Math 2025, Homework, Due Tuesday, Sept. 17

Name: ______________________________

We denote by $\varphi_{[a,b)}(t)$ the indicator function of the interval $[a,b)$. Thus $\varphi_{[a,b)}(t) = \begin{cases} 1 & t \in [a,b) \\ 0 & \text{otherwise} \end{cases}$.
We let $\varphi(t) = \varphi_{[0,1)}(t)$ be the Haar scaling function and $\psi(t) = \varphi(2t) - \varphi(2t-1)$.

For the solution of (1) and (2) you use the following facts:

- $\varphi(\lambda t - r) = \varphi_{[r/\lambda,(r+1)/\lambda}(t)$
- $\varphi_{[a,b)}(t) = \varphi\left(\frac{1}{b-a}(t-a)\right)$

1) Write the following translate and dilate $\varphi(\lambda t - r)$ of the Haar scaling function as an indicator function $\varphi_{[a,b)}(t)$ of an interval.

(1) $\varphi(8t-5) = \varphi_{[\frac{5}{8},\frac{3}{4})}(t)$
(2) $\varphi(\frac{t}{5}-1) = \varphi_{[5,10)}$

2) Write the following indicator functions as translate and dilate of the Haar scaling function:

(1) $\varphi_{[1,3)}(t) = \varphi\left(\frac{1}{2}(t-1)\right) = \varphi\left(\frac{1}{2}t - \frac{1}{2}\right)$
(2) $\varphi_{[0.5,2)}(t) = \varphi\left(\frac{2}{3}\left(t-\frac{1}{2}\right)\right) = \varphi\left(\frac{2}{3}t - \frac{1}{3}\right)$

3) Write the step function $f(t) = 3\varphi(4t) + \varphi(4t-1) + 2\varphi(4t-2) + 4\varphi(4t-3)$ as a linear combination of the function $\varphi(t), \psi(t)$, $\psi(2t)$, and $\psi(2t-1)$.

Solution: We apply the ordered Haar wavelet transform to the vector $(3,1,2,4)$ and get

$$(3,1,2,4) \mapsto (2,3;1,-1) \mapsto (\frac{5}{2};-\frac{1}{2};1,-1)\ .$$

Thus

$$3\varphi(4t) + \varphi(4t-1) + 2\varphi(4t-2) + 4\varphi(4t-3) = \frac{5}{2}\varphi(t) - \frac{1}{2}\psi(t) + \psi(2t) - \psi(2t-1)\ .$$

Notice, that if we use the notation $\varphi^i_j(t) = \varphi(2^i t - j)$ and $\psi^i_j(t) = \psi(2t-j)$ then we can write this as

$$3\varphi^2_0 + \varphi^2_1 + 2\varphi^2_2 + 4\varphi^2_3 = \frac{5}{2}\varphi - \frac{1}{2}\psi + \psi^1_0 - \psi^1_1\ .$$

4) Calculate the ordered fast Haar wavelet transform of the data:

(1) $\vec{\mathbf{s}}^2 = (1,-1,2,0)$. **Answer:** $\vec{\mathbf{s}}^0 = (\frac{1}{2};-\frac{1}{2};1,1)$

Solution:

$$\vec{\mathbf{s}}^2 = (1,-1,2,0) \mapsto \vec{\mathbf{s}}^1 = (0,1;1,1) \mapsto \vec{\mathbf{s}}^0 = (\frac{1}{2};-\frac{1}{2};1,1)$$

(2) $\overrightarrow{\mathbf{s}}^3 = (2,4,6,4,3,1,2,0)$. **Answer:** $\overrightarrow{\mathbf{s}}^0 = (\frac{11}{4}; \frac{5}{4}; -1, \frac{1}{2}; -1,1,1,1)$

Solution:

$$\begin{aligned}(2,4,6,4,3,1,2,0) &\mapsto (3,5,2,1;-1,1,1,1)\\ &\mapsto (4,\frac{3}{2};-1,\frac{1}{2};-1,1,1,1)\\ &\mapsto (\frac{11}{4};\frac{5}{4};-1,\frac{1}{2};-1,1,1,1)\end{aligned}$$

(3) $\overrightarrow{\mathbf{s}}^2 = (1, \frac{1}{2}, \frac{3}{2}, 1)$. **Answer:** $\overrightarrow{\mathbf{s}}^0 = \big(1; -\frac{1}{4}; \frac{1}{4}, \frac{1}{4}\big)$

Solution:

$$(1,\frac{1}{2},\frac{3}{2},1) \mapsto \left(\frac{3}{4},\frac{5}{4};\frac{1}{4},\frac{1}{4}\right) \mapsto \left(1;-\frac{1}{4};\frac{1}{4},\frac{1}{4}\right)$$

5) Assume that the Haar wavelet transform of the initial data $\overrightarrow{\mathbf{s}}^2 = (a_0^{(2)}, a_1^{(2)}, a_2^{(2)}, a_3^{(2)})$ produces the result $\overrightarrow{\mathbf{s}}^0 = (\mathbf{3}, 0, -2, 1)$. Find the initial array $\overrightarrow{\mathbf{s}}^2 = (a_0^{(2)}, a_1^{(2)}, a_2^{(2)}, a_3^{(2)})$.

Solution:

$$\overrightarrow{\mathbf{s}}^0 = (3,0;-2,1) \mapsto \overrightarrow{\mathbf{s}}^1 = (3,3,-2,1) \mapsto \overrightarrow{\mathbf{s}}^2 = (1,5,-1,-3)$$

So the final answer is: $\overrightarrow{\mathbf{s}}^2 == (1,5,-1,-3)$