MATH 2065-4, Fall 2010 Name: Solutions

Test 2, Thursday, Sept 26, 2010. For partial credit, show all your work!

1[24P]) Compute the Laplace transform of each of the following functions:

a) $\mathcal{L}\left(t^3 e^{-2t}\right)(s) = \dfrac{6}{(S+2)^4}$

b) $\mathcal{L}\left(t\sin(4t) + e^{-3t}\cos(t)\right)(s) = \dfrac{8s}{(s^2+16)^2} + \dfrac{S+3}{(S+3)^2+1}$

c) $\mathcal{L}\left((2te^t+1)(t^3+2)\right)(s) = \dfrac{48}{(s-1)^5} + \dfrac{6}{s^4} + \dfrac{4}{(s-1)^2} + \dfrac{2}{s}$

2[24P]) Find the partial fraction decomposition of each of the following rational functions. You may use the convolution formula if you prefer.

a) $\dfrac{7s+9}{(s-1)(s+3)} = \dfrac{4}{s-1} + \dfrac{3}{s+3}$

Set

$$\frac{7s+9}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}.$$ Then

$$(A+B)s + 3A - B = 7s + 9$$ or

$$A+B = 7$$
$$3A - B = 9$$
$$4A = 16$$ or $A = 16/4 = 4$

$$B = 7 - A = 7 - \cancel{16}\,4 = \underline{3}$$

b) $\frac{1}{(s-1)(s^2+1)} = \frac{1}{2}\left[\frac{1}{s-1} - \frac{s+1}{s^2+1}\right]$

write

$\frac{1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+1}{s^2+1}$. Then $\begin{cases} A+B=0 \\ -B+C=0 \\ A-\cancel{B}=1 \end{cases}$

Thus $B=C$ and $\begin{matrix} A+B=0 \\ A-B=1 \end{matrix}$ or $A=1/2$, $B=C=-1/2$.

c) $\frac{1}{s(s-1)} = -\frac{1}{s} + \frac{1}{s-1}$

3[16P]) Evaluate the following convolutions. You may use the Laplace transform if you prefer.

a) $t^2 * t = \frac{1}{12}t^4$.

Use the Laplace transform: $\mathcal{L}(t^2 * t) = \frac{2}{s^3}\cdot\frac{1}{s^2} = \frac{2}{s^5}$

$= \frac{1}{12}\cdot\frac{4!}{s^5}$

b) $t * e^{-4t} = \frac{1}{4}t - \frac{1}{16} + \frac{1}{16}e^{-4t}$

$\int_0^t u e^{-4(t-u)}\,du = e^{-4t}\left[\int_0^t u e^{4u}\,du\right] = e^{-4t}\left[\frac{1}{4}ue^{4u}\Big|_0^t - \frac{1}{4}\int_0^t e^{4u}\,du\right] = e^{-4t}\left[\frac{1}{4}te^{4t} - \frac{1}{16}e^{4u}\Big|_0^t\right] = \frac{1}{4}t - \frac{1}{16} + \frac{1}{16}e^{-4t}$

3[36P]) Compute the inverse Laplace transform for each of the following functions:

a) $\mathcal{L}^{-1}\left(\frac{5}{s^2+6s+9}\right) = 5te^{-3t}$

b) $\mathcal{L}^{-1}\left(\frac{2}{(s-1)(s^2+1)}\right) = e^{t} - \cos t - \sin t$

$$\frac{2}{(s-1)(s^2+1)} = -\frac{s+1}{s^2+1} + \frac{1}{s-1} = \frac{-s}{s^2+1} - \frac{1}{s^2+1} + \frac{1}{s-1}$$

(use problem #2-b)

c) $\mathcal{L}^{-1}\left(\frac{s}{s^2+4s+5}\right)(t) = e^{-2t}\left[\cos t - \sin t\right]$

$$\frac{s}{s^2+4s+5} = \frac{s}{(s+2)^2+1} = \frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}$$

d) $\mathcal{L}^{-1}\left(\frac{s+1}{((s+1)^2+1)^2}\right) = \frac{te^{-t}\sin t}{2}$

(use the last formula in the table)

A short table of Laplace transforms and inverse Laplace transform

$$
\begin{aligned}
\mathcal{L}(af(t)+bg(t))(s) &= aF(s)+bG(s)\\
\mathcal{L}(e^{at}f(t))(s) &= F(s-a)\\
\mathcal{L}(-tf(t))(s) &= \frac{d}{ds}F(s)\\
\mathcal{L}(1)(s) &= \frac{1}{2}\\
\mathcal{L}(t^n)(s) &= \frac{n!}{s^{n+1}}\\
\mathcal{L}(e^{at})(s) &= \frac{1}{s-a}\\
\mathcal{L}(\cos(bt))(s) &= \frac{s}{s^2+b^2}\\
\mathcal{L}(\sin(bt))(s) &= \frac{b}{s^2+b^2}\\
\mathcal{L}(f'(t))(s) &= sF(s)-f(0)\\
\mathcal{L}(f*g(t))(s) &= F(s)G(s)\\
\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)(t) &= \frac{1}{2}(\sin(t)-t\cos(t))\\
\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right)(t) &= \frac{1}{2}t\sin(t)
\end{aligned}
$$