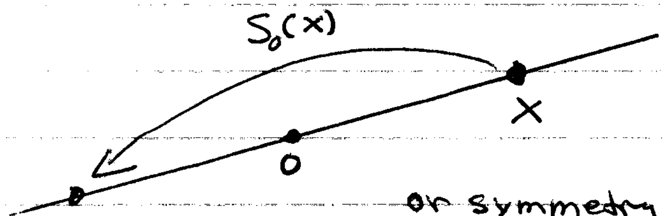
Harmonic Analysis on Symmetric Spaces

§1 Symmetric spaces

Let us start with two simple examples

a) $X = \mathbb{R}^n$. Define a map $s_0 : \mathbb{R}^n \to \mathbb{R}^n$ by

$$s_0(x) = -x$$



Thus s_0 is the reflection or symmetry around the point $0 \in \mathbb{R}^n$.

We can also describe s_0 in the following way:

Let $\gamma(t) = t\vec{v}$ be a line (geodesic) in $\mathbb{R}^n$. Then

$$s_0(\gamma(t)) = \gamma(-t).$$

There is – from the point of view of geometry – nothing special about the point $\vec{0}$, and we can do the same for any other point $\vec{y}$ instead of $\vec{0}$. So, let $\gamma(t) = \vec{y} + t\vec{v}$ be a line in $\mathbb{R}^n$, then the symmetry s_y around y is given by

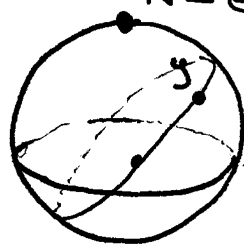
$$s_y(\gamma(t)) = \gamma(-t) = \vec{y} - t\vec{v}$$

We can also use

$$s_y(\vec{x}) = \underbrace{s_0(\underbrace{x-y}_{\text{Translating to } \vec{0}}) + y}_{\text{Translating back to } y} = 2y - x.$$

b) $S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$

$N = e_n$



~~We define first~~

Let $y \in S^{n-1}$. Let $x \in S^{n-1}$. Then there exists an unique $\underbrace{\text{big circle}}_{\text{geodesic}}$ γ containing y and x.

Write the circle as $\gamma(t)$, with $\gamma(1) = 1$. Then

$$s_y(x) = \gamma(-t).$$

<u>Definition</u> A manifold X is called locally symmetric if for all $y \in X$ there exist an open neighborhood U and a differentiable map $s_y : U \Rightarrow U$ such that $(ds_y)_y = -Id$.

X is a symmetric space if s_y extends to all of X for every $y \in X$.

§ 2 Lie groups

We will only use Lie groups that are closed subgroups of

$$GL(n, \mathbb{C}) = \{ g \mid g \; n \times n \text{ matrix}, \det g \neq 0 \}.$$

Examples:

- $SL(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) \mid \det g = 1\}$
- $U(n) = \{g \mid g^* g = I_n\}$ where

$$g^* = \overline{(g^t)}$$

or

$$(g_{ij})^* = (\overline{g_{ji}}).$$

Here $I_n = \mathrm{diag}(1, \dots, 1)$ is the identity matrix.

- $SU(n) = \{g \in U(n) \mid \det g = 1\}$

- $SO(n) = \{g \in SL(n,\mathbb{R}) \mid g^t = g^{-1}\}$

$$= SU(n) \cap SL(n,\mathbb{R}).$$

We will mainly be interested symmetric spaces of the form

$$G/K = X$$

where G is a Lie group. Furthermore, there exists an involution $\theta : G \to G$ $(\theta(ab) = \theta(a)\theta(b),\ \theta^2 = id)$ such that

$$(G^\theta)_o \subseteq K \subseteq G^\theta = \{g \in G \mid \theta(g) = g\}$$

where the subscript o stands for the connected component containing the identity element e.

Note that K is a closed subgroup, but in general <u>not</u> a normal subgroup. Define a map

$$\tilde{\theta} : X \to X$$

by

$$\tilde{\theta}(gK) = \theta(g)K$$

or

$$\begin{tikzcd} G \arrow[r, "\theta"] \arrow[d, "k"'] & G \arrow[d, "k"] \\ G/K \arrow[r, "\tilde{\theta}"] & G/K \end{tikzcd}$$

As $\theta(K) = K$ it follows that $\tilde{\theta}$ is well defined [If $gK = hK \Rightarrow \exists k \in K: gk = h \Rightarrow$ $\theta(h) = \theta(g)k$ and hence $\theta(h)K = \theta(g)K$.]

We also note that $\tilde{\theta}^2 = id$. Hence we can define $S_{x_0}(x) = \tilde{\theta}(x)$ $(x_0 = eK)$. Note, we still have to show that $(dS_{x_0})_{x_0} = id$!

~~For $y = gx_0$~~

Define for $g \in G$ a diffeomorphism $l_g : X \to X$ by

$$l_g(x) = g \cdot x \quad \text{or} \quad l_g(hK) = (gh)K.$$

Translation in the space X.

Let $y \in X$. Let $g \in G$ be such that
$$y = \ell_g(x_0)$$

Define $s_y(x) = \ell_g(s_{x_0}(\ell_{g^{-1}} x))$. It is easy to see that s_y is well defined, i.e. does not depend on whic g with $y = g x_0$, we use. Again $s_y^2 = \mathrm{id}$.

Examples

a) Let $G = \mathbb{R}^n \times SO(n)$ (as a set). We view elements of G as diffeomorphism (in fact affine linear maps) on $\mathbb{R}^n$ and use that to define the group multiplication. It then becomes

$$(x, g)(\xi) = g(\xi) + x$$

and

$$(x, g)(y, h)(\xi) = (x, g)(h(\xi) + y)$$

$$= gh(\xi) + g(y) + x$$
$$= (x + g(y), gh)(\xi)$$

In particular $(x, g)^{-1} = (-g^{-1}x, g^{-1})$. Thus G is the semidirect product $\mathbb{R}^n \rtimes SO(n)$. Note, the elements in G are exactly the orientpreseving maps that preserve also the distance
$$\|x - y\| = \|T(x) - T(y)\|$$

Define $\theta : G \to G$ by $\theta(x,g) = (-x,g)$. Then θ is an involution and $G^\theta = SO(n)$. Thus

$$\mathbb{R}^n = G/K$$

It is clear by construction that the symmetry introduced in the beginning is the one corresponding to θ.

b) For $p,q \in \mathbb{N}$, $p+q=n$ let

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

and note that $I_{p,q}^2 = I_n$. Define $\tau_{p,q} : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ by

$$\tau_{p,q}(g) = I_{p,q}\, g\, I_{p,q}.$$

Then $\tau_{p,q}$ is an involution. Write $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then

$$\tau_{p,q}(g) = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$$

Thus

$$GL^{\tau_{p,q}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \,\middle|\, A \in GL(p),\ B \in GL(q) \right\}$$

Let $G = SO(n)$. Then $\tau_{p,q}(G) = G$ and $\tau_{p,q}$ defines an involution on $SO(n)$. We have

$$SO(n)^{\tau_{p,q}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \,\middle|\, A \in O(p),\ B \in O(q),\ \det A \det B = 1 \right\}$$

$$= S(O(p) \times O(q)).$$

Let $G_{n,p}$ be the set of p-dimensional subspaces of $\mathbb{R}^n$. Let $x_0 = \{(x_1, \dots, x_p, 0, \dots, 0) \mid x_1, \dots, x_p \in \mathbb{R}\}$
$$= \sum_{j=1}^{p} \mathbb{R} e_j \in G_{n,p}.$$

a) Let $X \in G_{n,p}$. Let $f_1, \dots, f_p \in X$ be an orthonormal basis for X. Extend $f_1, \dots, f_p$ to an onb of $\mathbb{R}^n$, denoted by $f_1, \dots, f_n$. We may assume that $\{f_1, \dots, f_n\}$ is positively oriented. Define a linear map $A: \mathbb{R}^n \to \mathbb{R}^n$ by

$$A e_j = f_j$$

where $e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^n$. Then, as the colums of A are the vectors f_j we have

$$A^* A = (\langle f_i, f_j \rangle) = (\delta_{ij}) = I$$

As $\det A = 1$ we have $A \in SO(n)$. ~~Also~~ Furthermore $A x_0 = X$. Hence ~~A~~ $SO(n)$ acts transitively. It easy to see that

$$SO(n)^{x_0} = S(O(p) \times O(q)).$$

Hence

$$(*) \qquad G_{n,p} = SO(n) / \overbrace{S(O(p) \times O(q))}^{K}.$$

If $g = [f_1, \dots, f_n]$ (f_j the column vectors). Then the map in $(*)$ can be given as

$$gK \longmapsto \text{Subspace generated by } f_1, \dots, f_p$$
$$= \sum_{j=1}^{p} \mathbb{R} f_j.$$

$$= X[f_1, \dots, f_p].$$

The involution Θ is given by $f_j = \cancel{\$}\begin{bmatrix} f_{1j} \\ f_{2j} \\ \vdots \\ f_{nj} \end{bmatrix}$

$$\left[\begin{array}{ccc|c} f_{11} & \cdots & f_{1p} & \\ f_{21} & & \vdots & * \\ \vdots & & & \\ f_{p1} & & f_{pp} & \\ \hline -f_{p+1,1} & & -f_{p+1,p} & \\ \vdots & & \vdots & * \\ -f_{n,1} & & -f_{n,p} & \end{array}\right]$$

Thus, the symmetry around x_0 is given by

$$x = \{(x_1, \dots, x_p, x_{p+1}, \dots, x_n)\}$$
$$\mapsto \{(x_1, \dots, x_p, -x_{p+1}, \dots, -x_n)\} .$$

c) Take now $S^{n-1} = X$. Then $SO(n)$ acts transitively on X.
The stabilizer of $e_1 \in X$ is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \middle| A \in SO(n-1) \right\} \cong SO(n-1)$$

Thus $S^{n-1} = SO(n)/SO(n-1)$ and the map is given by

$$[f_1, \dots, f_n]\, SO(n) \mapsto f_1 \in S^{n-1}$$

As we have just seen, the symmetry is $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$ wich is exactly the same symmetry as considered earlier

d) H^+ or $\mathbb{C}^+ = \{ z = x+iy \in \mathbb{C} \mid y>0\}$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ define

$$g \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$
$$= \frac{(ax+b+iay)(cx+d-icy)}{|cz+d|^2}$$
$$= \frac{ac(x^2+y^2)+(ad+bc)x+bd}{|cz+d|^2} + i\, \frac{y}{|cz+d|^2}$$

Hence $\operatorname{Im}(g\cdot z) > 0$. It is easy to see that $(g_1 g_2)\cdot z = g_1(g_2\cdot z)$ and $I\cdot z = z$. Note that

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}\cdot i = \frac{ai + x}{a^{-1}} = a^2 i + x\cancel{a}$$

Hence

$$\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\cdot i = x + iy$$

so $SL(2,\mathbb{R})$ acts transitively on $\mathbb{C}^+$. Assume that

$$g\cdot i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} i = \frac{ai+b}{ci+d} = i$$

~~Then~~ This is if and only if

$$ai + b = -c + di$$

$$\Leftrightarrow \quad b + c = \cancel{\text{and}}\ d - a = 0$$

$$\Leftrightarrow \quad b = -c \text{ and } d = a$$

Thus

$$g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

As $\det g = a^2 + b^2 = 1$ it follows, that there exist a θ s.t.

$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2).$$

Thus $\mathbb{C}^+ \cong SL_2(\mathbb{R})/SO(2)$. Define $\theta : SL_2(\mathbb{R}) \to SL_2(\mathbb{R})$ by

$$\theta(g) = (g^{-1})^t$$

Then $SO(2) = SL_2(\mathbb{R})^\theta$. Thus, the symmetry around i is given by:

$$z = x+iy = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \cdot i$$

$$\mapsto \begin{pmatrix} \frac{1}{\sqrt{y}} & 0 \\ -\frac{x}{\sqrt{y}} & \sqrt{y} \end{pmatrix} \cdot i$$

$$= \frac{i/\sqrt{y}}{-i\, x/\sqrt{y} + \sqrt{y}} = \frac{-1}{\bar{z}}$$

$$= \frac{-x}{x^2+y^2} + i\,\frac{y}{x^2+y^2}.$$

e) Our final example is the space of positive definite matrices. Recall, A is positive definite if $A^t = A$ and for all $v \in \mathbb{R}^n, v \neq 0$,

$$(Av, v) > 0.$$

Define an action of $GL(n,\mathbb{R}) = G$ on $Sym^+(n,\mathbb{R})$ by

$$g\cdot X = gXg^t$$

Then the stabilizer of I_n is $SO(n)$. Let $X \in Sym^+(n,\mathbb{R})$ then there is $g \in GL(n,\mathbb{R})$ such that $X = gg^t = g\cdot I$

Hence the action is transitive.

[As X is symmetric it can by diagonalized. Thus, there is $k \in SO(n)$ such that

$$k^{-1}\cdot X = k^{-1}Xk = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = \operatorname{diag}(a_1, \dots, a_n).$$

As X is positive definite $a_j > 0$. Let

$$Y = k\cdot \operatorname{diag}(\sqrt{a_1}, \dots, \sqrt{a_n})$$

Then $Y^2 = X$. But $Y = Y^t$.

Let $\theta(g) = (g^{-1})^t$. Then $G^\theta = SO(n) = G^I = K$.
Hence $Sym^+(n,\mathbb{R})$ is a symmetric space and the symmetry is given by $X \mapsto X^{-1}$.

<u>Lemma</u> Let $g \in GL(n,\mathbb{R})$. Then there exists a unique $k \in SO(n)$ and $X \in Sym^+(n,\mathbb{R})$ such that
$$g = Xk.$$
Furthermore, there exists an unique symmetric matrix Y s.t.
$$X = \exp Y.$$

<u>Proof</u>. The first statement is just the polar decomposition of g, $g = \sqrt{gg^t}\, k$ for some $k \in SO(n)$. For the second part let
$$Y = k \operatorname{diag}(\log a_1, \ldots, \log a_n) k^{-1}$$
where k and a_j are as on page 10. ∎

§ Symmetric Lie algebras.

The Lie algebra of the linear Lie group $G \subseteq GL(n,\mathbb{C})$ is the set

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \forall t \in \mathbb{R} : e^{tX} \in G\}.$$

Here $\mathfrak{gl}(n,\mathbb{C}) = M_n(\mathbb{C})$ is the Lie algebra of $GL(n,\mathbb{C})$.

<u>Lemma</u> $\mathfrak{g}$ is a Lie algebra, i.e.

a) $\mathfrak{g}$ is a vectorspace (over $\mathbb{R}$)

b) $[X,Y] = XY - YX \in \mathfrak{g}$ if $X, Y \in \mathfrak{g}$.

Note that the Jacobi identity
$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$
holds. ∎

Note, $X \in \mathfrak{gl} \Rightarrow \det e^X = e^{\mathrm{Tr}(X)} > 0 \Rightarrow e^X \in GL$

Note also that we can view $\mathfrak{g}$ as the tangent space at $e = I$:

$$\mathfrak{g} = T_e G$$

Here are some simple facts:

Lemma There exists an open $V \subseteq \mathfrak{g}$, $0 \in V$, and open $U \subseteq G$, $e \in U$, such that

$$\exp : V \to U, \quad X \mapsto e^X,$$

is a diffeomorphism.

Proof. We note first that this is correct for $G = GL(n, \mathbb{R})$ because $D\exp = \mathrm{Id}$. ~~By~~ The claim the follows by the inverse function theorem.

As next step one shows:

Lemma Let $\mathfrak{g}_1, \dots, \mathfrak{g}_k \subseteq \mathfrak{gl}$ be subspaces such that $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k = \mathfrak{gl}(n)$. Then there exists open neighborhoods V_j of 0 in $\mathfrak{g}_j$ and open $e \in U \subset GL$ such that $\cancel{\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k \ni (X_1, \dots, X_k) \mapsto e^{X_1} \cdots e^{X_k}}$ C

$$V_1 \times \dots \times V_k \ni (X_1, \dots, X_k) \mapsto e^{X_1} \cdots e^{X_k} \in U$$

is a diffeomorphism.

Proof This follows again from the ~~implicit~~ inverse function theorem. ∎

Our Lemma now follows easily by taking ~~$\mathfrak{g}$~~ $k = 2$, $\mathfrak{g}_2 = \mathfrak{g}$ and $\mathfrak{g}_1$ as any complementare subspace of $\mathfrak{g}_2$ in $\mathfrak{gl}(n)$.

If $X \in \mathfrak{g}$ then we can view X as vector field on G by defining

$$X_g f := \frac{d}{dt} f(ge^{tX})\Big|_{t=0}$$

Define $l_g : G \to G$ by $x \mapsto gx$. Then l_g is a diffeomorphism with inverse $l_{g^{-1}}$. Hence

$$T_g G = (dl_g)_e T_e G = (dl_g)_e \mathfrak{g}$$

or

$$TG = G \times \mathfrak{g}.$$

<u>Def.</u> A vectorfield $X : G \Rightarrow TG$ is <u>left-invariant</u> if $X \circ l_g = dl_g \circ X$ for all $g \in G$.

Note, that this means that

$$X_{gh} = (dl_g)_h (X_h)$$

for all $g, h \in G$.

<u>Lemma</u> Denote by $\mathfrak{X}(G)^G$ the space of left invariant vectorfields ~~of~~ on G. Then

$$\mathfrak{X}(G)^G X \mapsto X_e \in \mathfrak{g}$$

is a linear isomorphism.

<u>Proof</u>: It is clear that the map is linear. If $X_e = 0$, then $X_g = (dl_g)_e (X_e) = 0$, so $X = 0$. Let $X \in \mathfrak{g}$ and define $X_g = (dl_g)_e (X)$ or

$$X_g f = \frac{d}{dt} f(ge^{tX})\Big|_{t=0}$$

as before. Then $X_e = X$ so the map is surjective. $\boxtimes$

Lemma Let G and H be Lie groups and $\varphi : G \to H$ a homomorphism. Then there exists an unique Lie algebra homomorphism $\dot{\varphi} : \mathfrak{g} \to \mathfrak{h}$ such that

$$\exp_H \dot{\varphi}(X) = \varphi(\exp_G X)$$

for all $X \in \mathfrak{g}$.

proof: φ is differentiable. Define $\dot{\varphi}(X) = \frac{d}{dt}\varphi(e^{tX})\Big|_{t=0}$

$$= D_e\varphi(e)X$$

Note that $\varphi(e^{(t_0+h)X}) = \varphi(e^{hX})\varphi(e^{t_0X})$. Hence

$$D\varphi(e^{t_0X})X = \varphi(e^{t_0X})D\varphi(e)X$$
$$= (D\varphi(e)X)\,\varphi(e^{t_0X}).$$

Define $\psi(t) = \varphi(e^{tX}) \Rightarrow \psi'(t) = (D\varphi(e)X)\psi(t)$, $\psi(0) = e$

$\Rightarrow \psi(t) = e^{tD\varphi(e)X} = e^{t\dot{\varphi}(X)}$ and the claim follows ∎

(we will next show that $\dot{\varphi}$ is a homomorphism)

Let $\tau : G \to G$ be a nontrivial involution. By the above there exists ~~an involution~~ homomorphism $\dot{\tau} : \mathfrak{g} \to \mathfrak{g}$ s.t.

$$\tau(e^X) = e^{\dot{\tau}X}$$

as $\tau^2(e^X) = e^X = \tau(e^{\dot{\tau}X}) = e^{\dot{\tau}^2X}$ and exp is a local diffeomorphism it follows that $\dot{\tau}^2 = \mathrm{id}$.

It follows that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$$

where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \dot{\tau}X = X\} \quad (+1\text{-eigenspace})$$
$$\mathfrak{q} = \{X \in \mathfrak{g} \mid \dot{\tau}X = -X\} \quad (-1 \text{ eigenspace})$$

<u>Lemma</u> $\mathfrak{k}$ is the Lie algebra of $K = G^{\tau}$ (or $(G^{\tau})_o$.

<u>Proof.</u> Let $X \in \mathfrak{k} \Rightarrow \tau(e^{tX}) = e^{t\dot{\tau}(X)} = e^{tX}$. Hence $e^{tX} \in K$ for all $t \in \mathbb{R}$ or $X \in \mathrm{Lie}(K)$. Similarly, if $\tau(e^{tX}) = e^{tX}$ for all $t \in \mathbb{R}$ ($e^{tX} \in K$ for all t), then

$$e^{t\dot{\tau}(X)} = e^{tX}$$

Differentiate at $t=0 \Rightarrow \dot{\tau}(X) = X$. $\boxtimes$

<u>Lemma</u> There exist a neighborhood of $\mathcal{U} \subseteq \mathfrak{q}$ and $V \subseteq G/K$, $eK \in V$ such that the map

$$\mathrm{Exp} : \mathcal{U} \to V$$
$$X \mapsto e^X \cdot K$$

is a diffeomaphism.

<u>Proof.</u> Let $\tilde{\mathcal{U}} \subseteq \mathfrak{g}$ be such that $\exp : 2\tilde{\mathcal{U}} \to \exp 2\tilde{\mathcal{U}}$ is a diffeomaphism. Let $\mathcal{U} \subseteq \mathfrak{q}$ and $\mathcal{U}_1 \subseteq \mathfrak{k}$ be such that $\mathcal{U} + \mathcal{U}_1 \subseteq \tilde{\mathcal{U}}$ and such that

$$\mathcal{U} \times \mathcal{U}_1 \ni (X, Y) \mapsto e^X e^Y \in G$$

is a diffeomorphism onto its image $\tilde{V}$. Let $V = \tilde{V}K = \kappa(\tilde{V})$ open. Then $\mathrm{Exp} : \mathcal{U} \to V$ is surjective. Assume that $X, Y \in \mathcal{U}$ are such that

$$e^X \cdot K = e^Y K$$

$\Rightarrow \exists k \in K : \; e^X k = e^Y$. Apply τ to both sides

$$e^{-X} k = e^{-Y}.$$ Thus

$e^{2Y} = e^Y \tau(e^Y) = e^X k k^{-1} e^{+X} = e^{2X}$. But then $X = Y$ $\boxtimes$

It follows that with $x_0 = eK$, we have

$$\mathfrak{p} \simeq T_{x_0} G/K$$

and each $X \in$ ~~$\mathfrak{g}$~~ $\mathfrak{p}$ defines a derivative of $C^\infty(G/K)$ ~~by~~ at the point x_0 by

$$X_0 f = \frac{d}{dt} f(e^{tX} \cdot x_0)\Big|_{t=0}$$

If we - again - define $l_g : G/K \to G/K$ by $l_g(aK) = (ga)K$, then

$$dl_g(X_0) f = \frac{d}{dt} f(g e^{tX} \cdot x_0)\Big|_{t=0}$$

<u>Exercise</u> Show that $T(G/K) \simeq G \times_K \mathfrak{p}$.

<u>Ex</u> $GL_n/SO(n)$

$\mathfrak{p} \simeq GL_n/SO(n)$

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~~§3 Riemannian~~

<u>Corollary</u> The involution $\tilde{\Theta} : G/K \to G/K$, $gK \mapsto \Theta(g)K$ makes G/K into a symmetric space.

~~§3 Riemannian Symmetric spaces~~

§ 3 Semisimple symmetric spaces

We will mainly be interested in connected Riemannian symmetric spaces of the form G/K. It turns out, that up to coverings and products they are (depending on the curvature)

- U/K $\quad U$ compact, $K = (G^\theta)_0$ for an involution θ on U (curvature >0)
- $\mathbb{R}^n$ (curvature $=0$)
- G/K where G is a semisimple Lie group and K a maximal compact subgroup.

Def. Let $G \subseteq GL(n,\mathbb{C})$. Then

1) G is reductive if there exists a $g \in GL(n,\mathbb{C})$ such that gGg^{-1} is invariant under the anti-homomorphism
$$X \mapsto X^*$$
2) G is semisimple if $\dim G > 1$ and the center of G is discrete.

Ex $GL(n,\mathbb{R})$ is reductive but not semisimple because the group
$$\mathbb{R}^+ I \subseteq Z(G(n,\mathbb{R}))$$
On the other hand the groups $SL(n,\mathbb{R})$ and $SO(n,\mathbb{R})$ are semisimple. $\boxtimes$

~~Let G~~

From now on we will assume that $G^* = G$ if G is reductive (semisimple). Define $\theta(g) = (g^*)^{-1}$. Note that $\theta|_{U(n)} = \mathrm{id}$, and we always have $\dot{\theta}(X) = X^*$.

Lemma If G is reductive, then $\mathfrak{g}^* = \mathfrak{g}$. $\boxtimes$

Lemma Let $K \subseteq GL(n,\mathbb{C})$. Then there exist a $g \in GL(n,\mathbb{C})$, g positive definite, such that

$$gKg^{-1} \subseteq U(n).$$

Proof. Let $(\cdot,\cdot)$ denote the standard inner product on $\mathbb{C}^n$. Denote by dk a normalized Haar measure on K $(\int_K dk = 1)$. Define a new inner product on $\mathbb{C}^n$ by

$$(v,w)^\sim = \int_K (kv, kw)\,dk$$

Then $(\,,\,)^\sim$ is K-invariant, i.e. for all $\tilde{k} \in K$, $v,w \in \mathbb{C}^n$ we have

$$(\tilde{k}v, \tilde{k}w)^\sim = \int_K (k\tilde{k}v, k\tilde{k}w)\,dk$$

$$= \int_K (kv, kw)\,dk$$

$$= (v,w)^\sim.$$

As $(\,,\,)^\sim$ is an inner product, there exist a positive definite A such that

$$(Av,w) = (v,w)^\sim$$

for all $v,w \in \mathbb{C}^n$. Let $k \in K$. Then

$$\begin{aligned}(Av,w) &= (v,w)^\sim \\ &= (kv,kw)^\sim \\ &= (Akv,kw) \\ &= (k^*Akv,w)\end{aligned}$$

$\Rightarrow$ $A = k^*Ak$ or $k^* = Ak^{-1}A^{-1}$. Let $g = \sqrt{A}$.

Then
$$\begin{aligned}(gkg^{-1})^* &= (g^{-1})^* k^* g^* \\ &= g^{-1}Ak^{-1}A^{-1}g \\ &= g\,k^{-1}g^{-1} \\ &= (gkg^{-1})^{-1}\end{aligned}$$

or $gKg^{-1} \subseteq U(n)$ $\boxtimes$