PROBLEMS - VECTOR SPACES.

1) Which of the following sets is a vector space and which are not.

a) $\{(x,y,z) \in \mathbb{R}^3 \mid 2xy + z = 0\}$ No, because of xy

b) $\{(x,y) \in \mathbb{R}^2 \mid 2x + 3y = 0\}$ Yes. This is the line $y = -\frac{2}{3}x$

c) $\{(x,y,z) \in \mathbb{R}^3 \mid 2x + 3y + z + 2 = 0\}$ No

d) $\left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x + 2y + z = 0 \\ 3x - y + 2z = 0 \end{matrix}\right\}$ Yes

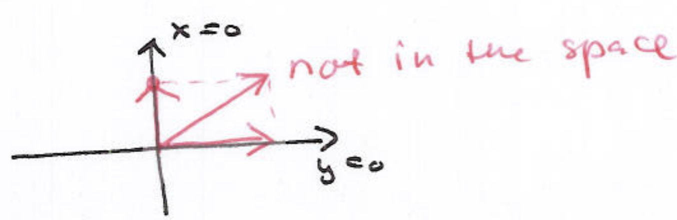
e) $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0\}$ Yes

f) $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 5\}$ no
$(0,0,\dots,0)$ is not in there

g) $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$ no

h) $\{(x,y) \in \mathbb{R}^2 \mid x\cos(y) = 0\}$ No

g)



2) Which of the following sets is a vector space and which are not.

a) $\{f \in C^1(\mathbb{R}) \mid f(0) = 0\}$ Yes

b) $\{f \in C^1(\mathbb{R}) \mid f(0) = 1\}$ No

c) $\{f \in C([0,1]) \mid \int_0^1 f(t)dt = 0\}$ Yes

d) $\{f \in C([0,1]) \mid \int_0^1 f(t)dt = 3\}$ No

e) $\{f \in C^1(\mathbb{R}) \mid f(0)f'(0) = 0\}$ No

f) $\{f \in C([-\pi,\pi]) \mid \int_{-\pi}^{\pi} f(t)\cos(t)dt = 0\}$ Yes

g) The set of all polynomials. Yes

h) $\{f \in C^4(\mathbb{R}) \mid 3f'''' + 2f''' - 3f'' + 2f' - f = 0\}$ Yes

i) $\{f \in C^4(\mathbb{R}) \mid 3f'''' + 2f''' - 3f'' + 2f' - f = 1\}$ No

j) V and W vector spaces, $T : V \to W$ linear. Is the space $\{v \in V \mid T(v) = 0\}$ a vector space? Yes

3) What is the definition of a linear map? [Look it up in the book]

4) Which of the following maps are linear and which are not?

a) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + 2y + z \\ xy + z \end{pmatrix}$ NO

b) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + 2y + z \\ x + y + z \end{pmatrix}$ Yes

c) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + 2y + z - 1 \\ x + y + z \end{pmatrix}$ NO

d) $T: \mathbb{R}^3 \to \mathbb{R}$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3x + 2y - z$ Yes

e) $T: \mathbb{R}^3 \to \mathbb{R}$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xyz$ NO

f) $T: \mathbb{R} \to \mathbb{R}$, $T(x) = x^2$ No

g) $T: \mathbb{R}^3 \to \mathbb{R}$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dfrac{1}{x + 3y + z}$ No

5) What are the linear maps $\mathbb{R}^n \to \mathbb{R}^m$?

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} a_{11} x_1 + \cdots + a_{1n} x_n \\ a_{21} x_1 + \cdots + a_{2n} x_n \\ \vdots \\ a_{m1} x_1 + \cdots + a_{mn} x_n \end{pmatrix}$$

6) Which of the following maps are linear and which are not:

a) $T: C^1(\mathbb{R}) \to C(\mathbb{R}),\ T(f) = f' + f$ Yes

b) $T: C^1(\mathbb{R}) \to C(\mathbb{R}),\ T(f) = ff'$ No

c) $T: C([-\pi,\pi]) \to \mathbb{R},\ T(f) = \int_{-\pi}^{\pi} f(t)\,dt$ Yes

d) $T: C([-\pi,\pi]) \to \mathbb{R},\ T(f) = \int_{-\pi}^{\pi} f(t)\sin(t)\,dt$ Yes

e) $T: C([-1,1]) \to \mathbb{R},\ T(f) = f(0.5)$ Yes

f) $T: C((-1,1)) \to \mathbb{R},\ T(f) = f(0)^2$ No

g) $T: C^4(\mathbb{R}) \to C(\mathbb{R}),\ T(f) = 3f'''' - 2f' + f$ Yes

h) $T: C(\mathbb{R}) \to C(\mathbb{R}),\ T(f) = e^{f(x)}$ No

i) $T: C(\mathbb{R}) \to C(\mathbb{R}),\ T(f) = f^2 + f$ No

7) Evaluate the inner product

a) $((x,y,z),(u,v,w)) = 2xu + yv + 3zw$

- $(x,y,z) = (1,2,-1)$, $(u,v,w) = (2,-3,4)$
- What is now $\|(2,1,-1)\|$

b) $V = C([0,\pi])$, $(f,g) = \int_0^\pi f(t)g(t)dt$

$f(t) = \cos(t)$, $g(t) = t$

$(f,g) =$ ______________________

c) $V = C([-\pi,\pi])$, $(f,g) = \int_{-\pi}^{\pi} f(t)g(t)dt$

- $(\cos(t), \sin(t)) =$ ______________
- $\|\cos(t)\| =$ ______________
- $\|x^2\| =$ ______________
- $(x^2, x) =$ ______________

See next page

7)

a) • $2\cdot(1\cdot 2)+2\cdot(-3)+3(-1)\cdot 4 =$

$= 4-6-12 = \underline{\underline{-14}}$

• $\|(2,1,-1)\|^2 = ((2,1,-1),(2,1,-1))$

$= 2\cdot 2\cdot 2 + 1\cdot 1 + 3(-1)\cdot(-1)$

$= 8+1+3 = 12$

$\|(2,1,-1)\| = \sqrt{12} = 2\sqrt{3}$

b) $(\cos(t), t) = \int_0^{\pi} t\cos(t)\,dt =$

$= t\sin(t)\Big]_0^{\pi} - \int_0^{\pi}\sin(t)\,dt$

$= \cos(t)\Big]_0^{\pi} = 0$

c) • $(\cos(t), \sin(t)) = \int_{-\pi}^{\pi}\cos(t)\sin(t)\,dt = 0$

• $\|\cos(t)\|^2 = \int_{-\pi}^{\pi}\cos^2(t)\,dt = \pi$

So $\|\cos(t)\| = \sqrt{\pi}$

• $\|x^2\|^2 = \int_{-\pi}^{\pi} x^4\,dx = \frac{x^5}{5}\Big]_{-\pi}^{\pi} = \frac{2\pi^5}{5}$, $\|x^2\| = \sqrt{\frac{2\pi^5}{5}}$

• $(x^2, x) = \int_{-\pi}^{\pi} x^3\,dx = 0$