Chapter 1

Vector Spaces over $\mathbb{R}$
**Definition.** A vector space over \( \mathbb{R} \) is a set \( V \) with operations of addition \( + \) and scalar multiplication \( \cdot \) satisfying the following properties:
**Definition.** vector space over $\mathbb{R}$ is a set $V$ with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- **A1 (Closure of addition)**
  
  For all $u, v \in V$, $u + v$ is defined and $u + v \in V$. 
**Definition.** vector space over \( \mathbb{R} \) is a set \( V \) with operations of addition \( + \) and scalar multiplication \( \cdot \) satisfying the following properties:

- **A1 (Closure of addition)**
  For all \( u, v \in V, u + v \) is defined and \( u + v \in V \).

- **A2 (Commutativity for addition)**
  \( u + v = v + u \) for all \( u, v \in V \).
Definition. A vector space over $\mathbb{R}$ is a set $V$ with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- **A1 (Closure of addition)**
  
  For all $u, v \in V$, $u + v$ is defined and $u + v \in V$.

- **A2 (Commutativity for addition)**
  
  $u + v = v + u$ for all $u, v \in V$.

- **A3 (Associativity for addition)**
  
  $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$. 
Definition. A vector space over \( \mathbb{R} \) is a set \( V \) with operations of addition \( + \) and scalar multiplication \( \cdot \) satisfying the following properties:

- **A1 (Closure of addition)**
  For all \( u, v \in V \), \( u + v \) is defined and \( u + v \in V \).

- **A2 (Commutativity for addition)**
  \( u + v = v + u \) for all \( u, v \in V \).

- **A3 (Associativity for addition)**
  \( u + (v + w) = (u + v) + w \) for all \( u, v, w \in V \).

- **A4 (Existence of additive identity)**
  There exists an element \( \vec{0} \) such that \( u + \vec{0} = u \) for all \( u \in V \).
**Definition.** vector space over \( \mathbb{R} \) is a set \( V \) with operations of addition \( + \) and scalar multiplication \( \cdot \) satisfying the following properties:

- **A1** (Closure of addition)
  For all \( u, v \in V \), \( u + v \) is defined and \( u + v \in V \).

- **A2** (Commutativity for addition)
  \( u + v = v + u \) for all \( u, v \in V \).

- **A3** (Associativity for addition)
  \( u + (v + w) = (u + v) + w \) for all \( u, v, w \in V \).

- **A4** (Existence of additive identity)
  There exists an element \( \vec{0} \) such that \( u + \vec{0} = u \) for all \( u \in V \).

- **A5** (Existence of additive inverse)
  For each \( u \in V \), there exists an element \(-u\) such that \( u + (-u) = \vec{0} \).
**Definition.** A vector space over \( \mathbb{R} \) is a set \( V \) with operations of addition \( + \) and scalar multiplication \( \cdot \) satisfying the following properties:
Definition. A vector space over \( \mathbb{R} \) is a set \( V \) with operations of addition \( + \) and scalar multiplication \( \cdot \) satisfying the following properties:

- **M1 (Closure for scalar multiplication)**
  
  For each number \( r \) and each \( u \in V \), \( r \cdot u \) is defined and \( r \cdot u \in V \).
**Definition.** A vector space over \( \mathbb{R} \) is a set \( V \) with operations of addition \( + \) and scalar multiplication \( \cdot \), satisfying the following properties:

- **M1 (Closure for scalar multiplication)**
  For each number \( r \) and each \( u \in V \), \( r \cdot u \) is defined and \( r \cdot u \in V \).

- **M2 (Multiplication by 1)**
  \( 1 \cdot u = u \) for all \( u \in V \).
**Definition.** A vector space over \( \mathbb{R} \) is a set \( V \) with operations of addition \( + \) and scalar multiplication \( \cdot \) satisfying the following properties:

- **M1 (Closure for scalar multiplication)**
  For each number \( r \) and each \( u \in V \), \( r \cdot u \) is defined and \( r \cdot u \in V \).

- **M2 (Multiplication by 1)**
  \( 1 \cdot u = u \) for all \( u \in V \).

- **M3 (Associativity for multiplication)**
  \( r \cdot (s \cdot u) = (r \cdot s) \cdot u \) for \( r, s \in \mathbb{R} \) and all \( u \in V \).
Definition. A vector space over $\mathbb{R}$ is a set $V$ with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- **M1** (Closure for scalar multiplication)
  For each number $r$ and each $u \in V$, $r \cdot u$ is defined and $r \cdot u \in V$.

- **M2** (Multiplication by 1)
  $1 \cdot u = u$ for all $u \in V$.

- **M3** (Associativity for multiplication)
  $r \cdot (s \cdot u) = (r \cdot s) \cdot u$ for $r, s \in \mathbb{R}$ and all $u \in V$.

- **D1** (First distributive property)
  $r \cdot (u + v) = r \cdot u + r \cdot v$ for all $r \in \mathbb{R}$ and all $u, v \in V$. 

Definition. A vector space over $\mathbb{R}$ is a set $V$ with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

- **M1 (Closure for scalar multiplication)**
  For each number $r$ and each $u \in V$, $r \cdot u$ is defined and $r \cdot u \in V$.

- **M2 (Multiplication by 1)**
  $1 \cdot u = u$ for all $u \in V$.

- **M3 (Associativity for multiplication)**
  $r \cdot (s \cdot u) = (r \cdot s) \cdot u$ for $r, s \in \mathbb{R}$ and all $u \in V$.

- **D1 (First distributive property)**
  $r \cdot (u + v) = r \cdot u + r \cdot v$ for all $r \in \mathbb{R}$ and all $u, v \in V$.

- **D2 (Second distributive property)**
  $(r + s) \cdot u = r \cdot u + s \cdot u$ for all $r, s \in \mathbb{R}$ and all $u \in V$. 
Some immediate results

**Remark.** The zero element $\vec{0}$ is unique, i.e., if $\vec{0}_1, \vec{0}_2 \in V$ are such that

$$u + \vec{0}_1 = u + \vec{0}_2 = u, \forall u \in V$$

then $\vec{0}_1 = \vec{0}_2$. 
Remark. The zero element \( \vec{0} \) is unique, i.e., if \( \vec{0}_1, \vec{0}_2 \in V \) are such that

\[
  u + \vec{0}_1 = u + \vec{0}_2 = u, \quad \forall u \in V
\]

then \( \vec{0}_1 = \vec{0}_2 \).

Proof. We have \( \vec{0}_1 = \vec{0}_1 + \vec{0}_2 = \vec{0}_2 + \vec{0}_1 = \vec{0}_2 \). \( \square \)
**Remark.** The zero element $\vec{0}$ is unique, i.e., if $\vec{0}_1, \vec{0}_2 \in V$ are such that

$$u + \vec{0}_1 = u + \vec{0}_2 = u, \forall u \in V$$

then $\vec{0}_1 = \vec{0}_2$.

**Proof.** We have $\vec{0}_1 = \vec{0}_1 + \vec{0}_2 = \vec{0}_2 + \vec{0}_1 = \vec{0}_2$.

**Lemma.** Let $u \in V$, then $0 \cdot u = \vec{0}$.
**Remark.** The zero element $\vec{0}$ is unique, i.e., if $\vec{0}_1, \vec{0}_2 \in V$ are such that

$$u + \vec{0}_1 = u + \vec{0}_2 = u, \forall u \in V$$

then $\vec{0}_1 = \vec{0}_2$.

**Proof.** We have

$$\vec{0}_1 = \vec{0}_1 + \vec{0}_2 = \vec{0}_2 + \vec{0}_1 = \vec{0}_2$$

**Lemma.** Let $u \in V$, then $0 \cdot u = \vec{0}$.

**Proof.**

$$u + 0 \cdot u = 1 \cdot u + 0 \cdot u$$

$$= (1 + 0) \cdot u$$

$$= 1 \cdot u$$

$$= u$$
**Remark.** The zero element $\vec{0}$ is unique, i.e., if $\vec{0}_1, \vec{0}_2 \in V$ are such that
\[
u + \vec{0}_1 = u + \vec{0}_2 = u, \forall u \in V\]
then $\vec{0}_1 = \vec{0}_2$.

**Proof.** We have $\vec{0}_1 = \vec{0}_1 + \vec{0}_2 = \vec{0}_2 + \vec{0}_1 = \vec{0}_2$ \qed

**Lemma.** Let $u \in V$, then $0 \cdot u = \vec{0}$.

**Proof.**

Thus
\[
\vec{0} = u + (-u) = (0 \cdot u + u) + (-u) = 0 \cdot u + (u + (-u)) = 0 \cdot u + \vec{0} = 0 \cdot u \quad \square
\]
Lemma. a) The element $-u$ is unique.

b) $-u = (-1) \cdot u$. 
Lemma. a) The element $-u$ is unique.

b) $-u = (-1) \cdot u$.

Proof of part (b).

\[
\begin{align*}
    u + (-1) \cdot u &= 1 \cdot u + (-1) \cdot u \\
    &= (1 + (-1)) \cdot u \\
    &= 0 \cdot u \\
    &= \vec{0}
\end{align*}
\]
Examples

- $\mathbb{R}^n$ (columns)
- $\mathbb{R}^n$ (rows)
- $\mathbb{R}^A$
- $V^A$

Exercises
Before examining the axioms in more detail, let us discuss two examples.
Example. Let $V = \mathbb{R}^n$, considered as column vectors

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mid x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}$$
Example. Let $V = \mathbb{R}^n$. Then for

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad r \in \mathbb{R}:$$
Example. Let $V = \mathbb{R}^n$. Then for

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad r \in \mathbb{R}:$$

Define

$$u + v = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \quad \text{and} \quad r \cdot u = \begin{pmatrix} rx_1 \\ \vdots \\ rx_n \end{pmatrix}$$
Example. Let $V = \mathbb{R}^n$. Then for

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad r \in \mathbb{R} :$$

Note that the zero vector and the additive inverse of $u$ are given by:

$$\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad -u = \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix}$$
Remark. \(\mathbb{R}^n\) can be considered as the space of all row vectors.

\[
\mathbb{R}^n = \left\{ (x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R} \right\}
\]
Remark. $\mathbb{R}^n$ can be considered as the space of all row vectors.

$$\mathbb{R}^n = \{ (x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R} \}$$

The addition and scalar multiplication is again given coordinate wise

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

$$r \cdot (x_1, \ldots, x_n) = (rx_1, \ldots, rx_n)$$
Example. If \( \vec{x} = (2, 1, 3), \vec{y} = (-1, 2, -2) \) and \( r = -4 \) find \( \vec{x} + \vec{y} \) and \( r \cdot \vec{x} \).
Example. If \( \vec{x} = (2, 1, 3), \vec{y} = (-1, 2, -2) \) and \( r = -4 \) find \( \vec{x} + \vec{y} \) and \( r \cdot \vec{x} \).

Solution.

\[
\begin{align*}
\vec{x} + \vec{y} & = (2, 1, 3) + (-1, 2, -2) \\
& = (2 - 1, 1 + 2, 3 - 2) \\
& = (1, 3, 1)
\end{align*}
\]
Example. If \( \vec{x} = (2, 1, 3) \), \( \vec{y} = (-1, 2, -2) \) and \( r = -4 \) find \( \vec{x} + \vec{y} \) and \( r \cdot \vec{x} \).

Solution.

\[
\vec{x} + \vec{y} = (2, 1, 3) + (-1, 2, -2) = (2 - 1, 1 + 2, 3 - 2) = (1, 3, 1)
\]

\[
r \cdot \vec{x} = -4 \cdot (2, 1, 3) = (-8, -4, -12).
\]
Remark.

\[(x_1, \ldots, x_n) + (0, \ldots, 0) = (x_1 + 0, \ldots, x_n + 0) = (x_1, \ldots, x_n)\]

So the additive identity is \(\vec{0} = (0, \ldots, 0)\).

Note also that

\[0 \cdot (x_1, \ldots, x_n) = (0 \cdot x_1, \ldots, 0 \cdot x_n) = (0, \ldots, 0)\]

for all \((x_1, \ldots, x_n) \in \mathbb{R}^n\).
Example. Let $A$ be the interval $[0, 1)$ and $V$ be the space of functions $f : A \rightarrow \mathbb{R}$, i.e.,

$$V = \{ f : [0, 1) \rightarrow \mathbb{R} \}$$

Define addition and scalar multiplication by

$$(f + g)(x) = f(x) + g(x)$$
$$(r \cdot f)(x) = rf(x)$$
Example. Let $A$ be the interval $[0, 1)$ and $V$ be the space of functions $f : A \rightarrow \mathbb{R}$, i.e.,

$$V = \{ f : [0, 1) \rightarrow \mathbb{R} \}$$

Define addition and scalar multiplication by

$$(f + g)(x) = f(x) + g(x)$$

$$(r \cdot f)(x) = rf(x)$$

For instance, the function $f(x) = x^4$ is an element of $V$ and so are

$$g(x) = x + 2x^2, \quad h(x) = \cos x, \quad k(x) = e^x$$

We have $(f + g)(x) = x + 2x^2 + x^4$. 
**Vector space of real-valued functions**

**Example.** Let $A$ be the interval $[0, 1)$ and $V$ be the space of functions $f : A \rightarrow \mathbb{R}$, i.e.,

$$V = \{ f : [0, 1) \rightarrow \mathbb{R} \}$$

Define addition and scalar multiplication by

$$(f + g)(x) = f(x) + g(x)$$
$$(r \cdot f)(x) = rf(x)$$

**Remark.** (a) The zero element is the function $\vec{0}$ which associates to each $x$ the number 0:

$$\vec{0}(x) = 0 \text{ for all } x \in [0, 1)$$

**Proof.**

$$(f + \vec{0})(x) = f(x) + \vec{0}(x) = f(x) + 0 = f(x).$$
Example. Let $A$ be the interval $[0, 1)$ and $V$ be the space of functions $f : A \rightarrow \mathbb{R}$, i.e.,

$$V = \{ f : [0, 1) \rightarrow \mathbb{R} \}$$

Define addition and scalar multiplication by

$$(f + g)(x) = f(x) + g(x)$$

$$(r \cdot f)(x) = rf(x)$$

Remark. (b) The additive inverse is the function $-f : x \mapsto -f(x)$.

Proof. $(f + (-f))(x) = f(x) - f(x) = 0$ for all $x$. □
Example. Instead of $A = [0, 1)$ we can take any set $A \neq \emptyset$, and we can replace $\mathbb{R}$ by any vector space $V$. We set

$$V^A = \{ f : A \rightarrow V \}$$

and set

$$(f + g)(x) = f(x) + g(x)$$

$$(r \cdot f)(x) = r \cdot f(x)$$
Example. Instead of $A = [0, 1)$ we can take any set $A \neq \emptyset$, and we can replace $\mathbb{R}$ by any vector space $V$. We set

$$V^A = \{ f : A \rightarrow V \}$$

and set

$$(f + g)(x) = f(x) + g(x)$$

$$(r \cdot f)(x) = r \cdot f(x)$$

addition in $V$
multiplication in $V$
**The vector space** $V^A$

**Example.** Instead of $A = [0, 1)$ we can take any set $A \neq \emptyset$, and we can replace $\mathbb{R}$ by any vector space $V$. We set

$$V^A = \{ f : A \rightarrow V \}$$

and set

$$(f + g)(x) = f(x) + g(x)$$

$$(r \cdot f)(x) = r \cdot f(x)$$

**Remark.** (a) The zero element is the function which associates to each $x$ the vector $\vec{0}$:

$0 : x \mapsto \vec{0}$

**Proof**

$$(f + 0)(x) = f(x) + 0(x)$$

$$= f(x) + \vec{0} = f(x) \quad \square$$
Remark.

(b) Here we prove that $+$ is associative:

Proof. Let $f, g, h \in V^A$. Then

\[
[(f + g) + h](x) = (f + g)(x) + h(x) \\
= (f(x) + g(x)) + h(x) \\
= f(x) + (g(x) + h(x)) \quad \text{associativity in } V \\
= f(x) + (g + h)(x) \\
= [f + (g + h)](x)
\]

$\square$
Let $V = \mathbb{R}^4$. Evaluate the following:

a) $(2, -1, 3, 1) + (3, -1, 1, -1)$.

b) $(2, 1, 5, -1) - (3, 1, 2, -2)$.

c) $10 \cdot (2, 0, -1, 1)$.

d) $(1, -2, 3, 1) + 10 \cdot (1, -1, 0, 1) - 3 \cdot (0, 2, 1, -2)$.

e) $x_1 \cdot (1, 0, 0, 0) + x_2 \cdot (0, 1, 0, 0) + x_3 \cdot (0, 0, 1, 0) + x_4 \cdot (0, 0, 0, 1)$. 

Chapter 2

Subspaces
In most applications we will be working with a subset $W$ of a vector space $V$ such that $W$ itself is a vector space.
Subspace of a vector space

In most applications we will be working with a subset $W$ of a vector space $V$ such that $W$ itself is a vector space.

Question: Do we have to test all the axioms to find out if $W$ is a vector space?
Subspace of a vector space

In most applications we will be working with a subset $W$ of a vector space $V$ such that $W$ itself is a vector space.

Question: Do we have to test all the axioms to find out if $W$ is a vector space?

The answer is NO.
Subspace of a vector space

In most applications we will be working with a subset $W$ of a vector space $V$ such that $W$ itself is a vector space.

Question: Do we have to test all the axioms to find out if $W$ is a vector space?

The answer is NO.

**Theorem.** Let $W \neq \emptyset$ be a subset of a vector space $V$. Then $W$, with the addition and scalar multiplication as $V$, is a vector space if and only if:

- $u + v \in W$ for all $u, v \in W$ (or $W + W \subseteq W$)
- $r \cdot u \in W$ for all $r \in \mathbb{R}$ and all $u \in W$ (or $\mathbb{R}W \subseteq W$).
In most applications we will be working with a subset $W$ of a vector space $V$ such that $W$ itself is a vector space.

Question: Do we have to test all the axioms to find out if $W$ is a vector space?

The answer is NO.

**Theorem.** Let $W \neq \emptyset$ be a subset of a vector space $V$. Then $W$, with the addition and scalar multiplication as $V$, is a vector space if and only if:

- $u + v \in W$ for all $u, v \in W$ (or $W + W \subseteq W$)
- $r \cdot u \in W$ for all $r \in \mathbb{R}$ and all $u \in W$ (or $\mathbb{R}W \subseteq W$).

In this case we say that $W$ is a **subspace** of $V$. 
Proof. Assume that \( W + W \subseteq W \) and \( \mathbb{R}W \subseteq W \).

To show that \( W \) is a vector space we have to show that all the 10 axioms hold for \( W \). But that follows because the axioms hold for \( V \) and \( W \) is a subset of \( V \):
Proof. Assume that \( W + W \subseteq W \) and \( RW \subseteq W \).

To show that \( W \) is a vector space we have to show that all the 10 axioms hold for \( W \). But that follows because the axioms hold for \( V \) and \( W \) is a subset of \( V \): 

- A1 (Commutativity of addition)
  
  For \( u, v \in W \), we have \( u + v = v + u \). This is because \( u, v \) are also in \( V \) and commutativity holds in \( V \).
Proof. Assume that $W + W \subseteq W$ and $RW \subseteq W$. To show that $W$ is a vector space we have to show that all the 10 axioms hold for $W$. But that follows because the axioms hold for $V$ and $W$ is a subset of $V$:

- **A1 (Commutativity of addition)**
  For $u, v \in W$, we have $u + v = v + u$. This is because $u, v$ are also in $V$ and commutativity holds in $V$.

- **A4 (Existence of additive identity)**
  Take any vector $u \in W$. Then by assumption $0 \cdot u = \vec{0} \in W$. Hence $\vec{0} \in W$. 


Proof. Assume that $W + W \subseteq W$ and $\mathbb{R}W \subseteq W$. To show that $W$ is a vector space we have to show that all the 10 axioms hold for $W$. But that follows because the axioms hold for $V$ and $W$ is a subset of $V$:

- **A1 (Commutativity of addition)**
  For $u, v \in W$, we have $u + v = v + u$. This is because $u, v$ are also in $V$ and commutativity holds in $V$.

- **A4 (Existence of additive identity)**
  Take any vector $u \in W$. Then by assumption $0 \cdot u = \vec{0} \in W$. Hence $\vec{0} \in W$.

- **A5 (Existence of additive inverse)**
  If $u \in W$ then $-u = (-1) \cdot u \in W$. 

□
Proof. Assume that $W + W \subseteq W$ and $\mathbb{R}W \subseteq W$.
To show that $W$ is a vector space we have to show that all the 10 axioms hold for $W$. But that follows because the axioms hold for $V$ and $W$ is a subset of $V$:

- **A1** (Commutativity of addition)
  For $u, v \in W$, we have $u + v = v + u$. This is because $u, v$ are also in $V$ and commutativity holds in $V$.

- **A4** (Existence of additive identity)
  Take any vector $u \in W$. Then by assumption $0 \cdot u = \vec{0} \in W$. Hence $\vec{0} \in W$.

- **A5** (Existence of additive inverse)
  If $u \in W$ then $-u = (-1) \cdot u \in W$.

One can check that the other axioms follow in the same way.
Usually the situation is that we are given a vector space $V$ and a subset of vectors $W$ satisfying some conditions and we need to see if $W$ is a subspace of $V$. 
Examples

Usually the situation is that we are given a vector space $V$ and a subset of vectors $W$ satisfying some conditions and we need to see if $W$ is a subspace of $V$.

$$W = \{ v \in V : \text{some conditions on } v \}$$
Usually the situation is that we are given a vector space $V$ and a subset of vectors $W$ satisfying some conditions and we need to see if $W$ is a subspace of $V$.

$$W = \{ v \in V : \text{some conditions on } v \}$$

We will then have to show that

$$u, v \in W \quad u + v \quad r \in \mathbb{R} \quad r \cdot u$$

Satisfy the same conditions.
Lines through the origin as subspaces of $\mathbb{R}^2$

Example.

\[ V = \mathbb{R}^2, \]
\[ W = \{(x, y) | y = kx\} \text{ for a given } k \]
\[ = \text{ line through } (0, 0) \text{ with slope } k. \]
**Example.**

\[ V = \mathbb{R}^2, \]
\[ W = \{(x, y) | y = kx\} \quad \text{for a given } k \]
\[ = \text{line through } (0, 0) \text{ with slope } k. \]

*To see that \( W \) is in fact a subspace of \( \mathbb{R}^2 \):*

Let \( u = (x_1, y_1), \ v = (x_2, y_2) \in W. \) Then \( y_1 = kx_1 \) and \( y_2 = kx_2 \)
Lines through the origin as subspaces of $\mathbb{R}^2$

Example.

$V = \mathbb{R}^2$,

$W = \{(x, y) | y = kx\}$ for a given $k$

= line through $(0, 0)$ with slope $k$.

To see that $W$ is in fact a subspace of $\mathbb{R}^2$:

Let $u = (x_1, y_1), v = (x_2, y_2) \in W$. Then $y_1 = kx_1$ and $y_2 = kx_2$

and

$u + v = (x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2, kx_1 + kx_2)$

$= (x_1 + x_2, k(x_1 + x_2)) \in W$
Lines through the origin as subspaces of $\mathbb{R}^2$

Example.

$$V = \mathbb{R}^2,$$
$$W = \{ (x, y) | y = kx \} \quad \text{for a given } k$$
$$= \text{line through } (0, 0) \text{ with slope } k.$$

To see that $W$ is in fact a subspace of $\mathbb{R}^2$:
Let $u = (x_1, y_1), \ v = (x_2, y_2) \in W$. Then $y_1 = kx_1$ and $y_2 = kx_2$
and

$$u + v = (x_1 + x_2, y_1 + y_2)$$
$$= (x_1 + x_2, kx_1 + kx_2)$$
$$= (x_1 + x_2, k(x_1 + x_2)) \in W$$

Similarly, $r \cdot u = (rx_1, ry_1) = (rx_1, kr x_1) \in W$
So what are the subspaces of \( \mathbb{R}^2 \)?
So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$
So what are the subspaces of $\mathbb{R}^2$?
1. \{0\}
2. Lines. But only those that contain $(0, 0)$. Why?
So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$

2. Lines. But only those that contain $(0, 0)$. Why?

3. $\mathbb{R}^2$
So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$

2. Lines. But only those that contain $(0, 0)$. Why?

3. $\mathbb{R}^2$

*Remark (First test).* If $W$ is a subspace, then $\vec{0} \in W$.

*Thus:* If $\vec{0} \notin W$, then $W$ is not a subspace.
So what are the subspaces of $\mathbb{R}^2$?

1. $\{0\}$
2. Lines. But only those that contain $(0, 0)$. Why?
3. $\mathbb{R}^2$

*Remark (First test).* If $W$ is a subspace, then $\vec{0} \in W$.

**Thus:** If $\vec{0} \notin W$, then $W$ is not a subspace.

This is why a line not passing through $(0, 0)$ can not be a subspace of $\mathbb{R}^2$. 
A subset of $\mathbb{R}^2$ that is not a subspace

*Warning.* We can not conclude from the fact that $\vec{0} \in W$, that $W$ is a subspace.
A subset of $\mathbb{R}^2$ that is not a subspace

Warning. We can not conclude from the fact that $\vec{0} \in W$, that $W$ is a subspace.

Example. Let's consider the following subset of $\mathbb{R}^2$:

$$W = \{(x, y) | x^2 - y^2 = 0\}$$

Is $W$ a subspace of $\mathbb{R}^2$? Why?
Warning. We can not conclude from the fact that $\vec{0} \in W$, that $W$ is a subspace.

**Example.** Let's consider the following subset of $\mathbb{R}^2$:

$$W = \{(x, y)|x^2 - y^2 = 0\}$$

Is $W$ a subspace of $\mathbb{R}^2$? Why?

The answer is NO.

We have $(1, 1)$ and $(1, -1) \in W$ but $(1, 1) + (1, -1) = (2, 0) \notin W$. i.e., $W$ is not closed under addition.
A subset of $\mathbb{R}^2$ that is not a subspace

**Warning.** We cannot conclude from the fact that $\vec{0} \in W$, that $W$ is a subspace.

**Example.** Let's consider the following subset of $\mathbb{R}^2$:

$$W = \{(x, y) | x^2 - y^2 = 0\}$$

Is $W$ a subspace of $\mathbb{R}^2$? Why?

The answer is NO.

We have $(1, 1)$ and $(1, -1) \in W$ but $(1, 1) + (1, -1) = (2, 0) \notin W$. i.e., $W$ is not closed under addition.

Notice that $(0, 0) \in W$ and $W$ is closed under multiplication by scalars.
What are the subspaces of $\mathbb{R}^3$?
What are the subspaces of $\mathbb{R}^3$?

1. $\{0\}$ and $\mathbb{R}^3$. 
What are the subspaces of $\mathbb{R}^3$?

1. \{0\} and $\mathbb{R}^3$.

2. Planes: A plane $W \subseteq \mathbb{R}^3$ is given by a normal vector $(a, b, c)$ and its distance from $(0, 0, 0)$ or

$$W = \{(x, y, z) | ax + by + cz = p\}$$

condition on $(x, y, z)$
What are the subspaces of $\mathbb{R}^3$?

1. $\{0\}$ and $\mathbb{R}^3$.

2. Planes: A plane $W \subseteq \mathbb{R}^3$ is given by a normal vector $(a, b, c)$ and its distance from $(0, 0, 0)$ or

$$W = \{(x, y, z) \mid ax + by + cz = p\}$$

condition on $(x, y, z)$

For $W$ to be a subspace, $(0, 0, 0)$ must be in $W$ by the first test. Thus

$$p = a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$$

or

$$p = 0$$
Planes containing the origin

A plane containing \((0, 0, 0)\) is indeed a subspace of \(\mathbb{R}^3\).
A plane containing \((0, 0, 0)\) is indeed a subspace of \(\mathbb{R}^3\).

**Proof.** Let \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) \(\in W\). Then

\[
ax_1 + by_1 + cz_1 = 0 \\
ax_2 + by_2 + cz_2 = 0
\]
A plane containing \((0, 0, 0)\) is indeed a subspace of \(\mathbb{R}^3\).

**Proof.** Let \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) \(\in W\). Then

\[
\begin{align*}
ax_1 + by_1 + cz_1 &= 0 \\
ax_2 + by_2 + cz_2 &= 0
\end{align*}
\]

Then we have

\[
a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2)
\]

\[
= (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2)
\]

\[
= 0 + 0 = 0
\]
A plane containing \((0, 0, 0)\) is indeed a subspace of \(\mathbb{R}^3\).

**Proof.** Let \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) \(\in W\). Then

\[
ax_1 + by_1 + cz_1 = 0 \\
ax_2 + by_2 + cz_2 = 0
\]

Then we have

\[
\begin{align*}
    a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) &= (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) \\
    &= 0 + 0 \\
    &= 0
\end{align*}
\]

and \(a(rx_1) + b(ry_1) + c(rz_1) = r(ax_1 + by_1 + cz_1) = 0\) \(\square\)
Summary of subspaces of $\mathbb{R}^3$

1. $\{0\}$ and $\mathbb{R}^3$.

2. Planes containing $(0, 0, 0)$. 
Summary of subspaces of $\mathbb{R}^3$

1. \( \{0\} \) and $\mathbb{R}^3$.

2. Planes containing \((0, 0, 0)\).

3. Lines containing \((0, 0, 0)\).
   (Intersection of two planes containing \((0, 0, 0)\))
Exercises

Determine whether the given subset of $\mathbb{R}^n$ is a subspace or not (Explain):

a) $W = \{(x, y) \in \mathbb{R}^2 | xy = 0\}$.

b) $W = \{(x, y, z) \in \mathbb{R}^3 | 3x + 2y^2 + z = 0\}$.

c) $W = \{(x, y, z) \in \mathbb{R}^3 | 2x + 3y - z = 0\}$.

d) The set of all vectors $(x_1, x_2, x_3)$ satisfying

\[2x_3 = x_1 - 10x_2\]
Exercises

Determine whether the given subset of $\mathbb{R}^n$ is a subspace or not (Explain):

e) The set of all vectors in $\mathbb{R}^4$ satisfying the system of linear equations

\[
\begin{align*}
2x_1 + 3x_2 + 5x_4 &= 0 \\
x_1 + x_2 - 3x_3 &= 0
\end{align*}
\]

f) The set of all points $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ satisfying

\[x_1 + 2x_2 + 3x_3 + x_4 = -1\]
Chapter 3

Vector Spaces of Functions
Let $I \subseteq \mathbb{R}$ be an interval.
Let \( I \subseteq \mathbb{R} \) be an interval. Then \( I \) is of the form (for some \( a < b \))

\[
I = \begin{cases} 
\{ x \in \mathbb{R} \mid a < x < b \}, & \text{an open interval;} \\
\{ x \in \mathbb{R} \mid a \leq x \leq b \}, & \text{a closed interval;} \\
\{ x \in \mathbb{R} \mid a < x < b \} \\
\{ x \in \mathbb{R} \mid a < x \leq b \} \\
\{ x \in \mathbb{R} \mid a \leq x < b \}. 
\end{cases}
\]
Let $I \subseteq \mathbb{R}$ be an interval. Then recall that the space of all functions $f : I \rightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:
Let $I \subseteq \mathbb{R}$ be an interval. Then recall that the space of all functions $f : I \rightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:

**Example (1).** Let $C(I)$ be the space of continuous functions. If $f$ and $g$ are continuous, so are the functions $f + g$ and $rf$ ($r \in \mathbb{R}$). Hence $C(I)$ is a vector space.
Let $I \subseteq \mathbb{R}$ be an interval. Then recall that the space of all functions $f : I \rightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:

**Example (1).** Let $C(I)$ be the space of continuous functions. If $f$ and $g$ are continuous, so are the functions $f + g$ and $rf$ ($r \in \mathbb{R}$). Hence $C(I)$ is a vector space.

Recall, that a function is continuous, if the graph has no gaps. This can be formulated in different ways:
Let $I \subseteq \mathbb{R}$ be an interval. Then recall that the space of all functions $f : I \rightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:

**Example (1).** Let $C(I)$ be the space of continuous functions. If $f$ and $g$ are continuous, so are the functions $f + g$ and $rf$ ($r \in \mathbb{R}$). Hence $C(I)$ is a vector space.

Recall, that a function is continuous, if the graph has no gaps. This can be formulated in different ways:

a) Let $x_0 \in I$ and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that for all $x \in I \cap (x_0 - \delta, x_0 + \delta)$ we have

$$|f(x) - f(x_0)| < \epsilon$$

This tells us that the value of $f$ at nearby points is arbitrarily close to the value of $f$ at $x_0$. 
Space of Continuous Functions

Let $I \subseteq \mathbb{R}$ be an interval. Then recall that the space of all functions $f : I \rightarrow \mathbb{R}$ is a vector space. We will now list some important subspaces:

**Example (1).** Let $C(I)$ be the space of continuous functions. If $f$ and $g$ are continuous, so are the functions $f + g$ and $rf$ ($r \in \mathbb{R}$). Hence $C(I)$ is a vector space.

Recall, that a function is continuous, if the graph has no gaps. This can be formulated in different ways:

b) A reformulation of (a) is:

$$\lim_{x \to x_0} f(x) = f(x_0)$$
Example (2). The space $C^1(I)$. Here we assume that $I$ is open.
Space of continuously differentiable functions.

Example (2). The space $C^1(I)$. Here we assume that $I$ is open. Recall that $f$ is differentiable at $x_0$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} =: f'(x_0)$$

exists.
Space of continuously differentiable functs.

**Example (2).** The space $C^1(I)$. Here we assume that $I$ is open. Recall that $f$ is differentiable at $x_0$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} =: f'(x_0)$$

exists. If $f'(x_0)$ exists for all $x_0 \in I$, then we say that $f$ is differentiable on $I$. In this case we get a new function on $I$

$$x \mapsto f'(x)$$

We say that $f$ is continuously differentiable on $I$ if $f'$ exists and is continuous on $I$. 

Spaces of Functions

- $C(I)$
- $C^r(I)$
- $PC(I)$

Indicator functions

- $\chi_{A \cap B}$
- $\chi_{A \cup B}$ (disjoint)
- $\chi_{A \cup B}$
Example (2). The space $C^1(I)$. Here we assume that $I$ is open. Recall that $f$ is differentiable at $x_0$ if

$$
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} =: f'(x_0)
$$

exists. We say that $f$ is continuously differentiable on $I$ if $f'$ exists and is continuous on $I$. Recall that if $f$ and $g$ are differentiable, then so are

$$f + g$$

and

$$rf \quad (r \in \mathbb{R})$$

moreover

$$(f + g)' = f' + g' ; \quad (rf)' = rf'$$
**Example (2).** The space $C^1(I)$. Here we assume that $I$ is open. Recall that $f$ is differentiable at $x_0$ if

$$
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} =: f'(x_0)
$$

exists. We say that $f$ is continuously differentiable on $I$ if $f'$ exists and is continuous on $I$. Recall that if $f$ and $g$ are differentiable, then so are

$$f + g$$

and

$$rf \quad (r \in \mathbb{R})$$

moreover

$$(f + g)' = f' + g' ; \quad (rf)' = rf'$$

As $f' + g'$ and $rf'$ are continuous by Example (1), it follows that $C^1(I)$ is a vector space.
A continuous but not differentiable function

Let \( f(x) = |x| \) for \( x \in \mathbb{R} \). Then \( f \) is continuous on \( \mathbb{R} \) but it is not differentiable on \( \mathbb{R} \).
Let \( f(x) = |x| \) for \( x \in \mathbb{R} \). Then \( f \) is continuous on \( \mathbb{R} \) but it is not differentiable on \( \mathbb{R} \). We show that \( f \) is not differentiable at \( x_0 = 0 \).
Let \( f(x) = |x| \) for \( x \in \mathbb{R} \). Then \( f \) is continuous on \( \mathbb{R} \) but it is not differentiable on \( \mathbb{R} \). We show that \( f \) is not differentiable at \( x_0 = 0 \). For \( h > 0 \) we have

\[
\frac{f(x_0 + h) - f(x_0)}{h} = \frac{|h| - 0}{h} = \frac{h}{h} = 1
\]

hence

\[
\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = 1
\]
A continuous but not differentiable function

Let \( f(x) = |x| \) for \( x \in \mathbb{R} \). Then \( f \) is continuous on \( \mathbb{R} \) but it is not differentiable on \( \mathbb{R} \). We show that \( f \) is not differentiable at \( x_0 = 0 \). For \( h > 0 \) we have

\[
\frac{f(x_0 + h) - f(x_0)}{h} = \frac{|h| - 0}{h} = \frac{h}{h} = 1
\]

hence

\[
\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = 1
\]

But if \( h < 0 \), then

\[
\frac{f(x_0 + h) - f(x_0)}{h} = \frac{|h| - 0}{h} = \frac{-h}{h} = -1
\]

hence

\[
\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = -1
\]
A continuous but not differentiable function

Let \( f(x) = |x| \) for \( x \in \mathbb{R} \). Then \( f \) is continuous on \( \mathbb{R} \) but it is not differentiable on \( \mathbb{R} \). We show that \( f \) is not differentiable at \( x_0 = 0 \).

Therefore,

\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}
\]

does not exist.
Space of \( r \)-times continuously diff. functs.

**Example (3).** *The space \( C^r(I) \)*

Let \( I = (a, b) \) be an open interval. and let \( r \in \mathbb{N} = \{1, 2, 3, \ldots \} \).

**Definition.** The function \( f : I \to \mathbb{R} \) is said to be \( r \)-times continuously differentiable if all the derivatives \( f', f'', \ldots, f^{(r)} \) exist and \( f^{(r)} : I \to \mathbb{R} \) is continuous.

We denote by \( C^r(I) \) the space of \( r \)-times continuously differentiable functions on \( I \). \( C^r(I) \) is a subspace of \( C(I) \).
Example (3). The space $C^r(I)$
Let $I = (a, b)$ be an open interval. and let
$r \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Definition. The function $f : I \rightarrow \mathbb{R}$ is said to be
$r$-times continuously differentiable if all the derivatives
$f', f'', \cdots, f^{(r)}$ exist and $f^{(r)} : I \rightarrow \mathbb{R}$ is continuous.

We denote by $C^r(I)$ the space of $r$-times continuously
differentiable functions on $I$. $C^r(I)$ is a subspace of $C(I)$.

We have
$$C^r(I) \not\subseteq C^{r-1}(I) \not\subseteq \cdots \not\subseteq C^1(I) \not\subseteq C(I).$$
We have seen that $C^1(I) \neq C(I)$. Let us try to find a function that is in $C^1(I)$ but not in $C^2(I)$.\"
We have seen that \( C^1(I) \neq C(I) \). Let us try to find a function that is in \( C^1(I) \) but not in \( C^2(I) \).

Assume \( 0 \in I \) and let \( f(x) = x^{\frac{5}{3}} \). Then \( f \) is differentiable and

\[
f'(x) = \frac{5}{3} x^{\frac{2}{3}}
\]

which is continuous.
We have seen that \( C^1(I) \neq C(I) \). Let us try to find a function that is in \( C^1(I) \) but not in \( C^2(I) \).

Assume \( 0 \in I \) and let \( f(x) = x^{\frac{5}{3}} \). Then \( f \) is differentiable and

\[
 f'(x) = \frac{5}{3} x^{\frac{2}{3}}
\]

which is continuous.

If \( x \neq 0 \), then \( f' \) is differentiable and

\[
 f''(x) = \frac{10}{3} x^{-\frac{1}{3}}
\]
But for \( x = 0 \) we have

\[
\lim_{h \to 0} \frac{f'(h) - 0}{h} = \lim_{h \to 0} \frac{5}{3} \frac{h^2}{h} = \lim_{h \to 0} \frac{5}{3} h^{-\frac{1}{3}}
\]

which does not exist.
But for $x = 0$ we have

$$\lim_{h \to 0} \frac{f'(h) - 0}{h} = \lim_{h \to 0} \frac{5}{3} \frac{h^2}{h} = \lim_{h \to 0} \frac{5}{3} h^{-\frac{1}{3}}$$

which does not exist.

Remark. One can show that the function

$$f(x) = x^{\frac{3r-1}{3}}$$

is in $C^{r-1}(\mathbb{R})$, but not in $C^r(\mathbb{R})$. 
But for \( x = 0 \) we have

\[
\lim_{h \to 0} \frac{f'(h) - 0}{h} = \lim_{h \to 0} \frac{5}{3} \frac{h^3}{h} = \lim_{h \to 0} \frac{5}{3} h^{-\frac{1}{3}}
\]

which does not exist.

**Remark.** One can show that the function

\[
f(x) = x^{\frac{3r-1}{3}}
\]

is in \( C^{r-1}(\mathbb{R}) \), but not in \( C^r(\mathbb{R}) \).

Thus, as stated before, we have

\[
C^r(I) \subsetneq C^{r-1}(I) \subsetneq \cdots \subsetneq C^1(I) \subsetneq C(I).
\]
**Example (4). Piecewise-continuous functions**

**Definition.** Let $I = [a, b)$. A function $f : I \rightarrow \mathbb{R}$ is called **piecewise-continuous** if there exists finitely many points

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that $f$ is continuous on each of the sub-intervals $(x_i, x_{i+1})$ for $i = 0, 1, \cdots, n - 1$. 
Example (4). Piecewise-continuous functions

Definition. Let $I = [a, b)$. A function $f : I \rightarrow \mathbb{R}$ is called 

piecewise-continuous if there exists finitely many points 

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that $f$ is continuous on each of the sub-intervals 

$$(x_i, x_{i+1})$$ for $i = 0, 1, \cdots, n - 1$.

Remark. If $f$ and $g$ are both piecewise-continuous, then 

$$f + g$$ and 

$$rf \ (r \in \mathbb{R})$$

are piecewise-continuous.
Example (4). Piecewise-continuous functions

Definition. Let $I = [a, b)$. A function $f : I \rightarrow \mathbb{R}$ is called piecewise-continuous if there exists finitely many points $a = x_0 < x_1 < \cdots < x_n = b$ such that $f$ is continuous on each of the sub-intervals $(x_i, x_{i+1})$ for $i = 0, 1, \cdots, n - 1$.

Remark. If $f$ and $g$ are both piecewise-continuous, then $f + g$ and $rf$ ($r \in \mathbb{R}$) are piecewise-continuous.

Hence the space of piecewise-continuous functions is a vector space. Denote this vector space by $PC(I)$. 
The indicator function $\chi_A$

Important elements of $PC(I)$ are the indicator functions $\chi_A$, where $A \subseteq I$ a sub-interval.
Important elements of $PC(I)$ are the indicator functions $\chi_A$, where $A \subseteq I$ a sub-interval.

Let $A \subseteq \mathbb{R}$ be a set. Define

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$
Important elements of $PC(I)$ are the indicator functions $\chi_A$, where $A \subseteq I$ a sub-interval.

Let $A \subseteq \mathbb{R}$ be a set. Define

$$\chi_A(x) = \begin{cases} 
1, & \text{if } x \in A \\
0, & \text{if } x \notin A 
\end{cases}$$

So the values of $\chi_A$ tell us whether $x$ is in $A$ or not.

If $x \in A$, then $\chi_A(x) = 1$ and if $x \notin A$, then $\chi_A(x) = 0$. 
Important elements of $PC(I)$ are the indicator functions $\chi_A$, where $A \subseteq I$ a sub-interval.

Let $A \subseteq \mathbb{R}$ be a set. Define

$$\chi_A(x) = \begin{cases} 
1, & \text{if } x \in A \\
0, & \text{if } x \notin A
\end{cases}$$

So the values of $\chi_A$ tell us whether $x$ is in $A$ or not.

If $x \in A$, then $\chi_A(x) = 1$ and if $x \notin A$, then $\chi_A(x) = 0$.

We will work a lot with indicator functions so let us look at some of their properties.
Some properties of $\chi_A$

**Lemma.** Let $A, B \subseteq I$. Then

$$\chi_{A \cap B}(x) = \chi_A(x) \chi_B(x) \quad (\ast)$$
Lemma. Let $A, B \subseteq I$. Then

$$\chi_{A \cap B}(x) = \chi_A(x) \chi_B(x) \quad (\ast)$$

Proof. We have to show that the two functions

$$x \mapsto \chi_{A \cap B}(x) \quad \text{and} \quad x \mapsto \chi_A(x) \chi_B(x)$$

take the same values at every point $x \in I$. So let’s evaluate both functions:
**Lemma.** Let $A, B \subseteq I$. Then

$$\chi_{A \cap B}(x) = \chi_A(x) \chi_B(x) \quad (\ast)$$

**Proof.** We have to show that the two functions

$$x \mapsto \chi_{A \cap B}(x) \quad \text{and} \quad x \mapsto \chi_A(x) \chi_B(x)$$

take the same values at every point $x \in I$. So let's evaluate both functions:

If $x \in A$ and $x \in B$, that is $x \in A \cap B$, then

$$\chi_{A \cap B}(x) = 1 \quad \text{and},$$
**Lemma.** Let $A, B \subseteq I$. Then

$$\chi_{A \cap B}(x) = \chi_A(x) \chi_B(x) \quad (\ast)$$

**Proof.** We have to show that the two functions

$$x \mapsto \chi_{A \cap B}(x) \quad \text{and} \quad x \mapsto \chi_A(x) \chi_B(x)$$

take the same values at every point $x \in I$. So lets evaluate both functions:

If $x \in A$ and $x \in B$, that is $x \in A \cap B$, then

$$\chi_{A \cap B}(x) = 1 \quad \text{and,}$$

since $\chi_A(x) = 1$ and $\chi_B(x) = 1$, we also have

$$\chi_A(x) \chi_B(x) = 1.$$  

Thus, the left and the right hand sides of $(\ast)$ agree.
Lemma. Let $A, B \subseteq I$. Then

$$\chi_{A \cap B}(x) = \chi_A(x) \chi_B(x) \quad (*)$$

Proof. We have to show that the two functions

$$x \mapsto \chi_{A \cap B}(x) \quad \text{and} \quad x \mapsto \chi_A(x) \chi_B(x)$$

take the same values at every point $x \in I$. So let's evaluate both functions:

On the other hand, if $x \notin A \cap B$, then there are two possibilities:
**Lemma.** Let $A, B \subseteq I$. Then

$$\chi_{A \cap B}(x) = \chi_A(x) \chi_B(x) \quad (*)$$

**Proof.** We have to show that the two functions

$$x \mapsto \chi_{A \cap B}(x) \quad \text{and} \quad x \mapsto \chi_A(x) \chi_B(x)$$

take the same values at every point $x \in I$. So let's evaluate both functions:

On the other hand, if $x \notin A \cap B$, then there are two possibilities:

- $x \notin A$ then $\chi_A(x) = 0$, so $\chi_A(x) \chi_B(x) = 0$. 

Lemma. Let \( A, B \subseteq I \). Then

\[
\chi_{A \cap B}(x) = \chi_A(x) \chi_B(x) \quad (*)
\]

Proof. We have to show that the two functions

\[
x \mapsto \chi_{A \cap B}(x) \quad \text{and} \quad x \mapsto \chi_A(x) \chi_B(x)
\]

take the same values at every point \( x \in I \). So let’s evaluate both functions:

On the other hand, if \( x \notin A \cap B \), then there are two possibilities:

- \( x \notin A \) then \( \chi_A(x) = 0 \), so \( \chi_A(x) \chi_B(x) = 0 \).
- \( x \notin B \) then \( \chi_B(x) = 0 \), so \( \chi_A(x) \chi_B(x) = 0 \).

It follows that

\[
0 = \chi_{A \cap B}(x) = \chi_A(x) \chi_B(x) \quad \square
\]
What about $\chi_{A \cup B}$? Can we express it in terms of $\chi_A, \chi_B$?
Some properties of $\chi_A$

What about $\chi_{A \cup B}$? Can we express it in terms of $\chi_A, \chi_B$? If $A$ and $B$ are disjoint, that is $A \cap B = \emptyset$ then

$$\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x).$$
Some properties of $\chi_A$

What about $\chi_{A \cup B}$? Can we express it in terms of $\chi_A$, $\chi_B$?

If $A$ and $B$ are disjoint, that is $A \cap B = \emptyset$ then

$$\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x).$$

Let us prove this:

- If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$. Thus the LHS (left hand side) and the RHS (right hand side) are both zero.
Some properties of $\chi_A$

What about $\chi_{A \cup B}$? Can we express it in terms of $\chi_A$, $\chi_B$?

If $A$ and $B$ are disjoint, that is $A \cap B = \emptyset$ then

$$\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x).$$

Let us prove this:

- If $x \not\in A \cup B$, then $x \not\in A$ and $x \not\in B$. Thus the LHS (left hand side) and the RHS (right hand side) are both zero.

- If $x \in A \cup B$ then either
Some properties of $\chi_A$

What about $\chi_{A \cup B}$? Can we express it in terms of $\chi_A$, $\chi_B$?

If $A$ and $B$ are disjoint, that is $A \cap B = \emptyset$ then

$$\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x).$$

Let us prove this:

- If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$. Thus the LHS (left hand side) and the RHS (right hand side) are both zero.

- If $x \in A \cup B$ then either
  - $x$ is in $A$ but not in $B$. In this case
    $$\chi_{A \cup B}(x) = 1 \text{ and } \chi_A(x) + \chi_B(x) = 1 + 0 = 1$$
Some properties of $\chi_A$

What about $\chi_{A \cup B}$? Can we express it in terms of $\chi_A$, $\chi_B$?

If $A$ and $B$ are disjoint, that is $A \cap B = \emptyset$ then

$$\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x).$$

Let us prove this:

- If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$. Thus the LHS (left hand side) and the RHS (right hand side) are both zero.

- If $x \in A \cup B$ then either
  - $x$ is in $A$ but not in $B$. In this case
    $$\chi_{A \cup B}(x) = 1 \text{ and } \chi_A(x) + \chi_B(x) = 1 + 0 = 1$$
  - or
  - $x$ is in $B$ but not in $A$. In this case
    $$\chi_{A \cup B}(x) = 1 \text{ and } \chi_A(x) + \chi_B(x) = 0 + 1 = 1$$
Thus we have, 

If \( A \cap B = \emptyset \), then \( \chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) \).
Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?
Thus we have,

If \( A \cap B = \emptyset \), then \( \chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) \).

Now, what if \( A \cap B \neq \emptyset \)?

**Lemma.** \( \chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x) \).
Thus we have,

If \( A \cap B = \emptyset \), then \( \chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) \).

Now, what if \( A \cap B \neq \emptyset \)?

**Lemma.** \( \chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x) \).

**Proof.**

- If \( x \notin A \cup B \), then both of the LHS and the RHS take the value 0.
Thus we have,

If \( A \cap B = \emptyset \), then \( \chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) \).

Now, what if \( A \cap B \neq \emptyset \)?

**Lemma.** \( \chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x) \).

**Proof.**
- If \( x \notin A \cup B \), then both of the LHS and the RHS take the value 0.
- If \( x \in A \cup B \), then we have the following possibilities:
Spaces of Functions

\[ C(I) \]
\[ C^1(I) \]
\[ C^r(I) \]
\[ PC(I) \]

Indicator functions

\[ \chi_A \]
\[ \chi_{A \cap B} \]
\[ \chi_{A \cup B} \text{ (disjoint)} \]

---

Some properties of \( \chi_A \)

Thus we have,

If \( A \cap B = \emptyset \), then \( \chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) \).

Now, what if \( A \cap B \neq \emptyset \)?

**Lemma.** \( \chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x) \).

**Proof.**

- If \( x \notin A \cup B \), then both of the LHS and the RHS take the value 0.

- If \( x \in A \cup B \), then we have the following possibilities:
  1. If \( x \in A, x \notin B \), then
    \[
    \chi_{A \cup B}(x) = 1 \\
    \chi_A(x) + \chi_B(x) - \chi_{A \cap B} = 1 + 0 - 0 = 1
    \]
Some properties of $\chi_A$

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

**Lemma.** $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x)$.

**Proof.**
- If $x \notin A \cup B$, then both of the LHS and the RHS take the value 0.
- If $x \in A \cup B$, then we have the following possibilities:
  1. If $x \in A$, $x \notin B$, then
     \[ \chi_{A \cup B}(x) = 1 \]
     \[ \chi_A(x) + \chi_B(x) - \chi_{A \cap B} = 1 + 0 - 0 = 1 \]
  2. Similarly for the case $x \in B$, $x \notin A$: LHS equals the RHS.
Some properties of $\chi_A$

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

Lemma. $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x)$.

Proof.

- If $x \notin A \cup B$, then both of the LHS and the RHS take the value 0.
- If $x \in A \cup B$, then we have the following possibilities:
  
  3. If $x \in A \cap B$, then

  $$\chi_{A \cup B}(x) = 1$$

  $$\chi_A(x) + \chi_B(x) - \chi_{A \cap B} = 1 + 1 - 1 = 1$$
Some properties of $\chi_A$

Thus we have,

If $A \cap B = \emptyset$, then $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x)$.

Now, what if $A \cap B \neq \emptyset$?

**Lemma.** $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x)$.

**Proof.**

- If $x \notin A \cup B$, then both of the LHS and the RHS take the value 0.

- If $x \in A \cup B$, then we have the following possibilities:
  1. If $x \notin A \cap B$, then $\chi_{A \cup B}(x) = 1$
  2. If $x \in A \cap B$, then
     
     $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x) = 1 + 1 - 1 = 1$

As we have checked all possibilities, we have shown that the statement in the lemma is correct.