{f}(\lambda))(x)$$

I^α is called the Riesz potential.
Recall, the following

$$\mathcal{F}\left(\frac{\partial f}{\partial x_j}\right)(\lambda) = i\lambda_j\,\mathcal{F}f(\lambda)$$

Let $\Delta = \sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j}\right)^2$ be the Laplacian.

Then

$$\mathcal{F}(\Delta f)(\lambda) = -|\lambda|^2 \mathcal{F}f(\lambda).$$

In particular, if $\alpha = -2m \in -2\mathbb{N}$, then

$$I^\alpha f(x) = (-\Delta)^m f(x)$$

and

$$I^0 f(x) = f(x).$$

We note also, that (formally)

$$I^{\alpha+\beta} = I^\alpha I^\beta.$$

Finally, $\alpha \mapsto I^\alpha f(x)$ has a meromorphic continuation to all of $\mathbb{C}$.

<u>Theorem</u> Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then $\exists\, c \neq 0$:

$$f = c\, I^{1-d} (\hat{f})^\vee$$

$$= c\,(-\Delta)^{\frac{d-1}{2}} (\hat{f})^\vee$$

<u>Remark</u>: Note the difference between d even and d odd. If d is odd, then $(d-1)/2$ is an integer and hence

$$(-\Delta)^{\frac{d-1}{2}}$$

is a differential operator and hence <u>local</u>

On the other hand, if d is even, then $\frac{d-1}{2}$ is a half-integer and $(-\Delta)^{\frac{d-1}{2}}$ is a pseudo-differential operator and involves <u>global</u> information.

There are other ways to write the inversion formula. We can consider the 1-D Riesz potential and write

$$f = c R^{\vee} I^{1-d} \hat{f}.$$

For $g \in S(\mathbb{R})$ we have

$$\begin{aligned}(I^{1-d} g)^{\wedge}(\lambda) &= |\lambda|^{d-1} \hat{g}(\lambda) \\ &= (\mathrm{sign}(\lambda))^{d-1} \lambda^{d-1} \hat{g}(\lambda) \\ &= (\mathrm{sign}(\lambda))^{d-1} \left(\widehat{(-i)^{d-1} g^{(d-1)}}\right)(\lambda),\end{aligned}$$

where

$$\mathrm{sign}(\lambda) = \begin{cases} 1 & \lambda > 0 \\ -1 & \lambda < 0. \end{cases}$$

Define the Hilbert transform by

$$\begin{aligned}(Hg)(x) &= \mathcal{F}^{-1}(-i\, \mathrm{sign}(\lambda) \hat{g}(\lambda))(x) \\ &= C \int_{-\infty}^{\infty} \frac{h(t)}{s-t}\, dt.\end{aligned}$$

The inversion formula can now be rewritten as

d even:

$$f(x) = c \int_{S^{d-1}} H\hat{f}^{(d-1)}(\omega, x\cdot\omega)\, d\sigma$$

$$= c \int_{S^{d-1}} \int_{-\infty}^{\infty} \frac{\hat{f}^{(n-1)}(\omega, t)}{x\cdot\omega - t}\, dt\, d\omega$$

$$= c \int_{S^{d-1}} \int_{-\infty}^{\infty} \frac{\hat{f}^{(n-1)}(\omega, x\cdot\omega + q)}{q}\, dq\, d\omega$$

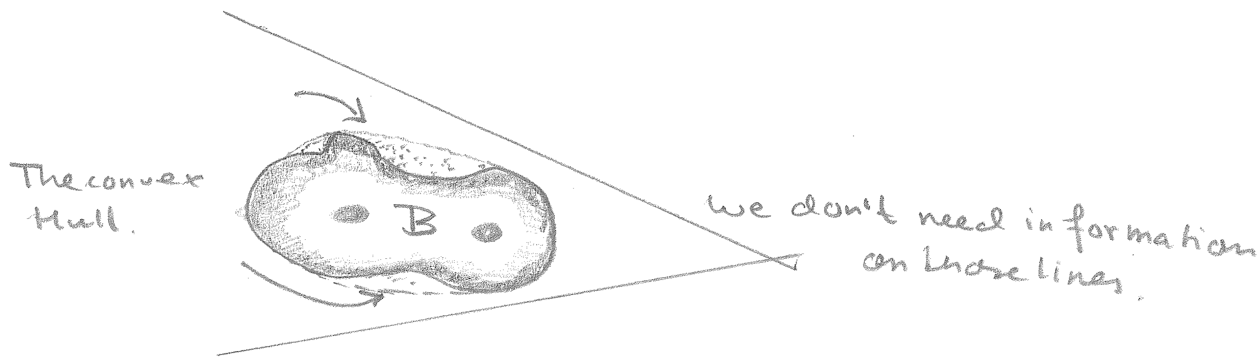
d-odd:

$$f(x) = c \int_{S^{d-1}} \hat{f}^{(n-1)}(\omega, x\cdot\omega)\, d\omega$$

This form of the inversion formula shows clearly the difference between the even and odd dimension. In odd dimension we only need to know $\hat{f}$ on hyperplanes in a neighborhood of $\check{x}$ whereas in even dimension we need information on $\hat{f}$ on all hyperplanes.

§6 Support Theorems

X-ray tomography is the 2-D Radon transfor and we have seen, that the reconstruction formula involves global information on $\hat{f}$. The following shows, that under the assumption $\mu = 0$ outside the body of interest, we only need information on $\hat{f}$ on lines, that actually go throug the body.



Theorem Let $f \in \mathcal{S}(\mathbb{R}^d)$ and $C \subseteq \mathbb{R}^d$ be convex and compact. If $\hat{f}(L) = 0$ for all lines L s.t. $L \cap C = \emptyset$ then $f|_{\mathbb{R}^d \setminus C} = 0$.

Remark: It is not neccessary to assume, that $f \in S(\mathbb{R}^d)$ but one has to assume that f vanishes "fast" at ∞.