Math 2057, Section 5

Test #1 is on Tuesday, Sept. 27. Material: Section 14.1–14.5, everything except partial differential equations.

Material covered until Sept. 13–22

Section 14.4: Tangent Planes and Linear Approximations.

- For function of one variable: Recall that the tangent line at the point \((a, b)\) is given by \(y - b = f'(a)(x - a)\).
- This can also be read as linear approximation 
  \[ f(x) \sim b + f'(a)(x - a). \]
- In two variables we need to replace line by plane. The equation of a plane, containing the point \((x_0, y_0, z_0)\) in three dimensions is given by

  \[ A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \]

  If \(C \neq 0\) then we can solve for \(z - z_0\) and write 
  \[ z - z_0 = a(x - x_0) + b(y - y_0). \]

**Definition 0.1.** Suppose the function \(f\) has continuous partial derivatives. An equation of the tangent plane to the surface \(z = f(x, y)\) at the point \(P(x_0, y_0, z_0)\) is given by

  \[ z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \]

The tangent plane at the point \((x_0, y_0, f(x_0, y_0))\) is close to the graph of the function \(f(x, y)\) as long as \((x, y)\) is close to \((x_0, y_0)\).

We therefore call the function

\[ (x, y) \mapsto z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]

the linear approximation to \(f(x, y)\).

The change in \(z\) is

\[ \Delta z = z - z_0 = f(x, y) - f(a, b). \]

The differential is the change in the linear approximation and is given by

\[ dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \]
Note also the definition of differentiable on p. 926 and the similar definition for functions of more than two variables (p. 929).

**Exercises from Section 14.4:** 1–5, 11–19, 29–33 odd.

Section 14.5 The chain rule:

Recall first the chain rule in one variable: If \( y \) is a function of the variable \( u \) and \( u \) is a function of \( x \), then \( y(u) \) depends on \( x \) and the derivative with respect to \( x \) is given by:

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

or

\[
\begin{align*}
x & \rightarrow \frac{du}{dx} \rightarrow u \\
& \rightarrow \frac{dy}{du} \rightarrow y
\end{align*}
\]

We can have similar situation in several variables.

**Case 1** \( z \) depends on \( x \) and \( y \), and \( x \) and \( y \) depend on the variable \( t \). Then

\[
z(x, y) = z(x(t), y(t))
\]

depends only on the variable \( t \). If \( z \) is differentiable then we get

\[
\Delta z = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y
\]

where \( \epsilon_1, \epsilon_2 \to 0 \) if \( \Delta x, \Delta y \to 0 \). Now, inserting for \( \Delta x \) and \( \Delta y \) (if differentiable) we get

\[
\Delta x = \frac{dx}{dt} \Delta t + \epsilon_2 \Delta t
\]

and

\[
\Delta y = \frac{dy}{dt} \Delta t + \epsilon_2 \Delta t.
\]

Dividing by \( \Delta t \) and taking the limit \( \Delta t \to 0 \) we get

\[
\frac{dz}{dt} = \lim_{t \to 0} \frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.
\]

**Case 2** If \( z \) depends on \( x \) and \( y \) and \( x \) and \( y \) depend on two variables \( s \) and \( t \), \( z(x, y) = z(x(s, t), y(s, t)) \) depends on \( s \) and \( t \) and we have
\[ \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}, \]
\[ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}. \]

**Case 3, the general case:** If \( z \) depends on the variables \( x_1, \ldots, x_n \) and each of the variables \( x_j \) depends on \( t_1, \ldots, t_m \). Then we have for each \( j = 1, \ldots, m \):
\[ \frac{\partial z}{\partial t_j} = \sum_{k=1}^{n} \frac{\partial z}{\partial x_k} \frac{\partial x_k}{\partial t_j}. \]

**Implicit differentiation:** The book lists two forms of this. Assume that the function \( F \) is differentiable and that \( F(a, b) = 0 \) and \( F_y(a, b) \neq 0 \). Then we can (in principle) solve the equation \( F(x, y) = 0 \) for \( y \) around \( x = a \) such that \( y(a) = b \) to define \( y \) as a function of \( x \). Note, that in most cases it is impossible to write an explicit formula for the function \( y \). In this case the function \( y \) is differentiable and we have
\[ \frac{dy}{dx} = -\frac{F_x}{F_y}. \]

If \( F \) depends on three variables \( x, y, z \) and \( F_z \neq 0 \). Then we can (in principle) solve for \( z \) (depending on \( x \) and \( y \)) and we get:
\[ \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \]
\[ \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}. \]

**Exercises from Section 14.5:** 1–11, 19–25 odd and 27, 31, and 43.
We did discuss Section 4.6, Directional Derivatives and the gradient vector on Thursday, Sept. 22. We will discuss that material next time.