Math 2057, Section 5

Test #1 is on Tuesday, Sept. 27. Material: Section 14.1–14.5, everything except partial differential equations.

Material covered until Sept. 13–22

Section 14.4: Tangent Planes and Linear Approximations.

- For function of one variable: Recall that the tangent line at the point (a, b) is given by $y - b = f'(a)(x - a)$.
- This can also be read **as linear approximation**

$$f(x) \sim b + f'(a)(x - a)\,.$$

- In two variables we need to replace **line** by **plane**. The equation of a plane, containing the point (x_0, y_0, z_0) in three dimensions is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\,.$$

  If $C \neq 0$ then we can solve for $z - z_0$ and write

$$z - z_0 = a(x - x_0) + b(y - y_0)\,.$$

Definition 0.1. Suppose the function f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)\,.$$

The tangent plane at the point $(x_0, y_0, f(x_0, y_0))$ is close to the graph of the function $f(x, y)$ as long as (x, y) is close to (x_0, y_0).
We therefore call the function

$$(x, y) \mapsto z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

the linear approximation to $f(x, y)$.
The change in z is

$$\Delta z = z - z_0 = f(x, y) - f(a, b)\,.$$

The differential is the change in the linear approximation and is given by

$$dz = \partial f/\partial x dx + \partial f/\partial y\,.$$

Note also the definition of differentiable on p. 926 and the similar definition for functions of more than two variables (p. 929).

Excersises from Section 14.4: 1–5, 11–19, 29–33 odd.
Section 14.5 The chain rule:

Recall first the chain rule in one variable: If y is a function of the variable u and u is a function of x, then $y(u)$ depends on x and the derivative with respect to x is given by:

$$\frac{d}{dx}y = \frac{dy}{du} \cdot \frac{du}{dx}$$

or

$$x \xrightarrow{du/dx} u \xrightarrow{dy/du} y$$

$$x \xrightarrow{\frac{du}{dx} = \frac{du}{du}\frac{du}{dx}} y$$

We can have similar situation in several variables.
Case 1 z depends on x and y, and x and y depend on the variable t. Then

$$z(x, y) = z(x(t), y(t))$$

depends only on the variable t. If z is differentiable then we get

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1, \epsilon_2 \to 0$ if $\Delta x, \Delta y \to 0$. Now, inserting for Δx and Δy (if differentiable) we get

$$\Delta x = \frac{dx}{dt}\Delta t + \epsilon_2 \Delta t$$

and

$$\Delta y = \frac{dy}{dt}\Delta t + \epsilon_2 \Delta t\,.$$

Dividing by Δt and taking the limit $\Delta t \to 0$ we get

$$\frac{dz}{dt} = \lim_{t \to 0} \frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\,.$$

Case 2 If z depends on x and y and x and y depend on two variables s and t, $z(x, y) = z(x(s, t), y(s, t))$ depends on s and t and we have

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\,.\end{aligned}$$

Case 3, the general case: If z depends on the variables $x_1, \ldots, x_n$ and each of the variables x_j depends on $t_1, \ldots, t_m$. Then we have for each $j = 1, \ldots, m$:

$$\frac{\partial z}{\partial t_j} = \sum_{k=1}^{n} \frac{\partial z}{\partial x_k}\frac{\partial x_k}{\partial t_j}\,.$$

Implicit differentiation: The book list two forms of this.
Asssume that the function F is differentiable and that $F(a,b) = 0$ and $F_y(a,b) \neq 0$. Then we can (in principle) solve the equation $F(x,y) = 0$ for y around $x = a$ such that $y(a) = b$ to define y as a function of x. Note, that in most cases it is impossible to write an explict formula for the function y. In this case the function y is differentiable and we have

$$\frac{dy}{dx} = -\frac{F_x}{F_x}\,.$$

If F depends on three variables x, y, z and $F_z \neq 0$. Then we can (in principle) solve for z (depending on x and y) and we get:

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z}\\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z}\,.\end{aligned}$$

Excersises from Section 14. 5: 1–11, 19–25 odd and 27,31, and 43.
We did discuss Section 4.6, Directional Derivatives and the gradient vector on Thursday, Sept. 22. We will discuss that material next time.