8.1 A SIMPLE MODEL

The Radon Transform has become popular because of its application in image reconstruction, in particular X-ray tomography.

The object that we are interested in is described by a function \( g : \mathbb{R}^2 \rightarrow \mathbb{R} \) (or \( \mathbb{R}^3 \rightarrow \mathbb{R} \)), called the attenuation coefficient. It describes the absorption (or scattering) of X-rays of given energy. The first assumption is that \( \mu = 0 \) outside the body. The aim is to reconstruct \( \mu \) from the measurements. One way to reconstruct the function \( \mu(x,y,z) \) of 3-variables by two-dimensional measurements is to fix the \( z \)-direction and consider \( f(x,y) = \mu(x,y,z) \) for several values of \( z \).
The assumptions that are made are:

- X-rays travel along straight lines;
- All the waves are of the same frequency;
- The intensity, $I$, of the x-ray beam satisfies Beer's law,

\[
\frac{dI}{ds} = -\rho(x)I \tag{1}
\]

where $s$ is the arc-length along the line $L$ given by (1).

From (1), it follows, that

\[
\log \left( \frac{I_{\text{in}}}{I_{\text{out}}} \right) = \int_L \rho(x) \, dx = \hat{\rho}(L)
\]

where $I_{\text{in}}$ denotes the intensity of the beam before entering the body and $I_{\text{out}}$ is the measured intensity after the beam leaves the body.

The integral transform $\rho \mapsto \hat{\rho}$ is exactly the Radon transform introduced by Radon in 1917. It transforms "suitable" functions on $\mathbb{R}^d$ to functions on the space of all affine lines (or more generally $k$-dimensional hypersurfaces in $\mathbb{R}^d$).
It should be noted, that the 3 assumptions that have been made do not quite reflect the true situation, in particular, the x-ray is never monochromatic, even if it might be localized around certain spectrum. Denote by \( f(E) \) the spectral function. Then a more correct model is given by the equation:

\[
\int_0^\infty f(E) e^{-S_L \mu(x, E)} dx dE
\]

which is non-linear in \( \mu \)!

There are other forms of applications like emission tomography

\[
T \mu f(L) = \int L f(x) e^{-S_L \mu(y) \delta y} dy dx
\]

around. I refer to the forthcoming volume in "Proceedings of Symposia in Applied Mathematics" for further discussion.

It is also a "fact of life" that we can only use finitely many measurements to reconstruct \( \mu \) from \( \hat{\mu} \), but \( \mu \) is never determined by finitely many values of \( \hat{\mu} \)!

This brings in topics like approximation and sampling.

If time allows, then we will also discuss 3D-versions like cone-beam tomography.
§ 2 The Radon Transform

Let $S(C^0)$ be the space of rapidly decreasing functions on $\mathbb{R}^n$, i.e., for each $N \in \mathbb{N}$

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |f(x)| \leq C_{N,f}. $$

That is $f(x) \to 0$, $x \to \infty$, faster than any $1/x^N$. Let $L^p(\mathbb{R}^n)$ be the space of measurable functions $f: \mathbb{R}^n \to \mathbb{C}$ s.t.

$$\|f\|_p = (\int_{\mathbb{R}^n} |f(x)|^p \, dx)^{1/p} < \infty. $$

Let $E$ be the manifold of $n(n-1)/2$-dimensional affine hyperplanes in $\mathbb{R}^n$. In 2-dimensional this is just the set of lines in $\mathbb{R}^2$. A hyperplane is determined by its normal vector

$$v \in S^{d-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\},$$

and its signed distance from the origin.

$$L = \{x \in \mathbb{R}^d \mid x \cdot v = r\} = \mathcal{F}(v, r).$$
Note that
\[ \Psi(v, r) = \Psi(-v, -r) \]

Thus \( \Xi = S^{d-1} \times \mathbb{Z}_2 \), where the subscript \( \mathbb{Z}_2 \) indicates that \((v, r)\) and \((-v, r) = (-v, -r)\) leads to the same hyperplane. Thus we view functions on \( \Xi \) as even functions on \( S^{d-1} \times \mathbb{R} \), i.e., \( \Psi(v, r) = \Psi(-v, -r) \).

The Radon transform \( f \mapsto \hat{f} = Rf \) maps functions on \( \mathbb{R}^d \) linearly to functions on \( \Xi \):

\[
\hat{f}(L) = \hat{f}(\Psi(v, r)) = \\
= \int_{L} f(x) \, dx \\
= \int_{x \cdot v = r} f(x) \, dx, \quad f \in S(\mathbb{R}^d).
\]

In 2-dimensions we can write this more explicitly as:

\((-\sin \theta, \cos \theta) = v^\perp \quad v = (\cos \theta, \sin \theta) \)

\[ \hat{\theta} = \tilde{v} = (\cos \theta, \sin \theta). \] Then \( v^\perp = \tilde{\theta}^\perp = (-\sin \theta, \cos \theta) \) is orthogonal to \( \tilde{\theta} \) and

\[ L = t_{\tilde{\theta}^\perp} + t_{\tilde{\theta}^\perp} \]
We have therefore

\[ \hat{f}(L) = \hat{f}(\theta, r) = \int_{-\infty}^{\infty} f(r \hat{\theta} + t \hat{\theta}^\perp) dt. \]

We have also that, for any \( x \in \mathbb{R}^d \),

\[ f(x) = \int_{-\infty}^{\infty} f(x + t \theta) dt, \quad \theta \in S^{d-1}. \]

Let \( d\sigma(v) \) be the normalized surface measure on \( S^{d-1} \).

In 2D this is just

\[ \int_{S^1} g(\sigma) d\sigma = \frac{1}{2\pi} \int_{0}^{2\pi} g(\cos \theta, \sin \theta) d\theta \]
\[ = \frac{1}{2\pi} \int_{0}^{2\pi} g(\hat{\theta}) d\theta. \]

We can then integrate function on \( \mathbb{R}^d \) by

\[ \int_{\mathbb{R}^d} g(x) dx := \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{S^{d-1}} g(x, r) d\sigma(v) dr 
\[ = \int_{S^d} \int_{S^{d-1}} g(\sigma, r) d\sigma(v) dr, \]
The following questions are now natural:

1. Injectivity of the Radon Transform (on given function spaces)
2. The Image of \( R \)
3. Inversion formula.

§ 3 The Central Slice Theorem and the injectivity

Define the Fourier transform of \( f \in L^1(\mathbb{R}^d) \) by

\[
\hat{f}(\lambda) = \mathcal{F}(f) = \int_{\mathbb{R}^d} f(x) e^{-i\lambda \cdot x} \, dx
\]

where \( e_\lambda(x) = e^{-i\lambda \cdot x} \).

Theorem 1 (The central slice theorem). Let \( f \in SC(\mathbb{R}^d) \), \( r \in \mathbb{R} \) and \( \omega \in S^{d-1} \). Then

\[
\frac{1}{(2\pi)^d} \int_{\mathbb{R}} Rf(\omega, t) e^{-itr} \, dt = \frac{1}{(2\pi)^d} \left[ \mathcal{F}(Rf(\omega, \cdot)) \right](r)
\]

Proof. Write \( x = tw + y \) where \( t \in \mathbb{R} \) and \( y \perp \omega \).

Then \( dx = dy \, dt \). Furthermore

\[
(rw) \cdot x = t
\]

as \( ||w|| = 1 \) and \( \omega \cdot y = 0 \). Thus

\[
(2\pi)^d \frac{1}{\sqrt{t}} \mathcal{F}(f)(rw) = \int \hat{f}(tw + y) e^{-itr} \, dy \, dt
\]

\[
= \int \mathcal{F}(f)(w, t) e^{-itr} \, dt.
\]
Corollary  The Radon transform is injective on $L^1(\mathbb{R}^d)$

Proof.  It follows from the injectivity of the Fourier transform.

The central slice theorem implies also the following
filtered back-projection (inversion) formula. Note, that if
f is rapidly decreasing (i.e. $f \in \mathcal{S}(\mathbb{R}^d)$) then all integrals
make sense. We will discuss the back-projection in more detail
in the next section. For $f \in \mathcal{S}(\mathbb{R}^d)$ define

$$GRF(f)(w, t) = \frac{1}{t} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} f(r) e^{-ir \cdot r \cdot w} \, dr$$

Then a simple calculation shows:

Theorem  (Filtered Back-projection formula)

Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then $\exists c$

$$f(x) = c \int_{\mathbb{R}^d-1} GRF(f)(w, w \cdot x) \, \text{d}w.$$ 

§ 4  The Back-projection

For $x \in \mathbb{R}^d$ let $\mathcal{X}$ denote the set of hyperplanes
containing $x$. Then

$$\mathcal{X} = \{ (w, w \cdot x) \mid w \in S^{d-1} \} \sim S^{d-1}.$$
Hence, if \( q : \mathbb{E} \rightarrow \mathbb{C} \) is continuous, then we can define a function on \( \mathbb{R}^d \) by average of \( q \) over all hyperplanes containing \( x \):

\[
R^v \hat{q}(x) = \hat{q}(x) = \int_{S^{d-1}} q(\omega, x \cdot \omega) d\sigma(\omega)
\]

The integral operator \( R^v \) is called the back-projection or dual Radon transform. We have

\[
\int_{\mathbb{R}^d} f(x) \hat{q}(x) dx = \int_{\mathbb{R}^d} \hat{f}(\xi) q(\xi) d\xi.
\]

Denote by \( \Omega_k = \frac{2 \pi^{k/2}}{\Gamma(k/2)} \) the area of the unit sphere in \( \mathbb{R}^k \).

**Lemma** Let \( f \in SC(\mathbb{R}^d) \). Then \( \exists C_n \) such that

\[
(\hat{f})^V(x) = C_n \int_{\mathbb{R}^d} (x - y)^{-1} f(y) dy
\]

**Proof** We have

\[
(\hat{f})^V(x) = \int_{S^{d-1}} \hat{f}(\omega, x \cdot \omega) d\sigma(\omega)
\]
\[
\int_{S^{d-1}} f((x \wedge \omega) + y) dy d\omega
\]

Next, note that \( x = (x \wedge \omega) \in \omega^\perp \) as \( \omega (x - (x \wedge \omega)) = \omega \times x - x \times \omega \| \omega \|^2 = 0 \). Hence

\[
\int_{S^{d-1}} f((x \wedge \omega) + y) dy d\omega = \int_{S^{d-1}} f(x + y) dy d\omega
\]

\[
= c \int_{\mathbb{R}^d} f(x + y) |y|^{-d} dy
\]

\[
= c \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|} dy.
\]

**Lemma**  Let \( f \in \mathcal{S}(\mathbb{R}^d) \), then

\[
\mathcal{F}^{-1} f(x) = c \int_{\mathbb{R}^d} \frac{f(x + y)}{|x - y|} dy.
\]

Note the occurrence of the \( y \mapsto |y|^{-\sigma} \) in both cases.

§ 5 Inversion formulas involving powers of the Laplacian

For \( \Re \sigma < n \) define

\[
\mathcal{I}^\sigma f(x) = \mathcal{F}^{-1} \left( |\lambda|^{-\sigma} \mathcal{F} f(\lambda) \right)(x)
\]

\( \mathcal{I}^\sigma \) is called the Riesz potential.

Recall the following

\[
\mathcal{F} \left( \frac{\partial f}{\partial x_j} \right)(\lambda) = i \lambda_j \mathcal{F} f(\lambda)
\]

Let \( \Delta = \sum_{j=1}^n \left( \frac{\partial}{\partial x_j} \right)^2 \) be the Laplacian.
Then
\[ \mathcal{F}(\Delta^s f)(x) = -i\lambda^2 \mathcal{F}f(x). \]

In particular, if \( s = -2m \in -2\mathbb{N} \), then
\[ I^s f(x) = (-\Delta)^m f(x) \]
and
\[ I^s f(x) = f(x). \]

We note also, that (formally)
\[ I^s + I^s = I^s I^s. \]

Finally, \( x \rightarrow I^s f(x) \) has a meromorphic continuation to all of \( \mathbb{C} \).

**Theorem** Let \( f \in \mathcal{S}(\mathbb{R}^d) \). Then \( \exists C > 0 \):
\[
\begin{align*}
I^s f &= C I^{1-d} \left( \frac{\partial}{\partial \xi} \right)^s \left( f \right) \\
&= C (-\Delta)^{\frac{d-1}{2}} \left( f \right) \nabla
\end{align*}
\]

**Remark:** Note the difference between \( d \) even and \( d \) odd. If \( d \) is odd, then \((d-1)/2\) is an integer and hence \( (-\Delta)^{\frac{d-1}{2}} \) is a differential operator and hence **local**
On the other hand, if \( d \) is even, then \( \frac{d-1}{2} \) is a half-integer and \( (-\Delta)^{d-1} \) is a pseudo-differential operator and involves global information.

There are other ways to write the inversion formula. We can consider the 1-D Riesz potential and write

\[
f = c \mathcal{R}^{\nu} I^{1/\alpha} \hat{f}.
\]

For \( g \in \mathcal{S}(\mathbb{R}) \) we have

\[
(I^{1/\alpha} g)(x) = |x|^{d-1} \hat{g}(\lambda)
\]

\[
= (\text{sign}(\lambda))^{d-1} \frac{\lambda^{d-1}}{|\lambda|^d} \hat{g}(\lambda)
\]

\[
= (\text{sign}(\lambda))^{d-1} \left( \frac{g^{d-1}}{(-i)^{d-1}} \right) (\lambda),
\]

where

\[
\text{sign}(\lambda) = \begin{cases} 1 & \lambda > 0 \\ -1 & \lambda < 0. \end{cases}
\]

Define the Hilbert transform by

\[
(hg)(x) = \mathcal{F}^{-1} (-i \text{sign}(\lambda) \hat{g}(\lambda))(x)
\]

\[
= c \int_{-\infty}^{\infty} \frac{h(t)}{s-t} \, dt.
\]
The inversion formula can now be rewritten as:

\[ f(x) = c \int_{\mathbb{R}^{d-1}} \hat{f}(x \cdot \omega) \, d\omega \]

\[ = c \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \frac{\hat{f}(x \cdot \omega)}{x \cdot \omega - t} \, dt \, d\omega \]

\[ = c \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \frac{\hat{f}(x \cdot \omega + t)}{t} \, dq \, d\omega \]

\text{d-odd:}

\[ f(x) = c \int_{\mathbb{R}^{d-1}} \hat{f}(x \cdot \omega) \, d\omega \]

This form of the inversion formula shows clearly the difference between the even and odd dimension. In odd dimension, we only need to know \( \hat{f} \) on hyperplanes in a neighborhood of \( x \) whereas in even dimension we need information on \( \hat{f} \) on all hyperplanes.
§ 6 Support Theorems

X-ray tomography is the 2-D Radon transform and we have seen, that the reconstruction formula involves global information on \( \hat{f} \). The following shows, that under the assumption \( \mu = 0 \) outside the body of interest, we only need information on \( \hat{f} \) on lines, that actually go through the body.

\[ \text{The convex hull} \]

\[ \text{we don't need information on those lines} \]

**Theorem** Let \( f \in SC(R^d) \) and \( C \subseteq R^d \) be convex and compact. If \( \hat{f}(L) = 0 \) for all lines \( L \) o.t. \( L \cap C = \emptyset \) then \( f|_{R^d \setminus C} = 0 \).

**Remark:** It is not necessary to assume, that \( f \in SC(R^d) \) but one has to assume that \( f \) vanishes "fast" at \( \infty \).