What to study for test # 3

* Do the problems posted for homework # 3 and quiz # 3.

* Inner product and norms. Orhogonal vectors

problems:

1) which of the following defines an inner product on the given space?

a) $V = \mathbb{R}^3$, $((x,y,z),(u,v,w)) = xu + yw + zv$

b) $V = \mathbb{R}^3$, $((x,y,z),(u,v,w)) = 2xu + 3yv + zw$

c) $V = C([-1,1])$, $(f,g) = \int_{-1}^{1} f(x)g(x)e^{x}\,dx$

d) $V =$ polynomial of degree $\leq n$

$$\left(\sum_{j=0}^{n} a_j x^j, \sum_{j=0}^{n} b_j x^j\right) = \sum_{j=0}^{n} a_j b_j .$$

2) Consider the inner product $\int_{-1}^{1} f(x)g(x)\,dx = (f,g)$ on $C([-1,1])$. Evaluate the following inner products and norms:

a) $(x^2-1, x+1) =$

b) $\| x^2+2x+1 \| =$

c) $\| \cos(\pi x) \| =$

d) $(\cos(\pi x), \sin(\pi x)) =$

e) $\| \cancel{x} e^x \| =$

3) With the usual inner product
$((z_1, z_2, z_3), (w_1, w_2, w_3)) = z_1\overline{w_1} + z_2\overline{w_2} + z_3\overline{w_3}$
on $\mathbb{C}^3$ evaluate the following inner products and norms

a) $((i, 1+i, 2-i), (i, 2, 1-i)) =$

b) $((1+i, 0, 1-i), (1+2i, 2+i, 3i)) =$

c) $\|(2-i, 3+i, i)\| =$

d) $\|(5+2i, i, 1+i)\| =$

⊛ <u>Linear span</u>, <u>linear independent</u>, <u>basis</u>

- If $v_1, \dots, v_k \in V$, then the set
$$W = \{c_1v_1 + \dots + c_kv_k \mid c_1, \dots, c_k \in \mathbb{F}\}$$
is called the (linear) span of the vectors. It is the smallest subspace of V containing all the vectors $v_1, \dots, v_k$.

- The vectors $v_1, \dots, v_k$ are <u>linearly independent</u> if a relation
$$c_1v_1 + \dots + c_kv_k = 0$$

implies $c_1 = c_2 = \cdots = c_k = 0$.

- The vectors $v_1, \ldots, v_k$ are linearly dependent if they are not linearly independent. This is the same as saying: There are numbers $c_1, \ldots, c_k$ not all equal to zero such that

$$c_1 v_1 + \cdots + c_k v_k = 0.$$

If say $c_\ell \neq 0$, then we can solve for v_ℓ

$$v_\ell = -\frac{c_1}{c_\ell} v_1 - \cdots \frac{c_{\ell-1}}{c_\ell} v_{\ell-1} - \frac{c_{\ell+1}}{c_\ell} v_{\ell+1} - \cdots - \frac{c_k}{c_\ell} v_k.$$

The vectors are linearly dependent if we can express one of them as a combination of the other vectors.

- The vectors $v_1, \ldots, v_k$ form a basis for W if they are
  a) Linearly Independent

and

  b) Span the spase.

$v_1, \ldots, v_k$ is a basis if every vector $v \in W$ can be written as a combination $v = c_1 v_1 + \cdots + c_k v_k$ in a unique way.

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- Two vectors are linearly dependent if they are both on the same line.
- Three vectors are linearly dependent if they are contained in a plane.

- Orthonal $\neq 0$ vectors are linearly independent
- We need two vector to form a basis for $\mathbb{R}^2$.
- In general we need n vectors for a basis for $\mathbb{R}^n$.
- More than n vectors in $\mathbb{R}^n$ are always linearly dependent.

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problems

4) Decide if the following sets of vectors are linearly independent or not.

a) $(1,2,-3)$, $(4,8,-12)$

b) $(1,0,1)$, $(1,1,-1)$, $(1,-2,-1)$.

c) $(1,2)$, $(2,-1)$, $(4,1)$

d) $1, x, x^2$ in $C([0,1])$

e) $\cos(\pi x)$ and $\sin(\pi x)$ in $C([0,1])$

f) $(2,1,3)$, $(-1,1,2)$, $(0,3,9)$.

Rule: If you have n vectors in $\mathbb{R}^n$, then they are linearly dependent if and only if

$$\det[V_1, \dots, V_n] = 0$$

where $[V_1, \dots, V_n]$ is the $n \times n$ matrix with columns $V_1, \dots, V_n$.

<u>Rule</u>: If $v_1, \dots, v_k$ are orthogonal and

$$v = c_1 v_1 + \cdots + c_k v_k$$

Then $$c_j = \frac{(v, v_j)}{\|v_j\|^2}.$$

5) Write the vector v as a combination of the given vectors.

a) $(1,2), (1,1)$; $v = (4,0)$, $v = (1,1)$

b) $(2,3), (6,-4)$; $v = (4,0)$, $v = (3,1)$.

c) $(1,0,1), (1,1,-1), (1,-2,-1)$; $v = (1,1,1)$, $v = (2,0,0)$, $v = (5,6,10)$.

d) $(1,2,0), (1,1,1), (1,0,3)$; $v = (3,0,1)$.

6) Which of the following sets of vectors is a basis for the given vector space?

a) $\mathbb{R}^2$; $(1,-2), (2,-4)$

b) $\mathbb{R}^3$; $(1,0,1), (1,1,-1)$

c) $\mathbb{R}^3$, $(1,0,1), (1,1,-1), (2,0,1), (1,0,-2)$

d) $\mathbb{R}^3$, $(1,0,1), (0,1,0), (1,0,-1)$.

e) $1, 1+x, 1+x^2$ the space of polynomials of degree ≤ 2.

GRAM-SCHMIDT: Given linearly independent vectors $V_1, \dots, V_k$ then we can construct a set of orthogonal vectors $w_1, \dots, w_k$ spanning the same space as $V_1, \dots, V_k$. This goes as follows

- $W_1 = V_1$
- $W_2 = V_2 - \dfrac{(V_2, W_1)}{\|W_1\|^2} W_1$.
- $W_3 = V_3 - \dfrac{(V_3, W_1)}{\|W_1\|^2} W_1 - \dfrac{(V_3, W_2)}{\|W_2\|^2} W_2$

$\vdots$

- $W_k = V_k - \dfrac{(V_k, W_1)}{\|W_1\|^2} - \dots - \dfrac{(V_k, W_{k-1})}{\|W_{k-1}\|^2} W_{k-1}$.

7) Apply Gram-Schmidt to the following set of vectors. The inner product on $\mathbb{R}^n$ is $((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{j=1}^{n} x_j y_j$. The function space is $C([0,1])$ with inner product $(f, g) = \int_0^1 f(x) g(x) dx$.

a) $(1,1), (1,2)$

b) $(1,0,1), (1,0,-1), (1,1,1)$

c) $(2,1,1), (1,0,1), (2,1,-1)$

d) $1, x$

e) $1, x, x^2$

f) x, e^x

If V is a vector space with inner product $(\cdot,\cdot)$ and $W \subseteq V$ is a subspace, then the orthogonal projection $P: V \to W$ is defined in the following way: Let $w_1, \dots, w_k$ be an orthogonal basis for W. If $v \in V$, then

$$P(v) = \sum_{j=1}^{k} \frac{(v, w_j)}{\|w_j\|^2} w_j$$

The vector v has the properties

i: $v - P(v) \perp W$

ii: $P(v)$ is the vector in W closest to v ($\|v - P(v)\|$ is as small as possible).

8) a) Let $V = \mathbb{R}^2$, $W = \{(x,y) \in \mathbb{R}^2 \mid 2x - y = 0\}$. If $v = (5,1)$ find the point (vector) on the line W closest to v.

b) Let $V = \mathbb{R}^3$ and $W = \{c_1(1,2,1) + c_2(1,1,0) \mid c_1, c_2 \in \mathbb{R}\}$. Find the vector in the plane W closest to the point $(1,1,1)$.

c) Let $W = \{c_1(1,1,-1) + c_2(1,0,1) \mid c_1, c_2 \in \mathbb{R}\}$.

i) Write a formula for the orthogonal projection $P: \mathbb{R}^3 \to W$.

ii) Find the point in the plane W closest to $(4,5,1)$.

9) Let $V = C([0,1])$ and let W be the subspace of polynomials of degree ≤ 1. The inner product is as usually $\int_0^1 f(x)g(x)dx$.

Find the polynomial of degree ≤ 1 closest to

a) x^2

b) $\cos(\pi x)$

c) e^x

d) x^3.

10) Let V be the space of piecewise continuous functions on $[0,1]$ and let

$$W^{(2)} = \{ c_0 \varphi_0^{(2)} + c_1 \varphi_1^{(2)} + c_2 \varphi_2^{(2)} + c_3 \varphi_3^{(2)} \mid c_1, \dots, c_3 \in \mathbb{R} \}$$

a) Show that $\varphi_0^{(2)}, \dots, \varphi_3^{(2)}$ is an orthogonal basis for $W^{(2)}$.

b) Find the function in $W^{(2)}$ closest to

a) x

b) x^2

c) $2x + 3x^2$

c) Show that $\varphi_0^1, \varphi_1^1, \psi_0^1, \psi_1^1$ is an orthogonal basis for $W^{(2)}$.

Solutions to problems on
"What to study for test # 3"

1) a) Not an inner product $((0,1,0),(0,1,0)) = 0$

b) Is an inner product

c) Is an inner product because $e^x > 0$ for all x.

- $(f,f) = \int_{-1}^{1} f(x)^2 e^x \, dx \geq 0$ as $f(x)^2 e^x \geq 0$.
- If $(f,f) = 0$, then $f(x)^2 e^x = 0$ for all x. $e^x > 0$, so $f(x) \equiv 0$.
- $(f,g) = \int_{-1}^{1} f(x) g(x) e^x \, dx = \int g(x) f(x) e^x \, dx = (g,f)$
- Linear in the first argument follows from the linearity of the integral.

d) Yes, is an inner product (why).

2) a) $(x^2-1, x+1) = \int_{-1}^{1} (x^2-1)(x+1)\,dx$

$$= \int_{-1}^{1} x^3 + x^2 - x - 1 \, dx = \int_{-1}^{1} x^2 - 1 \, dx$$

$$= 2 \int_0^1 x^2 - 1 \, dx = 2\left(\frac{1}{3} - 1\right) = \underline{\underline{-\frac{4}{3}}}$$

b) $\| x^2 + 2x + 1 \|^2 = \int_{-1}^{1} (x^2 + 2x + 1)^2 \, dx$

$$= \int_{-1}^{1} x^4 + 4x^3 + 6x^2 + 3x + 1 \, dx$$

$$= 2 \int_0^1 x^4 + 6x^2 + 1 \, dx$$

$$= 2\left(\frac{1}{5} + 2 + 1\right) = \frac{32}{5}$$

$\| x^2 + 2x + 1 \| = \underline{\underline{4\sqrt{\frac{2}{5}}}}$

c) $\|\cos(\pi x)\|^2 = 2\int_0^1 \cos^2(\pi x)\,dx$

$$= \int_0^1 1+\cos(2\pi x)\,dx$$

$$= 1 + \frac{1}{2\pi}\sin(2\pi x)\Big]_0^1 = \underline{\underline{1}}$$

d) $(\cos(\pi x), \sin(\pi x)) = 0.$

3) a) $6+3i$

b) $-4i$

c) 4

d) $4\cdot\sqrt{2}$

4) a) Linearly dependent $(4,8,-12) = 4\cdot(1,2,-3).$

b) Linearly independent.

If $a(1,0,1)+b(1,1,-1)+c(1,-2,-1) = (0,0,0)$. Then

$$\begin{aligned} a + b + c &= 0 \\ b - 2c &= 0 \\ a - b - c &= 0 \end{aligned}$$

Add first and third: $2a = 0$ so $a=0$. First and second then give

$$\begin{aligned} b+c&=0 \\ b-2c&=0 \end{aligned}$$

Substract the second from the first $3c=0$, or $c=0$. But then $b=0$.

c) Three vectors in $\mathbb{R}^2$ are always linearly dependent.

d) Linearly independent:

$a\cdot 1 + bx + cx^2 = 0$. Take $x=0$. Then $a=0$.

Then divide out x: $b + cx = 0$. Take $x=0$, then $b=0$

Then by taking $x=1$ we see that $c=0$.

e) $\cos(\pi x)$ and $\sin(\pi x)$ are orthogonal and hence linearly independent.

f) Linearly independent.

5) a) $(4,0) = a(1,2) + b(1,1)$ gives

$$\left.\begin{array}{r} a + b = 4 \\ 2a + b = 0 \end{array}\right\} \text{ subtract: } a = -4, \quad -4 + b = 4,\ b = 8$$

Answer:

$$(4,0) = -4(1,2) + 8(1,1).$$

$$(1,1) = 0 \cdot (1,2) + 1(1,1).$$

b) $$(4,0) = \frac{8}{13}(2,3) + \frac{24}{52}(6,-4)$$

$$= \frac{8}{13}(2,3) + \frac{6}{13}(6,-4)$$

(Note that $(2,3) \perp (6,-4)$.)

$$(3,1) = \frac{9}{13}(2,3) + \frac{7}{26}(6,-4)$$

c) Note first that the vectors are orthogonal which makes the solution easier. A simple calculation shows that

$$(x,y,z) = \frac{x+z}{2}(1,0,1) + \frac{x+y-z}{3}(1,1,-1) + \frac{x-2y-z}{6}(1,-2,-1)$$

- $(1,1,1) = (1,0,1) + \frac{1}{3}(1,1,-1) - \frac{1}{3}(1,-2,-1)$.
- $(2,0,0) = (1,0,1) + \frac{2}{3}(1,1,-1) + \frac{1}{3}(1,-2,-1)$
- $(5,6,0) = \frac{15}{2}(1,0,1) + \frac{1}{3}(1,1,-1) - \frac{17}{6}(1,-2,-1)$.

6) a) No $2(1,-2) = (2,-4)$.

b) No, we need 3 vectors in $\mathbb{R}^3$ to have a basis.

c) No, too many.

d) Yes, they are orthogonal and hence linearly independent. 3 linearly independent vectors in $\mathbb{R}^3$ form a basis. In fact

$$(x,y,z) = \frac{x+z}{2}(1,0,1) + y(0,1,0) + \frac{x-z}{2}(1,0,-1).$$

e) Basis

$$a+bx+cx^2 = (a-b)\cdot 1 + (b-c)(1+x) + c(1+x^2).$$

7) a) $w_1 = (1,1)$

$$w_2 = (1,2) - \frac{3}{2}(1,1) = \left(-\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{2}(1,-1).$$

b) $w_1 = (1,0,1)$, $w_2 = (1,0,-1)$, $w_3 = (0,1,0)$

c) $(2,1,1)$, $\frac{1}{2}(0,-1,1)$, $\frac{2}{3}(1,-1,-1)$.

d) 1, $x - \frac{1}{2}$

e) 1, $x-\frac{1}{2}$, $x^2 - x + \frac{1}{6}$

f) x, $e^x - 3x$.

8) a) $(1,-2)$ is a basis for W. The orthogonal projection is

$$P((x,y)) = \frac{x-2y}{5}(1,-2).$$

$$P((5,1)) = \frac{5-2}{5}(1,-2) = \frac{3}{5}(1,-2).$$

b) Use $(1,1,0)$, $(-1,1,2)$ as an orthogonal basis. The orthogonal projection

$$P((x,y,z)) = \frac{x+y}{2}(1,1,0) + \frac{-x+y+2z}{6}(-1,1,2)$$

$$= \left(\frac{2x+y-z}{3}, \frac{x+2y+z}{3}, \frac{-x+y+2z}{3}\right)$$

$$P((1,1,1)) = \frac{1}{3}(2,4,2) = \frac{2}{3}(1,2,1).$$

c) $(1,1,-1), (1,0,1)$ is an orthogonal basis for W.

i) $P((x,y,z)) = \frac{x+y-z}{3}(1,1,-1) + \frac{x+z}{2}(1,0,1)$

$$= \frac{1}{6}(5x+2y+z,\ 3x+3z,\ x-2y+5z)$$

ii) $P((4,5,1)) = \frac{1}{6}(31, 18, -1)$.

9) a) $1,\ x-\frac{1}{2}$ is an orthonormal basis, $\|1\|^2 = 1$, $\|x-\frac{1}{2}\|^2 = \frac{1}{12}$

$(1, x^2) = \frac{1}{3}$, $(x^2, x-\frac{1}{2}) = \int_0^1 x^3 - \frac{1}{2}x^2\,dx$

$$= \frac{1}{4} - \frac{1}{6} = +\frac{1}{12}$$

$$\frac{1}{3}\cdot 1 + \frac{1/12}{1/12}(x-\tfrac{1}{2}) = \boxed{x - \tfrac{1}{6}}$$

b) $(1, \cos(\pi x)) = 0$; $(x-\frac{1}{2}, \cos(\pi x)) = \frac{2-\pi}{\pi^2}$

$$\boxed{12\cdot\frac{2-\pi}{\pi^2}(x-\tfrac{1}{2})}$$

c) $(e^x, 1) = e-1$, $(e^x, x-\frac{1}{2}) = \frac{3-e}{2}$

$$(e-1)\cdot 1 + 6(3-e)(x-\tfrac{1}{2}) = \boxed{6(3-e)x + 4e - 10.}$$

d) $(1, x^3) = \frac{1}{4}$; $(x^3, x-\frac{1}{2}) = \frac{3}{40}$

$$\frac{1}{4}\cdot 1 + \frac{3}{40}(x-\tfrac{1}{2}) = \boxed{\frac{3}{40}x + \frac{17}{80}}$$

10) a) Will be done in class, $\|\varphi_j^{(2)}\| = \frac{1}{4}$. c) Will also be done in class

a$_1$) $(\varphi_0^{(2)} + 3\varphi_1^{(2)} + 5\varphi_2^{(2)} + 7\varphi_2^{(2)})\cdot\frac{1}{8}$

b$_1$) $\frac{1}{54}(\varphi_0^{(2)} + 7\varphi_1^{(2)} + 19\varphi_2^{(2)} + 37\varphi_3^{(2)})$.

c$_1$) Use (a$_1$) + (b$_1$): $\frac{1}{16}(5\varphi_0^{(2)} + 19\varphi_1^{(2)} + 39\varphi_2^{(2)} + 65\varphi_3^{(2)})$