Integral Transforms
Math 2025

Gestur Olafsson

Mathematics Department

Louisiana State University

Chapter 1

Vector Spaces over $\mathbb{R}$

Definition. vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

Definition. vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- A1 (Closure of addition)
  For all $u, v \in V, u + v$ is defined and $u + v \in V$.

Vector Spaces over $\mathbb{R}$

Definition. vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- A1 (Closure of addition)
  For all $u, v \in V, u + v$ is defined and $u + v \in V$.
- A2 (Commutativity for addition)
  $u + v = v + u$ for all $u, v \in V$.

Definition. vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- A1 (Closure of addition)
  For all $u, v \in V, u + v$ is defined and $u + v \in V$.
- A2 (Commutativity for addition)
  $u + v = v + u$ for all $u, v \in V$.
- A3 (Associativity for addition)
  $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.

Definition. vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- A1 (Closure of addition)
  For all $u, v \in V, u + v$ is defined and $u + v \in V$.
- A2 (Commutativity for addition)
  $u + v = v + u$ for all $u, v \in V$.
- A3 (Associativity for addition)
  $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.
- A4 (Existence of additive identity)
  There exists an element $\vec{0}$ such that $u + \vec{0} = u$ for all $u \in V$.

Vector Spaces over $\mathbb{R}$

Definition. vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- A1 (Closure of addition)
  For all $u, v \in V$, $u + v$ is defined and $u + v \in V$.
- A2 (Commutativity for addition)
  $u + v = v + u$ for all $u, v \in V$.
- A3 (Associativity for addition)
  $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.
- A4 (Existence of additive identity)
  There exists an element $\vec{0}$ such that $u + \vec{0} = u$ for all $u \in V$.
- A5 (Existence of additive inverse)
  For each $u \in V$, there exists an element -denoted by $-u$- such that $u + (-u) = \vec{0}$.

Definition. A vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

Definition. A vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- M1 (Closure for scalar multiplication)
  For each number r and each $u \in V$, $r \cdot u$ is defined and $r \cdot u \in V$.

Definition. A vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- M1 (Closure for scalar multiplication)
  For each number r and each $u \in V$, $r \cdot u$ is defined and $r \cdot u \in V$.
- M2 (Multiplication by 1)
  $1 \cdot u = u$ for all $u \in V$.

Definition. A vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- M1 (Closure for scalar multiplication)
  For each number r and each $u \in V$, $r \cdot u$ is defined and $r \cdot u \in V$.
- M2 (Multiplication by 1)
  $1 \cdot u = u$ for all $u \in V$.
- M3 (Associativity for multiplication)
  $r \cdot (s \cdot u) = (r \cdot s) \cdot u$ for $r, s \in \mathbb{R}$ and all $u \in V$.

Definition. A vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- M1 (Closure for scalar multiplication)
  For each number r and each $u \in V$, $r \cdot u$ is defined and $r \cdot u \in V$.
- M2 (Multiplication by 1)
  $1 \cdot u = u$ for all $u \in V$.
- M3 (Associativity for multiplication)
  $r \cdot (s \cdot u) = (r \cdot s) \cdot u$ for $r, s \in \mathbb{R}$ and all $u \in V$.
- D1 (First distributive property)
  $r \cdot (u + v) = r \cdot u + r \cdot v$ for all $r \in \mathbb{R}$ and all $u, v \in V$.

Definition. A vector space over $\mathbb{R}$ is a set V with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- M1 (Closure for scalar multiplication)
  For each number r and each $u \in V$, $r \cdot u$ is defined and $r \cdot u \in V$.
- M2 (Multiplication by 1)
  $1 \cdot u = u$ for all $u \in V$.
- M3 (Associativity for multiplication)
  $r \cdot (s \cdot u) = (r \cdot s) \cdot u$ for $r, s \in \mathbb{R}$ and all $u \in V$.

- D1 (First distributive property)
  $r \cdot (u + v) = r \cdot u + r \cdot v$ for all $r \in \mathbb{R}$ and all $u, v \in V$.
- D2 (Second distributive property)
  $(r + s) \cdot u = r \cdot u + s \cdot u$ for all $r, s \in \mathbb{R}$ and all $u \in V$.

Some immediate results

Remark. The zero element $\vec{0}$ is unique, i.e., if $\vec{0_1}, \vec{0_2} \in V$ are such that

$$u + \vec{0_1} = u + \vec{0_2} = u, \forall u \in V$$

then $\vec{0_1} = \vec{0_2}$.

Some immediate results

Remark. The zero element $\vec{0}$ is unique, i.e., if $\vec{0_1}, \vec{0_2} \in V$ are such that

$$u + \vec{0_1} = u + \vec{0_2} = u, \forall u \in V$$

then $\vec{0_1} = \vec{0_2}$.

Proof. We have $\vec{0_1} = \vec{0_1} + \vec{0_2} = \vec{0_2} + \vec{0_1} = \vec{0_2}$ □

Some immediate results

Remark. The zero element $\vec{0}$ is unique, i.e., if $\vec{0}_1, \vec{0}_2 \in V$ are such that

$$u + \vec{0}_1 = u + \vec{0}_2 = u, \forall u \in V$$

then $\vec{0}_1 = \vec{0}_2$.

Proof. We have $\vec{0}_1 = \vec{0}_1 + \vec{0}_2 = \vec{0}_2 + \vec{0}_1 = \vec{0}_2$ □

Lemma. *Let $u \in V$, then $0 \cdot u = \vec{0}$.*

Some immediate results

Remark. The zero element $\vec{0}$ is unique, i.e., if $\vec{0_1}, \vec{0_2} \in V$ are such that

$$u + \vec{0_1} = u + \vec{0_2} = u, \forall u \in V$$

then $\vec{0_1} = \vec{0_2}$.

Proof. We have $\vec{0_1} = \vec{0_1} + \vec{0_2} = \vec{0_2} + \vec{0_1} = \vec{0_2}$ □

Lemma. *Let $u \in V$, then $0 \cdot u = \vec{0}$.*

Proof.

$$\begin{aligned} u + 0 \cdot u &= 1 \cdot u + 0 \cdot u \\ &= (1+0) \cdot u \\ &= 1 \cdot u \\ &= u \end{aligned}$$

Remark. The zero element $\vec{0}$ is unique, i.e., if $\vec{0}_1, \vec{0}_2 \in V$ are such that

$$u + \vec{0}_1 = u + \vec{0}_2 = u, \forall u \in V$$

then $\vec{0}_1 = \vec{0}_2$.

Proof. We have $\vec{0}_1 = \vec{0}_1 + \vec{0}_2 = \vec{0}_2 + \vec{0}_1 = \vec{0}_2$ □

Lemma. *Let $u \in V$, then $0 \cdot u = \vec{0}$.*

Proof.

Thus

$$\begin{aligned} \vec{0} = u + (-u) &= (0 \cdot u + u) + (-u) \\ &= 0 \cdot u + (u + (-u)) \\ &= 0 \cdot u + \vec{0} \\ &= 0 \cdot u \end{aligned}$$

□

Lemma. *a) The element $-u$ is unique.*

b) $-u = (-1) \cdot u$*.*

Lemma. *a) The element $-u$ is unique.*

b) $-u = (-1) \cdot u$.

Proof of part (b).

$$\begin{aligned} u + (-1) \cdot u &= 1 \cdot u + (-1) \cdot u \\ &= (1 + (-1)) \cdot u \\ &= 0 \cdot u \\ &= \vec{0} \end{aligned}$$

□

Examples

$\mathbb{R}^n$ as column vectors

Before examining the axioms in more detail, let us discuss two examples.

$\mathbb{R}^n$ as column vectors

Example. *Let* $V = \mathbb{R}^n$ *,considered as column vectors*

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}$$

$\mathbb{R}^n$ as column vectors

Example. *Let* $V = \mathbb{R}^n$ *.Then for*

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \textit{and} \quad r \in \mathbb{R} :$$

$\mathbb{R}^n$ as column vectors

Example. *Let* $V = \mathbb{R}^n$ *.Then for*

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \textit{and} \quad r \in \mathbb{R} :$$

Define

$$u + v = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \quad \textit{and} \quad r \cdot u = \begin{pmatrix} rx_1 \\ \vdots \\ rx_n \end{pmatrix}$$

$\mathbb{R}^n$ as column vectors

Example. *Let $V = \mathbb{R}^n$ .Then for*

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \text{ and } r \in \mathbb{R} :$$

Note that the zero vector and the additive inverse of u are given by:

$$\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad -u = \begin{pmatrix} -x_1 \\ \vdots \\ -x_2 \end{pmatrix}$$

$\mathbb{R}^n$ as row vectors

Remark. $\mathbb{R}^n$ can be considered as the space of all row vectors.

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}$$

$\mathbb{R}^n$ as row vectors

Remark. $\mathbb{R}^n$ can be considered as the space of all row vectors.

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}$$

The addition and scalar multiplication is again given coordinate wise

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

$$r \cdot (x_1, \ldots, x_n) = (rx_1, \ldots, rx_n)$$

Example. *If* $\vec{x} = (2, 1, 3)$, $\vec{y} = (-1, 2, -2)$ *and* $r = -4$ *find* $\vec{x} + \vec{y}$ *and* $r \cdot \vec{x}$.

Example. *If* $\vec{x} = (2,1,3)$, $\vec{y} = (-1,2,-2)$ *and* $r = -4$ *find* $\vec{x} + \vec{y}$ *and* $r \cdot \vec{x}$.

Solution.

$$\begin{aligned} \vec{x} + \vec{y} &= (2,1,3) + (-1,2,-2) \\ &= (2-1, 1+2, 3-2) \\ &= (1,3,1) \end{aligned}$$

Example. *If* $\vec{x} = (2, 1, 3)$, $\vec{y} = (-1, 2, -2)$ *and* $r = -4$ *find* $\vec{x} + \vec{y}$ *and* $r \cdot \vec{x}$.

Solution.

$$\begin{aligned} \vec{x} + \vec{y} &= (2, 1, 3) + (-1, 2, -2) \\ &= (2 - 1, 1 + 2, 3 - 2) \\ &= (1, 3, 1) \end{aligned}$$

$$r \cdot \vec{x} = -4 \cdot (2, 1, 3) = (-8, -4, -12).$$

Remark.

$$\begin{aligned}(x_1,\dots,x_n)+(0,\dots,0) &= (x_1+0,\dots,x_n+0)\\ &= (x_1,\dots,x_n)\end{aligned}$$

So the additive identity is $\vec{0}=(0,\dots,0)$.

Note also that

$$\begin{aligned}0\cdot(x_1,\dots,x_n) &= (0x_1,\dots,0x_n)\\ &= (0,\dots,0)\end{aligned}$$

for all $(x_1,\dots,x_n)\in\mathbb{R}^n$.

Vector space of real-valued functions

Example. *Let A be the interval $[0,1)$ and V be the space of functions $f : A \longrightarrow \mathbb{R}$, i.e.,*

$$V = \{f : [0,1) \longrightarrow \mathbb{R}\}$$

Define addition and scalar multiplication by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (r \cdot f)(x) &= r f(x) \end{aligned}$$

Vector space of real-valued functions

Example. *Let* A *be the interval* $[0,1)$ *and* V *be the space of functions* $f : A \longrightarrow \mathbb{R}$*,i.e.,*

$$V = \{f : [0,1) \longrightarrow \mathbb{R}\}$$

Define addition and scalar multiplication by

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (r\cdot f)(x) &= r f(x)\end{aligned}$$

For instance, the function $f(x) = x^4$ *is an element of* V *and so are*

$$g(x) = x + 2x^2, \qquad h(x) = \cos x, \qquad k(x) = e^x$$

We have $(f+g)(x) = x + 2x^2 + x^4$*.*

Vector space of real-valued functions

Example. *Let* A *be the interval* $[0,1)$ *and* V *be the space of functions* $f : A \longrightarrow \mathbb{R}$*,i.e.,*

$$V = \{f : [0,1) \longrightarrow \mathbb{R}\}$$

Define addition and scalar multiplication by

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (r\cdot f)(x) &= rf(x)\end{aligned}$$

Remark. (a) The zero element is the function $\vec{0}$ which associates to each x the number 0:

$$\vec{0}(x) = 0 \text{ for all } x \in [0,1)$$

Proof.
$(f+\vec{0})(x) = f(x) + \vec{0}(x) = f(x) + 0 = f(x).$ □

Vector space of real-valued functions

Example. *Let* A *be the interval* $[0, 1)$ *and* V *be the space of functions* $f : A \longrightarrow \mathbb{R}$*,i.e.,*

$$V = \{f : [0, 1) \longrightarrow \mathbb{R}\}$$

Define addition and scalar multiplication by

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ (r \cdot f)(x) &= r f(x)\end{aligned}$$

Remark. (b) The additive inverse is the function $-f : x \mapsto -f(x)$.

Proof. $(f + (-f))(x) = f(x) - f(x) = 0$ for all x. □

Example. *Instead of $A = [0, 1)$ we can take any set $A \neq \emptyset$, and we can replace $\mathbb{R}$ by any vector space V. We set*

$$V^A = \{f : A \longrightarrow V\}$$

and set

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ (r \cdot f)(x) &= r \cdot f(x) \end{aligned}$$

The vector space V^A

Example. *Instead of $A = [0, 1)$ we can take any set $A \neq \emptyset$, and we can replace $\mathbb{R}$ by any vector space V. We set*

$$V^A = \{f : A \longrightarrow V\}$$

and set

$$\begin{aligned}(f+g)(x) &= f(x)+g(x) \\ (r \cdot f)(x) &= r \cdot f(x)\end{aligned}$$

multiplication in V *addition in V*

Example. *Instead of* $A = [0, 1)$ *we can take any set* $A \neq \emptyset$*, and we can replace* $\mathbb{R}$ *by any vector space* V*. We set*

$$V^A = \{f : A \longrightarrow V\}$$

and set

$$\begin{aligned}(f + g)(x) &= f(x)+g(x)\\(r \cdot f)(x) &= r \cdot f(x)\end{aligned}$$

Remark. (a) The zero element is the function which associates to each x the vector $\vec{0}$:

$$0 : x \mapsto \vec{0}$$

Proof

$$\begin{aligned}(f + 0)(x) &= f(x) + 0(x)\\&= f(x) + \vec{0} = f(x) \quad \square\end{aligned}$$

Remark.

(b) Here we prove that $+$ is associative:

Proof. Let $f, g, h \in V^A$. Then

$$\begin{aligned}[(f+g)+h](x) &= (f+g)(x)+h(x) \\ &= (f(x)+g(x))+h(x) \\ &= f(x)+(g(x)+h(x)) \quad \text{associativity in} V \\ &= f(x)+(g+h)(x) \\ &= [f+(g+h)](x)\end{aligned}$$

□

Exercises

Let $V = \mathbb{R}^4$. Evaluate the following:

a) $(2, -1, 3, 1) + (3, -1, 1, -1)$.

b) $(2, 1, 5, -1) - (3, 1, 2, -2)$.

c) $10 \cdot (2, 0, -1, 1)$.

d) $(1, -2, 3, 1) + 10 \cdot (1, -1, 0, 1) - 3 \cdot (0, 2, 1, -2)$.

e) $x_1 \cdot (1, 0, 0, 0) + x_2 \cdot (0, 1, 0, 0) + x_3 \cdot (0, 0, 1, 0) + x_4 \cdot (0, 0, 0, 1)$.

Chapter 2

Subspaces

Subspace of a vector space

In most applications we will be working with a subset W of a vector space V such that W itself is a vector space.

Subspace of a vector space

In most applications we will be working with a subset W of a vector space V such that W itself is a vector space.

Question: Do we have to test all the axioms to find out if W is a vector space?

Subspace of a vector space

In most applications we will be working with a subset W of a vector space V such that W itself is a vector space.

Question: Do we have to test all the axioms to find out if W is a vector space?

The answer is NO.

Subspace of a vector space

In most applications we will be working with a subset W of a vector space V such that W itself is a vector space.

Question: Do we have to test all the axioms to find out if W is a vector space?

The answer is NO.

Theorem. *Let $W \neq \emptyset$ be a subset of a vector space V. Then W, with the addition and scalar multiplication as V, is a vector space if and only if:*

- $u + v \in W$ *for all* $u, v \in W$ *(or* $W + W \subseteq W$*)*
- $r \cdot u \in W$ *for all* $r \in \mathbb{R}$ *and all* $u \in W$ *(or* $\mathbb{R}W \subseteq W$*).*

Subspace of a vector space

In most applications we will be working with a subset W of a vector space V such that W itself is a vector space.

Question: Do we have to test all the axioms to find out if W is a vector space?

The answer is NO.

Theorem. *Let $W \neq \emptyset$ be a subset of a vector space V. Then W, with the addition and scalar multiplication as V, is a vector space if and only if:*

- $u + v \in W$ *for all* $u, v \in W$ *(or* $W + W \subseteq W$*)*
- $r \cdot u \in W$ *for all* $r \in \mathbb{R}$ *and all* $u \in W$ *(or* $\mathbb{R}W \subseteq W$*).*

In this case we say that W is a *subspace* of V.

Proof. Assume that $W + W \subseteq W$ and $\mathbb{R}W \subseteq W$.
To show that W is a vector space we have to show that all the 10 axioms hold for W. But that follows because the axioms hold for V and W is a subset of V: □

Proof. Assume that $W + W \subseteq W$ and $\mathbb{R}W \subseteq W$.
To show that W is a vector space we have to show that all the 10 axioms hold for W. But that follows because the axioms hold for V and W is a subset of V:

- A1 (Commutativity of addition)
  For $u, v \in W$, we have $u + v = v + u$. This is because u, v are also in V and commutativity holds in V.

□

Proof. Assume that $W + W \subseteq W$ and $\mathbb{R}W \subseteq W$.

To show that W is a vector space we have to show that all the 10 axioms hold for W. But that follows because the axioms hold for V and W is a subset of V:

- A1 (Commutativity of addition)
  For $u, v \in W$, we have $u + v = v + u$. This is because u, v are also in V and commutativity holds in V.
- A4 (Existence of additive identity)
  Take any vector $u \in W$. Then by assumption $0 \cdot u = \vec{0} \in W$. Hence $\vec{0} \in W$.

□

Proof. Assume that $W + W \subseteq W$ and $\mathbb{R}W \subseteq W$.

To show that W is a vector space we have to show that all the 10 axioms hold for W. But that follows because the axioms hold for V and W is a subset of V:

- A1 (Commutativity of addition)
  For $u, v \in W$, we have $u + v = v + u$. This is because u, v are also in V and commutativity holds in V.
- A4 (Existence of additive identity)
  Take any vector $u \in W$. Then by assumption $0 \cdot u = \vec{0} \in W$. Hence $\vec{0} \in W$.
- A5 (Existence of additive inverse)
  If $u \in W$ then $-u = (-1) \cdot u \in W$.

□

Proof. Assume that $W + W \subseteq W$ and $\mathbb{R}W \subseteq W$.

To show that W is a vector space we have to show that all the 10 axioms hold for W. But that follows because the axioms hold for V and W is a subset of V:

- A1 (Commutativity of addition)
  For $u, v \in W$, we have $u + v = v + u$. This is because u, v are also in V and commutativity holds in V.
- A4 (Existence of additive identity)
  Take any vector $u \in W$. Then by assumption $0 \cdot u = \vec{0} \in W$. Hence $\vec{0} \in W$.
- A5 (Existence of additive inverse)
  If $u \in W$ then $-u = (-1) \cdot u \in W$.

- One can check that the other axioms follow in the same way.

□

Examples

Usually the situation is that we are given a vector space V and a subset of vectors W satisfying some conditions and we need to see if W is a subspace of V.

Usually the situation is that we are given a vector space V and a subset of vectors W satisfying some conditions and we need to see if W is a subspace of V.

$$W = \{v \in V : \underline{\text{some conditions}} \text{ on } v\}$$

Usually the situation is that we are given a vector space V and a subset of vectors W satisfying some conditions and we need to see if W is a subspace of V.

$$W = \{v \in V : \underline{\text{some conditions}} \text{ on } v\}$$

We will then have to show that

$$\begin{array}{ll} u, v \in W & u + v \\ r \in \mathbb{R} & r \cdot u \end{array} \Bigg\} \text{ Satisfy the } \underline{\text{same conditions}}.$$

Lines through the origin as subspaces of $\mathbb{R}^2$

Example.

$$\begin{aligned} V &= \mathbb{R}^2, \\ W &= \{(x,y) | y = kx\} \qquad \textit{for a given } k \\ &= \textit{line through } (0,0) \textit{ with slope } k. \end{aligned}$$

Lines through the origin as subspaces of $\mathbb{R}^2$

Example.

$$\begin{aligned} V &= \mathbb{R}^2, \\ W &= \{(x,y) | y = kx\} \qquad \textit{for a given } k \\ &= \textit{line through } (0,0) \textit{ with slope } k. \end{aligned}$$

To see that W is in fact a subspace of $\mathbb{R}^2$:
Let $u = (x_1, y_1),\ v = (x_2, y_2) \in W$. Then $y_1 = kx_1$ and $y_2 = kx_2$

Lines through the origin as subspaces of $\mathbb{R}^2$

Example.

$$\begin{aligned} V &= \mathbb{R}^2, \\ W &= \{(x,y) | y = kx\} \qquad \textit{for a given } k \\ &= \textit{line through } (0,0) \textit{ with slope } k. \end{aligned}$$

To see that W is in fact a subspace of $\mathbb{R}^2$:
Let $u = (x_1, y_1),\ v = (x_2, y_2) \in W$. Then $y_1 = kx_1$ and
$y_2 = kx_2$
and

$$\begin{aligned} u + v &= (x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2, kx_1 + kx_2) \\ &= (x_1 + x_2, k(x_1 + x_2)) \in W \end{aligned}$$

Lines through the origin as subspaces of $\mathbb{R}^2$

Example.

$$\begin{aligned} V &= \mathbb{R}^2, \\ W &= \{(x,y) | y = kx\} \qquad \text{for a given } k \\ &= \text{line through } (0,0) \text{ with slope } k. \end{aligned}$$

To see that W is in fact a subspace of $\mathbb{R}^2$:
Let $u = (x_1, y_1),\ v = (x_2, y_2) \in W$. Then $y_1 = kx_1$ and $y_2 = kx_2$
and

$$\begin{aligned} u + v &= (x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2, kx_1 + kx_2) \\ &= (x_1 + x_2, k(x_1 + x_2)) \in W \end{aligned}$$

Similarly, $r \cdot u = (rx_1, ry_1) = (rx_1, krx_1) \in W$

So what are the subspaces of $\mathbb{R}^2$?

So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$

Subspaces of $\mathbb{R}^2$

So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$
2. Lines. But only those that contain $(0, 0)$. Why?

So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$
2. Lines. But only those that contain $(0, 0)$. Why?
3. $\mathbb{R}^2$

So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$
2. Lines. But only those that contain $(0,0)$. Why?
3. $\mathbb{R}^2$

Remark (First test). If W is a subspace, then $\vec{0} \in W$.

Thus: If $\vec{0} \notin W$, then W is not a subspace.

So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$
2. Lines. But only those that contain $(0, 0)$. Why?
3. $\mathbb{R}^2$

Remark (First test). If W is a subspace, then $\vec{0} \in W$.

Thus: If $\vec{0} \notin W$, then W is not a subspace.

This is why a line not passing through $(0, 0)$ can not be a subspace of $\mathbb{R}^2$.

A subset of $\mathbb{R}^2$ that is not a subspace

Warning. We can not conclude from the fact that $\vec{0} \in W$, that W is a subspace.

A subset of $\mathbb{R}^2$ that is not a subspace

Warning. We can not conclude from the fact that $\vec{0} \in W$, that W is a subspace.

Example. *Lets consider the following subset of $\mathbb{R}^2$:*

$$W = \{(x, y) | x^2 - y^2 = 0\}$$

Is W a subspace of $\mathbb{R}^2$? Why?

A subset of $\mathbb{R}^2$ that is not a subspace

Warning. We can not conclude from the fact that $\vec{0} \in W$, that W is a subspace.

Example. *Lets consider the following subset of $\mathbb{R}^2$:*

$$W = \{(x, y)|x^2 - y^2 = 0\}$$

Is W a subspace of $\mathbb{R}^2$? Why?

The answer is NO.

We have $(1, 1)$ and $(1, -1) \in W$ but $(1, 1) + (1, -1) = (2, 0) \notin W$. i.e., W is not closed under addition.

A subset of $\mathbb{R}^2$ that is not a subspace

Warning. We can not conclude from the fact that $\vec{0} \in W$, that W is a subspace.

Example. *Lets consider the following subset of* $\mathbb{R}^2$*:*

$$W = \{(x, y) | x^2 - y^2 = 0\}$$

Is W *a subspace of* $\mathbb{R}^2$*? Why?*

The answer is NO.

We have $(1, 1)$ *and* $(1, -1) \in W$ *but* $(1, 1) + (1, -1) = (2, 0) \notin W$*. i.e.,* W *is not closed under addition.*

Notice that $(0, 0) \in W$ *and* W *is closed under multiplication by scalars.*

What are the subspaces of $\mathbb{R}^3$?

What are the subspaces of $\mathbb{R}^3$?

1. $\{0\}$ and $\mathbb{R}^3$.

Subspaces of $\mathbb{R}^3$

What are the subspaces of $\mathbb{R}^3$?

1. $\{0\}$ and $\mathbb{R}^3$.
2. Planes: A plane $W \subseteq \mathbb{R}^3$ is given by a normal vector (a, b, c) and its distance from $(0, 0, 0)$ or

$$W = \{(x, y, z) | \underbrace{ax + by + cz = p}_{\text{condition on } (x,y,z)}\}$$

What are the subspaces of $\mathbb{R}^3$?

1. $\{0\}$ and $\mathbb{R}^3$.
2. Planes: A plane $W \subseteq \mathbb{R}^3$ is given by a normal vector (a, b, c) and its distance from $(0, 0, 0)$ or

$$W = \{(x, y, z) | \underbrace{ax + by + cz = p}_{\text{condition on } (x,y,z)}\}$$

For W to be a subspace, $(0, 0, 0)$ must be in W by the *first test*. Thus

$$p = a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$$

or

$$p = 0$$

Planes containing the origin

A plane containing $(0, 0, 0)$ is indeed a subspace of $\mathbb{R}^3$.

Planes containing the origin

A plane containing $(0,0,0)$ is indeed a subspace of $\mathbb{R}^3$.

Proof. Let (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in W$. Then

$$\begin{aligned} ax_1 + by_1 + cz_1 &= 0 \\ ax_2 + by_2 + cz_2 &= 0 \end{aligned}$$

Planes containing the origin

A plane containing $(0,0,0)$ is indeed a subspace of $\mathbb{R}^3$.

Proof. Let (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in W$. Then

$$\begin{aligned} ax_1 + by_1 + cz_1 &= 0 \\ ax_2 + by_2 + cz_2 &= 0 \end{aligned}$$

Then we have

$$\begin{aligned} &a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) \\ &\qquad = \underbrace{(ax_1 + by_1 + cz_1)}_{0} + \underbrace{(ax_2 + by_2 + cz_2)}_{0} \\ &\qquad = 0 \end{aligned}$$

A plane containing $(0,0,0)$ is indeed a subspace of $\mathbb{R}^3$.

Proof. Let (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in W$. Then

$$\begin{aligned} ax_1 + by_1 + cz_1 &= 0 \\ ax_2 + by_2 + cz_2 &= 0 \end{aligned}$$

Then we have

$$\begin{aligned} &a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) \\ &\quad = \underbrace{(ax_1 + by_1 + cz_1)}_{0} + \underbrace{(ax_2 + by_2 + cz_2)}_{0} \\ &\quad = 0 \end{aligned}$$

and $a(rx_1) + b(ry_1) + c(rz_1) = r(ax_1 + by_1 + cz_1) = 0 \quad \square$

Summary of subspaces of $\mathbb{R}^3$

1. $\{0\}$ and $\mathbb{R}^3$.
2. Planes containing $(0, 0, 0)$.

Summary of subspaces of $\mathbb{R}^3$

1. $\{0\}$ and $\mathbb{R}^3$.
2. Planes containing $(0, 0, 0)$.
3. Lines containing $(0, 0, 0)$.
   (Intersection of two planes containing $(0, 0, 0)$)

Determine whether the given subset of $\mathbb{R}^n$ is a subspace or not (Explain):

a) $W = \{(x, y) \in \mathbb{R}^2 |\ xy = 0\}$.

b) $W = \{(x, y, z) \in \mathbb{R}^3 |\ 3x + 2y^2 + z = 0\}$.

c) $W = \{(x, y, z) \in \mathbb{R}^3 |\ 2x + 3y - z = 0\}$.

d) The set of all vectors (x_1, x_2, x_3) satisfying

$$2x_3 = x_1 - 10x_2$$

Determine whether the given subset of $\mathbb{R}^n$ is a subspace or not (Explain):

e) The set of all vectors in $\mathbb{R}^4$ satisfying the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 + 5x_4 &= 0 \\ x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

f) The set of all points $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ satisfying

$$x_1 + 2x_2 + 3x_3 + x_4 = -1$$

Chapter 3

Vector Spaces of Functions

Space of Continuous Functions

Let $I \subseteq \mathbb{R}$ be an interval.

Space of Continuous Functions

Let $I \subseteq \mathbb{R}$ be an interval. Then I is of the form (for some $a < b$)

$$I = \begin{cases} \{x \in \mathbb{R} |\ a < x < b\}, & \text{an open interval;} \\ \{x \in \mathbb{R} |\ a \leq x \leq b\}, & \text{a closed interval;} \\ \{x \in \mathbb{R} |\ a \leq x < b\} & \\ \{x \in \mathbb{R} |\ a < x \leq b\}. & \end{cases}$$

Space of Continuous Functions

Let $I \subseteq \mathbb{R}$ be an interval. Then recall that the space of all functions $f : I \longrightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:

Space of Continuous Functions

Let $I \subseteq \mathbb{R}$ be an interval. Then recall that the space of all functions $f : I \longrightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:

Example (1). *Let $C(I)$ be the space of continuous functions. If f and g are continuous, so are the functions $f + g$ and $rf \ \ (r \in \mathbb{R})$. Hence $C(I)$ is a vector space.*

Space of Continuous Functions

Let $I \subseteq \mathbb{R}$ be an interval. Then recall that the space of all functions $f : I \longrightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:

Example (1). *Let $C(I)$ be the space of continuous functions. If f and g are continuous, so are the functions $f + g$ and $rf \ \ (r \in \mathbb{R})$. Hence $C(I)$ is a vector space.*

Recall, that a function is continuous, if the graph has no gaps. This can be formulated in different ways:

Space of Continuous Functions

Let $I \subseteq \mathbb{R}$ be an interval. Then recall that the space of all functions $f : I \longrightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:

Example (1). *Let $C(I)$ be the space of continuous functions. If f and g are continuous, so are the functions $f + g$ and $rf \ \ (r \in \mathbb{R})$. Hence $C(I)$ is a vector space.*

Recall, that a function is continuous, if the graph has no gaps. This can be formulated in different ways:

a) Let $x_0 \in I$ and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that for all $x \in I \cap (x_0 - \delta, x_0 + \delta)$ we have

$$|f(x) - f(x_0)| < \epsilon$$

This tells us that the value of f at nearby points is arbitrarily close to the value of f at x_0.

Space of Continuous Functions

Let $I \subseteq \mathbb{R}$ be an interval. Then recall that the space of <u>all</u> functions $f : I \longrightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:

Example (1). *Let $C(I)$ be the space of continuous functions. If f and g are continuous, so are the functions $f+g$ and $rf \ \ (r \in \mathbb{R})$. Hence $C(I)$ is a vector space.*

Recall, that a function is continuous, if the graph has no gaps. This can be formulated in different ways:

b) A reformulation of (a) is:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Space of continuously differentiable functs.

Example (2). *The space $C^1(I)$. Here we assume that I is open.*

Space of continuously differentiable functs.

Example (2). *The space $C^1(I)$. Here we assume that I is open. Recall that f is differentiable at x_0 if*

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} =: f'(x_0)$$

exists.

Space of continuously differentiable functs.

Example (2). *The space $C^1(I)$. Here we assume that I is open. Recall that f is differentiable at x_0 if*

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=:f'(x_0)$$

exists.If $f'(x_0)$ exists for all $x_0\in I$, then we say that f is differentiable on I. In this case we get a new function on I

$$x\mapsto f'(x)$$

We say that f is continuously differentiable on I if f' exists and is continuous on I.

Space of continuously differentiable functs.

Example (2). *The space $C^1(I)$. Here we assume that I is open. Recall that f is differentiable at x_0 if*

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=: f'(x_0)$$

exists. We say that f is continuously differentiable on I if f' exists and is continuous on I. Recall that if f and g are differentiable, then so are

$$f+g \text{ and } rf \quad (r\in\mathbb{R})$$

moreover

$$(f+g)'=f'+g'\,;\;\;(rf)'=rf'$$

Space of continuously differentiable functs.

Example (2). *The space $C^1(I)$. Here we assume that I is open. Recall that f is differentiable at x_0 if*

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} =: f'(x_0)$$

exists. We say that f is continuously differentiable on I if f' exists and is continuous on I. Recall that if f and g are differentiable, then so are

$$f+g \text{ and } rf \quad (r\in\mathbb{R})$$

moreover

$$(f+g)' = f'+g' \,;\; (rf)' = rf'$$

As $f'+g'$ and rf' are continuous by Example (1), it follows that $C^1(I)$ is a vector space.

A continuous but not differentiable function

Let $f(x) = |x|$ for $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$ but it is not differentiable on $\mathbb{R}$.

A continuous but not differentiable function

Let $f(x) = |x|$ for $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$ but it is not differentiable on $\mathbb{R}$. We show that f is not differentiable at $x_0 = 0$.

A continuous but not differentiable function

Let $f(x) = |x|$ for $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$ but it is not differentiable on $\mathbb{R}$. We show that f is not differentiable at $x_0 = 0$. For $h > 0$ we have

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{|h| - 0}{h} = \frac{h}{h} = 1$$

hence

$$\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = 1$$

A continuous but not differentiable function

Let $f(x) = |x|$ for $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$ but it is not differentiable on $\mathbb{R}$. We show that f is not differentiable at $x_0 = 0$. For $h > 0$ we have

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{|h| - 0}{h} = \frac{h}{h} = 1$$

hence

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = 1$$

But if $h < 0$, then

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{|h| - 0}{h} = \frac{-h}{h} = -1$$

hence

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = -1$$

A continuous but not differentiable function

Let $f(x) = |x|$ for $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$ but it is not differentiable on $\mathbb{R}$. We show that f is not differentiable at $x_0 = 0$.
Therefore,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

does not exist. $\square$

Space of r-times continuously diff. functs.

Example (3). *The space $C^r(I)$*
Let $I = (a, b)$ be an open interval. and let $r \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Definition. The function $f : I \longrightarrow \mathbb{R}$ is said to be r-times continuously differentiable if all the derivatives $f',\ f'',\ \cdots\ ,\ f^{(r)}$ exist and $f^{(r)} : I \longrightarrow \mathbb{R}$ is continuous.

We denote by $C^r(I)$ the space of r-times continuously differentiable functions on I. $C^r(I)$ is a subspace of $C(I)$.

Space of r-times continuously diff. functs.

Example (3). *The space $C^r(I)$*
Let $I = (a, b)$ be an open interval. and let $r \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Definition. The function $f : I \longrightarrow \mathbb{R}$ is said to be r-times continuously differentiable if all the derivatives $f',\ f'',\ \cdots\ ,\ f^{(r)}$ exist and $f^{(r)} : I \longrightarrow \mathbb{R}$ is continuous.

We denote by $C^r(I)$ the space of r-times continuously differentiable functions on I. $C^r(I)$ is a subspace of $C(I)$.

We have

$$C^r(I) \subsetneqq C^{r-1}(I) \subsetneqq \cdots \subsetneqq C^1(I) \subsetneqq C(I).$$

$C^r(I) \neq C^{r-1}(I)$

We have seen that $C^1(I) \neq C(I)$. Let us try to find a function that is in $C^1(I)$ but not in $C^2(I)$.

$C^r(I) \neq C^{r-1}(I)$

We have seen that $C^1(I) \neq C(I)$. Let us try to find a function that is in $C^1(I)$ but not in $C^2(I)$.

Assume $0 \in I$ and let $f(x) = x^{\frac{5}{3}}$. Then f is differentiable and

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}}$$

which is continuous.

We have seen that $C^1(I) \neq C(I)$. Let us try to find a function that is in $C^1(I)$ but not in $C^2(I)$.

Assume $0 \in I$ and let $f(x) = x^{\frac{5}{3}}$. Then f is differentiable and

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}}$$

which is continuous.

If $x \neq 0$, then f' is differentiable and

$$f''(x) = \frac{10}{3}x^{-\frac{1}{3}}$$

But for $x = 0$ we have

$$\lim_{h \to 0} \frac{f'(h) - 0}{h} = \lim_{h \to 0} \frac{5}{3} \frac{h^{\frac{2}{3}}}{h} = \lim_{h \to 0} \frac{5}{3} h^{-\frac{1}{3}}$$

which does not exist.

But for $x = 0$ we have

$$\lim_{h\to 0} \frac{f'(h) - 0}{h} = \lim_{h\to 0} \frac{5}{3} \frac{h^{\frac{2}{3}}}{h} = \lim_{h\to 0} \frac{5}{3} h^{-\frac{1}{3}}$$

which does not exist.

Remark. One can show that the function

$$f(x) = x^{\frac{3r-1}{3}}$$

is in $C^{r-1}(\mathbb{R})$, but not in $C^r(\mathbb{R})$.

But for $x = 0$ we have

$$\lim_{h\to 0} \frac{f'(h) - 0}{h} = \lim_{h\to 0} \frac{5}{3} \, \frac{h^{\frac{2}{3}}}{h} = \lim_{h\to 0} \frac{5}{3} \, h^{-\frac{1}{3}}$$

which does not exist.

Remark. One can show that the function

$$f(x) = x^{\frac{3r-1}{3}}$$

is in $C^{r-1}(\mathbb{R})$, but not in $C^r(\mathbb{R})$.

Thus, as stated before, we have

$$C^r(I) \subsetneq C^{r-1}(I) \subsetneq \cdots \subsetneq C^1(I) \subsetneq C(I).$$

Example (4). *Piecewise-continuous functions*

Definition. Let $I = [a, b)$. A function $f : I \longrightarrow \mathbb{R}$ is called piecewise-continuous if there exists finitely many points

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that f is continuous on each of the sub-intervals (x_i, x_{i+1}) for $i = 0, 1, \cdots, n-1$.

Piecewise-continuous functions

Example (4). *Piecewise-continuous functions*

Definition. Let $I = [a, b)$. A function $f : I \longrightarrow \mathbb{R}$ is called piecewise-continuous if there exists finitely many points

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that f is continuous on each of the sub-intervals (x_i, x_{i+1}) for $i = 0, 1, \cdots, n-1$.

Remark. If f and g are both piecewise-continuous, then

$$f + g \text{ and } rf \quad (r \in \mathbb{R})$$

are piecewise-continuous.

Piecewise-continuous functions

Example (4). *Piecewise-continuous functions*

Definition. Let $I = [a, b)$. A function $f : I \longrightarrow \mathbb{R}$ is called piecewise-continuous if there exists finitely many points

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that f is continuous on each of the sub-intervals (x_i, x_{i+1}) for $i = 0, 1, \cdots, n - 1$.

Remark. If f and g are both piecewise-continuous, then

$$f + g \text{ and } rf \quad (r \in \mathbb{R})$$

are piecewise-continuous.

Hence the space of piecewise-continuous functions is a vector space. Denote this vector space by $PC(I)$.

The indicator function χ_A

Important elements of $PC(I)$ are the indicator functions χ_A, where $A \subseteq I$ a sub-interval.

The indicator function χ_A

Important elements of $PC(I)$ are the indicator functions χ_A, where $A \subseteq I$ a sub-interval.

Let $A \subseteq \mathbb{R}$ be a set. Define

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

The indicator function χ_A

Important elements of $PC(I)$ are the indicator functions χ_A, where $A \subseteq I$ a sub-interval.

Let $A \subseteq \mathbb{R}$ be a set. Define

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

So the values of χ_A tell us whether x is in A or not.
If $x \in A$, then $\chi_A(x) = 1$ and if $x \notin A$, then $\chi_A(x) = 0$.

The indicator function χ_A

Important elements of $PC(I)$ are the indicator functions χ_A, where $A \subseteq I$ a sub-interval.

Let $A \subseteq \mathbb{R}$ be a set. Define

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

So the values of χ_A tell us whether x is in A or not.
If $x \in A$, then $\chi_A(x) = 1$ and if $x \notin A$, then $\chi_A(x) = 0$.

We will work a lot with indicator functions so let us look at some of their properties.

Lemma. *Let* $A,\ B \subseteq I$*. Then*

$$\chi_{A\cap B}(x) = \chi_A(x)\,\chi_B(x) \qquad (*)$$

Lemma. *Let* $A,\ B \subseteq I$. *Then*

$$\chi_{A\cap B}(x) = \chi_A(x)\,\chi_B(x) \qquad (*)$$

Proof. We have to show that the two functions

$$x \mapsto \chi_{A\cap B}(x) \text{ and } x \mapsto \chi_A(x)\chi_B(x)$$

take the same values at every point $x \in I$. So lets evaluate both functions:

Some properties of χ_A

Lemma. *Let* $A,\ B \subseteq I$. *Then*

$$\chi_{A\cap B}(x) = \chi_A(x)\,\chi_B(x) \qquad (*)$$

Proof. We have to show that the two functions

$$x \mapsto \chi_{A\cap B}(x) \text{ and } x \mapsto \chi_A(x)\chi_B(x)$$

take the same values at every point $x \in I$. So lets evaluate both functions:

If $x \in A$ and $x \in B$, that is $x \in A \cap B$, then $\chi_{A\cap B}(x) = 1$ and,

Some properties of χ_A

Lemma. *Let* $A,\ B \subseteq I$. *Then*

$$\chi_{A\cap B}(x) = \chi_A(x)\,\chi_B(x) \qquad (*)$$

Proof. We have to show that the two functions

$$x \mapsto \chi_{A\cap B}(x) \text{ and } x \mapsto \chi_A(x)\chi_B(x)$$

take the same values at every point $x \in I$. So lets evaluate both functions:

If $x \in A$ and $x \in B$, that is $x \in A \cap B$, then $\chi_{A\cap B}(x) = 1$ and,

since $\chi_A(x) = 1$ and $\chi_B(x) = 1$, we also have $\chi_A(x)\chi_B(x) = 1$.

Thus, the left and the right hand sides of $(*)$ agree.

Some properties of χ_A

Lemma. *Let $A,\ B \subseteq I$. Then*

$$\chi_{A\cap B}(x) = \chi_A(x)\,\chi_B(x) \qquad (*)$$

Proof. We have to show that the two functions

$$x \mapsto \chi_{A\cap B}(x) \text{ and } x \mapsto \chi_A(x)\chi_B(x)$$

take the same values at every point $x \in I$. So lets evaluate both functions:

On the other hand, if $x \notin A \cap B$, then there are two possibilities:

Lemma. *Let $A,\ B \subseteq I$. Then*

$$\chi_{A\cap B}(x) = \chi_A(x)\,\chi_B(x) \qquad (*)$$

Proof. We have to show that the two functions

$$x \mapsto \chi_{A\cap B}(x) \text{ and } x \mapsto \chi_A(x)\chi_B(x)$$

take the same values at every point $x \in I$. So lets evaluate both functions:

On the other hand, if $x \notin A \cap B$, then there are two possibilities:

- $x \notin A$ then $\chi_A(x) = 0$, so $\chi_A(x)\chi_B(x) = 0$.

Some properties of χ_A

Lemma. *Let $A,\ B \subseteq I$. Then*

$$\chi_{A\cap B}(x) = \chi_A(x)\,\chi_B(x) \qquad (*)$$

Proof. We have to show that the two functions

$$x \mapsto \chi_{A\cap B}(x) \text{ and } x \mapsto \chi_A(x)\chi_B(x)$$

take the same values at every point $x \in I$. So lets evaluate both functions:

On the other hand, if $x \notin A \cap B$, then there are two possibilities:

- $x \notin A$ then $\chi_A(x) = 0$, so $\chi_A(x)\chi_B(x) = 0$.
- $x \notin B$ then $\chi_B(x) = 0$, so $\chi_A(x)\chi_B(x) = 0$.

It follows that

$$0 = \chi_{A\cap B}(x) = \chi_A(x)\chi_B(x) \quad \square$$

Some properties of χ_A

What about $\chi_{A\cup B}$? Can we express it in terms of χ_A, χ_B?

Some properties of χ_A

What about $\chi_{A\cup B}$? Can we express it in terms of χ_A, χ_B?

If A and B are disjoint, that is $A \cap B = \emptyset$ then

$$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x).$$

Some properties of χ_A

What about $\chi_{A\cup B}$? Can we express it in terms of χ_A, χ_B?
If A and B are disjoint, that is $A \cap B = \emptyset$ then

$$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x).$$

Let us prove this:

- If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$. Thus the LHS (left hand side) and the RHS (right hand side) are both zero.

Some properties of χ_A

What about $\chi_{A\cup B}$? Can we express it in terms of χ_A, χ_B?
If A and B are disjoint, that is $A \cap B = \emptyset$ then

$$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x).$$

Let us prove this:

- If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$. Thus the LHS (left hand side) and the RHS (right hand side) are both zero.
- If $x \in A \cup B$ then either

What about $\chi_{A\cup B}$? Can we express it in terms of χ_A, χ_B?
If A and B are disjoint, that is $A \cap B = \emptyset$ then

$$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x).$$

Let us prove this:

- If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$. Thus the LHS (left hand side) and the RHS (right hand side) are both zero.
- If $x \in A \cup B$ then either
  - x is in A but not in B. In this case $\chi_{A\cup B}(x) = 1$ and $\chi_A(x) + \chi_B(x) = 1 + 0 = 1$

What about $\chi_{A\cup B}$? Can we express it in terms of χ_A, χ_B?
If A and B are disjoint, that is $A \cap B = \emptyset$ then

$$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x).$$

Let us prove this:

- If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$. Thus the LHS (left hand side) and the RHS (right hand side) are both zero.
- If $x \in A \cup B$ then either
  - x is in A but not in B. In this case $\chi_{A\cup B}(x) = 1$ and $\chi_A(x) + \chi_B(x) = 1 + 0 = 1$

or

  - x is in B but not in A. In this case $\chi_{A\cup B}(x) = 1$ and $\chi_A(x) + \chi_B(x) = 0 + 1 = 1$ $\square$

Some properties of χ_A

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x)$.

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

Some properties of χ_A

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

Lemma. $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$.

Some properties of χ_A

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

Lemma. $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$.

Proof.

- If $x \notin A \cup B$, then both of the LHS and the RHS take the value 0.

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

Lemma. $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$.

Proof.

- If $x \notin A \cup B$, then both of the LHS and the RHS take the value 0.
- If $x \in A \cup B$, then we have the following possibilities:

Some properties of χ_A

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

Lemma. $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$.

Proof.

- If $x \notin A \cup B$, then both of the LHS and the RHS take the value 0.
- If $x \in A \cup B$, then we have the following possibilities:
  1. If $x \in A,\ x \notin B$, then

$$\chi_{A\cup B}(x) = 1$$
$$\chi_A(x) + \chi_B(x) - \chi_{A\cap B} = 1 + 0 - 0 = 1$$

Some properties of χ_A

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

Lemma. $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$.

Proof.

- If $x \notin A \cup B$, then both of the LHS and the RHS take the value 0.
- If $x \in A \cup B$, then we have the following possibilities:

1. If $x \in A,\ x \notin B$, then

$$\chi_{A\cup B}(x) = 1$$
$$\chi_A(x) + \chi_B(x) - \chi_{A\cap B} = 1 + 0 - 0 = 1$$

2. Similarly for the case $x \in B,\ x \notin A$: LHS equals the RHS.

Some properties of χ_A

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

Lemma. $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$.

Proof.

- If $x \notin A \cup B$, then both of the LHS and the RHS take the value 0.
- If $x \in A \cup B$, then we have the following possibilities:

3. If $x \in A \cap B$, then

$$\chi_{A\cup B}(x) = 1$$
$$\chi_A(x) + \chi_B(x) - \chi_{A\cap B} = 1 + 1 - 1 = 1$$

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

Lemma. $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$.

Proof.

- If $x \notin A \cup B$, then both of the LHS and the RHS take the value 0.
- If $x \in A \cup B$, then we have the following possibilities:

3. If $x \in A \cap B$, then

$$\chi_{A\cup B}(x) = 1$$
$$\chi_A(x) + \chi_B(x) - \chi_{A\cap B} = 1 + 1 - 1 = 1$$

As we have checked all possibilities, we have shown that the statement in the lemma is correct □