MATH 2065-4, Fall 2010 Name: SOLUTION

Test 1, Thursday, Sept 23, 2010

. For partial credit, show all your work!

1[6P]) Which of the following functions $y(t)$ is a solution (**S**)/not a solution (**N**) to the differential equation $y' = y - t$?

$y(t)$	Y	N
$t+1$	✓	
e^t+1+t	✓	
$2e^t-t-1$		✓

$y=t+1,\ y'=1,\ y-t=t+1-t=1=y'$

$y=e^t+t+1,\ y'=e^t+1,\ y-t=e^t+1$

$y=2e^t-t-1,\ y'=2e^t-1,\ y-t=2e^t-2t-1 \neq y'$

2[9P]) Determine and mark with **Y** for yes, and **N** for no, if each of the following differential equation is separable (**S**), linear (**L**), and/or homogeneous (**H**). Note, that in each case, more than one might be correct.

$y' = \frac{y-t}{y+t} = \frac{(y/t)-1}{(y/t)+1} \left(= \frac{v-1}{v+1}\right)$ H.

Equation	S	L	H
$y' = \dfrac{y-t}{y+t}$			✓
$y' - ty = t^3y$	✓	✓	
$(y^2+2t^2) + 2tyy' = 0$			✓

$y' = t^3y + ty = (t^3+t)y$

separable

$y' - (t+t^3)y = 0$, linear

standard form

$y' = \frac{y^2+2t^2}{2ty} = \frac{\left(\frac{y}{t}\right)^2+2}{2(y/t)}$ homogeneous

3) Solve the following differential equation:

a[10P]) $y' = -2yt$. **Solution** $y(t) =$ Ce^{-t^2}

you can use that this is separable and also linear $y'+2ty=0$.

separable

$$\frac{dy}{y} = -2t\,dt,\quad \ln|y| = -t^2 - c$$

$$y = Ce^{-t^2}$$

b[10P]) $y' - y = ty^2$. Solution $y(t) = \dfrac{1}{-t+1+Ce^{-t}}$

Bernoulli equation with $n=2$, $y^{-2}y' - \frac{1}{y} = t$.

$z = 1/y$, $z' = -\frac{1}{y^2}y'$

$z' + z = -t$, $p=1$, $P(t)=t$, $\mu(t) = e^t$

$-\int te^t dt = -te^t + e^t + C$, $z = \frac{1}{\mu}(-te^t+e^t+C) = -t+1+Ce^{-t}$

$y = \frac{1}{z} = \dfrac{1}{-t+1+Ce^{-t}}$

5) Solve the following initial value problems.

a[15P]) $y' - \frac{3}{t}y = t^3e^t$, $y(1) = 3$. Solution $y(t) = t^3(e^t + 3 - e)$

Linear: $\mu = e^{-3\ln t} = t^{-3}$

$\int e^t dt = e^t + C$

$y(t) = t^3(e^t + C)$

$y(1) = e + C = 3$, $C = 3 - e$

b[15P]) $t^2y' = -y^2 + yt$, $y(1) = 1$. Solution $y(t) = \dfrac{t}{\ln(t)+1}$

This is a homogeneous equation

$y' = -\left(\frac{y}{t}\right)^2 + \frac{y}{t} = -v^2 + v$ if $v = \frac{y}{t}$

$tv' + v = -v^2 + v$ or $tv' = -v^2$

$-\frac{dv}{v^2} = \frac{dt}{t}$, $\frac{1}{v} = \ln(t) + C \; (t>0)$

$v = \dfrac{1}{\ln(t)+C}$, $y = tv = \dfrac{t}{\ln(t)+C}$

$y(1) = 1 = \frac{1}{C}$ so $C = 1$

3) A tank contains 100 gal of brine made by dissolving 40 lb of salt in water. Pure water runs into the tank at the rate of 4 gal/min, and the mixture. which is kept uniform by stirring, runs out at the same rate. Denote by $y(t)$ the amount of salt in the tank at time t.

a[8P])) Write an initial value problem that $y(t)$ must satisfy.

Solution: $y' = -\frac{1}{25}y$ and $y(0) = 40$

y' = input − output. There is no salt in the input so, output $\frac{4}{100}y(t)$

$y' = -\frac{1}{25}y$. It is given, that at time $t=0$ the amount of salt is 40lb.

b[10P]) Solve the initial value problem. **Solution:** $y(t) = 40e^{-\frac{t}{25}}$

$$\frac{dy}{y} = -\frac{dt}{25}, \quad \ln y = -\frac{t}{25} + C \quad (y>0)$$

$$y = Ce^{-t/25}. \quad \text{Taking } t=0,\ y(0) = 40 = C$$

c[5P]) How much salt is in the tank after 10 min? **Solution:** $40e^{-\frac{2}{5}} \approx 26.8$

$$y(10) = 40e^{-\frac{10}{25}} = 40e^{-2/5} \approx 26.8 \text{ lb}$$

4[12P]) Apply Picard's to compute the approximations $y_0(t)$, $y_1(t)$, and $y_2(t)$ to the solution of the initial value problem $y' = y^2 + 1, \quad y(0) = 0$.

$$y_0 = 0$$

$$y_1 = 0 + \int_0^t 0^2 + 1\, du = \int_0^t du = t$$

$$y_2 = 0 + \int_0^t u^2 + 1\, du = \frac{1}{3}t^3 + t.$$