Noncommutative Harmonic Analysis
An Introduction

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Contents

Introduction

We hope in this text to introduce the student to harmonic analysis and then set up the framework to allow a transition from classical Fourier analysis to the realm of noncommutative harmonic analysis. Indeed, readers may approach the text from several perspectives. For those students wanting to know basic Fourier analysis of periodic functions, Chapter 1 suffices as a firm introduction. If one is interested in the Fourier integral, then Chapter 2 provides one with more than sufficient background to handle the material in Chapter 3 on the Fourier integral and in Chapter 4 on extensions and applications of the Fourier integral. These first four chapters provide the essentials of standard Fourier analysis.

In Chapter 5 we begin by developing the groundwork on which noncommutative harmonic analysis rests. There we cover topological groups and homogeneous spaces from a general and abstract perspective. The chapter gives important examples coming from classical matrix groups and their homogeneous spaces. After this material one can begin Chapter 6 on representation theory. This material is the framework from which to attack harmonic analysis on more general spaces. Here the Haar measure is presented along with basic representation theory and how function theory on homogeneous spaces give rise to important representations. Understanding the decomposition of these representations is equivalent to decomposing functions and their generalizations into simple pieces or harmonics.

In Chapter 7 we apply the tools of Fourier analysis and representation theory to the extremely important Heisenberg group. This group which encodes the commutation relations of quantum mechanics and the uncertainty principle offers a superb example of the large step one must take when dealing with the noncommutative side of harmonic analysis. Indeed, one must step exclusively into the realm of infinite dimensional representation theory. The advantage of this group, however, is how close it is to classical Fourier space and for this reason the tools of Fourier analysis developed in Chapters 3 and 4 are used so successfully.

Finally, Chapter 8 deals with the harmonic analysis associated with compact groups. In this case compactness affords the complete reducibility of representations into finite dimensional representations. Moreover compactness simplifies much of the representation theory. We introduce special traces to obtain projections giving the decompositions of homogeneous representations. One need only read Chapter 6 to peruse this chapter. However, the suggestive interplay between Chapters 7 and Chapters 8 is enlightening, and thus the authors feel much would be gained if the reader did these as a pair.

A substantial part of the material can be approached by well grounded undergraduates. The entire text should fit the needs of an introductory graduate course. The authors welcome feedback on errors and presentation as well as suggestions on problems and additions to material.

Chapter 1

Fourier Series

1. Harmonic Functions on the Disk

In this section we discuss one of the problems that motivated the beginning of the theory of Fourier series and is close to Fourier's original work. Let $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_n)^2$ be the Laplace operator on $\mathbb{R}^n$. It is one of the most interesting differential operators on $\mathbb{R}^n$, in part because of the role it plays in partial differential equations arising in physics:

- **The heat equation:** $\Delta u = a^2 u_t$. Here $u(x,t)$ is a function of $n+1$ variables, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $t > 0$, and the subscript $_t$ denotes the partial derivative with respect to t.
- **The wave equation:** $a^2 \Delta u = u_{tt}$.
- **Schrödinger's equation:** $\frac{1}{i}\Delta u = u_t$.
- **Helmholtz's equation:** $-\Delta u = \lambda u$.

Fourier analysis is *one* of the main tools used to deal with the solutions to these equations; these are discussed later in the book. As motivation we start with the equation $\Delta u = 0$ on the unit disc

$$D := \{z \in \mathbb{C} \mid |z| < 1\} = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\right\}$$

where u takes prescribed values on the boundary. Thus we would like to solve the following *Dirichlet problem:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (x,y) \in D \tag{1.1}$$

$$u(x,y) = f(x,y) \qquad x^2 + y^2 = 1\,. \tag{1.2}$$

Here f is a continuous function on the boundary and we will assume that $u \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$. That is u is twice continuously differentiable in D and continuous on the closed domain $\bar{D} = \big\{(x,y) \mid x^2 + y^2 \leqslant 1\big\}$.

Definition 1.1. *Let Ω be an open subset of $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{C}$ is* ***harmonic*** *on Ω if $\Delta u = 0$.*

Notice that a harmonic function can be viewed as a time independent solution to the heat equation. Let us rewrite (1.1) using polar-coordinates

$$x = r\cos(\theta)\,, \quad y = r\sin(\theta)\,.$$

The Laplacian becomes

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\right)$$

and $u(r,\theta)$ is periodic in θ with period 2π, i.e., $u(r,\theta+2\pi) = u(r,\theta)$. The Dirichlet's problem (1.1) and (1.2) is now

$$\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0\,, \quad u(1,\theta) = f(\theta)\,. \tag{1.3}$$

One approach to this problem is to use *separation of variables*, that is begin by finding solutions of the form:

$$u(r,\theta) = F(r)G(\theta).$$

Then Laplace's equation (1.1) can be rewritten as:

$$\frac{1}{G(\theta)}\frac{d^2G}{d\theta^2}(\theta) = -\frac{r}{F(r)}\frac{d}{dr}\left(r\frac{dF}{dr}(r)\right)\,.$$

The left hand side is independent of r and the right hand side is independent of θ. Hence there is a constant k such that

$$\frac{1}{G(\theta)}\frac{d^2G}{d\theta^2}(\theta) = -\frac{r}{F(r)}\frac{d}{dr}\left(r\frac{dF}{dr}(r)\right) = k.$$

This gives two ordinary differential equations:

$$\frac{d^2G}{d\theta^2}(\theta) = kG(\theta) \text{ and}$$

$$r\frac{d}{dr}\left(r\frac{dF}{dr}(r)\right) = r^2F''(r) + rF'(r) = -kF(r)\,.$$

The general solution to these equations are:

$$G_k(\theta) = \begin{cases} a_0 + b_0\theta & \text{if} \quad k = 0; \\ a_k e^{\sqrt{k}\theta} + b_k e^{-\sqrt{k}\theta} & \text{if} \quad k \neq 0 \end{cases} \tag{1.4}$$

$$F_k(r) = \begin{cases} A_0 + B_0\log(r) & \text{if} \quad k = 0; \\ A_k r^{\sqrt{-k}} + B_k r^{-\sqrt{-k}} & \text{if} \quad k \neq 0 \end{cases} \tag{1.5}$$

where we have indicated the dependence of F and G on the constant k by the index k. The function $G_k(\theta)$ has period 2π if and only if $k = -n^2 < 0$ or $k = 0$ and $b_0 = 0$. The function F_k is defined on all of D if and only if $B_k = 0$ for all k. on the disk. We hence have $F_k(r) = A_k r^n$. Concluding we obtain solutions:

$$u_n(r,\theta) = r^n \left(a_n e^{in\theta} + b_n e^{-in\theta}\right), \qquad n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$$

Writing $b_n = a_{-n}$ and noticing formally that sums of solutions are solutions, we can tentatively write a solution as:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \tag{1.6}$$

$$u(1,\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} = f(\theta)\,. \tag{1.7}$$

To make the step from this formal solution to an actual solution one still needs to resolve the following issues:

(a) Is it possible to choose the constants a_n such that the given function f can be written as $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$?

(b) If the answer to (a) is yes, how can we actually find the constants a_n?

(c) In what sense (pointwise, in L^p, ...) does the series in 1.7 represent the function f?

(d) Does the equation (1.6) then give a smooth function on the disk such that $\lim_{r\to 1-} u(r,\theta) = f(\theta)$?

(e) Is the solution to our problem unique?

(f) Is every harmonic function in the disk given by a series as in 1.6?

To look for an answer, we continue our formal calculations. Later we show these calculations can be justified. First multiply f by $e^{-im\theta}$ and then integrate. We interchange the summation and integration and use

$$\int_0^{2\pi} e^{ik\theta}\,d\theta = \begin{cases} 2\pi & \text{if} \quad k = 0 \\ 0 & \text{if} \quad n \neq 0 \end{cases}$$

to obtain

$$\begin{aligned}\int_0^{2\pi} f(\theta)e^{-im\theta}\,d\theta &= \int_0^{2\pi}\sum_{n=-\infty}^{\infty} a_n e^{in\theta}e^{-im\theta}\,d\theta \\ &= \sum_{n=-\infty}^{\infty} a_n \int_0^{2\pi} e^{i(n-m)\theta}\,d\theta \\ &= 2\pi a_m\,.\end{aligned}$$

One of our first results on Fourier series states that if one defines a_m by

$$a_m = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-im\theta}\,d\theta,$$

then $f(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ holds in $\mathrm{L}^2([0,2\pi])$. The constant function $\theta \mapsto 1$ is in $\mathrm{L}^2([0,2\pi],\frac{d\theta}{2\pi})$ with norm one. By Hölder's inequality for L^2-functions one has

$$|f|_1 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|\,d\theta \leqslant |f|_2|1|_2 = |f|_2.$$

Consequently

$$|a_n| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|\ d\theta \leqslant |f|_2 < \infty\,.$$

Note for $0 < r \leqslant R < 1$,

$$\begin{aligned}\left|\sum_{n\in\mathbb{Z}} a_n r^{|n|}e^{in\theta}\right| &\leqslant \sum_{n\in\mathbb{Z}} |a_n|\,R^{|n|} \\ &\leqslant |f|_1\left(\sum_{n=1}^{\infty} R^n + \sum_{n=0}^{\infty} R^n\right) \\ &\leqslant \frac{2\,|f|_2}{1-R}\,.\end{aligned}$$

Therefore the series defining $u(r,\theta)$ converges uniformly on compact subsets of D. The derivatives of this series can also be shown to converge uniformly on compact subsets of D. Thus the series defines a smooth function on the disk. To evaluate the limit $\lim_{r\to 1-} u(r,\theta)$, we rewrite $u(r,\theta)$ as an integral over $[0,2\pi]$. This will be done by formally interchanging summation and integration and using the following simple fact

$$(1.8)\quad \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} = \sum_{n=0}^{\infty}\left(re^{i\theta}\right)^n + \sum_{n=0}^{\infty}\left(re^{-i\theta}\right)^n - 1 = \frac{1-r^2}{1-2r\cos(\theta)+r^2}\,.$$

The function

$$(1.9)\qquad P(r,\theta) := \frac{1-r^2}{1-2r\cos(\theta)+r^2}\,,\quad 0 \leqslant r < 1,\ \theta\in\mathbb{R}$$

is called the *Poisson kernel* for the unit disk. We will point out some properties of the Poisson kernel in the following exercises. See figure 1 for the graph of the Poisson kernel for $r = 0.5$ (blue), $r = 0.7$ (green), and $r = 0.9$ (red).

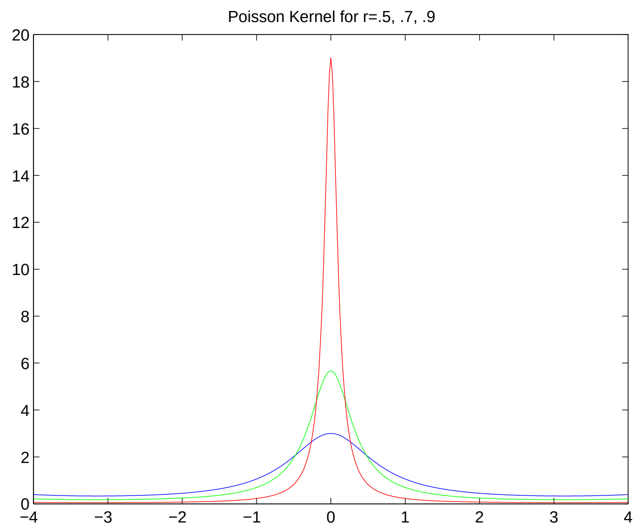

Figure 1. Poisson Kernels for $r = .5, \quad .7, \quad .9$

Inserting the definition of a_n and formally interchanging summation and integration, we see:

$$\begin{aligned} u(r,\theta) &= \sum_{n\in\mathbb{Z}} a_n r^{|n|} e^{in\theta} \\ &= \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-in\phi}\, d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \sum_{n\in\mathbb{Z}} r^{|n|} e^{in(\theta-\phi)}\, d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) P(r,\theta-\phi)\, d\phi\,. \end{aligned} \tag{1.10}$$

Hence u is given by convolving f with the Poisson kernel. This can be used to show that if f is continuous, then $u(r,\theta) \to f(\theta)$ uniformly.

Exercise Set 1.1

1. Prove equation (1.9): $\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos(\theta)+r^2}$.

2. Prove the following:

(a) $P(r,\theta) \geqslant 0$ and $\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta)\, d\theta = 1$ for all $r \geqslant 0$.

(b) The maximum of $\theta \mapsto P(r,\theta)$ occurs at $\theta = 0$ and $\max_\theta P(r,\theta) = \frac{1+r}{1-r}$. In particular $P(r,0) \to \infty$ as $r \to 1-$.

(c) The function $\theta \mapsto P(r,\theta)$ takes its minimum at $\theta = \pi$. Evaluate $P(r,\pi)$.

3. Suppose that f is 2π-periodic and piecewise continuous. Show that

$$\lim_{r\to 1-} \frac{1}{2\pi}\int_0^{2\pi} f(\phi)P(r,\theta-\phi)\,d\phi = f(\theta)$$

if f is continuous at θ.

4. Write $u(x,y) = u(z)$ where $z = x+iy$. Suppose that f is continuous. Show that u is holomorphic on D if and only if $a_n = 0$ for all $n < 0$.

2. Periodic Functions

Definition 1.2. *A function $f : \mathbb{R} \to \mathbb{C}$ is periodic if there exists a number $L > 0$ such that $f(x+L) = f(x)$ for all $x \in \mathbb{R}$. The number L is called a period of f.*

Let P_f be the set of periods of f. Then $P_f \neq \varnothing$ if and only if f is periodic. Note P_f need not have a smallest element for the characteristic function of the irrationals has all positive rationals for periods.

If f is periodic, let $L_f := \inf P_f \geqslant 0$. The next lemma states that $P_f - P_f$ is an additive subgroup of $\mathbb{R}$.

Lemma 1.3. *Let L and M be periods of f. Then $f(x+jL+kM) = f(x)$ for all $j,\, k \in \mathbb{Z}$.*

Proof. We have $f(x-L) = f((x-L)+L) = f(x)$. The other statements follow by induction. □

Lemma 1.4. *Suppose $L_f > 0$. Then L_f is a period for f and then $P_f = L_f\mathbb{N}$.*

Proof. If L_f is not a period and $L_f > 0$, we can choose periods M and L with $L_f < M < L < 2L_f$. Thus $0 < L - M < L_f$. By Lemma 1.3, $L - M$ is a period strictly less than L_f. This is a contradiction. Thus L_f is a period.

Now $L_f\mathbb{N} \subseteq P_f$. Let $L \in P_f$. Then we can choose $n \in \mathbb{N}$ such that

$$nL_f \leqslant L < (n+1)L_f.$$

Hence $0 \leqslant L - nL_f < L_f$. If $L - nL_f > 0$, then Lemma 1.3 implies $L - nL_f$ is a smaller period than L_f which is clearly untrue. Hence $L_f\mathbb{N} = P_f$. □

Lemma 1.5. *If $L_f = 0$ and f is continuous, then f is constant.*

Proof. Fix x. Choose a sequence L_n of periods converging to 0. Choose integers k_n so that $k_nL_n \leqslant x < (k_n+1)L_n$. Then $k_nL_n \to x$. Hence $f(k_nL_n) \to f(x)$. But $f(k_nL_n) = f(0 + k_nL_n) = f(0)$. Consequently, $f(0) = f(x)$ and f is a constant function. □

The functions $x \mapsto \cos(2\pi mx)$, $\sin(2\pi kx)$, and $e^{2\pi inx} = \cos(2\pi nx) + i\sin(2\pi nx)$ all have period 1. We will show in a certain sense that "each" periodic function with period 1 can be written as an infinite linear combination of $e^{2\pi inx}$ and hence also of $\cos(2\pi nx)$ and $\sin(2\pi nx)$. If f has period $L > 0$, then $g(x) = f(Lx)$ has period 1. Hence f can be written as a linear combinations of functions of form $e^{2\pi inx/L}$. We therefore restrict ourselves to functions of period 1. Let

$$\mathbb{T} := \{z \in \mathbb{C} \mid \ |z| = 1\}$$

be the one-dimensional *torus*. Then $\mathbb{T}$ is a closed and bounded subset of $\mathbb{C}$ and hence is compact. Furthermore $\mathbb{T}$ is an abelian group under multiplication and the map

$$\mathbb{R} \ni x \overset{\kappa}{\mapsto} e^{2\pi ix} \in \mathbb{T}$$

is a surjective group homomorphism of $(\mathbb{R}, +)$ onto $(\mathbb{T}, \cdot)$ with kernel $\mathbb{Z}$. The torus $\mathbb{T}$ has a natural topology as a subset of $\mathbb{C}$.

Let $z, w, z_0, w_0 \in \mathbb{T}$. Then $|z| = |w| = |z_0| = |w_0| = 1$ and

$$|zw - z_0w_0| \leqslant |w|\,|z - z_0| + |z_0|\,|w - w_0| = |z - z_0| + |w - w_0|$$

and

$$\left|z^{-1} - z_0^{-1}\right| = |\bar{z} - \bar{z}_o| = |z - z_0| \ .$$

Hence it follows that both the multiplication and the inverse map are continuous maps in this topology. These are conditions defining a *topological group.*

Lemma 1.6. *The mapping $\kappa : \mathbb{R} \to \mathbb{T}$ is a continuous periodic open mapping from $\mathbb{R}$ onto $\mathbb{T}$ satisfying $\kappa(\theta+\phi) = \kappa(\theta)\kappa(\phi)$ for all θ and ϕ in $\mathbb{R}$. Moreover, every complex function f on $\mathbb{R}$ having period 1 has form $f = F \circ \kappa$ for a unique function F on $\mathbb{T}$, and f is continuous iff F is continuous.*

Proof. Clearly κ is continuous, onto, and has period 1. Let $I = (a, b)$ be an open interval. Then if $b - a > 1$, $\kappa(I)$ equals $\mathbb{T}$ and if $b - a \leqslant 1$, then $\kappa(I)$ is an 'open arc' in $\mathbb{T}$ and thus is open in $\mathbb{T}$ in the relative topology from $\mathbb{C}$. Since every open subset of $\mathbb{R}$ is a countable union of open intervals, we see $\kappa(U)$ is open in $\mathbb{T}$ for any open subset U of $\mathbb{R}$.

Let f be a function on $\mathbb{R}$ with period 1. Define $F(e^{2\pi ix}) = f(x)$. F is well defined and is clearly the only function with $F \circ \kappa = f$. Note f is continuous if F is continuous. If f is continuous and U is open, then

$F^{-1}(U) = \kappa(f^{-1}(U))$ is an open set in $\mathbb{T}$, for κ is an open mapping. Thus F is continuous. $\square$

3. Integration on the Torus

Let X be a topological space. Denote the space of complex valued continuous functions on X by $C(X)$. The last lemma can be used to integrate and differentiate functions on $\mathbb{T}$. Define a Borel measure μ on $\mathbb{T}$ by

$$\mu(E) = m(\kappa^{-1}(E) \cap [0,1))$$

where m is Lebesgue measure on $\mathbb{R}$. Then $g \in L^1(\mathbb{T},\mu)$ iff $g \circ \kappa \in L^1[0,1]$ and then

$$\int g(z)\,d\mu(z) = \int_0^1 g(e^{2\pi i x})\,dm(x).$$

Note by a change in variables, one also has

$$\int g(z)\,d\mu(z) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})\,d\theta.$$

The measure μ is left and right invariant; i.e.,

$$\mu(aE) = \mu(Ea) = \mu(E)$$

for all Borel subsets E of $\mathbb{T}$ and $a \in \mathbb{T}$.

The left and right invariance of the measure μ implies

$$\int g(y^{-1}z)\,d\mu(z) = \int g(zy)\,d\mu(z) = \int g(z)\,d\mu(z)$$

for all $g \in L^1(\mathbb{T},\mu)$ and $y \in \mathbb{T}$. The measure is also invariant under the inverse-mapping. Thus

$$\int g(z^{-1})\,d\mu(z) = \int g(z)\,d\mu(z)\,.$$

Indeed,

$$\int g(z^{-1})\,d\mu(z) = \int_0^1 g(e^{-2\pi i x})\,dx = \int_0^1 g(e^{2\pi i x})\,dx.$$

Denote the linear space of p-integrable complex valued functions by $L^p(\mathbb{T})$. Recall the norm is given by

$$|f|_p = \left(\int |f(z)|^p\,d\mu(z)\right)^{\frac{1}{p}}.$$

This space is the same as the space of L^p functions on $[0,1]$ or the space of Lebesgue measurable functions on $\mathbb{R}$ that are periodic with period 1 and p-integrable over $[0,1]$. Let us recall the following two well known facts on integration.

Theorem 1.7. *Let $1 \leqslant p < \infty$. Then $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$.*

Theorem 1.8 (Hölder Inequality)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $p, q \geqslant 1$ satisfy $1/p + 1/q = 1$. Let $f \in L^p(X)$ and $g \in L^q(X)$. Then $fg \in L^1(X)$ and*

$$|fg|_1 \leqslant |f|_p \, |g|_q \, .$$

Lemma 1.9. *Suppose $f \in L^2(\mathbb{T})$ satisfies $\int f(z)\overline{g(z)}\, d\mu(z) = 0$ for all continuous functions g. Then $f = 0$.*

Proof. Since $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, we can choose a sequence $g_n \in C(\mathbb{T})$ with $|g_n - f|_2 \to 0$. Hence

$$\int f\bar{f} \leqslant \lim \left(\int |f(\bar{f} - \bar{g}_n)|\, d\mu + |\int f\bar{g}_n \, d\mu| \right) \leqslant \lim |f|_2 |f - g_n|_2 = 0.$$

Hence $\int f\bar{f}\, d\mu = |f|_2^2 = 0$, which implies that $f = 0$ a.e. □

Finally, by the Riesz–Fischer Theorem, we know the spaces $L^p(T)$ are complete if equipped with norm $f \mapsto |f|_p$. In particular, $L^2(T)$ is a Hilbert space with inner product

$$(f, g)_2 = \int_{\mathbb{T}} f(z)\bar{g}(z)\, d\mu(z).$$

Theorem 1.10. *Let $1 \leqslant p \leqslant \infty$. Then $L^p(\mathbb{T}) \subset \mathbb{L}^1(\mathbb{T})$ and $|f|_1 \leqslant |f|_p$ for all $f \in L^p(\mathbb{T})$.*

Proof. Let q be such that $1/p + 1/q = 1$. Then the constant function $z \mapsto 1$ is in $L^q(\mathbb{T})$ as $\mu(\mathbb{T}) = 1 < \infty$. By Hölder's-inequality, one has

$$\int |f| \; 1 \, d\mu \leqslant |f|_p \, |1|_q = |f|_p \, .$$

□

Definition 1.11. *Let $1 \leqslant p \leqslant \infty$. For $a \in \mathbb{T}$, define linear operators $\lambda(a)$ and $\rho(a)$ on $L^p(T)$ by*

$$\begin{aligned} \lambda(a)f(z) &= f(a^{-1}z) \\ \rho(a)f(z) &= f(za). \end{aligned}$$

Then λ and ρ are called the left and right regular representations of $\mathbb{T}$ on $L^p(T)$.

Suppose f is a complex valued function on $\mathbb{T}$. Then $\check{f}$ will be the function defined by

$$\check{f}(z) = f(z^{-1}).$$

Lemma 1.12. *The mappings $a \mapsto \lambda(a)$ and $a \mapsto \rho(a)$ are homomorphisms of $\mathbb{T}$ into the group of invertible linear isometries of $L^p(\mathbb{T})$. Moreover, $f \mapsto \check{f}$ is a linear isometry of $L^p(\mathbb{T})$ satisfying*

$$(\lambda(a)f)\check{} = \rho(a)\check{f}.$$

Proof. Note

$$\begin{aligned}\lambda(ab)f(x) &= f((ab)^{-1}x) = f(b^{-1}a^{-1}x)\\ &= \lambda(b)f(a^{-1}x) = \lambda(a)\lambda(b)f(x).\end{aligned}$$

and thus $\lambda(ab) = \lambda(a)\lambda(b)$ on L^p. Clearly $\lambda(1) = I$; and since $\lambda(a)\lambda(a^{-1}) = \lambda(1) = \lambda(a^{-1})\lambda(a)$, we have $\lambda(a)^{-1} = \lambda(a^{-1})$. Thus $a \mapsto \lambda(a)$ is a group homomorphism.

Suppose $p = \infty$. Then $|\lambda(a)f|_\infty = \operatorname{ess\,sup}|f(az)| = \operatorname{ess\,sup}|f(z)| = |f|_\infty$. For $1 \leqslant p < \infty$, we have

$$|\lambda(a)f|_p^p = \int |f(a^{-1}z)|^p \, d\mu(z) = \int |f(z)|^p \, d\mu(z)$$

and thus $|\lambda(a)f|_p = |f|_p$.

Note

$$\begin{aligned}(\lambda(a)f)\check{}(z) &= \lambda(a)f(z^{-1})\\ &= f(a^{-1}z^{-1})\\ &= \check{f}(za)\\ &= \rho(a)\check{f}(z)\end{aligned}$$

and if $1 \leqslant p < \infty$, then

$$|\check{f}|_p^p = \int |f(z^{-1})|^p \, d\mu(z) = \int |f(z)|^p \, d\mu(z) = |f|_p^p.$$

One easily checks $|\check{f}|_\infty = |f|_\infty$. Thus $f \mapsto \check{f}$ is a linear isometry and it is onto for $(\check{f})\check{} = f$. $\square$

Definition 1.13 (Convolution). *Let f and g be in $L^1(\mathbb{T})$. The convolution $f * g$ of f and g is defined by*

$$f * g(x) = \int_{\mathbb{T}} f(y)g(y^{-1}x) \, d\mu(y).$$

Note $(y, x) \mapsto f(y)g(y^{-1}x)$ is a measurable function on $\mathbb{T} \times \mathbb{T}$ and by Fubini's Theorem,

$$\begin{aligned}\int\int |f(y)g(y^{-1}x)|\, d\mu(y)\, d\mu(x) &= \int\int \left|f(y)g(y^{-1}x)\right|\, d\mu(x)\, d\mu(y)\\ &= \int\int |f(y)g(x)|\, d\mu(x)\, d\mu(y)\\ &= |f|_1\, |g|_1 < \infty\,.\end{aligned}$$

It follows that for almost all $x \in \mathbb{T}$, the function $y \mapsto f(y)g(y^{-1}x)$ is integrable and

$$\begin{aligned}\int |f * g(x)|\, d\mu(x) &= \int \left|\int f(y)g(y^{-1}x)\, d\mu(y)\right|\, d\mu(x)\\ &\leqslant \int\int |f(y)g(y^{-1}x)|\, d\mu(y)\, d\mu(x)\\ &= |f|_1\, |g|_1.\end{aligned}$$

As $L^p(\mathbb{T})$ and $L^q(\mathbb{T})$ are subspaces of $L^1(\mathbb{T})$ with larger norms, one has

$$|f * g|_1 \leqslant |f|_1 |g|_1 \leqslant |f|_p |g|_q$$

whenever $f \in L^p$ and $g \in L^q$.

Lemma 1.14. *Let $f, g, h \in L^1(\mathbb{T})$. Then the following hold:*

(a) $f * g \in L^1(\mathbb{T})$ *and* $|f * g|_1 \leqslant |f|_1\, |g|_1$.

(b) $f * g = g * f$.

(c) $f * (g * h) = (f * g) * h$.

(d) $[\lambda(a)f] * g = f * [\lambda(a)g] = \lambda(a)(f * g)$

Proof. Note (a) was proved just before we stated the lemma.

To see (b), note

$$\begin{aligned}f * g(z) &= \int f(y)g(y^{-1}z)\, d\mu(y)\\ &= \int f(zy)g((zy)^{-1}z)\, d\mu(y)\\ &= \int g(y^{-1})f(yz)\, d\mu(y)\\ &= \int g(y)f(y^{-1}z)\, d\mu(y)\end{aligned}$$

where we have used invariance of integration under transformations $y \mapsto zy$ and $y \mapsto y^{-1}$.

For (c) we have

$$\begin{aligned}
(f * g) * h(z) &= \int (f * g)(y) h(y^{-1}z)\, d\mu(y) \\
&= \int\int f(x) g(x^{-1}y) h(y^{-1}z)\, d\mu(x)\, d\mu(y) \\
&= \int f(x) \int g(x^{-1}y) h(y^{-1}z)\, d\mu(y)\, d\mu(x) \\
&= \int f(x) \int g(x^{-1}xy) h((xy)^{-1}z)\, d\mu(y)\, d\mu(x) \\
&= \int f(x) \int g(y) h(y^{-1}x^{-1}z)\, d\mu(y)\, d\mu(x) \\
&= \int f(x)\, (g * h)(x^{-1}z)\, d\mu(x) \\
&= f * (g * h)(z)
\end{aligned}$$

where the changes in the order of integration follow by Fubini's theorem and we have used the invariance of the measure $d\mu$ under left translation by x^{-1}.

For (d) note

$$\begin{aligned}
[\lambda(a)f] * g(y) &= \int [\lambda(a)f](x) g(x^{-1}y)\, d\mu(x) \\
&= \int f(a^{-1}x) g(x^{-1}y)\, d\mu(x) \\
&= \int f(a^{-1}ax) g((ax)^{-1}y)\, d\mu(x) \\
&= \int f(x) g(x^{-1}a^{-1}y)\, d\mu(x) \\
&= f * g(a^{-1}y) \\
&= \lambda(a)(f * g)(y) \\
&= \lambda(a)(g * f)(y) \\
&= [\lambda(a)g] * f(y) \\
&= f * [\lambda(a)g](y)
\end{aligned}$$

where we have used the commutativity of convolution. □

Proposition 1.15. *Suppose* $1 \leqslant p \leqslant q \leqslant \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $f * g \in C(\mathbb{T})$ *whenever* $f \in L^p(\mathbb{T})$ *and* $g \in L^q(\mathbb{T})$.

Proof. Note if $f \in C(\mathbb{T})$ and $g \in L^q(\mathbb{T})$ and $z_n \to z$ in $\mathbb{T}$, then

$$\begin{aligned} f * g(z_n) &= g * f(z_n) = \int g(x) f(x^{-1} z_n)\, d\mu(x) \\ &\to \int g(x) f(x^{-1} z)\, d\mu(x) \\ &= g * f(z) \\ &= f * g(z) \end{aligned}$$

as $n \to \infty$ by the Lebesgue dominated convergence theorem. Indeed, $x \mapsto g(x)f(x^{-1}z_n)$ is dominated pointwise by $|g|\,|f|_\infty$ which is in $L^1(\mathbb{T})$ and converges pointwise to $g(x)f(x^{-1}z)$. Hence $f * g \in C(\mathbb{T})$ if $f \in C(\mathbb{T})$.

Now suppose $f \in L^p(\mathbb{T})$ and $\epsilon > 0$. Let $z_n \to z$. Since $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$, we know we can choose $f_0 \in C(\mathbb{T})$ satisfying $|f - f_0|_p \leqslant \frac{\epsilon}{3(|g|_q+1)}$. Choose N so that $|f_0 * g(z_n) - f_0 * g(z)| < \frac{\epsilon}{3}$ for $n \geqslant N$. Then for $n \geqslant N$, we have

$$\begin{aligned} |f * g(z_n) - f * g(z)| &\leqslant |(f - f_0) * g(z_n)| + |f_0 * g(z_n) - f_0 * g(z)| + |(f_0 - f) * g(z)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

since by Hölder's inequality

$$\begin{aligned} |(f - f_0) * g(y)| &\leqslant \int |(f - f_0)(x) g(x^{-1}y)|\, d\mu(x) \\ &\leqslant |f - f_0|_p |\lambda(y)\check{g}|_q \\ &= |f - f_0|_p |g|_q \\ &\leqslant \frac{\epsilon}{3(|g|_q + 1)} \cdot |g|_q \\ &\leqslant \frac{\epsilon}{3} \end{aligned}$$

for all $y \in \mathbb{T}$. □

Exercise Set 1.2

1. Show if f is a function on $\mathbb{T}$ such that $f \circ \kappa$ is a simple measurable function on $\mathbb{R}$, then

$$\int f(z)\, d\mu(z) = \int_0^1 f(e^{2\pi i x})\, dx.$$

2. Show $L^p(\mathbb{T}) \subseteq L^q(\mathbb{T})$ and $|f|_p \geqslant |f|_q$ for $p \geqslant q$.

3. Consider $C(\mathbb{T})$ with norm $|\cdot|_\infty$. For $w \in \mathbb{T}$, define $\lambda(w)f(z) = f(w^{-1}z)$ for $f \in C(\mathbb{T})$.

(a) Show $\lambda(w)$ is a linear isometry for each w.

(b) Show λ is a homomorphism of $\mathbb{T}$ into the isometry group of $C(\mathbb{T})$.

(c) Show λ is not continuous from $\mathbb{T}$ into the Banach space of bounded linear operators on $C(\mathbb{T})$.

(d) Show for each $f \in C(\mathbb{T})$, the mapping $w \mapsto \lambda(w)f$ is continuous from $\mathbb{T}$ into $C(\mathbb{T})$

4. Let $1 \leqslant p < \infty$. Let $\lambda(z)f(w) = f(z^{-1}w)$ for $f \in L^p(\mathbb{T})$. Show for each f and each $\epsilon > 0$, there is a $\delta > 0$ such that $|\lambda(z)f - f|_p < \epsilon$ if $|z - 1| < \delta$.

5. For $1 \leqslant p < q$, find $f \in L^p(\mathbb{T})$ such that $f \notin L^q(\mathbb{T})$.

6. Let $h \in L^1(\mathbb{T})$.

(a) Let $g \in L^2(\mathbb{T})$. Show that $\lambda(h)g := h * g$ is in $L^2(\mathbb{T})$. (**Hint:** Let $f \in L^2(\mathbb{T})$. Then $f\overline{\lambda(h)g}$ is integrable and $f \mapsto \int f(z)\overline{\lambda(h)g(z)}\,d\mu(z)$ is a continuous linear form on $L^2(\mathbb{T})$.)

(b) Show that $\lambda(h) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a bounded linear map with $|\lambda(h)| \leqslant |h|_1$.

4. The Fourier Transform

The functions of the form

$$p(\theta) = \sum_{n=-N}^{M} a_n e^{2\pi i n\theta}$$

on $\mathbb{R}$ are called *trigonometric polynomials*. The trigonometric polynomials are periodic with period 1 and as functions on $\mathbb{T}$ they can simply be written as

$$p(z) = \sum_{n=-N}^{M} a_n z^n.$$

The trigonometric polynomials form an algebra of continuous functions on $\mathbb{T}$ which separate points, contain the constants, and are closed under conjugation. By the Stone–Weierstrass Theorem, this algebra is dense in $C(\mathbb{T})$ under the $|\cdot|_\infty$ norm. Since $C(\mathbb{T})$ is dense in every $L^p(\mathbb{T})$ except $L^\infty(\mathbb{T})$, one can show the algebra of trigonometric polynomials is dense in every $L^p(\mathbb{T})$ where $1 \leqslant p < \infty$. We shall be interested in those trigonometric polynomials which are the partial sums of Fourier series.

Let $e_n(e^{2\pi i\theta}) = e^{2\pi i n\theta}$. Thus this is the function $z \mapsto z^n$ on $\mathbb{T}$. If $f \in L^1(\mathbb{T})$, then the *Fourier transform* $\hat{f} : \mathbb{Z} \to \mathbb{C}$ of f is defined by

$$\hat{f}(n) := \int f(z)z^{-n}\,d\mu(z) = \int_I f(e^{2\pi ix})\,e^{-2\pi inx}dx \tag{1.11}$$

where I is any interval in $\mathbb{R}$ having length 1. By the change of variables $\theta = 2\pi x$, we also have

$$\hat{f}(n) := \int f(z) z^{-n} \, d\mu(z) = \frac{1}{2\pi} \int_J f(e^{i\theta}) \, e^{-in\theta} d\theta \tag{1.12}$$

over any interval J of length 2π.

Note that $\hat{f}(n) = (f, e_n)$ when $f \in L^2(\mathbb{T})$. The *Fourier series* corresponding to f is

$$\sum \hat{f}(n) \, e^{2\pi i n\theta} = \sum \hat{f}(n) z^n$$

or

$$\sum_{n=-\infty}^{\infty} (f, e_n) e_n .$$

Notice that

$$(e_n, e_m) = \int_0^1 e^{2\pi i (n-m)\theta} \, d\theta = \delta_{n,m} \, . \tag{1.13}$$

Hence $\{e_n \mid n \in \mathbb{Z}\}$ is a orthonormal subset of $L^2(\mathbb{T})$.

Definition 1.16. *Let $g \in C(\mathbb{T})$ and $r \in \mathbb{N}_0$. Then g is r-times continuously differentiable if the continuous periodic function $g \circ \kappa$ is r-times continuously differentiable on $\mathbb{R}$.*

We denote the space of r-times continuously differentiable functions by $C^r(\mathbb{T})$. The space $C^\infty(\mathbb{T})$ of smooth functions is $\cap_r C^r(\mathbb{T})$; i.e., the space of functions that are r-times continuously differentiable for all r. The *derivative operator* D on $C^1(\mathbb{T})$ is defined by

$$[Df](e^{2\pi i x}) = (f \circ \kappa)'(x) = \frac{d}{dx} f(e^{2\pi i x}) \, .$$

Notice that $Dz^n = 2\pi i n z^n$ for all n. Recall $\lambda(w) f(z) = f(w^{-1} z)$.

Lemma 1.17. *Let $f, g \in L^1(\mathbb{T})$. Let $p = \sum_{n=-N}^{M} a_n e_n$ be a trigonometric polynomial. Then the following hold:*

(a) $\left|\hat{f}(n)\right| \leqslant |f|_1$ *for all* $n \in \mathbb{Z}$.

(b) $\widehat{\lambda(w) f}(n) = w^{-n} \hat{f}(n)$.

(c) *Then* $\widehat{f^\vee}(n) = \hat{f}(-n)$.

(d) $\lim_{n \to \infty} \hat{f}(n) = 0$ *(Lebesgue Lemma).*

(e) $\widehat{f * g} = \hat{f} \cdot \hat{g}$.

(f) $f * p = \sum_{n=-N}^{M} a_n \hat{f}(n) e_n$.

(g) *Assume that* $f \in C^r(\mathbb{T})$. *Then* $\widehat{D^r f}(n) = (2\pi i n)^r \, \hat{f}(n)$.

Proof. For (a) note $\left|\hat{f}(n)\right| = \left|\int f(z)z^{-n}\,d\mu(z)\right| \leqslant \int |f(z)|\,d\mu(z) = |f|_1$ because $|z| = 1$ for $z \in \mathbb{T}$.

For (b) one has

$$\begin{aligned}
\widehat{\lambda(w)f}(n) &= \int \lambda(w)f(z)z^{-n}\,d\mu(z)\\
&= \int f(w^{-1}z)z^{-n}\,d\mu(z)\\
&= \int f(z)[wz]^{-n}\,d\mu(z)\\
&= w^{-n}\int f(z)z^{-n}\,d\mu(z)\\
&= w^{-n}\hat{f}(n).
\end{aligned}$$

To see (c) one has

$$\begin{aligned}
\widehat{f^{\vee}}(n) &= \int f(z^{-1})z^{-n}\,d\mu(z)\\
&= \int f(z)z^{n}\,d\mu(z)\\
&= \hat{f}(-n)\,.
\end{aligned}$$

To do (d), we first do the case where $F = f \circ \kappa|_{[0,1)} = C\chi_{[a,b]}$ with $0 \leqslant a < b < 1$. Then

$$\begin{aligned}
\hat{f}(n) &= C\int_a^b e^{-2\pi i n\theta}\,d\theta\\
&= \frac{Ci}{2\pi n}\left[e^{-2\pi i n b} - e^{-2\pi i n a}\right] \to 0 \qquad n \to \infty\,.
\end{aligned}$$

It follows that the claim holds for any F which is a step function on $[0, 1)$. But these define f's which are dense in $L^1(\mathbb{T})$. Hence if f is in $L^1(\mathbb{T})$ and $\epsilon > 0$, one can choose a step function $F_0 \in L^2[0, 1)$ satisfying

$$|f \circ \kappa - F_0|_1 < \frac{\epsilon}{2}.$$

Setting $f_0(e^{2\pi i\theta}) = F_0(\theta)$, we have

$$\begin{aligned}
|\hat{f}(n)| &\leqslant |\hat{f}(n) - \hat{f}_0(n)| + |\hat{f}_0(n)|\\
&\leqslant |f - f_0|_1 + |\hat{f}_0(n)|\\
&< \frac{\epsilon}{2} + |\hat{f}_0(n)|\\
&< \epsilon
\end{aligned}$$

for large n.

For (e), by left invariance of the measure μ and Fubini's Theorem, we obtain:

$$\begin{aligned}
\widehat{f * g}(n) &= \int f * g(z)\, z^{-n}\, d\mu(z) \\
&= \int \left[\int f(w) g(w^{-1}z)\, d\mu(w) \right] z^{-n}\, d\mu(z) \\
&= \int \int f(w) g(w^{-1}z) z^{-n}\, d\mu(z) d\mu(w) \\
&= \int f(w) \int g(z)\, [wz]^{-n}\ d\mu(z) d\mu(w) \\
&= \int f(w) w^{-n} \int g(z) z^{-n}\, d\mu(z) d\mu(w) \\
&= \widehat{f}(n)\widehat{g}(n)\,.
\end{aligned}$$

To do (f), we have by the definition of $\hat{f}(n)$ that:

$$\begin{aligned}
f * p(z) &= \sum a_n \int f(w)(w^{-1}z)^n\, d\mu(w) \\
&= \sum a_n z^n \int f(w) w^{-n}\, d\mu(w) \\
&= \sum a_n \hat{f}(n) e_n(z)\,.
\end{aligned}$$

Finally for (g), note for $r = 1$ the statement follows by integration by parts:

$$\widehat{Df}(n) = \int_0^1 (f \circ \kappa)'(\theta)\, e^{-2\pi i n\theta}\, d\theta = -\int_0^1 f(e^{2\pi i\theta}) \frac{d}{d\theta} e^{-2\pi i n\theta}\, d\theta = 2\pi i n \hat{f}(n).$$

Repeat this argument for general r. □

Corollary 1.18. *For $f \in L^1(\mathbb{T})$, $f * e_n = \hat{f}(n) e_n$. Moreover, $e_m * e_n = \delta_{m,n} e_n$.*

Proof. Note e_n is a trigonometric polynomial. Hence by (f), $f * e_n = \hat{f}(n) e_n$. Since

$$\hat{e}_m(n) = \int e_m(z) z^{-n}\, d\mu(z) = \int e_m(z) \overline{e_n(z)}\, d\mu(z) = \delta_{m,n},$$

we see $e_m * e_n = \delta_{m,n} e_n$. □

Lemma 1.19. *Let $g \in C(\mathbb{T})$ and $\epsilon > 0$. Then there exists a trigonometric polynomial p such that*

$$|g - p|_\infty < \epsilon\,.$$

Proof. $\mathbb{T}$ is a compact Hausdorff space, and the space $\mathcal{A}$ of all trigonometric polynomials $p(z) = \sum_{n=-N}^{M} a_n z^n$ form an algebra of continuous functions on $\mathbb{T}$ which contain the constants and separate points. Moreover $\mathcal{A}$ is closed under conjugation for $\overline{z^n} = z^{-n}$. By the Stone–Weierstrass Theorem for continuous complex valued functions on a compact Hausdorff space, one has for each $f \in C(\mathbb{T})$ and each $\epsilon > 0$, there is a $p \in \mathcal{A}$ with $|f - p|_\infty < \epsilon$. □

Theorem 1.20 (Plancherel Theorem)**.** *The set of functions* $\{e_n \mid n \in \mathbb{Z}\}$ *is an orthonormal basis for* $L^2(\mathbb{T})$*. In particular, if* f *is in* $L^2(\mathbb{T})$*, then*

(a) $f = \sum_{n=-\infty}^{\infty} (f, e_n) e_n$ *in* $L^2(\mathbb{T})$ *and*

(b) $|f|^2 = \sum_{n=-\infty}^{\infty} |(f, e_n)|^2$.

Proof. By equation 1.13, the set $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal. Let $\epsilon > 0$ and let $f \in L^2(\mathbb{T})$. Choose $g \in C(\mathbb{T})$ such that $|f - g|_2 < \epsilon/2$. By Lemma 1.19 there is a trigonometric polynomial p such that

$$|g - p|_\infty < \epsilon/2\,.$$

Thus

$$\begin{aligned} |g - p|_2^2 &= \int |g(z) - p(z)|^2 \, d\mu(z) \\ &\leqslant \int |g - p|_\infty^2 \, d\mu(z) \\ &< (\epsilon/2)^2 \,. \end{aligned}$$

Thus $|g - p|_2 < \epsilon/2$. It follows that

$$|f - p|_2 \leqslant |f - g|_2 + |g - p|_2 < \epsilon\,.$$

Thus e_n form a complete orthonormal basis and the theorem follows. □

Let ℓ^2 be the space of bi-infinite complex sequences $\{a_n\}_{n=-\infty}^{\infty}$ which satisfy $\sum |a_n|^2 < \infty$. This space is a Hilbert space with inner product

$$(\{a_n\}, \{b_n\}) = \sum a_n \bar{b}_n$$

and norm

$$|\{a_n\}| = \sqrt{\sum |a_n|^2}.$$

One can give a direct proof of this fact; however, it follows easily from measure theory. Namely, let ν be counting measure on $\mathbb{Z}$; thus every subset of $\mathbb{Z}$ is measurable, and $\nu(E)$ is the number of elements in E. Then ν is a measure, every function is measurable, and a function $a : \mathbb{Z} \to \mathbb{C}$ is in $L^2(\mathbb{Z}, \nu)$ iff $\sum |a(n)|^2 = \int |a(n)|^2 \, d\nu(n) < \infty$.

We now easily reformulate the Plancherel Theorem:

Theorem 1.21. *The Fourier transform $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2$ is an isomorphism of Hilbert spaces.*

Proof. We have by the Plancherel Theorem that

$$\sum \left|\hat{f}(n)\right|^2 = |f|^2 < \infty\,.$$

Hence $\mathcal{F}f \in \ell^2$ and $\mathcal{F}$ is an isometry into ℓ^2. Let $A = \{a_n\} \in \ell^2$. Define a sequence

$$f_n(z) = \sum_{j=-n}^{n} a_n z^n \in L^2(\mathbb{T}).$$

Thus $f_n = \sum_{j=-n}^{n} a_n e_n$. For $m \geqslant n$ one has

$$|f_n - f_m|^2 = \sum_{n<|j|\leqslant m} |a_j|^2\,.$$

But $\sum |a_j|^2 < \infty$. Hence if $\epsilon > 0$, we can find an $N \in \mathbb{N}$ such that for all $n \geqslant N$

$$\sum_{n\leqslant|j|} |a_j|^2 < \epsilon.$$

But this then implies that $\{f_n\}$ is a Cauchy sequence in $L^2(\mathbb{T})$. Thus $\sum a_j e_j$ converges in L^2 to an L^2 function f and

$$\begin{aligned}\hat{f}(n) &= (f, e_n)\\ &= \sum a_j(e_j, e_n)\\ &= a_n\,.\end{aligned}$$

Hence $\mathcal{F}f = A$ and $\mathcal{F}$ is surjective. $\square$

Let $C^\infty(\mathbb{T})$ be the space of smooth function on $\mathbb{T}$; i.e., $C^\infty(\mathbb{T}) = \cap_{r\in\mathbb{N}} C^r(\mathbb{T})$. Define a vector space topology on $C^\infty(\mathbb{T})$ by the seminorms

$$\sigma_k(f) := \left|D^k f\right|_\infty\,.$$

We leave it as an exercise to show that $C^\infty(\mathbb{T})$ with this topology is a locally convex complete topological vector space. (Basic concepts in locally convex topological vector spaces are covered in Section 1 in the next chapter.) This topology is called the Schwartz topology on $C^\infty(\mathbb{T})$ and the space $C^\infty(\mathbb{T})$ with this topology is denoted by $\mathcal{D}(\mathbb{T})$. To find the image of $\mathcal{D}(\mathbb{T})$ under the Fourier transform, let $\mathcal{S}(\mathbb{Z})$ be the space of sequences $a = \{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that for each k

$$\rho_k(a) := \sup_n (1 + |n|)^k\, |a_n| < \infty\,.$$

The ρ_k are seminorms. With these seminorms $\mathcal{S}(\mathbb{Z})$ becomes a locally convex complete topological vector space. Sequences $\{a_n\}$ which satisfy $\rho_k(a) < \infty$ for all k are said to be *rapidly decreasing.* Notice that

$$\frac{|n|^k}{(1+|n|)^k} \to 1 \text{ as } n \to \infty\,.$$

Hence there are positive constants C_k such that for all $n \neq 0$,

$$C_k(1+|n|)^k\,|a_n| \leqslant |n|^k\,|a_n| \leqslant (1+|n|)^k\,|a_k|\ .$$

The topology on $\mathcal{S}(\mathbb{Z})$ can therefore also be defined by the seminorms ρ'_k where

$$\rho'_k(a) := \sup |n|^k\,|a_n|\ , \quad k \neq 0\,.$$

In these formulas, expression 0^0 is given value 1. This topology can also be defined by using the seminorms

$$\rho''_k(a) = \sum |n|^k\,|a_n|\ .$$

Theorem 1.22. *The Fourier transform $f \mapsto \hat{f}$ is a topological isomorphism of $\mathcal{D}(\mathbb{T})$ onto $\mathcal{S}(\mathbb{Z})$.*

Proof. We have the Fourier transform of $D^k f$ is $n \mapsto (2\pi i n)^k \hat{f}(n) \in \ell^2$. Thus $(2pi)^k \sum |n|^{2k}|\hat{f}(n)|^2 < \infty$ for each k. Hence $\sup_n |n^{2k}\hat{f}(n)|^2$ is finite for all k. Consequently $\sup_n |n^k \hat{f}(n)| < \infty$ for all k. Thus $\hat{f} \in \mathcal{S}(\mathbb{T})$.

Clearly $f \mapsto \hat{f}$ is linear. It is one-to-one, for $\hat{f} = 0$ implies $f = 0$ in $L^2(\mathbb{T})$, and thus $f = 0$ in $\mathcal{D}(\mathbb{T})$.

We show this mapping is onto. Let $\{a_n\} \in \mathcal{S}(\mathbb{Z})$. For each k, define $g_k(z) = \sum_n (2\pi i n)^k a_n z^n$. We note this series converges uniformly for each k. Indeed,

$$\begin{aligned}
\sum |(2\pi)^k i^k n^k a_n z^n| &= (2\pi)^k \sum |n^k a_n| \\
&\leqslant (2\pi)^k \sum_{n\neq 0} |n^{-2}(1+|n|^{k+2})|a_n| \\
&\leqslant (2\pi)^k \rho_{k+2}(a) \sum |n^{-2}| \\
&< \infty.
\end{aligned}$$

Thus each $g_k \in C(\mathbb{T})$. Hence $\sum_n (2\pi i n)^k a_n e^{2\pi i n\theta}$ converges uniformly on $\mathbb{R}$ and since

$$D(\sum_{n=-N}^{N} (2\pi i n)^k a_n e^{2\pi i n\theta}) = \sum_{n=-N}^{N} (2\pi i n)^{k+1} a_n e^{in\theta}$$

converges uniformly to $g_{k+1}(e^{2\pi i\theta})$ on $\mathbb{R}$, we see $g_k(e^{2\pi i\theta})$ is differentiable and has derivative $g_{k+1}(e^{2\pi i\theta})$. Thus $g_0 \in \mathcal{D}(\mathbb{T})$ and $D^k g_0 = g_k$. Moreover, $\hat{g}_0 = a$, and we see $f \mapsto \hat{f}$ is onto.

To show $f \mapsto \hat{f}$ and $\hat{f} \mapsto f$ are continuous, it is sufficient to show they are continuous at 0.

Now $f \mapsto \hat{f}$ is continuous at 0 iff $f \mapsto \rho'_k(\hat{f})$ are continuous at 0. But

$$\begin{aligned}\rho'_k(\hat{f}) &= \sup |n^k \hat{f}(n)| \\ &= \frac{1}{(2\pi)^k} |\widehat{D^k f}|_\infty \\ &< \epsilon\end{aligned}$$

if $|D^k f|_\infty < (2\pi)^k \epsilon$, for $|\hat{g}(n)| \leqslant |g|_1 \leqslant |g|_\infty$. Hence $f \mapsto \hat{f}$ is continuous.

We finally show $\hat{f} \mapsto f$ is continuous at 0. Note

$$\begin{aligned}|D^k f|_\infty &= \sup_{|z|=1} |\sum_n (2\pi i n)^k \hat{f}(n) z^n| \\ &\leqslant (2\pi)^k \sum |n^k \hat{f}(n)| \\ &\leqslant (2\pi)^k |\hat{f}(0)| + (2\pi)^k \sum_{n \neq 0} n^{-2} \cdot \sup(1+|n|)^{k+2} |\hat{f}(n)| \\ &\leqslant (2\pi)^k \left[\rho_0(\hat{f}) + \rho_{k+2}(\hat{f}) \cdot \sum_{n \neq 0} \frac{1}{n^2} \right] \\ &< \epsilon\end{aligned}$$

if $\rho_0(\hat{f}) < \frac{\epsilon}{2(2\pi)^k}$ and $\rho_{k+2}(\hat{f}) < \frac{\epsilon}{2(2\pi)^K} \left(\sum_{n \neq 0} \frac{1}{n^2} \right)^{-1}$. □

Corollary 1.23. *Suppose f is a periodic C^∞ function on $\mathbb{R}$ having period 1. Then the Fourier series $\sum \hat{f}(n) e^{2\pi i n \theta}$ converges uniformly to f and the derivatives of these series converge uniformly to the derivatives of f.*

Corollary 1.24. *Suppose $f \in L^2(\mathbb{T})$ and $\sum_{n \neq 0} |\hat{f}(n)|^2 n^{2k+2} < \infty$. Then $f \in C^k(\mathbb{T})$ and the series*

$$\sum_n (2\pi i n)^k \hat{f}(n) z^n$$

converges uniformly to $D^k f(z)$.

Proof. We show the series $\sum_n (2\pi i n)^r \hat{f}(n) z^n$ converges uniformly for each $0 \leqslant r \leqslant k$. Indeed, note by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\sum |2\pi i n|^r |\hat{f}(n)| &\leqslant (2\pi)^r \sum_{n\neq 0} |\hat{f}(n) n^{r+1}| \frac{1}{n} \\
&\leqslant (2\pi)^r \left(\sum_{n\neq 0} |\hat{f}(n)|^2 n^{2r+2} \right)^{\frac{1}{2}} \left(\sum_{n\neq 0} \frac{1}{n^2} \right)^{\frac{1}{2}} \\
&\leqslant (2\pi)^r \left(\sum_{n\neq 0} |\hat{f}(n)|^2 n^{2k+2} \right)^{\frac{1}{2}} \left(\sum_{n\neq 0} \frac{1}{n^2} \right)^{\frac{1}{2}} \\
&< \infty .
\end{aligned}$$

Thus $\sum (2\pi i n)^r \hat{f}(n) z^n$ converges uniformly for each $r \leqslant k$. Setting $g_r(z)$ to be this sum, we have each $g_r \in C(\mathbb{T})$; and since

$$D\Big(\sum_{n=-N}^{M} (2\pi i n)^r \hat{f}(n) e^{2\pi i n\theta} \Big) = \sum_{n=-N}^{M} (2\pi i n)^{r+1} \hat{f}(n) e^{2\pi i n\theta},$$

we see $Dg_r = g_{r+1}$ for $r+1 \leqslant k$. But $g_0 = f$. Hence $f \in C^k(\mathbb{T})$ and $D^k f = g_k$. □

Corollary 1.25. *Suppose $f \in C^{k+1}(\mathbb{T})$. Then for each $r \leqslant k$, the Fourier series $\sum (2\pi i n)^r \hat{f}(n) e^{2\pi i n\theta}$ converges uniformly to $D^r f(e^{2\pi i\theta})$.*

Proof. We know $(D^{k+1} f)\hat{}(n) = (2\pi i n)^{k+1} \hat{f}(n)$. Thus $\sum_n n^{2k+2} |\hat{f}(n)|^2 < \infty$ for $D^{k+1} f \in L^2(\mathbb{T})$. Hence $\sum n^{2r+2} |\hat{f}(n)|^2 < \infty$ for any $r \leqslant k$. □

Exercise Set 1.3

1. Suppose σ_k are seminorms on vector space X and ρ_k are seminorms on vector space Y. Give X and Y the topological vector space topologies defined by these seminorms. Show a linear transformation $T : X \to Y$ is continuous iff $\rho_k \circ T$ is continuous at 0 for each k.

(Hint: Recall a subset U of X will be open in the topology defined by the seminorms σ_k if for each $p \in U$, there is an $\epsilon > 0$ and finitely many seminorms $\sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_n}$ so that if $\sigma_{k_i}(q - p) < \epsilon$ for $i = 1, 2, \ldots, n$, then $q \in U$.)

2. Show that $f \in L^2([0, 2\pi], dx)$ can be written in the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) .$$

Find an expression for a_n and b_n.

3. Let g be the function on the torus given by $g(e^{i\theta}) = |\theta|$ for $\theta \in [-\pi, \pi)$.

(a) Find $\hat{g}(n)$.

(b) Show that the Fourier series converges uniformly.

4. Use the Fourier transform to evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

5. Suppose $f \in L^2[0,1]$ has Fourier series

$$f(x) = \sum_n c_n e^{2\pi i n x}.$$

Show

$$\int_0^x f(t)\,dt = i \sum_{n=-\infty}^{\infty} c_n \frac{1 - e^{2\pi i n x}}{n}$$

for $0 \leqslant x \leqslant 1$.

6. Let f be the periodic function with period 1 corresponding to $\chi_{[-\frac{1}{2},0)} - \chi_{[0,\frac{1}{2})}$. Evaluate $\hat{f}(n)$.

7. Let f be a C^∞ function of compact support on $\mathbb{R}$. Define F by

$$F(x) = \sum_{n \in \mathbb{Z}} f(x+n).$$

(a) Show F is a C^∞ function of period 1.

(b) Show

$$\sum_n f(n) = \sum_n \hat{F}(n).$$

8. A function f on the torus is even if $f(z) = f(z^{-1})$ and odd if $f(z) = -f(z^{-1})$. Suppose that $f \in C^2(\mathbb{T})$. Show the following:

(a) If f is even, then $f(e^{2\pi i\theta}) = \sum \hat{f}(n) \cos(2\pi n\theta)$;

(b) If f is odd, then $f(e^{2\pi i\theta}) = i \sum \hat{f}(n) \sin(2\pi n\theta)$.

9. Let $g \in C^1(\mathbb{T})$ and $f \in L^1(\mathbb{T})$. Then $f * g \in C^1(\mathbb{T})$ and $D(f * g) = f * Dg$.

10. Let $L > 0$. Let f be a L-periodic function such that $\int_0^L |f(t)|^2\,dt < \infty$. Show that there are constants $a_n \in \mathbb{C}$ such that in $L^2([0,L))$ we have

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n t/L}.$$

Find an expression for a_n.

11. Show that the seminorms ρ_k, ρ'_k, and ρ''_k all define the same topology on $\mathcal{S}(\mathbb{Z})$.

12. Let $\mathbb{T}^k := \{\mathbf{z} = (z_1, \ldots, z_k) \mid z_j \in \mathbb{T}\}$ with the product topology. For $\mathbf{z}, \mathbf{w} \in \mathbb{T}^k$, let $\mathbf{zw} = (z_1 w_1, \ldots, z_k w_k)$. For $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, let $e_{\mathbf{n}}(\mathbf{z}) = z_1^{n_1} \cdots z_k^{n_k}$. Show the following:

(a) $\mathbb{T}^k$ is a topological group, i.e., the map $\mathbb{T}^k \times \mathbb{T}^k \ni (\mathbf{z}, \mathbf{w}) \mapsto \mathbf{z}^{-1}\mathbf{w} \in \mathbb{T}^k$ is continuous.

(b) $e_{\mathbf{n}} : \mathbb{T}^k \to \mathbb{T}$ is a continuous homomorphism.

(c) If $\chi : \mathbb{T}^k \to \mathbb{T}$ is a continuous homomorphism, then there exists a $\mathbf{n} \in \mathbb{Z}^k$ such that $\chi = e_{\mathbf{n}}$.

13. Let $\mu_k = \mu \times \cdots \times \mu$ be the product measure on $\mathbb{T}^k$. For $f \in L^1(\mathbb{T}^k)$, define $\hat{f} : \mathbb{Z}^k \to \mathbb{C}$ by

$$\hat{f}(\mathbf{n}) := \int_{\mathbb{T}^k} f(\mathbf{z})\, e_{-\mathbf{n}}(\mathbf{z})\, d\mu_k \,.$$

Show the following:

(a) If $f \in L^2(\mathbb{T}^k)$, then $|f|_2 = \sqrt{\sum_{\mathbf{n} \in \mathbb{Z}^k} \left|\hat{f}(\mathbf{n})\right|^2}$.

(b) If $f \in L^2(\mathbb{T}^k)$, then $f = \sum_{\mathbf{n} \in \mathbb{Z}^k} \hat{f}(\mathbf{n})\, e_{\mathbf{n}}$ in $L^2(\mathbb{T}^k)$.

14. Generalize Corollary 1.23 to $\mathbb{R}^k$. Namely let f be a C^∞ function on $\mathbb{R}^k$ satisfying $f(\mathbf{x} + \mathbf{n}) = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{n} \in \mathbb{Z}^k$. Define $\hat{f}(\mathbf{n}) = \int_{[0,1]^k} f(\mathbf{x}) e^{-2\pi i \mathbf{x} \cdot \mathbf{n}}\, d\mathbf{x}$. Show the series $\sum_{\mathbf{n} \in \mathbb{Z}^k} \hat{f}(\mathbf{n}) e^{2\pi i \mathbf{n} \cdot \mathbf{x}}$ and its derivatives converge uniformly to f and f's derivatives on $\mathbb{R}^k$.

15. Show if $f \in \mathcal{C}^\infty(\mathbb{R}^k)$ and $f(\mathbf{x}+\mathbf{n}) = f(\mathbf{x})$ for all $\mathbf{n} \in \mathbb{Z}^n$**,** then $\hat{f} : \mathbb{Z}^k \to \mathbb{C}$ is rapidly decreasing; i.e.,

$$\sup_{\mathbf{n} \in \mathbb{Z}^k} |p(\mathbf{n}) \hat{f}(\mathbf{n})|$$

is finite for every polynomial p. Use this to show the absolute uniform convergence of the series $\sum_{\mathbf{n}} \hat{f}(n) D^\alpha(e^{2\pi i \mathbf{x} \cdot \mathbf{n}})$ for each $\alpha \in \mathbb{N}_0^n$.

16. Show that there is no differentiable function f on $\mathbb{T}$ such that $Df = 1$.

17. Let $p(z) = \sum_{n=0}^k a_n z^n$ be a polynomial. Define $p(D) : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ by

$$p(D) f = \sum_{n=0}^{k} a_n D^n f \,.$$

Show that if $g \in C^\infty(\mathbb{T})$ is such that $\hat{g}(n) = 0$ if $p(2\pi i n) = 0$, then the differential equation $p(D) f = g$ has a solution.

18. Let $f \in L^2(\mathbb{T})$. Show there exists a unique $g \in L^2(\mathbb{T})$ such that

$$\hat{g}(n) = -i \operatorname{sgn}(n)\, \hat{f}(n) \,, \quad \forall n \in \mathbb{Z} \,.$$

Define $Hf = g$. Then $H : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is linear. Prove the following:

(a) $|Hf|_2 = |f|_2$ if and only if $\int f\,d\mu = 0$.

(b) If $f \in C^\infty(\mathbb{T})$, then $Hf \in C^\infty(\mathbb{T})$.

(c) If $f \in C^\infty(\mathbb{T})$, then $Hf(1) = -\frac{1}{\pi}\sum_{n=1}^\infty \int_0^{2\pi} f(e^{i\theta})\,\sin(n\theta)\,d\theta$.

5. Approximate Units

Lemma 1.26. *Let $f \in L^1(\mathbb{T})$ and $g \in L^p(\mathbb{T})$ where $1 \leqslant p \leqslant \infty$. Then $f*g \in L^p(\mathbb{T})$ and $|f*g|_p \leqslant |f|_1|g|_p$. Moreover, if $g \in C(\mathbb{T})$, then $f*g \in C(\mathbb{T})$.*

Proof. We have already seen this is true if $p = 1$. So we may assume $p > 1$. If $p = \infty$, $|f*g(x)| \leqslant \int |f(y)|\,|g(y^{-1}x)|\,d\mu(y) \leqslant |g|_\infty \int |f(y)|\,d\mu(y) = |f|_1|g|_\infty$. Suppose $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned}
|\langle f*g, h\rangle| &= \left|\int f*g(x)h(x)\,d\mu(x)\right| \\
&\leqslant \int\int |f(y)g(y^{-1}x)h(x)|\,d\mu(y)\,d\mu(x) \\
&\leqslant \int |f(y)| \int |\lambda(y)g(x)h(x)|\;d\mu(x)\,d\mu(y) \\
&\leqslant \int |f(y)|\,|\lambda(y)g|_p|h|_q\,d\mu(y) \\
&= |g|_p \int |f(y)|\,d\mu(y)\,|h|_q \\
&= |f|_1|g|_p|h|_q.
\end{aligned}$$

This implies $f*g$ defines a bounded linear functional on L^q and consequently must be in L^p. Moreover, the norm of this bounded linear functional is at most $|f|_1|g|_p$. Thus $f*g \in L^p$ and $|f*g|_p \leqslant |f|_1|g|_p$.

Finally suppose $g \in C(\mathbb{T})$ and $x_n \to x$ as $n \to \infty$. Then $g(y^{-1}x_n) \to g(y^{-1}x)$ for all y and $|f(y)g(y^{-1}x_n)| \leqslant |f(y)|\,|g|_\infty$. Hence by the Lebesgue dominated convergence theorem, $\int f(y)g(y^{-1}x_n)\,d\mu(y) \to \int f(y)g(y^{-1}x)\,d\mu(x)$ as $n \to \infty$. Thus $f*g$ is continuous function. $\square$

Definition 1.27. *An approximate unit in $L^1(\mathbb{T})$ will be a sequence ϕ_n of nonnegative measurable functions satisfying*

(a) $\int \phi_n\,d\mu = 1$ *for each* n

(b) *if* U *is a neighborhood of* 1, *then* $\sup_{x\notin U} \phi_n(x) \to 0$ *as* $n \to \infty$.

Proposition 1.28. *Let ϕ_n be an approximate unit in $L^1(\mathbb{T})$, and suppose $f \in L^p$ where $1 \leqslant p < \infty$. Then $\phi_n * f \to f$ in $L^p(\mathbb{T})$. Moreover, if $f \in C(\mathbb{T})$, then $\phi_n * f$ converges uniformly to f.*

Proof. We note by the Hahn–Banach Theorem that there is always an $h \in L^q$ satisfying $|h|_q = 1$ and $\langle \phi_n * f - f, h \rangle = |\phi_n * f - f|_p$. Now by Hölder's inequality,

$$\begin{aligned}
|\langle \phi_n * f - f, h \rangle| &\leqslant \int\int \phi_n(y) |f(y^{-1}x) - f(x)|\, |h(x)|\, d\mu(y)\, d\mu(x) \\
&= \int \phi_n(y) \int |\lambda(y) f(x) - f(x)|\, |h(x)|\, d\mu(x)\, d\mu(y) \\
&\leqslant \int \phi_n(y)\, |\lambda(y) f - f|_p |h|_q\, d\mu(y).
\end{aligned}$$

Now if $\epsilon > 0$, we can choose a neighborhood U of 1 in $\mathbb{T}$ such that $|\lambda(y) f - f|_p < \frac{\epsilon}{2}$ if $y \in U$. (See Exercise 1.2.4.) But $\phi_n \to 0$ uniformly off U. Hence for large n, $\int_{\mathbb{T}-U} \phi_n(y)\, d\mu(y) < \frac{\epsilon}{4|f|_p}$. Thus for large n, we have

$$|\langle \phi_n * f - f, h \rangle| \leqslant \int_{\mathbb{T}-U} \phi_n(y)(2|f|_p)\, d\mu(y) + \int_U \frac{\epsilon}{2} \phi_n(y)\, dy < \epsilon$$

for any h with $|h|_q = 1$. Consequently $|\phi_n * f - f|_p < \epsilon$ for large n.

Finally, suppose $f \in C(\mathbb{T})$. First choose a neighborhood U of 1 such that $|f(y^{-1}x) - f(x)| < \frac{\epsilon}{2}$ whenever $y \in U$, and then choose N such that $\sup_{x \in \mathbb{T}-U} \phi_n(x) < \frac{\epsilon}{4|f|_\infty}$ for $n \geqslant N$. Then if $n \geqslant N$, one has

$$\begin{aligned}
|\phi_n * f(x) - f(x)| &\leqslant \int \phi_n(y) |f(y^{-1}x) - f(x)|\, dy \\
&< \int_{\mathbb{T}-U} 2\phi_n(y) |f|_\infty\, d\mu(y) + \int_U \phi_n(y) \frac{\epsilon}{2}\, d\mu(y) \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

Lemma 1.29. *Let $f \in L^1(\mathbb{T})$ and $g \in C^\infty(\mathbb{T})$. Then $f * g \in C^\infty(\mathbb{T})$. Moreover,*

$$D^k(f * g) = f * D^k g.$$

Proof. We have $|\hat{f}|_\infty \leqslant |f|_1$ and $|\hat{g}|_\infty \leqslant |g|_1$. Using the seminorms ρ_k'' and Theorem 1.22, we know $\sum_n |n|^k |\hat{g}(n)| < \infty$ for all k. By (e) of Lemma 1.17, $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$. But

$$\sum |\hat{f}(n)\hat{g}(n)|^2 n^{2k+2} \leqslant |f|_1^2 |g|_1 \sum |\hat{g}(n)|\, n^{2k+2} < \infty.$$

Thus by Corollary 1.24, $f * g \in C^k(\mathbb{T})$ for all k. Moreover,

$$\begin{aligned}\mathcal{F}(D^k(f * g))(n) &= (2\pi i n)^k \mathcal{F}(f * g)(n)\\ &= (2\pi i n)^k \hat{f}(n)\hat{g}(n)\\ &= \hat{f}(n)\mathcal{F}(D^k g)(n)\\ &= \mathcal{F}(f * D^k g)(n).\end{aligned}$$

Since $\mathcal{F}$ is one-to-one on $L^2(\mathbb{T})$,

$$D^k(f * g) = f * D^k g.$$

□

This result also follows from Exercise 1.3.9.

6. Convergence of the Fourier Series

We saw in a previous section that the Fourier transform converges in L^2 if f is an L^2-function. Also if f is in $C^2(\mathbb{T})$, then the Fourier series converges uniformly to f. But in general it does not hold that the Fourier series $\sum a_n z^n$ converges to $f(z)$. In this section we will deal with the question of how to recover a function from its Fourier series. We start by stating two negative results; however, first we have a few preliminaries.

Define the partial Fourier sum $s_N(f)$ to be

$$s_N(f) = \sum_{n=-N}^{N} \hat{f}(n) e_n\,.$$

Note $s_N(f) = f * D_N$ where $D_N = e_{-N} + e_{-N+1} + \cdots + e_{N-1} + e_N$. D_N is a trigonometric polynomial that is an idempotent under convolution. It is called the Dirichlet kernel.

Lemma 1.30. *Let $N \in \mathbb{N}$. Then the following hold:*

(a) $D_N(z) = \frac{z^{N+1/2} - z^{-(N+1/2)}}{z^{1/2} - z^{-1/2}}$ *if $z \neq 1$ and $D_N(1) = 2N + 1$.*

(b) $D_N(e^{2\pi i x}) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$ *if $x \notin \mathbb{Z}$ and $D_N(1) = 2N + 1$.*

(c) $D_N(z) = D_N(z^{-1})$.

(d) $\int_{\mathbb{T}} D_N(z)\, d\mu(z) = 1$.

(e) $s_N(f)(z) = f * D_N(z) = D_N * f(z)$.

Proof. (a) We have

$$
\begin{aligned}
D_N(z) &= z^{-N} \sum_{n=0}^{2N} z^n \\
&= z^{-N} \frac{1 - z^{2N+1}}{1 - z} \\
&= \frac{z^{N+1/2} - z^{-(N+1/2)}}{z^{1/2} - z^{-1/2}} .
\end{aligned}
$$

Also $D_N(1) = \sum_{n=-N}^{N} 1 = 2N + 1$.

(b) This follows immediately by using that $\sin(\psi) = (e^{i\psi} - e^{-i\psi})/2i$.

(c) This follows from (a).

(d) Using $\int_{\mathbb{T}} z^n \, d\mu = 0$ for $n \neq 0$ one has $\int_{\mathbb{T}} D_N(z) \, d\mu(z) = \sum_{n=-N}^{N} \int z^n \, d\mu(z) = \int 1 \, d\mu(z) = 1$.

(e) This follows by (f) of Lemma 1.17.

$\square$

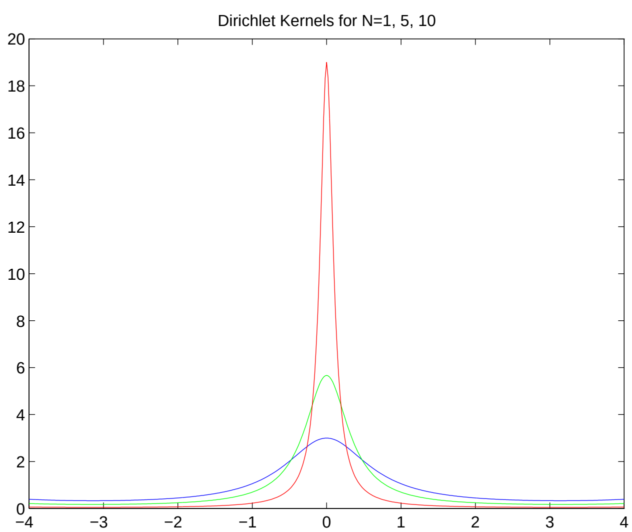

Figure 2. Dirichlet Kernels for $N = 1$(blue), 5(green), 10(red)

Figure 1.2 shows the functions D_N become more and more localized around $z = 1$ and then oscillates.

Lemma 1.31. *There exists a dense G_δ subset $D \subset L^1(\mathbb{T})$ such that the Fourier series does not converges in $L^1(\mathbb{T})$ for $f \in D$.*

Proof. Define linear transformations $\Lambda_N : L^1(\mathbb{T}) \to L^1(\mathbb{T})$ by $\Lambda_N f = D_N * f$. We note individually they are bounded for $||\Lambda_N|| \leqslant |D_N|_1$. They, however, are not uniformly bounded.

Define f_k on $\mathbb{T}$ by $f_k(e^{i\theta}) = \pi k \chi_{[-\frac{1}{k},\frac{1}{k}]}(\theta)$. Note each f_k has length 1 in $L^1(\mathbb{T})$. Moreover,

$$\hat{f}_k(n) = \frac{1}{2\pi}\int_{-\frac{1}{k}}^{\frac{1}{k}} k\pi e^{-in\theta}\, d\theta = -\frac{k}{2in} e^{-in\theta}|_{-\frac{1}{k}}^{\frac{1}{k}} = \frac{k}{2in}\left(e^{i\frac{n}{k}} - e^{-i\frac{n}{k}}\right) = \frac{k}{n}\sin(\frac{n}{k}).$$

Hence for fixed n, $\hat{f}_k(n) \to 1$ as $k \to \infty$. Thus $\Lambda_N(f_k) = f_k * D_N = \sum_{l=-N}^{N} \hat{f}_k(l) e_l \to \sum_{l=-N}^{N} e_l = D_N$ as $k \to \infty$.

Recall

$$D_N(e^{2\pi i x}) = \frac{\sin(\pi(2N+1)x)}{\sin(\pi x)}.$$

Now $|D_N|_1 = \int_0^1 \frac{|\sin(\pi(2N+1)x)|}{\sin(\pi x)}\, dx \to \infty$ as $N \to \infty$. Indeed,

$$\begin{aligned}
|D_N|_1 &= \int_0^1 \left|\frac{\sin(\pi(2N+1)x)}{\sin(\pi x)}\right| dx \\
&= \sum_{k=1}^{4N+2} \int_{(k-1)/(4N+2)}^{k/(4N+2)} \left|\frac{\sin(\pi(2N+1)x)}{\sin(\pi x)}\right| dx \\
&> \sum_{k=1}^{4N+2} \int_{(k-1)/(4N+2)}^{k/(4N+2)} \left|\frac{\sin(\pi(2N+1)x)}{\pi x}\right| dx \\
&> \frac{1}{\pi}\sum_{k=1}^{4N+2} \int_{(k-1)/(4N+2)}^{k/(4N+2)} \left|\frac{\sin(\pi(2N+1)x)}{k/(4N+2)}\right| dx \\
&= \frac{(4N+2)}{\pi}\sum_{k=1}^{4N+2} \frac{1}{k}\int_{(k-1)/(4N+2)}^{k/(4N+2)} |\sin(\pi(2N+1)x)|\, dx \\
&= \frac{2}{\pi^2}\sum_{k=1}^{4N+2} \frac{1}{k}\int_{(k-1)\pi/2}^{k\pi/2} |\sin t|\, dt \quad (\text{where } t = \pi(2N+1)x) \\
&= \frac{2}{\pi^2}\sum_{k=1}^{4N+2} \frac{1}{k}\int_0^{\pi/2} \sin t\, dt \\
&= \frac{2}{\pi^2}\sum_{k=1}^{4N+2} \frac{1}{k} \to 0 \text{ as} N \to 0.
\end{aligned}$$

Hence the Λ_N are not uniformly bounded on the unit ball of $L^1(\mathbb{T})$. By the Banach–Steinhaus Theorem (i.e., the principle of uniform boundedness), there exists a dense G_δ set D such that

$$\sup_N \Big|\sum_{k=-N}^{N} \hat{f}(k)e_k\Big|_1 = \sup_N |\Lambda_N(f)|_1 = \infty \text{ for all } f \in D.$$

□

Remark. We showed $|D_N|_1 \geqslant \frac{2}{\pi^2}\sum_{k=1}^{4N+2}\frac{1}{k}$. Thus $|D_N|_1 \geqslant \frac{2}{\pi^2}\ln(4N+3) \geqslant \frac{2}{\pi^2}\ln N$. Thus the D_N's are not bounded in $L^1(\mathbb{T})$. It is known that $|D_N|_1 = \frac{4}{\pi^2}\ln N + O(1)$. This is the central reason they do not form an approximate unit in $L^1(\mathbb{T})$.

The following shows convergence can be a problem even for continuous functions. We state it without proof.

Lemma 1.32. *Let $\{z_k\}_{k=0}^{\infty}$ be a sequence in $\mathbb{T}$. Then there exists a function $f \in C(\mathbb{T})$ such that $\lim_{N\to\infty} |s_N(f)(z_k)| = \infty$ for all k.*

Remark 1.33. One of the most intriguing and long standing problems in analysis was when and where the Fourier series of a continuous function on $\mathbb{T}$ converges. In 1873 P. Dubois Reymond gave an example of a continuous function whose Fourier series fails to converge at a point.

In 1922 A. N. Kolmogorov gave an example of an integrable function whose Fourier series converges at no point.

This is clearly related to Lemma 1.31 in that the Fourier series of a function $f \in L^1(\mathbb{T})$ need not converge pointwise almost everywhere, for if it did it would converge in $L^1(\mathbb{T})$.

In 1966, L. Carleson showed there is a.e. pointwise convergence for every L^2 function. This was later extended by Hunt to the Fourier series of any function in $L^p(\mathbb{T})$ where $p > 1$.

Lemma 1.34 (Lebesgue Lemma)**.** *Suppose $T > 0$ and g is in $L^1[0,T]$. Then*

$$\int_0^T g(x)\sin ax\,dx \to 0$$

as $a \to \infty$.

Sketch. This is similar to the argument for (d) in Lemma 1.17. First check it works for $g(x) = \chi_{[c,d]}$. Then show it works for step functions and then for any L^1 function. □

Theorem 1.35. *Suppose $f(t) = F(e^{2\pi it})$ is a periodic function on $\mathbb{R}$ with period 1 and f is integrable on $[0,1]$. Let x be a point where $f(x+) = \lim_{t\to x+} f(t)$ and $f(x-) = \lim_{t\to x-} f(t)$ exist. If there exist $K > 0$, $\delta > 0$, and $\alpha > 0$ with $|f(x+t) - f(x+)| \leqslant Kt^\alpha$ and $|f(x-t) - f(x-)| \leqslant Kt^\alpha$ for*

$0 < t < \delta$, *then* $D_N * F(e^{2\pi ix}) \to \frac{1}{2}(f(x+) + f(x-))$ *as* $N \to \infty$*; i.e., one has*

$$\sum_{k=-N}^{N} \hat{f}(k)e^{2\pi ikx} \to \frac{1}{2}(f(x+) + f(x-)).$$

Proof. Note

$$\begin{aligned} D_N * F(e^{2\pi ix}) &= \int D_N(z)F(z^{-1}e^{2\pi ix})\,d\mu(z) \\ &= \int_{-1/2}^{1/2} D_N(e^{2\pi it})F(e^{2\pi i(x-t)})\,dt \\ &= \int_{-1/2}^{1/2} D_N(e^{2\pi it})f(x-t)\,dt. \end{aligned}$$

Hence $D_N * F(e^{2\pi ix}) = \int_0^{1/2} D_N(e^{2\pi it})f(x-t)\,dt + \int_{-1/2}^{0} D_N(e^{-2\pi it})f(x-t)\,dt$ for $D_N(e^{-2\pi it}) = D_N(e^{2\pi it})$. Changing variables in the second integral gives

$$D_N * F(e^{2\pi ix}) = \int_0^{1/2} D_N(e^{2\pi it})(f(x+t) + f(x-t))\,dt.$$

Hence the result will follow if we show

$$\int_0^{1/2} D_N(e^{2\pi it})f(x+t)\,dt \to \frac{1}{2}f(x+)$$

and

$$\int_0^{1/2} D_N(e^{2\pi it})f(x-t)\,dt \to \frac{1}{2}f(x-)$$

as $N \to \infty$.

We show the first, for the second follows by the same argument. Note $\int_0^{1/2} D_N(e^{2\pi it})\,dt = \frac{1}{2}\int D_N(z)\,d\mu(z) = \frac{1}{2}$. Hence

$$\begin{aligned} \int_0^{1/2} D_N(e^{2\pi it})f(x+t)\,dt - \frac{1}{2}f(x+) &= \int_0^{1/2} D_N(e^{2\pi it})(f(x+t) - f(x+))\,dt \\ &= \int_0^{1/2} \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}(f(x+t) - f(x+))\,dt. \end{aligned}$$

By the Lebesgue Lemma, this will converge to 0 as $N \to \infty$ if the function

$$\psi(t) = \frac{f(x+t) - f(x+)}{\sin \pi t}$$

is integrable on $[0, 1/2]$. Choose $\delta > 0$ so that $|f(x+t) - f(x+)| < Kt^\alpha$ and $\sin \pi t > \frac{\pi t}{2}$ for $0 < t < \delta$. Then

$$|\psi(t)| = \chi_{(0,\delta)}\frac{|f(x+t) - f(x+)|}{\sin \pi t} + \chi_{[\delta,1/2]}(t)\frac{|f(x+t) - f(x+)|}{\sin \pi t}.$$

The second of these two terms is clearly in $L^1[0, 1/2]$, and the first is less than $\frac{2Kt^\alpha}{\pi t}$ which is integrable on $[0, 1/2]$ since $\alpha > 0$. $\square$

Corollary 1.36. *Suppose $f(t)$ is periodic with period 1 and $f|_{[0,1]} \in L^1[0,1]$. If $f(x+)$ and $f(x-)$ exist and*

$$f'(x+) = \lim_{t \to 0+} \frac{f(x+t) - f(x+)}{t} \text{ and } f'(x-) = \lim_{t \to 0+} \frac{f(x-t) - f(x-)}{t}$$

exist, then

$$\lim_{N \to \infty} \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k x} = \frac{1}{2}(f(x+) + f(x-)).$$

Proof. We can choose $\delta > 0$ so that $\frac{|f(x \pm t) - f(x\pm)|}{t} < |f'(x\pm)| + 1$ if $0 < t < \delta$. Hence there is a K such that $|f(x \pm t) - f(x\pm)| < Kt$ for $0 < t < \delta$. □

Definition 1.37. *A sequence $\{a_n\}_{n=0}^{\infty}$ is **Cesáro summable to L** if the average $1/(N+1)\sum_{k=0}^{N} s_k$ of the partial sums $s_k = \sum_{n=0}^{k} a_n$ converges to L. It is **Abel summable to L** if $\sum_{n=0}^{\infty} a_k r^k$ exists for all $0 \leqslant r < 1$ and $\lim_{r \to 1-} \sum a_n r^n = L$.*

To recover $f \in L^1(\mathbb{T})$ from its Fourier transform one uses the average of partial sums, i.e., Cesáro summability. Define

$$\sigma_N(f)(z) = \frac{1}{N+1} \sum_{n=0}^{N} s_n(f)(z)\,.$$

Let

$$\Sigma_N(z) = \frac{1}{N+1} \sum_{n=0}^{N} D_N(z).$$

Then

$$\sigma_N(f) = f * \Sigma_N\,.$$

Lemma 1.38. *One has*

(a) $\Sigma_N(e^{2\pi i x}) = \frac{1}{N+1}\left[\frac{\sin((N+1)\pi x)}{\sin(\pi x)}\right]^2$ *if* $e^{2\pi i x} \neq 1$ *and* $\Sigma_N(1) = N+1$. *In particular we have* $\Sigma_N \geqslant 0$.

(b) $\Sigma(z) = \Sigma(z^{-1})$.

(c) $\int_{\mathbb{T}} \Sigma_N \, d\mu = 1$.

Proof. (a) Using $D_n(z) = (z^{n+1} - z^{-n})/(z-1)$, we see that

$$\begin{aligned}
\sum_{n=0}^{N} D_n(z) &= \frac{1}{z-1}\left[\frac{z^{N+2}-z}{z-1} - \frac{z^{-N-1}-1}{z^{-1}-1}\right] \\
&= \frac{1}{(z-1)^2}\left[z^{N+2} - z + z^{-N} - z\right] \\
&= \frac{z}{(z-1)^2}\left(z^{N+1} - 2 + z^{-N-1}\right) \\
&= \frac{1}{(z^{1/2}-z^{-1/2})^2}\left(z^{(N+1)/2} - z^{-(N+1)/2}\right)^2 \\
&= \left[\frac{z^{(N+1)/2} - z^{-(N+1)/2}}{z^{1/2}-z^{-1/2}}\right]^2 .
\end{aligned}$$

Hence if $z = e^{2\pi ix}$, one has

$$\Sigma_N(e^{2\pi ix}) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(e^{2\pi ix}) = \frac{1}{N+1}\left[\frac{\sin((N+1)\pi x)}{\sin(\pi x)}\right]^2 .$$

The claim for $z = 1$ follows either by continuity or by

$$\frac{1}{N+1}\sum_{n=0}^{N}(2n+1) = N+1 .$$

(b) This follows immediately from (a).

(c) By lemma 1.30 we have

$$\begin{aligned}
\int \Sigma_N(z)\, d\mu(z) &= \frac{1}{N+1}\sum_{n=0}^{N}\int D_n(z)\, d\mu(z) \\
&= \frac{1}{N+1}\sum_{n=0}^{N} 1 \\
&= 1
\end{aligned}$$

$\square$

Note that the integral of Σ_N is concentrated more and more around $z = 1$ as $N \to \infty$. Figure 1.3 shows the graphs of the Fejer kernels Σ_N for $N = 1$, $N = 5$, and $N = 9$.

Lemma 1.39. *Let $0 < \delta < \frac{1}{2}$. Then there exists a constant $C = C(\delta)$ independent of N such that $\left|\Sigma_N(e^{2\pi ix})\right| \leqslant \frac{C}{N+1}$ for $\delta \leqslant |\theta| \leqslant 1/2$.*

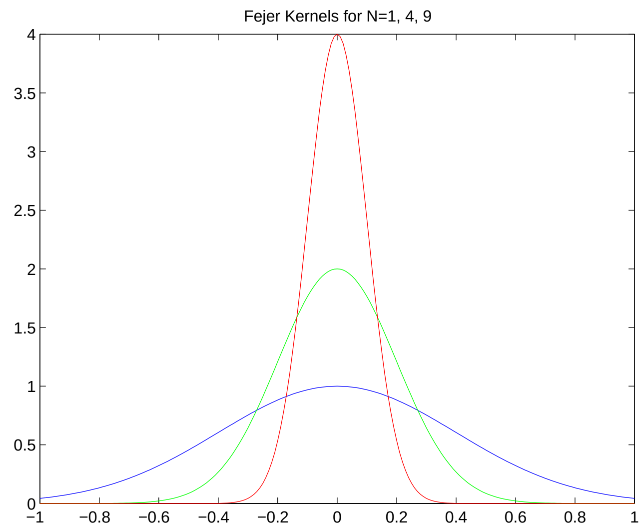

Figure 3. Fejer Kernels for $N = 1$(blue), 5(green), 9(red)

Proof. Choose $C > 0$ such that $|\sin(\pi x)| \geqslant 1/\sqrt{C}$ for $\delta < |x| \leqslant \frac{1}{2}$. As $|\sin((N+1)\pi x)| \leqslant 1$ it follows that

$$\left|\Sigma_N(e^{2\pi i x})\right| \leqslant \frac{C}{N+1}\,.$$

$\square$

Theorem 1.40 (Fejér). *Let $1 \leqslant p < \infty$ and $f \in L^p(\mathbb{T})$. Then $f * \Sigma_N$ is a trigonometric polynomial and*

$$\lim_{N\to\infty} |f * \Sigma_N - f|_p = 0\,.$$

If $f \in C(\mathbb{T})$, then

$$\lim_{N\to\infty} |f * \Sigma_N - f|_\infty = 0\,.$$

Thus for continuous f, the sequence $a_0 := \hat{f}(0)$, $a_n := \hat{f}(n)z^n + \hat{f}(-n)z^{-n}$ for $n > 0$ is uniformly Cesáro summable to $f(z)$.

Proof. By Lemmas 1.38 and 1.39, the Fejer kernels Σ_N form an approximate unit in $L^1(\mathbb{T})$. Hence the statements follow from Proposition 1.28. $\square$

7. The Poisson Kernel

In Section 1, we showed that by separation of variables using polar coordinates on the unit disc $|z| \leqslant 1$, Laplace's equation $\Delta u = 0$ produced solutions of form

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta}.$$

We note if $r = 1$, the resulting function would be a Fourier series and should represent the boundary condition $u(r, 1) = f(\theta)$. Hence we would hope

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta} = f(\theta)$$

in some sense, i.e., pointwise, uniformly, in L^2, etc.

Associated with this decomposition is the function $P(r, \theta)$ where we take all the $a_n's$ in the function $u(r, \theta)$ equal to one. As can be seen in the next chapter, $P(1, \theta)$ is the Fourier series" of the Dirac function δ and conceivably $P(r, \theta)$ is close in some sense to δ for r near one. The function $P(r, \theta)$ is called the Poisson kernel. Hence

$$P(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

This series converges uniformly on any subset S of $[0, 1] \times \mathbb{R}$ for which $\sup\{r : (r, \theta) \in S\} < 1$.

As seen in Section 1,

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Define $P_r(e^{i\theta}) = P(r, e^{i\theta})$ for $0 \leqslant r < 1$.

Lemma 1.41. *P_r satisfy the following conditions.*

(a) *$P_r(z) \geqslant 0$ for all $z \in \mathbb{T}$.*

(b) *$\int P_r(z)\, d\mu(z) = 1$ for $0 \leqslant r < 1$.*

(c) *if U is a neighborhood of 1 in $\mathbb{T}$, then $\sup_{z \notin U} |P_r(z)| \to 0$ as $r \to 1-$.*

Proof. Clearly we have (a) and since $\sum r^{|n|} e_n(e^{i\theta})$ converges uniformly on $\mathbb{T}$, $\int P_r(z)\, d\mu(z) = \sum r^{|n|} \int e_n(z)\, d\mu(z) = r^0 \int e_0(z)\, dz = 1$. Thus (b) holds.

For (c), choose $a > 0$ with $1 - \cos\theta < \frac{1}{2}$ if $|\theta| < a$. Then for any δ with $0 < \delta < a$, one has

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \leqslant \frac{1 - r^2}{1 - r + r^2}$$

if $\delta \leqslant |\theta| < \pi$. Since $\frac{1-r^2}{1-r+r^2} \to 0$ as $r \to 1-$, (c) follows. □

This lemma shows $P_r(z)$ form an 'approximate unit' in $L^1(\mathbb{T})$ and the argument in the proof of Proposition 1.28 shows the following are true:

(1) if $f \in L^p(\mathbb{T})$ where $1 \leqslant p < \infty$, $P_r * f \to f$ in $L^p(\mathbb{T})$ as $r \to 1-$.

(2) if $f \in C(\mathbb{T})$, then $P_r * f \to f$ uniformly on $\mathbb{T}$ as $r \to 1-$.

Theorem 1.42 (Poisson Theorem)**.**

(a) *Let* $f \in L^1(\mathbb{T})$*, then the function*

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta-\phi) f(e^{i\phi})\,d\phi$$

is harmonic on the open disk $|z| < 1$*.*

(b) *If* $f \in L^p(\mathbb{T})$ *where* $1 \leqslant p < \infty$*, then*

$$u(r,\theta) \to f(e^{i\theta}) \text{ in } L^p(\mathbb{T}) \text{ as } r \to 1- .$$

(c) *If* $f \in C(\mathbb{T})$*, then*

$$u(r,\theta) \to f(e^{i\theta}) \text{ uniformly on } \mathbb{T} \text{ as } r \to 1- .$$

Proof. We note we already have (b) and (c). For (a), we need only note since $u(r,\theta) = \sum \hat{f}(n) r^{|n|} e^{in\theta}$, that $|\hat{f}(n)| \leqslant |f|_1$ for all n; and thus both the series and the series for the r and θ derivatives of any order converge uniformly on any disk $|r| < a$ where $a < 1$. □

Exercise Set 1.4

1. Show that the sequence $a_n = (-1)^n$ is Abel and Cesáro summable to 1/2.

2. Show that if $\sum a_n$ converges to L, then $\{a_n\}$ is Abel and Cesáro summable to L. (**Hint**: To show $\sum a_k$ is Abel summable, use Abel's summation formula $\sum_{k=1}^n a_k b_k = (\sum_{k=1}^n a_k) b_n + \sum_{k=1}^{n-1}(\sum_{l=1}^k a_l)(b_k - b_{k+1})$ to show the series $\sum a_k r^k$ converges uniformly for $0 \leqslant r \leqslant 1$.)

3. Show if the series $\sum a_n$ is Cesáro summable, then $\lim_{n\to\infty} \frac{a_n}{n} = 0$.

4. Show the series $\sum_{n=0}^{\infty}(-1)^n \frac{(n+k)!}{n!} n^k$ is Abel summable and find its sum. (**Hint:** Consider $r^k \left(\frac{d}{dr}\right)^k (1+r)^{-1}$ expressed as a series.)

5. Let $\delta_0 \geqslant \delta_1 \geqslant \cdots \geqslant 0$ be a decreasing sequence with $\lim \delta_n = 0$. Define $a_0 := \delta_0$ and $a_n := \delta_{n-1} - \delta_n$ for $n \geqslant 1$. Let $g(z) := \sum a_n z^n$. Show that g is continuous, $g * D_N$ converges uniformly to g, and $|g * D_N - g|_\infty \leqslant \delta_N$.

6. Suppose $f \in L^1(\mathbb{T})$. Show if $\hat{f}(n) = 0$ for all n, then $f(z) = 0$ a.e. z.

7. Let F be the function on $\mathbb{T}$ defined by $F(e^{i\theta}) = \theta$ for $-\pi \leqslant \theta < \pi$.

(a) Evaluate $\hat{F}(n)$.

(b) Show the series $\sum \hat{F}(n) e^{in\pi}$ diverges.

(c) Show for each N, $\sum_{k=-N}^{N} \hat{F}(n) e^{in\pi} = 0$.

(d) Define f by $f(x) = F(e^{2\pi i x})$. Show f satisfies the conditions of Theorem 1.35 at $x = \frac{1}{2}$.

8. Show the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converges pointwise everywhere and determine its limit. Does this series converge uniformly? (**Hint:** Consider the odd function of period 2π equal to $\pi - x$ on the interval $0 < x \leqslant \pi$.)

9. Show

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$$

for $0 < x < \pi$.

10. Suppose $u(x+iy)$ is harmonic on a nonempty open subset U of $\mathbb{C}$; i.e., u is C^2 on this set and $u_{xx} + u_{yy} = 0$.

(a) Show if $u = 0$ on a circle $|z - z_0| = a$ contained inside U, then $u = 0$ inside the circle. (**Hint:** Use the divergence theorem (Stoke's Theorem) on this disk and its boundary applied to the vector field $u\nabla u$.)

(b) Show if $z_0 \in U$ and the circle $|z - z_0| = a$ is contained in U, then

$$u(z_0 + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(\frac{r}{a}, \theta - \phi)u(z_0 + ae^{i\phi})\, d\phi.$$

(**Hint:** Consider Theorem 1.42.)

(c) Conclude

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + ae^{i\phi})\, d\phi = \frac{1}{2\pi a} \oint_{|z-z_0|=a} u(z)\, |dz|.$$

(d) **(Maximum Principle)** Use (c) to show u cannot assume a local maximum or minimum inside U.

8. Applications

In this section we discuss some applications of the Fourier transform. The first two are examples illustrating how one uses the Fourier transform to solve differential equations, and the last is an example of its application to geometry.

Before starting we remark that Fourier series can be used to analyze functions on any finite interval. Indeed if I is an interval of positive length L and f is a complex valued function on I, then $F(e^{2\pi ix/L}) = f(x)$ for $x \in I$ defines a function on $\mathbb{T}$ and thus may have a Fourier series

$$f(x) = F(e^{2\pi ix/L}) = \sum \hat{f}(n)e^{2\pi inx/L}$$

where

$$\begin{aligned}\hat{f}(n) &:= \hat{F}(n)\\ &= \int_0^1 F(e^{it})e^{-2\pi int}\,dt\\ &= \frac{1}{L}\int_I F(e^{2\pi ix/L})e^{-2\pi inx/L}\,dx\\ &= \frac{1}{L}\int_I f(x)e^{-2\pi inx/L}\,dx.\end{aligned}$$

In this situation, the Fourier transform of Df satisfies

$$\widehat{Df}(n) = \frac{2\pi in}{L}\hat{f}(n)$$

and in this respect becomes simplest when $L = 2\pi$.

8.1. The Wave Equation. The general form of the wave equation is

$$a^2\partial_x^2 u(x,t) = \partial_t^2 u(x,t)\,,\quad u(x,0) = f(x)\,,\, u_t(x,0) = g(x)\,. \tag{1.14}$$

Here $x \in [0, L]$, f, g are functions on $[0, L]$, and $a > 0$ is a constant. In the case where $u(0,t) = u(L,t) = 0$ for all t and f and g are real valued, this equation describes the vibration of a homogeneous string, fastened at both ends and starting at position $u(x,0) = f(x)$ with initial velocity $u_t(x,0) = g(x)$. The constant $a^2 = T/\rho$ is given by the tension T and the linear density ρ.

As we are motivated by the vibration of a string, let us assume that f and g are continuous on $[0, L]$, $f(0) = f(L) = g(0) = g(L) = 0$, and f and g are C^2 on $(0, L)$. Let us look for a smooth solution $x \mapsto u(x,t)$. First we extend f and g and $u(\,\cdot\,,t)$ to be odd functions on $[-L, L]$ by $f(-x) = -f(x)$, $g(-x) = -g(x)$, and $u(-x,t) = -u(x,t)$. Then f and g are continuous on $[-L, L]$ and have continuous derivatives on $(-L, L)$. Taking the Fourier transform in the x-variable and denoting it by $\hat{u}(n,t)$, and using that all functions are real valued and odd, one has:

$$\begin{aligned}\hat{u}(n,t) &= \overline{\hat{u}(-n,t)} = \frac{-i}{L}\int_0^L u(x,t)\sin(\pi nx/L)\,dx\,;\\ \hat{f}(n) &= \overline{\hat{f}(-n)} = \frac{-i}{L}\int_0^L f(x)\sin(\pi nx/L)\,dx\,;\\ \hat{g}(n) &= \overline{\hat{g}(-n)} = \frac{-i}{L}\int_0^L g(x)\sin(\pi nx/L)\,dx\,.\end{aligned}$$

Next notice that differentiation in the t-variable commutes with taking the Fourier transform in the x-variable. Thus for all $n \in \mathbb{Z}$:

$$-\frac{\pi^2 a^2}{L^2}n^2\,\hat{u}(n,t) = \hat{u}_{tt}(n,t)\,,\quad \hat{u}(n,0) = \hat{f}(n),\quad \hat{u}_t(n,0) = \hat{g}(n)\,.$$

This is an ordinary second order linear initial value problem for the function $t \mapsto \hat{u}(n,t)$ with unique solution

$$\hat{u}(n,t) = \hat{f}(n)\cos\left(\frac{\pi nat}{L}\right) + \frac{L}{\pi na}\,\hat{g}(n)\sin\left(\frac{\pi nat}{L}\right).$$

Summing up we conclude
(1.15)

$$u(x,t) = 2i\sum_{n=1}^{\infty}\left[\hat{f}(n)\cos\left(\frac{\pi nat}{L}\right) + \frac{L}{\pi na}\,\hat{g}(n)\sin\left(\frac{\pi nat}{L}\right)\right]\sin\left(\frac{\pi nx}{L}\right).$$

This solution satisfies $u(0,t) = u(\pi,t) = 0$ for all $t \in \mathbb{R}$. We note this solution is periodic in t with period $\frac{L}{a}$.

8.2. The Heat Equation. In this section we discuss the heat equation

$$a^2\partial_x^2 u(x,t) = u_t(x,t)\,, \quad u(x,0) = f(x)\,,\ 0 \leqslant x \leqslant L\,.$$

In the case where f is real valued, this is the differential equation describing the heat flow in a homogeneous cylindrical rod of length L, whose lateral surface is insulated from the surrounding medium and where the initial temperature at the point $x \in [0,L]$ is $f(x)$. The constant a in this case is given by $a^2 = K/c\rho$ where K is the *thermal conductivity* of the material from which the rod is made, c is the *heat capacity*, and ρ is the *density*. We will only consider solutions that are fixed by the same constant at the endpoints $x = 0$ and $x = L$. We can then assume that $u(0,t) = u(L,t) = 0$ for all t. In this case, we will assume that $L = \pi$ and will accordingly extend all functions depending on the variable x to odd functions on the interval $[-\pi,\pi]$. Taking the Fourier transform in the x-variable we obtain:

$$-a^2n^2\hat{u}(n,t) = \hat{u}_t(n,t)\,, \quad \hat{u}(n,0) = \hat{f}(n)\,.$$

Thus $\hat{u}(n,t) = \hat{f}(n)e^{-a^2n^2t}$. Now using that f and $u(\cdot,t)$ are odd and real valued we have:

$$u(x,t) = \sum_{n\in\mathbb{Z}}\hat{f}(n)e^{inx-a^2n^2t} = 2i\sum_{n=1}^{\infty}\hat{f}(n)e^{-n^2a^2t}\sin(nx)\,.$$

One can do general L by using Fourier transform on $[-L,L]$ as for the wave equation or making appropriate changes in variables to return to the case when $L = \pi$. One obtains

$$u(x,t) = 2i\sum_{n=1}^{\infty}\hat{f}(n)e^{-n^2\pi^2a^2t/L^2}\sin\left(\frac{n\pi}{L}x\right)$$

where

$$\hat{f}(n) = -\frac{i}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

8.3. The Isoperimetric Problem. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be a continuous smooth simple and closed curve. Then $C := \gamma([0, 2\pi])$ defines a bounded domain $\Omega(\gamma) \subset \mathbb{C}$. Assume for simplicity that the length of C is one.
Question: For which curve is the area $a(\Omega(\gamma))$ maximum?

Theorem 1.43 (Hurwitz)**.** *We have $a(\Omega(\gamma)) \leqslant \frac{1}{4\pi}$ and $a(\Omega(\gamma)) = \frac{1}{4\pi}$ if and only if $\Omega(\gamma)$ is a circle.*

Proof. Let $\Omega = \Omega(\gamma)$. Set $G(e^{i\theta}) = \gamma(\theta) = x(\theta) + iy(\theta)$. Then G is a smooth function on the torus. Hence

$$G(z) = \sum_{n=-\infty}^{\infty} \widehat{G}(n) z^n$$

and if $z = e^{i\theta}$,

$$\gamma'(\theta) = DG(z) = i \sum_{n=-\infty}^{\infty} n\widehat{G}(n) z^n \,.$$

We may assume the parametrization satisfies

$$|\gamma'(\theta)|^2 = x'(\theta)^2 + y'(\theta)^2 = \frac{1}{2\pi}\,.$$

Then, by the Plancherel formula

$$\begin{aligned} \frac{1}{4\pi^2} &= |DG|_2^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} |\gamma'(\theta)|^2 \, dt \\ &= \sum_{n=-\infty}^{\infty} n^2 |\widehat{G}(n)|^2 \,. \end{aligned}$$

Or

$$\pi \sum_{n=-\infty}^{\infty} n^2 \left|\widehat{G}(n)\right|^2 = \frac{1}{4\pi}\,.$$

Now applying Stoke's Theorem to the boundary of Ω and using $xx' + yy' = 0$ and the Plancherel Theorem, one obtains

$$\begin{aligned}
a(D(\gamma)) &= \frac{1}{2}\int\int_{\Omega} d(x\,dy - y\,dx) \\
&= \frac{1}{2}\int_{\partial\Omega} xdy - ydx \\
&= \frac{1}{2}\int_0^{2\pi} \big(x(\theta)y'(\theta) - y(\theta)x'(\theta)\big)\, d\theta \\
&= \frac{-1}{2i}\int_0^{2\pi} \gamma(\theta)\overline{\gamma'(\theta)}\, d\theta \\
&= \pi \sum_{n=-\infty}^{\infty} n\widehat{G}(n)\overline{\widehat{G}(n)} \\
&= \pi \sum_{n=-\infty}^{\infty} n|\widehat{G}(n)|^2 .
\end{aligned}$$

It follows now that

$$\frac{1}{4\pi} - a(D(\gamma)) = \pi \sum_{n=-\infty}^{\infty} n(n-1)|\widehat{G}(n)|^2 .$$

Next note that $n(n-1) > 0$ for all $n \neq 0, 1$. Hence

$$\frac{1}{4\pi} - a(\Omega) = \pi \sum_{n=-\infty}^{\infty} n(n-1)|\widehat{G}(n)|^2 \geqslant 0$$

and

$$\frac{1}{4\pi} - a(\Omega) = 0$$

if and only if $\widehat{G}(n) = 0$ for $n \neq 0$ or $n \neq 1$. But in that case

$$G(e^{i\theta}) = \widehat{G}(0) + \widehat{G}(1)e^{-i\theta}$$

which is a circle. □

Chapter 2

Function Spaces on $\mathbb{R}^n$

Functions play a pivotal role in Fourier analysis and noncommutative harmonic analysis. Those with particularly nice properties form linear spaces which can be studied in the abstract. These function spaces, their integrable and differential properties, and their topological structure allow one to systematize how differential and integral operators can be analyzed. Those spaces particularly adapted to a linear integral transform or a partial differential operator provide the best starting point in inverting, solving, or diagonalizing these transformations. Several of the spaces described in this chapter are particularly suited for the Fourier transform on $\mathbb{R}^n$. This transform, to be introduced in the next chapter, may arguably be the most important transform in mathematics. To deal with the varieties of spaces we shall encounter, we shall reexamine and present afresh many topics dealing with general concepts in topology and analysis. To many readers this may be unnecessary. However, we hope the detail will make a ready source for students who need further experience with these topics. The chapter will emphasize the broad aspects of the theory more than the fine detail.

1. Locally Convex Topological Vector Spaces

In this chapter function spaces play a prominent role. These function spaces may be Banach spaces, Hilbert spaces, or topological vector spaces. However, whenever we consider function spaces, the topologies important for them will either by defined by a norm or a family of seminorms. Spaces whose topologies are defined by seminorms are locally convex. In this section the abstract structure of such spaces are developed. In the following discussion, vector spaces are assumed to be complex. The same results hold if the vector spaces are real. The arguments work identically in both cases.

A subset C of a vector space X is **convex** if $\lambda x + (1-\lambda)y \in C$ whenever $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$.

Definition 2.1. *A* **locally convex vector space topology** *on X is a topology on X having the following properties:*

- *$(x, y) \mapsto x + y$ is continuous from $X \times X$ into X*
- *$(\lambda, x) \mapsto \lambda x$ is continuous from $\mathbb{C} \times X$ into X*
- *for each point x in an open subset G of X, there is a convex open set U with $x \in U \subseteq G$.*

If the last condition is removed, X is said to be a **topological vector space**. The space X is said to be **separated** if for each $x \neq 0$ in X, there is a open neighborhood U of 0 with $x \notin U$. Exercise 2.1.1 shows this is equivalent to X being Hausdorff.

Definition 2.2. *A nonempty subset U of a vector space X is* **balanced** *if $\alpha U \subseteq U$ for any scalar α with $|\alpha| \leqslant 1$. It is* **saturating** *if for each $x \in X$ there is a $\lambda \neq 0$ with $x \in \lambda U$.*

Remark 2.3. If N is a neighborhood of 0 in a topological vector space X, then N saturates. Indeed, by continuity of scalar multiplication, if $x \in X$, we can find a nonzero λ near 0 such that $\lambda x \in N$. Thus $x \in \frac{1}{\lambda}N$.

A **seminorm** on a vector space X is a mapping $x \mapsto ||x||$ satisfying

- $||x|| \geqslant 0$ for all x
- $||\alpha x|| = |\alpha|\,||x||$ for scalars α and $x \in X$
- $||x + y|| \leqslant ||x|| + ||y||$ for x, y in X.

Note by Exercise 2.1.2, if $||\cdot||$ is a seminorm, then the set $U = \{x \mid ||x|| < 1\}$ is convex, balanced, and saturating.

Lemma 2.4. *Let U be a balanced, convex, and saturating subset of X. Define $||\cdot||$ by*

$$||x|| = \inf\{\lambda \mid \lambda > 0,\, x \in \lambda U\}.$$

Then $||\cdot||$ is a seminorm on X.

Proof. Note there is a $\lambda > 0$ with $\lambda x \in U$, for U is saturating and balanced. Thus $||x|| < \infty$ for all $x \in X$. Note $\lambda 0 \in U$ for all $\lambda > 0$ implies $||0|| = 0$.

Also for $\alpha \neq 0$ since $\frac{|\alpha|}{\alpha}U = U$,

$$\begin{aligned}
||\alpha x|| &= \inf_{\lambda > 0}\{\lambda \mid \alpha x \in \lambda U\} \\
&= \inf\{\lambda \mid |\alpha| \frac{\alpha}{|\alpha|} x \in \lambda U\} \\
&= \inf\{\lambda \mid x \in \frac{\lambda}{|\alpha|}\frac{|\alpha|}{\alpha} U\} \\
&= \inf\{\lambda \mid x \in \frac{\lambda}{|\alpha|} U\} \\
&= \inf\{|\alpha|\lambda \mid x \in \lambda U\} \\
&= |\alpha| \, ||x||.
\end{aligned}$$

Let $\epsilon > 0$. Then $||x|| + \epsilon$ is not a lower bound for $\{\lambda \mid \lambda > 0,\, x \in \lambda U\}$ and thus there is a $\lambda_1 > 0$ with with $\lambda_1 < ||x|| + \epsilon$ and $x \in \lambda_1 U$. Similarly, there is a $\lambda_2 > 0$ with $\lambda_2 < ||y|| + \epsilon$ and $y \in \lambda_2 U$. Now since U is convex, we see

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{1}{\lambda_1} x + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \frac{1}{\lambda_2} y \in U.$$

Hence $\frac{1}{\lambda_1 + \lambda_2}(x + y) \in U$. Thus $||x + y|| \leqslant \lambda_1 + \lambda_2 < ||x|| + ||y|| + 2\epsilon$. This implies $||x + y|| \leqslant ||x|| + ||y||$. □

Definition 2.5. *Let U be a subset of X. A point $x \in U$ is internal to U if for each $y \in X$, there is a $\delta_y > 0$ such that $x + \alpha y \in U$ whenever $|\alpha| < \delta_y$.*

Lemma 2.6. *Suppose U is a balanced, saturating, convex set and each point in U is internal. Let $|| \cdot ||$ be the seminorm defined by*

$$||x|| = \inf_{\lambda > 0,\, x \in \lambda U} \lambda.$$

Then $U = \{x \mid ||x|| < 1\}$.

Proof. If $x \in U$, then $x + \alpha x \in U$ for $|\alpha| < \delta_x$. Hence there is an $\alpha > 0$ with $(1 + \alpha)x \in U$, and we see $||x|| \leqslant \frac{1}{1+\alpha} < 1$.

If $||x|| < 1$, then there is a $0 \leqslant \lambda < 1$ with $x \in \lambda U$. Thus $x \in \lambda U \subseteq U$. □

Lemma 2.7. *Let G be a neighborhood of 0 in a locally convex topological vector space. Then there exists a balanced, convex, open set U contained in G.*

Proof. Since $(\alpha, x) \mapsto \alpha x$ is continuous and X is locally convex, there is a convex open set V containing 0 and a $\delta > 0$ with $\alpha V \subseteq G$ whenever $|\alpha| \leqslant \delta$. Hence $\alpha\delta V \subseteq G$ whenever $|\alpha| = 1$. Replace V by δV and set $W = \cap_{|\alpha|=1} \alpha V$. Note W is an intersection of convex sets and thus is convex. To see W is balanced, first note $\lambda W = W$ if $|\lambda| = 1$. Thus if $\beta \neq 0$ and $|\beta| \leqslant 1$, then since W is convex and $0 \in W$, we have

$$\beta w = |\beta|(\frac{\beta}{|\beta|}w) = |\beta|(\frac{\beta}{|\beta|}w) + (1 - |\beta|)0 \in W.$$

We now have W is a convex balanced set. Note its interior contains 0. Indeed, choose an open neighborhood N of 0 and a $\delta' > 0$ such that $\lambda N \subseteq V$ if $|\lambda| = \delta'$. Then $\alpha(\delta' N) \subseteq V$ if $|\alpha| = 1$. Set $O = \cup_{|\alpha|=1}(\alpha\delta' N)$. Note O is open and contains 0 and is a subset of V. Since $\alpha O = O$ if $|\alpha| = 1$, we see $O \subseteq W$. Thus 0 is an interior point of W.

Hence W is a balanced convex subset of G containing 0 as an interior point. Exercises 2.1.4 and 2.1.5 imply W° is open, convex, and balanced. □

Lemma 2.8. *Let X be a locally convex topological vector space. Then there is a collection U_α of open, balanced, convex sets which form a neighborhood subbase at* 0*; i.e., for each open subset V of X containing* 0*, there is a finite subset F of α's such that*

$$\cap_{\alpha \in F} U_\alpha \subseteq V.$$

Proof. Take the collection of all open balanced convex subsets of X. By Lemma 2.7, these sets form a neighborhood base at 0. □

Proposition 2.9. *Let X be a locally convex topological vector space. Suppose U_α, $\alpha \in A$, is a collection of open balanced convex sets satisfying if V is an open set containing* 0*, then there is a finite subset F of A and an $\delta > 0$ such that $\delta(\cap_{\alpha\in F} U_\alpha) \subseteq V$. Then:*

(a) *The seminorms $||\cdot||_\alpha$ defined by $||x||_\alpha = \inf\{\lambda \mid x \in \lambda U_\alpha\}$ are continuous on X.*

(b) *A subset V of X is open if and only if for each $x \in V$ there is a finite subset F of A and an $\delta > 0$ with $x + \{y \mid ||y||_\alpha < \delta \text{ for } \alpha \in F\} \subseteq V$. This is equivalent to $x + \delta(\cap_{\alpha\in F} U_\alpha) \subseteq V$.*

Moreover, the space X is separated if and only if $\cap_{\delta>0,\, \alpha\in A} \delta U_\alpha = \{0\}$.

Proof. (a) We show $||\cdot||_\alpha$ is continuous. Note $||x - x_0||_\alpha < 1$ if and only if $x - x_0 \in U_\alpha$. Thus $||x - x_0||_\alpha < \delta$ if and only if $x - x_0 \in \delta U_\alpha$. Thus $||x - x_0||_\alpha < \delta$ if $x \in x_0 + \delta U_\alpha$. So these seminorms are continuous.

(b) Note since $U_\alpha = \{y \mid ||y|| < 1\}$, the sets $\delta(\cap_F U_\alpha)$ and $\{y \mid ||y||_\alpha < \delta \text{ for } \alpha \in F\}$ are equal. Suppose V is open and $x \in V$. Then $V - x$ is an open set about 0 and thus contains $\delta(\cap_F U_\alpha)$ for some finite set F and some $\delta > 0$. Thus $x + \delta(\cap_F U_\alpha) \subseteq V$. The converse follows for if for each $x \in V$, there is an F and an $\delta > 0$ such that $x + \delta(\cap_F U_\alpha) \subseteq V$, then each x in V is interior, and thus V is open.

X is separated if and only if for each $x \neq 0$, there is an open neighborhood V of 0 with $x \notin V$ if and only if for each $x \neq 0$ there is a $\delta > 0$ and a finite set F of α's such that $x \notin \delta \cap_{\alpha \in F} U_\alpha$ if and only if $\cap_{\delta > 0, \alpha} \delta U_\alpha = \{0\}$. □

Theorem 2.10. *If X is a vector space and U_α, $\alpha \in A$, is a family of convex balanced saturating sets each having only internal points, then there is a locally convex vector space topology on X such that the sets δU_α where $\delta > 0$ and $\alpha \in A$ form a neighborhood subbase at 0.*

Proof. Let $|| \cdot ||_\alpha$ be the seminorm defined by U_α and define a set V to be open if for each $x \in V$ there is an $\delta > 0$ and a finite subset F of α's with $x + \cap_{\alpha \in F} \{y \mid ||y||_\alpha < \delta\} \subseteq V$. The collection of open sets is clearly closed under unions and finite intersections. Hence we have a topology. Moreover, the sets $x + \cap_F \{y \mid ||y||_\alpha < \delta\}$ are convex. They are open for if $x + y_0$ is in the set $U = x + \cap_F \{y \mid ||y||_\alpha < \delta\}$ and $\delta = \inf_F \{\delta - ||y_0||_\alpha\}$, then $x + y_0 + \cap_F \{y \mid ||y||_\alpha < \delta\} \subseteq U$

To see summation is continuous note $(x + \cap_F \{y \mid ||y||_\alpha < \frac{\delta}{2}\}) + (x' + \cap_F \{y \mid ||y||_\alpha < \frac{\delta}{2}\}) \subseteq x + x' + \cap_F \{y \mid ||y||_\alpha < \delta\}$.

The continuity of scalar multiplication is a consequence of

$$\begin{aligned} ||\lambda x - \lambda_0 x_0||_\alpha &\leqslant ||\lambda x - \lambda x_0||_\alpha + ||\lambda x_0 - \lambda_0 x_0||_\alpha \\ &\leqslant |\lambda| \, ||x - x_0||_\alpha + |\lambda - \lambda_0| \, ||x_0||_\alpha \\ &< (|\lambda_0| + 1) \frac{\delta}{2(|\lambda_0| + 1)} + \frac{\delta}{2(||x_0||_\alpha + 1)} ||x_0||_\alpha \\ &< \delta \end{aligned}$$

if $|\lambda - \lambda_0| < \min\{\frac{\delta}{2(||x_0||_\alpha + 1)}, 1\}$ and $||x - x_0||_\alpha < \frac{\delta}{2(|\lambda_0| + 1)}$. □

Definition 2.11. *Let X be a locally convex topological vector space. A sequence $\{x_n\}_{n=1}^\infty$ is* **Cauchy** *if for each neighborhood U of 0, there is an N such that $x_m - x_n \in U$ for all $m, n \geqslant N$. This sequence* **converges** *to x if for each neighborhood U of x, there is an N such that $x_n \in U$ for all $n \geqslant N$.*

Lemma 2.12. *Suppose the locally convex topology on X is defined using the family of seminorms $|| \cdot ||_\alpha$ where $\alpha \in A$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in X.*

(a) *This sequence is Cauchy if and only if for each α and each $\epsilon > 0$, there is an N such that*

$$||x_m - x_n||_\alpha < \epsilon$$

for $m, n \geqslant N$.

(b) *This sequence converges to x if and only if for each α and each $\epsilon > 0$, there is an N such that*

$$||x_n - x||_\alpha < \epsilon$$

for $n \geqslant N$.

Proof. Suppose x_n is Cauchy. Let $U = \{y \mid ||y||_\alpha < \epsilon\}$. There is an N such that $x_m - x_n \in U$ for $m, n \geqslant N$. So $||x_m - x_n||_\alpha < \epsilon$ for $m, n \geqslant N$.

For the converse, if U is a neighborhood of 0, then there is a finite set $F \subseteq A$ and an $\epsilon > 0$ so that $\cap_{\alpha \in F}\{y \mid ||y||_\alpha < \epsilon\} \subseteq U$. For each $\alpha \in F$, choose N_α so that $||x_m - x_n||_\alpha < \epsilon$ if $m, n \geqslant N_\alpha$. Set $N = \max\{N_\alpha \mid \alpha \in F\}$. Then $x_m - x_n \in U$ for $m, n \geqslant N$.

The argument for (b) is essentially the same. $\square$

A locally convex topological vector space is **complete** if every Cauchy sequence has a limit; i.e., if $\{x_n\}_{n=1}^\infty$ is Cauchy, then there exists an x such that $x_n \to x$ as $n \to \infty$. A **Fréchet space** is a locally convex topological vector space which is metrizable and complete. Note by Exercise 2.1.10 that a locally convex vector space topology is metrizable if and only if it is Hausdorff and its topology is defined by countably many seminorms. A subset E is said to be **bounded** if for each open neighborhood V of 0, there is a $r > 0$ such that $E \subseteq rV$.

Definition 2.13. *Let X be a vector space and let X_i be a collection of linear subspaces each having a locally convex topology. Assume $\cup X_i = X$. The strongest locally convex topology on X satisfying the relative topology of X on X_i is weaker than the topology on X_i for each i is called the* **inductive limit topology** *on X.*

Lemma 2.14. *The inductive limit topology on $X = \cup X_i$ exists.*

Proof. Let $\mathcal{T}_i$ be the locally convex topology on X_i. Consider the collection $\mathcal{C}$ of all locally convex topologies $\mathcal{T}$ on X with $\mathcal{T}|_{X_i} = \{G \cap X_i \mid G \in \mathcal{T}\} \subseteq \mathcal{T}_i$ for all i. Define a subset U of X to be open if and only if for each $x \in U$, there are subsets $V_1, V_2, \ldots, V_n$ with $V_k \in \cup_{\mathcal{T} \in \mathcal{C}} \mathcal{T}$ and $x \in \cap V_k \subseteq U$. This defines a locally convex topology on X which contains all the topologies in the collection $\mathcal{C}$. Moreover, $U \cap X_i \in \mathcal{T}_i$ for each i. Indeed, if $x \in U \cap X_i$, then $x \in \cap_{k=1}^{k=n}(V_k \cap X_i) \subseteq U \cap X_i$ and thus x is $\mathcal{T}_i$ interior to $U \cap X_i$. $\square$

Proposition 2.15. *Let $X = \cup X_i$ have the inductive limit topology. Then a linear transformation $T : X \to Y$ where Y is a locally convex topological vector space is continuous if and only if $T|_{X_i}$ is continuous for each i.*

Proof. Suppose $T|_{X_i}$ is continuous for each i. Let $\mathcal{T} = \{T^{-1}(U) \mid U \text{ open in } Y\}$. This is a locally convex topology on X. Moreover, the relative topology of $\mathcal{T}$ on X_i consists of the sets $T^{-1}(U) \cap X_i$ which are open in X_i by assumption. Thus $\mathcal{T}$ is a subset of the inductive limit topology. Thus T is continuous.

Conversely, suppose $T : X \to Y$ is continuous. Since the inclusion map $X_i \to X$ is continuous, we see $T|_{X_i}$ is continuous for each i. $\square$

A family X_i of linear subspaces of X is said to be **directed** if for each i_1 and i_2, there is an i_3 with $X_{i_1} \cup X_{i_2} \subseteq X_{i_3}$.

Proposition 2.16. *Let X be a vector space. Suppose X_i form a directed collection of linear locally convex topological vector subspaces whose union is X. Give X the inductive limit topology from the linear subspaces X_i. Then there is a one-to-one correspondence between all balanced convex sets U in X satisfying $U \cap X_i$ is open in X_i for all i and all continuous seminorms on X. This correspondence is given by*

$$U \mapsto ||\cdot||_U$$

where

$$||x||_U = \inf\{\lambda > 0 \mid x \in \lambda U\}.$$

In particular, a seminorm $||\cdot||$ on X is continuous if and only if its restriction to each space X_i is continuous.

Proof. Let U be a balanced convex subset of X with $U \cap X_i$ open for each i. Let $x \in U$. Then $x \in U \cap X_{i'}$ for some i'. If $y \in X$, there is an i'' with $y \in X_{i''}$. Since the family of subspaces X_i is directed, there is an i with both x and y in X_i. Now $U \cap X_i$ is open and thus x is interior. Exercise 2.1.3 shows x is internal in the set $U \cap X_i$ as a subset of X_i. Hence there is a $\delta_y > 0$ so that $x + \lambda y \in U$ for $|\lambda| < \delta_y$. Hence every point in U is internal. Next note if $z \in X$, then $z \in X_i$ for some i. Since $U \cap X_i$ is open in X_i, it is saturating in X_i. Thus there is a nonzero λ with $z \in \lambda U$. Hence U is saturating in X. By Lemma 2.4, $||\cdot||_U$ is a seminorm on X. The topology defined on X by the seminorm $||\cdot||_U$ is locally convex. Moreover, its restriction to each X_i defines a locally convex topology on X_i weaker than the given topology on X_i. Since the inductive limit topology on X is the strongest such topology, the seminorm $||\cdot||_U$ is continuous in the inductive limit topology. The mapping is one-to-one for by Lemma 2.6, $U = \{x \mid ||x||_U < 1\}$. Now let $||\cdot||$ be a continuous seminorm on X. Then $U = \{x \mid ||x|| < 1\}$ is open, balanced, and convex in X. Thus $U \cap X_i$ is open in each X_i. It follows that $||\cdot||_U = ||\cdot||$.

The last statement follows from the from $||\cdot|| = ||\cdot||_U$ where $U = \{x \mid ||x|| < 1\}$ and $U \cap X_i$ is open in X_i for all i if $||\cdot||$ is continuous on X. □

Corollary 2.17. *Let X_i be a directed family of locally convex topological subspaces of X whose union X. Then a convex balanced set U is open in X if and only if $U \cap X_i$ is open in X_i for all i.*

Proof. Let U be a balanced convex set. The mappings $X_i \hookrightarrow X$ are continuous. Thus if U is open in the inductive limit topology, $U \cap X_i$ is open for each i. Conversely, if $U \cap X_i$ is open for each i, the seminorm defined by

$$||x|| = \inf\{\lambda \mid \lambda > 0 \text{ and} x \in \lambda U\}$$

is continuous in the inductive limit topology. Thus the unit ball $W = \{x \mid ||x||_U < 1\}$ is open in the inductive limit topology. Note $W \subseteq U$ for if $x \in W$, then $x \in \lambda U$ for some λ with $0 < \lambda < 1$. Thus $x = \lambda x + (1-\lambda)0 \in U$. But $U \subseteq W$ for if $x \in U$, then $x \in U \cap X_i$ for some i. Since this set is open in X_i, $\frac{1}{\lambda}x \in U$ for some $\lambda \in (0,1)$. Hence $||x||_U \leqslant \lambda < 1$ and we see $U = W$. □

We conclude by presenting the following two basic principles for functional analysis on locally convex vector spaces.

Theorem 2.18 (Uniform Boundedness Principle)**.** *Let X be a Fréchet space and let Y be a locally convex topological vector space. Suppose T_n is a sequence of continuous linear transformations from X into Y such that $T_n x$ converges for each x in X. Then the linear transformation $Tx = \lim_{n\to\infty} T_n x$ is a continuous linear transformation of X into Y.*

Proof. Let $|\cdot|$ be a continuous seminorm on Y. It suffices to show the function $x \mapsto |Tx|$ is continuous at 0. Let $F_{m,n} = \{x \mid |T_n x| \leqslant m\}$. Set $F_m = \cap_{n=1}^{\infty} F_{m,n}$. Then F_m is a closed subset of X. Moreover, since for each x the sequence $T_n x$ converges, we see $|T_n x|$ is bounded. Hence $X = \cup_m F_m$. Since X is complete, the Baire Category Theorem implies F_m has interior for some m. Thus there is a nonempty open subset V of X such that for $x \in V$, $|T_n x| \leqslant m$ for all n. Fix $p \in V$ and choose an open neighborhood U of 0 with $U + p \subseteq V$. Then for $x \in U$, $|T_n x| = |T_n(x+p) - T_n p| \leqslant m + |T_n p|$. Taking $M = m + |Tp|$ and letting $n \to \infty$, we find $|Tx| \leqslant M$ for $x \in U$. Thus when $\epsilon > 0$, we have $|Tx| < \epsilon$ for $x \in \frac{\epsilon}{M+1}U$. □

Theorem 2.19 (Open Mapping)**.** *Let X and Y be Fréchet spaces. Every continuous linear transformation T of X onto Y is an open mapping.*

Proof. Using Exercise 2.1.11, there are complete metrics σ and ρ on X and Y satisfying $\sigma(x,x') = \sigma(x-x',0)$ and $\rho(y,y') = \rho(y-y',0)$ for x, $x' \in X$ and y, $y' \in Y$. Set $D_n = \{x \mid \sigma(x,0) < \frac{1}{2^n}\}$. Note since $\cup_{k=1}^{\infty} kD_{n+1} = X$, one has $\cup k\overline{T(D_{n+1})} = Y$. Since Y is complete, the Baire Category Theorem implies $\overline{T(D_{n+1})}$ has interior. For $\epsilon > 0$, let $B_\epsilon(y)$ be the ball $\{y' \mid \rho(y,y') < \epsilon\}$. We can choose $\epsilon_n > 0$ and $p_n \in \overline{T(D_{n+1})}$ such that $B_{\epsilon_n}(y_n) \subseteq \overline{T(D_{n+1})}$. Then $B_{\epsilon_n}(0) = B_{\epsilon_n}(y_n) - y_n \subseteq \overline{T(D_{n+1})} - \overline{T(D_{n+1})} = \overline{T(D_{n+1}) - T(D_{n+1})}$. But if $x, x' \in D_{n+1}$, then $x - x' \in D_n$ for $\rho(x-x',0) = \rho(x,x') \leqslant \rho(x,0) + \rho(0,x') < \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}$. Hence for each n, there is an $\epsilon_n > 0$ such that $B_{\epsilon_n}(0) \subseteq \overline{T(D_n)}$. We may assume $\epsilon_n \to 0$ as $n \to \infty$. Set $\epsilon = \epsilon_1$. Let $y \in B_\epsilon(0)$. Since $B_\epsilon(0) \subseteq \overline{T(D_1)}$, we can choose $x_1 \in D_1$ with $\rho(y - Tx_1, 0) = \rho(y, Tx_1) < \epsilon_2$. Thus we can find $x_2 \in D_2$ so that $\rho(y - Tx_1 - Tx_2, 0) = \rho(y - Tx_1, Tx_2) < \epsilon_3$. Repeating we find a sequence $\{x_k\}_{k=1}^{\infty}$ satisfying $x_k \in D_k$ for each k and $\rho(y - \sum_{k=1}^{n} Tx_n, 0) < \epsilon_{n+1}$ for all n. Note $x'_n = x_1 + x_2 + \cdots + x_n$ is Cauchy in X. Indeed, if $n < m$, $\sigma(x'_n, x'_m) = \sigma(0, x'_m - x'_n) = \sigma(0, x_{n+1} + x_{n+2} + \cdots + x_m)$. Thus

$$\begin{aligned}
\sigma(x'_n, x'_m) &\leqslant \sigma(0, x_{n+1}) + \sigma(x_{n+1}, x_{n+1} + x_{n+2} + \cdots + x_m) \\
&= \sigma(0, x_{n+1}) + \sigma(0, x_{n+2} + x_{n+3} + \cdots + x_m) \\
&\leqslant \sigma(0, x_{n+1}) + \sigma(0, x_{n+2}) + \sigma(x_{n+2}, x_{n+2} + x_{n+3} + \cdots + x_m) \\
&= \sigma(0, x_{n+1}) + \sigma(0, x_{n+2}) + \sigma(0, x_{n+3} + \cdots + x_m) \\
&\leqslant \cdots \\
&= \sum_{k=n+1}^{m} \sigma(0, x_k) \\
&< \sum_{k=n+1}^{m} \frac{1}{2^k} < \frac{1}{2^n}.
\end{aligned}$$

Since X is complete, there is an x with $x'_n \to x$. Note the above argument shows

$$\rho(0, x'_n) \leqslant \rho(0, x_1) + \rho(0, x_2) + \cdots + \rho(0, x_n) < \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$$

for all n. Thus $\rho(0, x) < 1$. Moreover, $Tx = y$ for T is continuous and $\rho(y, Tx'_n) = \rho(y - Tx'_n, 0) < \epsilon_n$. Hence $T(D_1) \supseteq B_\epsilon(0)$.

Now let U be a nonempty open subset of X and suppose $u \in U$. Choose $r > 0$ such that $rD_1 + u \subseteq U$. Thus $rT(D_1) + T(u) = T(rD_1 + u) \subseteq T(U)$. Thus $rB_\epsilon(0) + T(u) \subseteq T(U)$. Hence each $T(u)$ is an interior point in $T(U)$. Consequently, $T(U)$ is an open subset of Y. □

Exercise Set 2.1

1. Show a topological vector space X is separated if and only if it is Hausdorff.

2. Let $||\cdot||$ be a seminorm on a vector space X. Show $U = \{x \mid ||x|| < 1\}$ is balanced, saturating, and convex and each point in U is internal.

3. Let X be a topological vector space. Show every interior point of a subset U of X is internal.

4. Let X be a topological vector space. Show the interior of any balanced subset of X containing the zero vector as an interior point is balanced.

5. Let X be a topological vector space. Show the interior of any convex set is convex.

6. Give an example of an internal point of a subset of a normed space X which is not interior.

7. Let W be a convex, saturating, and balanced set in a vector space X with corresponding seminorm $||\cdot||$ where

$$||x|| = \inf\{\lambda > 0 \mid x \in \lambda W\}.$$

Show $\{x \mid ||x|| < 1\}$ equals the set of internal points in W. In particular, the set of internal points of W has only internal points and is convex, saturating, and balanced.

8. Show there is a unique Hausdorff vector space topology on $\mathbb{R}^n$. (**Hint:** Show all compact sets in the standard topology are compact and hence closed in any other topology. Then show if V is open and contains the zero vector and $\alpha V \subseteq \{x \mid ||x|| \neq 1\}$ where $|\alpha| \leqslant \delta$, then V is bounded.)

9. Let X be a locally convex topological vector space. Let $\mathcal{F}$ be the family of all continuous seminorms on X. Show the topology on X is the locally convex vector space topology on X defined by using all the seminorms in $\mathcal{F}$.

10. Let X be a locally convex topological vector space.

(a) Show $\frac{t}{1+t} \leqslant \frac{r}{1+r} + \frac{s}{1+s}$ whenever $r, s, t \geqslant 0$ and $t \leqslant r + s$.

(b) Suppose the locally convex topology on X is defined by a countable family $\{||\cdot||_i \mid i \in \mathbb{N}\}$ of seminorms. Define d on $X \times X$ by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{||x-y||_i}{1 + ||x-y||_i} \right).$$

Show d is a pseudometric on X; i.e., $d(x, x) = 0$, $d(x, y) = d(y, x)$, and $d(x, z) \leqslant d(x, y) + d(y, z)$ for all x, y and z in X.

(c) Show the topology on X defined by the pseudometric d is the topology on X.

(d) Show X is metrizable if and only if X is Hausdorff and the locally convex topology on X can be defined by countably many seminorms.

11. Show if X is a Fréchet space, there is a complete metric ρ on X satisfying $\rho(x, y) = \rho(x - y, 0)$ for all $x, y \in X$.

12. Let $X = C[0, 1]$ be the vector space of real valued continuous functions on the closed interval $[0, 1]$. For each $x \in [0, 1]$, define $||f||_x = |f(x)|$.

(a) Show the family of seminorms $||\cdot||_x$ define a locally convex vector space topology on X.

(b) Show this topology is not metrizable; i.e., show there is no metric whose topology defines the same topology as the seminorms $||\cdot||_x$.

(c) Show the space is not complete.

(d) Find a completion for X; i.e., a complete locally convex topological vector space Y such that X is a linear subspace of Y, X has the relative topology of Y, and X is dense in Y.

13. Let X be a locally convex topological vector space whose topology is defined by the seminorms $||\cdot||_\alpha$ for $\alpha \in A$.

(a) Show a subset E of X is bounded if and only if for each $\alpha \in A$, there is a $B_\alpha \geqslant 0$ such that

$$||x||_\alpha \leqslant B_\alpha \text{ for } x \in E.$$

(b) Show every compact subset of X is closed and bounded.

(c) Give an example of an X which has a closed bounded subset which is not compact.

(d) Show a subset E of X is bounded if and only if for each continuous seminorm $|\cdot|$ on X there is a constant $M \geqslant 0$ such that

$$|x| \leqslant M \text{ for all } x \in E.$$

14. Show every Cauchy sequence in a locally convex topological vector space is bounded.

15. Let X be a locally convex topological vector space. Show the closure of any bounded subset of X is bounded.

16. Let X and Y be locally convex topological vector spaces. Suppose T is a continuous linear transformation from X into Y. Show if B is a bounded subset of X, then $T(B)$ is a bounded subset of Y.

17. Let V be a complex locally convex topological Hausdorff topological space. Let V^* be the dual space of V. Thus V^* is the vector space whose elements are the continuous complex valued linear functionals on V. The topology on V defined by the seminorms $|\cdot|_f$ where $|v|_f = |f(v)|$ for $f \in V^*$ is called the weak topology on V while the topology on V^* defined by seminorms $|\cdot|_v$ on V^* where $|f|_v = |f(v)|$ for $f \in V^*$ and $v \in V$ is called the weak-$*$ topology on V^*.

(a) Show V^* is Hausdorff.

(b) Assume V is a Hilbert space. Show the conjugate linear isomorphism of V onto V^* given by $w \mapsto f_w$ where $f_w(v) = (v, w)$ is a homeomorphism of V with the weak topology onto V^* with the weak-$*$ topology.

18. Let X be a vector space and suppose X_i for $i \in I$ form a directed family of locally convex topological vector subspaces of X whose union is X. Give X the inductive limit topology. Show the relative inductive limit topology on each X_i is the topology on X_i for all i if and only if there is a family $|\cdot|_\alpha$,

$\alpha \in A$ of seminorms on X such that the restrictions of these seminorms to each X_i define the topology on X_i. An example is $\mathcal{D}(\Omega)$ and the seminorms

$$||\phi||_\alpha = \max_{x \in \operatorname{supp} \phi} |D^\alpha \phi(x)| \text{ for } \alpha \in \mathbb{N}_0^n.$$

19. Let X_n be an increasing sequence of locally convex topological vector subspaces of a vector space X whose union is X. Let X have the inductive limit topology. Suppose M is a vector subspace of X. Give M the inductive limit topology of the relative topologies of X_i on $X_i \cap M$.

(a) Show the relative topology of X on M is weaker than the topology on M.

(b) Give an example where the relative topology is strictly weaker.

2. The Space $\mathbb{R}^n$

The space $\mathbb{R}^n$ is an inner product space, the inner product being defined by

$$(x \mid y) = x \cdot y = \sum_{j=1}^{n} x_j y_j \,.$$

The corresponding norm on $\mathbb{R}^n$ is given by $|x| = \sqrt{\sum x_j^2}$. This norm defines a locally convex vector space topology on $\mathbb{R}^n$. Exercise 2.1.8 shows this is the only Hausdorff vector space topology on $\mathbb{R}^n$. In particular, a subset U in $\mathbb{R}^n$ is open if and only if for each $x \in U$, there is an $R > 0$ such that the ball $B_R(x) = x + \{y \mid |y| < R\} = \{y \mid |y - x| < R\} \subseteq U$. The space $\mathbb{R}^n$ has the **Heine-Borel Property**; that is a subset is compact if and only if it is a closed bounded set.

Differentiable Functions. Let $\Omega \subset \mathbb{R}^n$ be open and nonempty. A function $F = (f_1, f_2, \ldots, f_m) : \Omega \to \mathbb{R}^m$ is *differentiable* at $x \in \Omega$ if there exists a linear map $D : \mathbb{R}^n \to \mathbb{R}^m$ and an $R > 0$ with $B_R(x) \subset \Omega$ satisfying

$$F(y) = F(x) + D(y - x) + o(|y - x|).$$

for $y \in B_R(x)$. If F is differentiable at x, then the partial derivatives $D_i f_j(x) = \frac{\partial}{\partial x_i} f_j(x)$ all exist and the linear transformation D is given by

$$D(a_1, \ldots, a_n) = \left(\sum_{i=1}^{n} a_i D_i f_1(x), \sum_{i=1}^{n} a_i D_i f_2(x), \ldots, \sum_{i=1}^{m} a_i D_i f_m(x) \right).$$

Thus if $F'(x)$ is the **Jacobian matrix**

$$F'(x) = \begin{pmatrix} D_1 f_1(x) & D_1 f_2(x) & \cdots & D_1 f_m(x) \\ D_2 f_2(x) & D_2 f_2(x) & \cdots & D_2 f_m(x) \\ \vdots & \vdots & & \vdots \\ D_n f_1(x) & D_n f_2(x) & \cdots & D_n f_m(x) \end{pmatrix},$$

then D is the linear transformation defined by

$$D(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n)F'(x).$$

The function F is *differentiable on* Ω if F is differentiable at every point in Ω. It is continuously differentiable on Ω if it is differentiable at each point in Ω and the function $x \mapsto F'(x)$ is continuous; i.e., the functions $x \mapsto D_i f_j(x)$ are continuous on Ω for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Conversely, it is known that if the partial derivatives $D_i f_j$ exist and are continuous on Ω, then F is continuously differentiable on Ω; see Exercise 2.2.5.

For each finite sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonnegative integers, D^α will denote the partial differential operator $\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. The order $|\alpha|$ of α or the differential operator D^α is given by $|\alpha| = \sum \alpha_i$. If α and β are multiindices, then $\beta \leqslant \alpha$ will mean that $\beta_i \leqslant \alpha_i$ for $i = 1, \ldots, n$. Set $\beta!$ to be $\prod(\beta_i!)$ and for $\beta \leqslant \alpha$, define

$$\binom{\alpha}{\beta} := \prod_{i=1}^{n} \binom{\alpha_i}{\beta_i} = \prod_{i=1}^{n} \frac{\alpha_i!}{\beta_i!(\alpha_i - \beta_i)!}.$$

A real or complex valued function f defined on an open set $\Omega \subseteq \mathbb{R}^n$ is said to be C^k or k-times continuously differentiable on Ω if $D^\alpha f(x)$ exists and is continuous on Ω for each multiindex α with $|\alpha| \leqslant k$. The function f on Ω is said to be **smooth** if f is C^k for all k. The vector space consisting of all C^k functions on Ω is denoted by $C^k(\Omega)$ while the collection of all smooth functions is the vector space $C^\infty(\Omega)$.

A function $F = (f_1, f_2, \ldots, f_m)$ from an open set Ω in $\mathbb{R}^n$ into $\mathbb{R}^m$ is C^k if each of the functions f_i is C^k. An invertible function $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a C^k **diffeomorphism** if F and F^{-1} are C^k. A C^k diffeomorphism between two nonempty open subsets Ω_1 and Ω_2 of $\mathbb{R}^n$ is an invertible function $F : \Omega_1 \to \Omega_2$ such that F is C^k on Ω_1 and F^{-1} is C^k on Ω_2. If $k = \infty$, then F is said to be a diffeomorphism.

The **support** of a function f on Ω is the relatively closed subset $\operatorname{supp} f$ of Ω defined by

$$\operatorname{supp} f = \overline{\{x \mid f(x) \neq 0\}} \cap \Omega.$$

$C_c^k(\Omega)$ is used to denote the vector space of C^k functions on Ω with compact support. This is a subspace of the space $C_b^k(\Omega)$ of uniformly bounded functions in $C^k(\Omega)$. The product rule holds in these spaces and extends to higher orders. The following simple lemma is used to prove the general Leibniz product rule.

Lemma 2.20. *Let* $\beta \leqslant \alpha$ *and assume that* $\beta_i \geqslant 1$. *Then*

$$\binom{\alpha}{\beta - \varepsilon_i} + \binom{\alpha}{\beta} = \binom{\alpha + \varepsilon_i}{\beta}$$

where ε_i *is the multiindex with* i^{th} *entry* 1 *and all others zero.*

Proof. Clearly one assume that $n = 1$. Then

$$\begin{aligned}\binom{\alpha}{\beta-1} + \binom{\alpha}{\beta} &= \frac{\alpha!}{(\beta-1)!(\alpha+1-\beta)!} + \frac{\alpha!}{\beta!(\alpha-\beta)!} \\ &= \frac{\alpha!\beta + \alpha!(\alpha+1-\beta)}{\beta!(\alpha+1-\beta)!} \\ &= \binom{\alpha+1}{\beta}.\end{aligned}$$

□

Lemma 2.21 (Leibniz's Rule)**.** *Let* $f, g \in \mathcal{C}^k(\Omega)$ *and* $|\alpha| \leqslant k$. *Then*

$$D^\alpha(fg) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g\,.$$

Proof. We prove this by induction on $|\alpha|$. If $|\alpha| = 1$ then $D^\alpha = \partial_i$ for some i and the claim reduces to the usual Leibniz's rule. If $|\alpha| > 1$ then we can find $\alpha_i > 1$. Let $\alpha' := \alpha - \varepsilon_i$. Then $D^\alpha = \partial_i D^{\alpha'}$. The claim follows now from the induction hypothesis, the usual Leibniz's rule, and Lemma 2.20. □

Integration on $\mathbb{R}^n$**.** Let dx or sometimes $d\lambda_n$ or $d\lambda$ denote the usual Lebesgue measure on the Lebesgue measurable subsets of $\mathbb{R}^n$. Thus

$$\int f\,d\lambda = \int_{\mathbb{R}^n} f(x_1, \ldots, x_n)\,dx_1 \cdots dx_n$$

for $f \in C_c(\mathbb{R}^n)$. The corresponding L^p-spaces will be denoted by $L^p(\mathbb{R}^n)$, the L^p-*norm* being given by $|f|_p = \left(\int |f(x)|^p\,dx\right)^{\frac{1}{p}}$ for $1 \leqslant p < \infty$. Recall $L^\infty(\mathbb{R}^n)$ is the space of essentially bounded measurable functions identified when they are equal almost everywhere with norm defined by

$$|f|_\infty = \inf\{M > 0 \mid \lambda\{x \mid |f(x)| > M\} = 0\}.$$

Note that functions in L^p are zero if and only if they are equal to zero almost everywhere. Each space L^p is a Banach space and if $1 \leqslant p < \infty$ and q is determined by $1/p + 1/q = 1$, then the dual of $L^p(\mathbb{R}^n)$ is the space $L^q(\mathbb{R}^n)$. In particular $L^2(\mathbb{R}^n)$ is a Hilbert space with inner product

$$(f \mid g) = \int f(x)\overline{g(x)}\,dx\,.$$

We shall use the following change of coordinates formula.

Theorem 2.22 (Change of coordinates)**.** *Let* $F : \Omega_1 \to \Omega_2$ *be a* C^1 *diffeomorphism between open subsets of* $\mathbb{R}^n$. *Then a measurable function* f *is*

Lebesgue integrable on Ω_2 *if and only if* $x \mapsto f(F(x))\,|\det F'(x)|$ *is Lebesgue integrable on* Ω_1 *and then*

$$\int_{\Omega_1} f(F(x))\,|\det F'(x)|\,dx = \int_{\Omega_2} f(y)\,dy.$$

Let T be an $n \times n$ matrix. The linear transformation $F(x) = xT$ is C^∞ and satisfies $F'(x) = T$. F is invertible if and only if $\det T \neq 0$.

Corollary 2.23. *Let* T *be an invertible* $n \times n$ *matrix. Then*

$$|\det(T)| \int f(xT)\,d\lambda(x) = \int f\,d\lambda$$

for all $f \in L^1(\mathbb{R}^n)$.

Polar coordinates. Let $S^{n-1} = S_1 = \{x \mid |x| = 1\} \subset \mathbb{R}^n$. Define a map $\Psi : \mathbb{R}^+ \times S^{n-1} \to \mathbb{R}^n \backslash \{0\}$ by

$$\Psi(r, s) := rs.$$

Then Ψ is a homeomorphism with inverse

$$\Phi(x) = \left(|x|\,, \frac{x}{|x|}\right).$$

The pair (r, s) in $\mathbb{R}^+ \times S^{n-1}$ with $rs = x$ are called *polar coordinates* of the point x.

Lebesgue measure λ on $\mathbb{R}^n$ satisfies $\lambda(aE) = a^n\lambda(E)$ for any positive constant a. This property allows one to polarly decompose Lebesgue measure. First, define a Borel measure σ on S^{n-1} by

$$\sigma(W) = n\lambda\{rs \mid s \in W,\, 0 < r \leqslant 1\} = n\lambda(\psi((0,1] \times W)).$$

Note $\sigma(S^{n-1})$ is $n\lambda(B_1)$ where B_1 is the ball of radius 1 and thus is n times the volume of the ball of radius 1. One can show this is the usual hypersurface area of the sphere S^{n-1}; see Exercise 2.2.9.

Lemma 2.24. *Let* $0 < r_1 < r_2 < \infty$ *and* $W \subseteq S^{n-1}$ *be Borel measurable. Then* $\lambda(\psi((r_1, r_2] \times W)) = \sigma(W) \int_{r_1}^{r_2} r^{n-1}\,dr$.

Proof.

$$\begin{aligned}
\lambda(\psi((r_1, r_2] \times W)) &= r_2^n \lambda(\psi((\frac{r_1}{r_2}, 1] \times W)) \\
&= r_2^n \lambda(\psi((0,1] \times W) - \psi((0, \frac{r_1}{r_2}] \times W)) \\
&= r_2^n \lambda(\psi((0,1] \times W)) - r_2^n \lambda(\psi((0, \frac{r_1}{r_2}] \times W)) \\
&= \frac{1}{n} r_2^n \sigma(W) - r_1^n \lambda(\frac{r_2}{r_1} \psi((0, \frac{r_1}{r_2}] \times W)) \\
&= \frac{1}{n} r_2^n \sigma(W) - \frac{1}{n} r_1^n \sigma(W) \\
&= \frac{r_2^n - r_1^n}{n} \sigma(W) \\
&= \sigma(W) \int_{r_1}^{r_2} r^{n-1}\, dr.
\end{aligned}$$

□

Proposition 2.25.

$$\lambda(E) = \int_0^\infty \int_{S^{n-1}} r^{n-1} \chi_E(rs)\, d\sigma(s)\, dr$$

for each measurable subset E of $\mathbb{R}^n$.

Proof. Define a measure λ' on $\mathbb{R}^n$ by

$$\lambda'(E) = \int_0^\infty \int_{S^{n-1}} r^{n-1} \chi_E(rs)\, d\sigma(s)\, dr.$$

Note by Lemma 2.24, $\lambda'(\psi((r_1, r_2] \times W))) = \lambda(\psi((r_1, r_2] \times W)$. This implies λ' and λ agree on all open sets of form $\psi((r_1, r_2) \times W)$ where $r_1 < r_2$ and W is open in S^{n-1}. But these sets are a basis for the open sets in $\mathbb{R}^n \backslash \{0\}$. Thus λ' and λ agree on the σ-algebra generated by the open sets in $\mathbb{R}^n \backslash \{0\}$. Since λ is the completion of λ on the Borel sets, we see λ' and λ agree on the Lebesgue measurable sets. □

Corollary 2.26. *A f is Lebesgue integrable on $\mathbb{R}^n$ if and only if $(r, u) \mapsto r^{n-1} f(ru)$ is integrable relative to the product measure $\lambda_1|_{[0,\infty)} \times \sigma$. Moreover,*

$$\int f(x)\, d\lambda_n(x) = \int_0^\infty \int r^{n-1} f(ru)\, d\sigma(u)\, dr.$$

Remark 2.27. A consequence of this corollary is that $\int_{|x| \geqslant 1} |x|^{-s}\, ds$ is finite if and only if $\int_1^\infty r^{n-1-s}\, dr < \infty$ which happens if and only if $n - 1 - s < -1$ or $n < s$. In particular the function $(1 + |x|^2)^{-s}$ is integrable if and only if $n < 2s$ or $s > \frac{n}{2}$. Indeed,

$$2^{-s} |x|^{-s} = (|x|^2 + |x|^2)^{-s} \leqslant (1 + |x|^2)^{-s} \leqslant |x|^{-2s}$$

for $|x| \geqslant 1$ and $s \geqslant 0$. The functions $(1+|x|^2)^m$ will be used in defining seminorms on spaces of functions on $\mathbb{R}^n$.

3. Integral Operators

Integral operators on $\mathbb{R}^n$ are a central feature of Fourier analysis. Formally, an integral operator has form

$$Tf(x) = \int_Y K(x,y)f(y)\,dy.$$

The function $K(x,y)$ is called the *kernel* of the integral operator. We first present two general results, the first dealing with sufficient conditions for an integral operator to be bounded between L^p spaces; the second dealing with L^p integrals on product spaces. Then we turn our attention to Hilbert-Schmidt and trace class operators. Our applications of these will be on $\mathbb{R}^n$.

Let $K : X \times Y \to \mathbb{C}$ be measurable. For $x \in X$ and $y \in Y$, let $K_x : Y \to \mathbb{C}$ and $K^y : X \to \mathbb{C}$ be defined by $K_x(t) = K(x,t)$ and $K^y(s) = K(s,y)$.

Lemma 2.28. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Suppose $K : X \times Y \to \mathbb{C}$ is $\mathcal{A} \times \mathcal{B}$-measurable and assume that there exists a $C > 0$ such that $|K_x|_1, |K^y|_1 \leqslant C$ for almost all x and y. Let $1 \leqslant p \leqslant \infty$ and $f \in L^p(X,\mu)$. Then*

$$Tf(x) = \int K(x,y)f(y)\,d\nu(y)$$

exists for almost all $x \in X$. Furthermore $Tf \in L^p(X,\mu)$, $T : L^p(Y,\nu) \to L^p(X,\mu)$ is linear, and $|T| \leqslant C$.

Proof. For $p = \infty$, one has

$$\begin{aligned}
\int |K(x,y)|\,|f(y)|\,d\nu(y) &\leqslant |f|_\infty \int |K(x,y)|\,d\nu(y) \\
&= |f|_\infty |K_x|_1 \\
&\leqslant C|f|_\infty \text{ for a.e. } x.
\end{aligned}$$

Now for $1 \leqslant p < \infty$, note

$$\begin{aligned}
\int\int |K(x,y)|\,|f(y)|^p\,d\nu(y)\,d\mu(x) &= \int |f(y)|^p \int |K(x,y)|\,d\mu(x)\,d\nu(y) \\
&= \int |f(y)|^p |K^y|_1\,d\nu(y) \\
&\leqslant C|f|_p^p < \infty
\end{aligned}$$

for $f \in L^p(Y,\nu)$. This implies the result for $p=1$ and for $1<p<\infty$ shows $|K_x|^{\frac{1}{p}} f$ is in $L^p(Y,\nu)$ for μ a.e. x. Now take q where $1<q<\infty$ to be the conjugate exponent to p; i.e., $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\begin{aligned} |Tf|_p^p &= \int_X \left| \int_Y K(x,y)\, f(y)\, d\nu(y) \right|^p d\mu(x) \\ &\leqslant \int_X \left[\int_Y |K(x,y)|\; |f(y)|\; d\nu(y) \right]^p d\mu(x) \\ &= \int_X \left[\int_Y |K_x(y)|^{1/q}\, |K_x(y)|^{1/p}\, |f(y)|\; d\nu(y) \right]^p d\mu(x) \qquad (2.1) \end{aligned}$$

Now for μ a.e. x, $|K_x|^{1/q}$ is in $L^q(Y,\nu)$ with norm $\leqslant C^{1/q}$ and $|K_x|^{1/p} f$ is in $L^p(Y,\nu)$. Hence by Hölder's inequality

$$\int_Y |K_x(y)|^{1/q}\, |K_x(y)|^{1/p}\, |f(y)|\, d\nu(y) \leqslant C^{1/q} \left(\int_Y |K(x,y)|\; |f(y)|^p\; d\nu(y) \right)^{1/p}.$$

Hence by (2.1),

$$\begin{aligned} |Tf|_p^p &\leqslant C^{p/q} \int_X \int_Y |K(x,y)|\, |f(y)|^p\; d\nu(y) d\mu(x) \\ &= C^{p/q} \int_Y |f(y)|^p \int_X |K(x,y)|\; d\mu(x)\, d\nu(y) \\ &\leqslant C^{p/q+1}\, |f|_p^p\,. \end{aligned}$$

Hence $|T| \leqslant (C^{p/q+1})^{1/p} = C$. □

Lemma 2.29. *Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be σ-finite measure spaces. Let $f: X\times Y \to \mathbb{C}$ be $\mathcal{A}\times\mathcal{B}$-measurable, and let $1 \leqslant p < \infty$. Then*

$$\left(\int_X \left(\int_Y |f(x,y)|\; d\nu(y) \right)^p d\mu(x) \right)^{1/p} \leqslant \int_Y \left(\int_X |f(x,y)|^p\; d\mu(x) \right)^{1/p} d\nu(y)\,.$$

Proof. Clearly, we may assume $f \geqslant 0$. For $p=1$, the result is a consequence of Tonelli's Theorem. Thus suppose $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Define $F(x) = \int f(x,y)\, d\nu(y)$. If $F \in L^p(X,\mu)$, then

$$|F|_p = \sup_{|g|_q=1} | \int F(x)\bar{g}(x)\, d\mu(x)|.$$

Now $F(x) \geqslant 0$ for all x implies $|\int F(x)\bar{g}(x)\, d\mu(x)| \leqslant \int F(x)|g(x)|\, d\mu(x)$. Consequently,

$$|F|_p = \sup_{|g|_q=1,\, g\geqslant 0} \int F(x) g(x)\, d\mu(x).$$

But if $g \geqslant 0$ and $|g|_q = 1$, then

$$\begin{aligned}\int F(x)g(x)\,d\mu(x) &= \int\int f(x,y)g(x)\,d\nu(y)\,d\mu(x)\\ &= \int\int f(x,y)g(x)\,d\mu(x)\,d\nu(y)\\ &\leqslant \int\left(\int f(x,y)^p\,d\mu(x)\right)^{\frac{1}{p}}\left(\int g(x)^q\,d\mu(x)\right)^{\frac{1}{q}}\,d\nu(y)\\ &= \int\left(\int f(x,y)^p\,d\mu(x)\right)^{\frac{1}{p}}\,d\nu(y).\end{aligned}$$

Consequently,

$$\left(\int\left(\int f(x,y)\,d\nu(y)\right)^p\,d\mu(x)\right)^{\frac{1}{p}} \leqslant \int\left(\int f(x,y)^p\,d\mu(x)\right)^{\frac{1}{p}}\,d\nu(y)$$

and the result follows.

Hence we may suppose $\int F(x)^p\,d\mu(x) = \infty$. Let $E_\infty = \{x \mid F(x) = \infty\}$. If $\mu(E_\infty) > 0$, then there is a subset $E \subseteq E_\infty$ with $0 < \mu(E) < \infty$. Set $g = \frac{1}{\mu(E)^{\frac{1}{q}}}\chi_E$. Then $|g|_q = 1$ and the argument above shows

$$\infty = \int F(x)g(x)\,d\mu(x) \leqslant \int\left(\int f(x,y)^p\,d\mu(x)\right)^{\frac{1}{p}}\,d\nu(y)$$

and again the result holds. We may therefore assume $F(x) < \infty$ for all x. Take an increasing sequence W_n of subsets of X with $X = \cup W_n$ and $\mu(W_n) < \infty$. Set $E_n = \{x \in W_n \mid F(x) < n\}$.

Then $F\chi_{E_n} \in L^p(X,\mu)$ and if $g \geqslant 0$ and $|g|_q = 1$, then

$$\begin{aligned}\int F(x)\chi_{E_n}(x)g(x)\,d\mu(x) &= \int\int f(x,y)\chi_{E_n}(x)g(x)\,d\nu(y)\,d\mu(x)\\ &= \int\int \chi_{E_n}(x)f(x,y)g(x)\,d\mu(x)\,d\nu(y)\\ &\leqslant \int\left(\int_{E_n} f(x,y)^p\,d\mu(x)\right)^{\frac{1}{p}}\left(\int g(x)^q\,d\mu(x)\right)^{\frac{1}{q}}\,d\nu(y)\\ &= \int\left(\int_{E_n} f(x,y)^p\,d\mu(x)\right)^{\frac{1}{p}}\,d\nu(y).\end{aligned}$$

Consequently, $|F\chi_{E_n}|_p \leqslant \int\left(\int f(x,y)^p\,d\mu(x)\right)^{\frac{1}{p}}\,d\nu(y)$. Letting $n \to \infty$, one has

$$\left(\int\left(\int f(x,y)\,d\nu(y)\right)^p\,d\mu(x)\right)^{\frac{1}{p}} \leqslant \int\left(\int f(x,y)^p\,d\mu(x)\right)^{\frac{1}{p}}\,d\nu(y).$$

$\square$

Hilbert-Schmidt and Trace Class Operators. Integral operators which have square integrable kernels are called Hilbert-Schmidt. They can be defined in terms of their behavior on an orthonormal basis.

Lemma 2.30. *Let T be a bounded linear transformation from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$. Let $\{e_\alpha\}_{\alpha\in A}$ and $\{e'_\alpha\}_{\alpha\in A}$ be two orthonormal bases of $\mathcal{H}$. Then*

$$\sum_{\alpha,\beta} ||Te_\alpha||^2 = \sum_\alpha ||Te'_\alpha||^2.$$

Proof. Let $\{f_\beta\}_{\beta\in B}$ be an orthonormal basis of $\mathcal{K}$. Then by Parseval's equality,

$$\begin{aligned}\sum ||Te_\alpha||^2 &= \sum_\alpha\sum_\beta |(Te_\alpha, f_\beta)|^2\\ &= \sum_\beta\sum_\alpha |(e_\alpha, T^*f_\beta)|^2\\ &= \sum ||T^*f_\beta||^2.\end{aligned}$$

This shows the sum is independent of orthonormal basis. □

Definition 2.31. *A bounded linear transformation T from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ is said to be Hilbert-Schmidt if*

$$\sum_\alpha ||Te_\alpha||^2 < \infty$$

for any orthonormal basis $\{e_\alpha\}_{\alpha\in A}$ of $\mathcal{H}$. The Hilbert-Schmidt norm of T is defined by

$$||T||_2 = \left(\sum_\alpha ||Te_\alpha||^2\right)^{\frac{1}{2}}.$$

The proof of Lemma 2.30 shows if the bounded linear operator T is Hilbert-Schmidt, then its adjoint T^* is also Hilbert-Schmidt and

$$||T||_2 = ||T^*||_2. \tag{2.2}$$

Moreover, note if v is any unit vector, it is a member of an orthonormal basis and thus $||T||_2 \geqslant ||Tv||$. Consequently,

$$||T||_2 \geqslant ||T||. \tag{2.3}$$

We let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the Banach space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ with operator norm and we let $\mathcal{B}_2(\mathcal{H},\mathcal{K})$ denote the vector space of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$ with Hilbert-Schmidt norm $||\cdot||_2$. Set $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$ and $\mathcal{B}(\mathcal{K}) = \mathcal{B}(\mathcal{K},\mathcal{K})$.

Proposition 2.32. *$\mathcal{B}_2(\mathcal{H},\mathcal{K})$ is a Hilbert space with inner product*

$$(R,S)_2 = \sum (Re_\alpha, Se_\alpha)$$

where $\{e_\alpha\}$ is an orthonormal basis of $\mathcal{H}$. Moreover, AR, $RB \in \mathcal{B}_2(\mathcal{H},\mathcal{K})$ if $R \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $A \in \mathcal{B}(\mathcal{K})$, $B \in \mathcal{B}(\mathcal{H})$, and then

$$||AR||_2 \leqslant ||A||\,||R||_2, \quad ||RB||_2 \leqslant ||R||_2||B||.$$

Proof. We first note $(R,S)_2$ is defined for

$$\begin{aligned}\sum |(Re_\alpha, Se_\alpha)| &\leqslant \sum ||Re_\alpha||\,||Se_\alpha|| \\ &\leqslant \left(\sum ||Re_\alpha||^2\right)^{\frac{1}{2}} \left(\sum ||Se_\alpha||^2\right)^{\frac{1}{2}} \\ &= ||R||_2||S||_2.\end{aligned}$$

To see this is well defined, note if $\{f_\beta\}$ is an orthonormal basis of $\mathcal{K}$, then

$$\begin{aligned}\sum (Re_\alpha, Se_\alpha) &= \sum_{\alpha,\beta,\gamma} ((Re_\alpha, f_\beta)f_\beta, (Se_\alpha, f_\gamma)f_\gamma) \\ &= \sum_{\beta,\gamma,\alpha} (Re_\alpha, f_\beta)(f_\gamma, Se_\alpha)(f_\beta, f_\gamma) \\ &= \sum_{\beta,\alpha} (Re_\alpha, f_\beta)(f_\beta, Se_\alpha) \\ &= \sum_{\beta,\alpha} (S^* f_\beta, e_\alpha)(e_\alpha, R^* f_\beta) \\ &= \sum_\beta (S^* f_\beta, \sum_\alpha (R^* f_\beta, e_\alpha)e_\alpha) \\ &= \sum_\beta (S^* f_\beta, R^* f_\beta)\end{aligned}$$

where the rearrangements are allowed for

$$\begin{aligned}\sum_{\alpha,\beta} |(Re_\alpha, f_\beta)(Se_\alpha, f_\beta)| &\leqslant \sum_\alpha \sum_\beta |(Re_\alpha, f_\beta)|\,|(Se_\alpha, f_\beta)| \\ &\leqslant \sum_\alpha \left(\sum_\beta |(Re_\alpha, f_\beta)|^2\right)^{\frac{1}{2}} \left(\sum_\beta |(Se_\alpha, f_\beta)|^2\right)^{\frac{1}{2}} \\ &= \sum_\alpha ||Re_\alpha||\,||Se_\alpha|| \\ &\leqslant \left(\sum ||Re_\alpha||^2\right)^{\frac{1}{2}} \left(\sum ||Se_\alpha||^2\right)^{\frac{1}{2}} \\ &< \infty\end{aligned}$$

by Parseval's identity and the Cauchy-Schwarz inequality.

We thus have a complex inner product. We show it is complete. Let T_n be a Cauchy sequence in $\mathcal{B}_2(\mathcal{H}, \mathcal{K})$. Then by inequality 2.3, T_n is Cauchy in $\mathcal{B}(\mathcal{H}, \mathcal{K})$. Hence there is a $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with $T_n \to T$ in operator norm. Since Cauchy sequences are bounded, there is an $M > 0$ with

$$\sum_\alpha ||T_n e_\alpha||^2 \leqslant M^2 \text{ for all } n.$$

Taking a limit gives $T \in \mathcal{B}_2(\mathcal{H}, \mathcal{K})$. Let $\epsilon > 0$. Choose N so that $m, n \geqslant N$ implies

$$\sum_\alpha ||T_m e_\alpha - T_n e_\alpha||^2 \leqslant \epsilon^2.$$

Letting $n \to \infty$ gives $||T_m - T||_2 \leqslant \epsilon$ for $m \geqslant N$. Thus $\mathcal{B}_2(\mathcal{H}, \mathcal{K})$ is complete.

Now let $A \in \mathcal{B}(\mathcal{K})$ and $R \in \mathcal{B}_2(\mathcal{H}, \mathcal{K})$. Since

$$||AR||_2^2 = \sum_\alpha ||ARe_\alpha||^2 \leqslant ||A||^2 \sum_\alpha ||Re_\alpha||^2$$

we see $||AR||_2 \leqslant ||A||\,||R||_2$.

The second part follows for

$$||RB||_2 = ||(RB)^*||_2 = ||B^* R^*||_2 \leqslant ||B^*||\,||R^*||_2 = ||B||\,||R||_2.$$

$\square$

Corollary 2.33. *Let $R, S \in \mathcal{B}_2(\mathcal{H}, \mathcal{K})$. Then S^*, $R^* \in \mathcal{B}_2(\mathcal{K}, \mathcal{H})$ and*

$$(S^*, R^*)_2 = (R, S)_2.$$

Theorem 2.34. *Let (X, μ) and (Y, ν) be σ-finite measure spaces. Then the mapping $K \mapsto T_K$ where*

$$T_K f(x) = \int K(x, y) f(y)\, d\nu(y)$$

defines a linear one-to-one onto correspondence between the space $L^2(X \times Y, \mu \times \nu)$ and the space of Hilbert-Schmidt operators from $L^2(Y, \nu)$ into $L^2(X, \mu)$. Moreover, $||T_K||_2 = |K|_2$.

Proof. Note $\int_X \int_Y |K(x, y)|^2\, d\nu(y)\, d\mu(x) < \infty$. By Fubini's Theorem, for almost every x, $K_x(y) = K(x, y)$ is in $L^2(Y)$. Thus $y \mapsto K(x, y) f(y)$ is in $L^1(Y)$ for almost every x. This implies $T_K f(x)$ is defined almost everywhere.

Moreover, the Cauchy-Schwarz inequality implies

$$\begin{aligned}
|Kf|_2^2 &= \int \left| \int K(x,y) f(y)\, d\nu(y) \right|^2 d\mu(x) \\
&\leqslant \int \int |K(x,y)|^2 \, d\nu(y) \int |f(y)|^2 \, d\nu(y)\, d\mu(x) \\
&= |f|_2^2 \int_X \int_Y |K(x,y)|^2 \, d\nu(y)\, d\mu(x) \\
&= |K|_2^2 \, |f|_2^2.
\end{aligned}$$

Thus T_K is a bounded linear operator. Finally note if $e_\alpha \in L^2(Y,\nu)$ is an orthonormal basis, then

$$\begin{aligned}
\sum_\alpha |T_K e_\alpha|_2^2 &= \sum_\alpha \int_X \left| \int_Y K(x,y) e_\alpha(y)\, d\nu(y) \right|^2 d\mu(x) \\
&= \int_X \sum_\alpha |(K_x, \bar{e}_\alpha)|^2 \, d\mu(x) \\
&= \int_X \int_Y |K_x(y)|^2 \, d\nu(y)\, d\mu(x) \\
&= \int_{X\times Y} |K(x,y)|^2 \, d(\mu \times \nu)(x,y) \\
&= |K|_2.
\end{aligned}$$

Conversely, suppose T is Hilbert-Schmidt. Let $e_\alpha(y)$ be an orthonormal basis of $L^2(Y,\nu)$ and $f_\beta(x)$ be an orthonormal basis of $L^2(X,\mu)$. Then by Exercise 2.2.18, the functions $f_\beta \times \bar{e}_\alpha$ defined by

$$f_\beta \times e_\alpha(x,y) = f_\beta(x)\overline{e_\alpha(y)}$$

form an orthonormal basis of $L^2(X\times Y, \mu\times\nu)$. Define $K(x,y) = \sum (Te_\alpha, f_\beta) f_\beta \times \bar{e}_\alpha$. Then

$$\begin{aligned}
|K|_2^2 &= \sum_\alpha \sum_\beta |(Te_\alpha, f_\beta)|^2 \\
&= \sum ||Te_\alpha||^2 \\
&= ||T||_2^2.
\end{aligned}$$

To check K works, it suffices to check it on a complete orthonormal basis. Note

$$\begin{aligned}\int K(x,y)e_{\alpha'}(y)\,d\nu(y) &= \int \sum_{\alpha,\beta}(Te_\alpha, f_\beta)f_\beta \times \bar{e}_\alpha(x,y)e_{\alpha'}(y)\,d\nu(y)\\ &= \int \sum_{\alpha,\beta}(Te_\alpha, f_\beta)f_\beta(x)\,\bar{e}_\alpha(y)e_{\alpha'}(y)\,d\nu(y)\\ &= \sum_{\beta}(Te_{\alpha'}, f_\beta)f_\beta(x)\\ &= Te_{\alpha'}.\end{aligned}$$

□

In particular the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ is a Hilbert space isomorphic to $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

Definition 2.35. *A rank one operator on a Hilbert space $\mathcal{K}$ into a Hilbert space $\mathcal{H}$ is an operator having form $v \otimes \bar{w}$ where $v \in \mathcal{H}$, $w \in \mathcal{K}$ where both are nonzero and*

$$v \otimes \bar{w}(u) = (u,w)v \text{ for } u \in \mathcal{H}.$$

A finite rank operator is a bounded linear transformation of $\mathcal{H}$ into $\mathcal{K}$ having finite dimensional range.

One can easily verify the following facts about rank one operators. See Exercise 2.2.19.

$$||v \otimes \bar{w}|| = ||v||\,||w|| = ||v \otimes \bar{w}||_2 \tag{2.4}$$

$$(v \otimes \bar{w})^* = w \otimes \bar{v} \tag{2.5}$$

$$A(v \otimes \bar{w})B^* = Av \otimes \overline{Bw} \text{ for } A, B \in \mathcal{B}(\mathcal{H}) \tag{2.6}$$

Furthermore, note if f and g are in $L^2(\mathbb{R}^n)$, then $\bar{g} \in L^2$ and $f \otimes g$ is the operator defined by

$$f \otimes g(h) = f \otimes \bar{\bar{g}}(h) = (h, \bar{g})f = (\int h(y)\,g(y)\,dy)f. \tag{2.7}$$

Hence $f \otimes g$ has kernel $K(x,y) = f(x)g(y)$. This kernel is usually denoted by $(f \otimes g)(x,y)$. In particular

$$||f \otimes g||_2 = |f|_2|g|_2$$

in both the operator sense and the L^2 sense.

Proposition 2.36. *Suppose $\{v_i\}_{i=1}^{\infty}$ and $\{w_i\}_{i=1}^{\infty}$ are sequences of vectors in $\mathcal{H}$ with $\sum ||v_i||^2 < \infty$ and $\sum ||w_i||^2 < \infty$. Then $\sum_{i=1}^{\infty} v_i \otimes \bar{w}_i$ is a bounded operator on $\mathcal{H}$ with $|| \sum_{i=1}^{\infty} v_i \otimes \bar{w}_i|| \leqslant \left(\sum_{i=1}^{\infty} ||v_i||^2\right)^{1/2} \left(\sum_{i=1}^{\infty} ||w_i||^2\right)^{1/2}$. Its adjoint is $\sum_{i=1}^{\infty} w_i \otimes \bar{v}_i$.*

Proof. Let $v \in \mathcal{H}$. Then using the Cauchy–Schwarz inequality for both $\mathcal{H}$ and l_2, one has

$$\begin{aligned}
\sum ||v_i \otimes \bar{w}_i(v)|| &= \sum |\langle v, w_i\rangle| \, ||v_i|| \\
&\leqslant \sum ||v|| \, ||w_i|| \, ||v_i|| \\
&\leqslant ||v|| \left(\sum ||w_i||^2\right)^{1/2} \left(\sum ||v_i||^2\right))^{1/2}.
\end{aligned}$$

To check the adjoint, note

$$\begin{aligned}
(\sum (v_i \otimes \bar{w}_i)v, w) &= \sum (v, w_i)(v_i, w) \\
&= \sum (v, (w, v_i)w_i) \\
&= (v, \sum (w, v_i)w_i) \\
&= (v, \sum (w_i \otimes \bar{v}_i)w).
\end{aligned}$$

Thus $(\sum (v_i \otimes \bar{w}_i))^* = \sum (w_i \otimes \bar{v}_i)$. □

Definition 2.37. *A bounded operator T on a Hilbert space $\mathcal{H}$ is said to be trace class if it has form $\sum v_i \otimes \bar{w}_i$ where $\sum ||v_i||^2 < \infty$ and $\sum ||w_i||^2 < \infty$.*

Definition 2.38. *Let T be a trace class operator. Write T in form $T = \sum v_i \otimes \bar{w}_i$ where $\sum ||v_i||^2 < \infty$ and $\sum ||w_i||^2 < \infty$. Define* $\mathrm{Tr}(T) = \sum (v_i, w_i)$*.*

Proposition 2.39. *Let T be a trace class operator and let $\{e_\alpha\}$ be any orthonormal basis of $\mathcal{H}$. Then* $\mathrm{Tr}(T) = \sum (Te_\alpha, e_\alpha)$ *where this series converges absolutely. In particular* $\mathrm{Tr}(T)$ *is well defined.*

Proof. Note $\sum |(v_i, w_i)| \leqslant \sum |v_i| \, |w_i| \leqslant \left(\sum |v_i|^2\right)^{1/2} \left(\sum |w_i|^2\right)^{1/2} < \infty$. Thus the series defining $\mathrm{Tr}(T)$ is absolutely convergent.

Now $T(e_\alpha) = \sum (e_\alpha, w_i) v_i$ and thus $(Te_\alpha, e_\alpha) = \sum_i (e_\alpha, w_i)(v_i, e_\alpha)$. Hence Parseval's equality gives

$$\begin{aligned}\sum_{i,\alpha} |(e_\alpha, w_i)|\,|(v_i, e_\alpha)| &\leqslant \left(\sum_{i,\alpha} |(e_\alpha, w_i)|^2\right)^{1/2} \left(\sum_{i,\alpha} |(e_\alpha, v_i)|^2\right)^{1/2} \\ &= \left(\sum_i \sum_\alpha |(e_\alpha, w_i)|^2\right)^{1/2} \left(\sum_i \sum_\alpha |(e_\alpha, v_i)|^2\right)^{1/2} \\ &= \left(\sum_i ||w_i||^2\right)^{1/2} \left(\sum_i ||v_i||^2\right)^{1/2} < \infty.\end{aligned}$$

Thus we may rearrange the following summation:

$$\begin{aligned}\sum_\alpha (Te_\alpha, e_\alpha) &= \sum_i \sum_\alpha (e_\alpha, w_i)(v_i, e_\alpha) \\ &= \sum_i (\sum_\alpha (v_i, e_\alpha) e_\alpha, w_i) \\ &= \sum_i (v_i, w_i).\end{aligned}$$

Also note

$$\sum_\alpha |(Te_\alpha, e_\alpha)| = \sum_\alpha |\sum_i (w_i, e_\alpha)(v_i, e_\alpha)| < \infty$$

□

Proposition 2.40. *A bounded linear operator T on a Hilbert space $\mathcal{H}$ is trace class if and only if $T = RS$ where R and S are Hilbert-Schmidt operators on $\mathcal{H}$. Moreover,*

$$\mathrm{Tr}(RS) = (R, S^*)_2 = (S, R^*)_2 = \mathrm{Tr}(SR).$$

We thus see $R \in \mathcal{B}(\mathcal{H})$ is Hilbert-Schmidt if and only if RR^* (or R^*R) is trace class and then $||R||_2 = \sqrt{\mathrm{Tr}(RR^*)} = \sqrt{\mathrm{Tr}(R^*R)}$

Proof. Assume T is trace class. Then $T = \sum v_i \otimes \bar{w}_i$ where $\sum ||v_i||^2 < \infty$ and $\sum ||w_i||^2 < \infty$. We may assume the dimension of $\mathcal{H}$ is infinite; (the finite dimensional case is easier, or just extend $\mathcal{H}$ so that it is infinite dimensional.) Let $\{e_i\}$ be an orthonormal set. Define $R = \sum v_i \otimes \bar{e}_i$ and $S = \sum e_j \otimes \bar{w}_j$. By Exercise 2.2.15, R and S are Hilbert-Schmidt and $S^* = \sum w_j \otimes \bar{e}_j$. Since $(v_i \otimes \bar{e}_i) \circ (e_j \otimes \bar{w}_j)(x) = (x, w_j)(e_j, e_i) v_i = \delta_{i,j}(x, w_j) v_i = \delta_{i,j}(v_i \otimes w_i)(x)$, we see

$$\begin{aligned} RS &= \sum_i \sum_j (v_i \otimes \bar{e}_i)(e_j \otimes \bar{w}_j) \\ &= \sum_i v_i \otimes \bar{w}_i \\ &= T. \end{aligned}$$

Also by taking an orthonormal basis $\{e_\alpha\}$ containing the vectors e_i, we see

$$\begin{aligned} (R, S^*)_2 &= \sum_\alpha (Re_\alpha, S^* e_\alpha) \\ &= \sum_i (Re_i, S^* e_i) \\ &= \sum_i (v_i, w_i) \\ &= \mathrm{Tr}(RS). \end{aligned}$$

Since $(R, S^*)_2 = (S, R^*)_2$, we can conclude $(R, S^*)_2 = \mathrm{Tr}(SR)$.

Conversely, let $T = RS$ where R and S are Hilbert-Schmidt. Again using Exercise 2.2.15 we may write R and S in forms $R = \sum v_i \otimes \bar{e}_i$ and $S = \sum e_j \otimes \bar{w}_j$ where $\sum ||v_i||^2 < \infty$ and $\sum ||w_j||^2 < \infty$. As seen above $RS = \sum v_i \otimes \bar{w}_i$. Thus RS is trace class. □

Corollary 2.41. *Let T be a trace class operator on $\mathcal{H}$. If $A \in \mathcal{B}(\mathcal{H})$, then both AT and TA are trace class and*

$$\mathrm{Tr}(AT) = \mathrm{Tr}(TA).$$

Proof. We know $T = RS$ where R and S are Hilbert-Schmidt operators. Thus by Proposition 2.32, AR and SA are Hilbert-Schmidt. Consequently, $AT = (AR)S$ and $TA = R(SA)$ are trace class operators and

$$\begin{aligned} \mathrm{Tr}(AT) &= \mathrm{Tr}((AR)S) \\ &= \mathrm{Tr}(S(AR)) \\ &= \mathrm{Tr}((SA)R) \\ &= \mathrm{Tr}(RSA) \\ &= \mathrm{Tr}(TA). \end{aligned}$$

□

Proposition 2.42. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let $\mathcal{B}_2(\mathcal{H}, \mathcal{K})$ be the space of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$. Let $v, v' \in \mathcal{H}$ and $w, w' \in \mathcal{K}$ and suppose $\{f_\beta \mid \beta \in B\}$ is an orthonormal basis of $\mathcal{H}$ and $\{e_\alpha \mid \alpha \in A\}$ is an orthonormal basis in $\mathcal{K}$. Then:*

(a) $(w \otimes \bar{v}, w' \otimes \bar{v})_2 = (w, w')_{\mathcal{K}}(v', v)_{\mathcal{H}}$

(b) $\{e_\alpha \otimes \bar{f}_\beta \mid \alpha \in A,\ \beta \in B\}$ *is an orthonormal basis of* $\mathcal{B}_2(\mathcal{H}, \mathcal{K})$.

(c) *A bounded linear operator* T *from* $\mathcal{H}$ *into* $\mathcal{K}$ *is Hilbert-Schmidt if and only if* T^*T *is trace class on* $\mathcal{H}$ *if and only if* TT^* *is trace class on* $\mathcal{K}$.

(d) *Let* $R, S \in \mathcal{B}_2(\mathcal{H}, \mathcal{K})$. *Then* S^*R *is trace class on* $\mathcal{H}$, RS^* *is trace class on* $\mathcal{K}$, *and*

$$(R, S)_2 = \mathrm{Tr}(S^*R) = \mathrm{Tr}(RS^*).$$

Proof. For (a), we know

$$\begin{aligned}
(w \otimes \bar{v}, w' \otimes \bar{v}')_2 &= \sum_\beta ((w \otimes \bar{v})(f_\beta), (w' \otimes \bar{v}')(f_\beta))_\mathcal{K} \\
&= \sum_\beta ((f_\beta, v)_\mathcal{H} w, (f_\beta, v')_\mathcal{H} w')_\mathcal{K} \\
&= (w, w')_\mathcal{K} \sum_\beta (v', f_\beta)_\mathcal{H} (f_\beta, v)_\mathcal{H} \\
&= (w, w')_\mathcal{K} (\sum (v', f_\beta)_\mathcal{H} f_\beta, v)_\mathcal{H} \\
&= (w, w')_\mathcal{K} (v', v)_\mathcal{H}.
\end{aligned}$$

That the $e_\alpha \otimes \bar{f}_\beta$ form an orthonormal set follows from (a). We show it is complete. Let T be a Hilbert-Schmidt operator orthogonal to all $e_\alpha \otimes \bar{f}_\beta$. Thus $(T, e_\alpha \otimes \bar{f}_\beta)_2 = \sum_{\beta'} (Tf_{\beta'},\ e_\alpha \otimes \bar{f}_\beta(f_{\beta'}))_\mathcal{K} = 0$ for all α and β. Thus $(Tf_\beta, e_\alpha) = 0$ for all α and β. This implies $Tf_\beta = 0$ for all β and thus $T = 0$. Hence (b) holds.

For (c) and (d), we again use Exercise 2.2.15. If R and S are Hilbert-Schmidt from $\mathcal{H}$ into $\mathcal{K}$, we know $R = \sum Rf_\beta \otimes \bar{f}_\beta$ and $S = \sum Sf_\beta \otimes \bar{f}_\beta$ where $\sum ||Rf_\beta||^2 < \infty$ and $\sum ||Sf_\beta||^2 < \infty$. An easy computation shows $S^* = \sum f_\beta \otimes \overline{Sf_\beta}$ and from this it follows that

$$RS^* = \sum Rf_\beta \otimes \overline{Sf_\beta}.$$

But then since $\sum ||Rf_\beta||^2 < \infty$ and $\sum ||Sf_\beta||^2 < \infty$, we see RS^* is trace class on $\mathcal{K}$. Similarly using that R^* and S^* are Hilbert-Schmidt, we also have $S^*R = S^*(R^*)^*$ is trace class on $\mathcal{H}$. From Corollary 2.33 we then see

$$(R, S)_2 = \sum (Rf_\beta, Sf_\beta) = \sum (S^*Rf_\beta, f_\beta) = \mathrm{Tr}(S^*R) \text{ and}$$

$$(R, S)_2 = (S^*, R^*)_2 = \sum_\alpha (S^*e_\alpha, R^*e_\alpha) = \sum (RS^*e_\alpha, e_\alpha) = \mathrm{Tr}(RS^*).$$

In particular, if $T \in \mathcal{B}_2(\mathcal{H}, \mathcal{K})$, then T^*T is trace class on $\mathcal{H}$ and TT^* is trace class on $\mathcal{K}$. Thus we have (d) and the forward implication of (c).

For the converse of (c), assume TT^* is trace class on $\mathcal{K}$. Thus by Proposition 2.39, $\infty > \mathrm{Tr}(TT^*) = \sum(TT^* f_\beta, f_\beta) = \sum ||T^* f_\beta||^2$. So T^* is Hilbert-Schmidt. Thus $T = (T^*)^*$ is Hilbert-Schmidt. Similarly T^*T trace class implies T is Hilbert-Schmidt. $\square$

Because of the formal behavior of the *tensors* $v \otimes \bar{w}$, we will denote the Hilbert space $\mathcal{B}_2(\mathcal{H}, \mathcal{K})$ by $\mathcal{K} \otimes \overline{\mathcal{H}}$, i.e.,

$$\mathcal{B}_2(\mathcal{H}, \mathcal{K}) = \mathcal{K} \otimes \overline{\mathcal{H}}. \tag{2.8}$$

We will introduce the conjugate Hilbert space when we discuss representations and this notation will become even more appropriate.

In the following discussion, we will restrict ourselves to the case where $\mathcal{H} = L^2(\mathbb{R}^n)$ where here we are using Lebesgue measure. The following can easily be extended to the case where $\mathcal{H} = L^2(X, \mu)$ where X is a locally compact Hausdorff space and μ is a regular Borel measure on X. We also remark that by Theorem 2.34, the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ is norm isometric to the Hilbert space $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. This mapping preserves inner products. Thus

$$(R, S)_2 = \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} R(x,y)\overline{S(x,y)}\, dx\, dy$$

where $R(x, y)$ and $S(x, y)$ are the kernels defining the operators R and S.

Note if T is a trace class operator on $L^2(\mathbb{R}^n)$, then we know T has an L^2 kernel for it is Hilbert-Schmidt. (It is a product of two Hilbert-Schmidt operators.)

Lemma 2.43. *Let E be a measurable set in $\mathbb{R}^n$, and let Y be a complete separable metric space. Suppose $\lambda(E) < \infty$ and $f : E \to Y$ is Borel measurable. Then if $\epsilon > 0$, there is a compact subset K of E with $\lambda(E) - \lambda(K) < \epsilon$ such that $f|_K$ is continuous.*

Proof. We may assume E is compact. Let $y_1, y_2, \ldots$ be a countable dense subset of Y. For each x, choose the first k with $\rho(f(x), y_k) < \frac{1}{n}$. Define $f_n(x) = y_k$. Note f_n is measurable and $\rho(f_n(x), f(x)) < \frac{1}{n}$ for all x. Set $E_{n,k} = f_n^{-1}(y_k)$. For each n, the sets $E_{n,k}$ are pairwise disjoint and have union E. Moreover, for each n, there is a compact subset K_n of E on which f_n has finitely many values, f_n is continuous, and $m(E - K_n) < \frac{\epsilon}{2^n}$. Indeed, choose compact sets $K_{n,k} \subseteq E_{n,k}$ with $\lambda(E_{n,k} - K_{n,k}) < \frac{\epsilon}{2^{n+k}}$. Then $\lambda(E - \cup_k K_{n,k}) < \frac{\epsilon}{2^n}$. Hence there is a l such that $\lambda(E - \cup_{k=1}^l K_{n,k}) < \frac{\epsilon}{2^n}$. Take $K_n = \cup_{k=1}^l K_{n,k}$. Note $K = \cap K_n$ satisfies $\lambda(E - K) < \epsilon$ and $f_n \to f$ uniformly on K. Since each f_n is continuous on K, f is continuous on K. $\square$

Theorem 2.44. *Let T be a trace class operator on $L^2(\mathbb{R}^n)$ with L^2 kernel $K(x,y)$. If K is continuous at almost every point (x,x) on the diagonal, then $x \mapsto K(x,x)$ is L^1 and $\mathrm{Tr}(T) = \int K(x,x)\,dx$.*

Proof. Let T be trace class and suppose the kernel K for T is continuous on the diagonal. Now $T = RS$ where R and S are in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$; here we are using R and S to stand both for the Hilbert-Schmidt operators and their kernels. Now by Exercise 2.2.16, RS has kernel $(x,y) \mapsto \int R(x,z)S(z,y)\,dz$. Hence $K(x,y) = \int R(x,z)S(z,y)\,dz$ a.e. (x,y). Note for a.e. x, the functions $z \mapsto R_x(z) = R(x,z)$ is L^2. Also $S^y(\cdot) \in L^2$ for a.e. y. Here $S^y(z) = S(z,y)$. The functions $x \mapsto R_x$ and $y \mapsto S^y$ are Borel measurable into L^2 on conull measurable sets. Take a cube E in $\mathbb{R}^n$. An application of Lemma 2.43 shows for any $\epsilon > 0$, there is a measurable compact F in E with $m(E - F) < \epsilon$ such that $y \mapsto S^y$ and $x \mapsto R_x$ are continuous on F. Thus $(x,y) \mapsto \int R(x,z)S(z,y)\,dz$ is continuous on $F \times F$ and equals $K(x,y)$ a.e. on $F \times F$. Hence for a.e. x in F, $K(x,y) = \int R(x,z)S(z,y)\,dz$ for a.e. $y \in F$. But by Exercise 2.2.30, the set of x's in F for which there is an $\delta > 0$ such that $\lambda(B_\delta(x) \cap F) = 0$ has measure 0. Hence, by continuity $K(x,x) = \int R(x,z)S(z,x)\,dz$ at almost every point x in F. Since $\epsilon > 0$ was arbitrary and E was any cube, $K(x,x) = \int R(x,z)S(z,x)\,dz$ a.e. x.

By Proposition 2.40, we know

$$\begin{aligned}
\mathrm{Tr}(T) &= (R, S^*)_2 \\
&= \int R(x,y)\overline{S^*(x,y)}\,dx\,dy \\
&= \int\int R(x,y)S(y,x)\,dy\,dx \\
&= \int K(x,x)\,dx.
\end{aligned}$$

□

4. Compact Operators

The trace class and Hilbert-Schmidt operators are a subfamily of a larger class of well behaved operators. They are the class of compact linear operators.

Definition 2.45. *A compact linear operator on a Hilbert space $\mathcal{H}$ is a linear transformation T of $\mathcal{H}$ such that for any bounded sequence $\{v_n\}_{n=1}^\infty$ of vectors in $\mathcal{H}$, the sequence $\{Tv_n\}_{n=1}^\infty$ has a convergent subsequence in $\mathcal{H}$.*

We first note compact operators are bounded for if T is unbounded one can find a sequence v_n of unit vectors such that $||Tv_n||$ diverge to ∞. Thus no subsequence of $\{Tv_n\}$ could converge.

Theorem 2.46. *The norm limit of any sequence T_n of compact linear operators is compact.*

Proof. Let $T = \lim T_n$. Let u_n be a sequence of unit vectors. We do Cantor diagonalization. Let $v(n,1)$ be a subsequence of u_n such that $T_1 v(n,1)$ converges. Suppose subsequence $v(n,k)$ for $k = 1, \ldots, m$ have been defined so that $T_k(v(n,k))$ converges as $n \to \infty$ and $\{v(n,k)\}_{n=1}^{\infty}$ is a subsequence of $\{v(n,k-1)\}_{n=1}^{\infty}$. Let $\{v(n,k+1)\}_{n=1}^{\infty}$ be a subsequence of $\{v(n,k)\}_{n=1}^{\infty}$ such that $T_{k+1}(v(n,k+1))$ converges as $n \to \infty$. Set $w(n) = v(n,n)$. Then $T_k w(n)$ converges as $n \to \infty$ for each k.

Let $\epsilon > 0$. Pick K with $||T_K - T|| < \frac{\epsilon}{3}$. Choose N such that $||T_K w(k) - T_K w(l)|| < \frac{\epsilon}{3}$ when $k, l \geqslant N$. Then

$$\begin{aligned} ||Tw(k) - Tw(l)|| &\leqslant ||(T - T_K)w(k)|| + ||T_K w(k) - T_K w(l)|| + ||(T_K - T)w(l)|| \\ &< \frac{\epsilon}{3}||w(k)|| + \frac{\epsilon}{3} + \frac{\epsilon}{3}||w(l)|| \\ &= \epsilon. \end{aligned}$$

Thus the sequence $\{T(w(k))\}$ is Cauchy. □

Proposition 2.47. *Let T and S be compact linear operators. Then:*

(a) *$T + S$ is a compact linear operator;*

(b) *if A is a bounded linear operator, AT and TA are compact linear operators;*

(c) *Let $\mathcal{H}(\lambda) = \{v \mid Tv = \lambda v\}$. Then $\dim \mathcal{H}(\lambda) < \infty$ when $\lambda \neq 0$.*

Proof.

[(a)] Let v_n be bounded. Since T and S are compact, we can find a subsequence w_n of v_n such that Tw_n converges and then a subsequence u_n of w_n such that Su_n converges. Consequently $(T + S)u_n$ converges. Thus $T_1 + T_2$ is compact.

[(b)] Let v_n be bounded. Then Av_n is bounded. Thus there is a subsequence w_n of v_n such that TAw_n converges. Thus TA is compact. One can also find a subsequence u_n of v_n so that Tu_n converges. Since A is continuous, ATu_n converges. Thus AT is compact.

[(c)] If $\mathcal{H}(\lambda)$ is infinite dimensional, it contains an infinite orthonormal sequence $\{u_n\}$. Since $||Tu_n - Tu_m|| = ||\lambda u_n - \lambda u_m|| = \lambda\sqrt{2}$ for $n \neq m$, no subsequence of Tu_n could converge. Thus $\dim \mathcal{H}(\lambda) < \infty$ for all $\lambda \neq 0$. □

Proposition 2.48. *Finite rank operators are compact.*

Proof. Let T be a finite rank operator. Then $T\mathcal{H}$ is a finite dimensional vector space. Consequently, if u_n is a bounded sequence, then Tu_n is a bounded sequence in a finite dimensional vector space. Thus Tu_n must have a convergent sequences. (Recall finite dimensional vector spaces have only one Hausdorff vector space topology and in these topologies every bounded sequence has a convergent subsequence.) $\square$

Corollary 2.49. *Every Hilbert-Schmidt operator from $\mathcal{H}$ to $\mathcal{H}$ is compact.*

Proof. Assume T is Hilbert-Schmidt. Let e_α be an orthonormal basis. Let $\epsilon > 0$. Choose a finite set F such that $\sum_{\alpha \notin F} ||Te_\alpha||^2 < \epsilon^2$.

Let P be the orthogonal projection of $\mathcal{H}$ onto the linear span of the vectors e_α where $\alpha \in F$. Clearly TP has finite rank and thus is compact. Moreover, if $v \in \mathcal{H}$, we have $v = \sum (v, e_\alpha)e_\alpha$, and thus

$$\begin{aligned} ||Tv - TPv||^2 &= || \sum_{\alpha \notin F} (v, e_\alpha) Te_\alpha ||^2 \\ &\leqslant \left(\sum_{\alpha \notin F} |(v, e_\alpha)| \, ||Te_\alpha|| \right)^2 \\ &\leqslant \sum_{\alpha \notin F} |(v, e_\alpha)|^2 \sum_{\alpha \notin F} ||Te_\alpha||^2 \\ &\leqslant \epsilon^2 ||v||^2. \end{aligned}$$

Consequently, $||T - TP|| \leqslant \epsilon$. Hence T is a norm limit of compact operators. $\square$

Lemma 2.50. *Let T be a compact self adjoint linear operator. Then the eigenspaces are pairwise orthogonal and 0 is the only possible accumulation point of the nonzero eigenvalues of T.*

Proof. First note if $Tu = \lambda u$ and $Tv = \mu v$ where $\lambda \neq \mu$ and u and v are nonzero vectors, then:

$$\lambda(u, v) = (Tu, v) = (u, Tv) = (u, \mu v) = \bar{\mu}(u, v).$$

Hence if $(u, v) \neq 0$, then $\lambda = \bar{\mu}$. But $\mu(v, v) = (Tv, v) = (v, Tv) = (v, \mu v) = \bar{\mu}(v, v)$. So $\mu = \bar{\mu}$. Consequently, $(u, v) \neq 0$ implies $\lambda = \mu$. So the eigenspaces are pairwise orthogonal.

Assume $\mu = \lim \lambda_n$ where $\mu \neq 0$ and the λ_n are distinct eigenvalues. Choose a unit vector u_n with $Tu_n = \lambda_n u_n$. Then, by taking a subsequence, we may assume $Tu_n \to y$. Thus $\lambda_n u_n \to y$. But since $\lambda_n \to \mu \neq 0$, $u_n \to \frac{1}{\mu} y$. But then (u_n, u_{n+1}) and (u_n, u_n) have the same limit. Since $(u_n, u_{n+1}) = 0$ and $(u_n, u_n) = 1$, this is impossible. $\square$

Theorem 2.51 (Spectral Theorem for Compact Operators)**.** *Let T be a compact self adjoint linear operator on a Hilbert space $\mathcal{H}$. Then there is an orthonormal set $\{e_n\}$ in $\mathcal{H}$ and corresponding real numbers λ_n such that*

$T = \sum \lambda_n e_n \otimes \bar{e}_n$ where $||T|| = |\lambda_1|$ and $|\lambda_{n-1}| \geqslant |\lambda_n| > 0$. Moreover, if this collection is infinite, then $\lim \lambda_n = 0$.

Proof. Set $\lambda = ||T||$. We show λ or $-\lambda$ is an eigenvalue of T. Choose x_k of norm 1 with $||Tx_k|| \to ||T||$. Using the compactness of T and taking a subsequence, we may assume $Tx_k \to y$ where $||y|| = ||T||$. Note using the Cauchy-Schwarz inequality that:

$$\begin{aligned} ||T||^4 &= \lim ||Tx_k||^4 \\ &= \lim (Tx_k, Tx_k)^2 \\ &= \lim (T^2 x_k, x_k)^2 \\ &\leqslant \lim ||T^2 x_k||^2 ||x_k||^2 \\ &= \lim ||T(Tx_k)||^2 \\ &= ||T(y)||^2 \\ &\leqslant ||T||^2 ||y||^2 \\ &= ||T||^4. \end{aligned}$$

Consequently, $||T(y)||^2 = ||T||^2||y||^2 = ||T||^4$. Thus:

$$||T||^4 = (Ty, Ty) = (T^2 y, y) \leqslant ||T^2 y||\,||y|| \leqslant ||T||^2||y||^2 = ||T||^4.$$

So $||T^2 y|| = ||T||^3$ and $(T^2 y, y) = ||T^2 y||\,||y||$. Since one has equality in the Cauchy-Schwarz inequality only when $T^2 y$ and y are linearly dependent, we see

$$T^2 y = \mu y \text{ for some } \mu > 0.$$

But $||T^2 y|| = ||T||^3$ gives $\mu||y|| = ||T||^3$ or $\mu = ||T||^2$. Hence $T^2 y = \lambda^2 y$. Thus $(T - \lambda I)(T + \lambda I)y = 0$. If $(T + \lambda I)y \neq 0$, then $||T||$ is an eigenvalue of T; otherwise $(T + \lambda I)y = 0$ and $-||T||$ is an eigenvalue of T.

For each nonzero eigenvalue λ of T, let $\mathcal{H}(\lambda)$ be the corresponding space of eigenvectors. Note T leaves invariant the subspace $\mathcal{H}_0$ of $\mathcal{H}$ consisting of those vectors perpendicular to all the subspaces $\mathcal{H}(\lambda)$. Indeed, if $v \in \mathcal{H}_0$ and $w \in \mathcal{H}(\lambda)$, then $(Tv, w) = (v, Tw) = (v, \lambda w) = \lambda(v, w) = 0$. Moreover, $T_0 = T|_{\mathcal{H}_0}$ is still compact and self adjoint. Consequently, if T_0 is not zero, T_0 would have a nonzero eigenvector with nonzero eigenvalue. But clearly every eigenvector v_0 of T_0 with nonzero eigenvalue would be an eigenvector for T and thus would be in some $\mathcal{H}(\lambda)$. Consequently, $(v_0, v_0) = 0$ which is impossible. Thus $T = 0$ on $\mathcal{H}_0$.

Since $T^* = T$, the eigenvalues are all real. Lemma 2.50 implies the nonzero ones form either a finite set or infinite set with 0 as its only accumulation point. In either case we may enumerate the eigenvalues into either a finite or an infinite sequence $\mu_1, \mu_2, \ldots, \mu_n, \ldots$ with with $|\mu_{n-1}| \geqslant |\mu_n|$ and converging to 0 in the infinite case. Since the spaces $\mathcal{H}(\mu_k)$ are finite dimensional, we can find for each k an orthonormal basis $e_{k,1}, e_{k,2}, \ldots, e_{k,n_k}$ of $\mathcal{H}(\mu_k)$. List the vectors under order $e_{k,l} \leqslant e_{m,n}$ if $k < m$ or $k = m$ and $l \leqslant n$. One obtains a sequence e_n. Define $\lambda_n = \mu_k$ if $e_n = e_{k,l}$ for some l. Then the set e_n is an orthonormal set, $Te_n = \lambda_n e_n$ and $||T|| = |\lambda_1| \geqslant |\lambda_2| \geqslant \cdots$ with $|\lambda_n| > 0$ for all n and having limit 0 if this sequence is infinite.

Now let $v \in \mathcal{H}$. Set $v_0 = v - \sum(v, e_n)e_n$. Note $(v_0, e_n) = 0$ for all n. Thus $v_0 \in \mathcal{H}(\lambda)^\perp$ for all $\lambda \neq 0$. Consequently, $v_0 \in \mathcal{H}_0$. As seen above, $T(v_0) = T_0(v_0) = 0$. Thus

$$Tv = T(\sum(v, e_n)e_n) = \sum(v, e_n)Te_n = \sum(v, e_n)\lambda_n e_n = \sum_n \lambda_n e_n \otimes \bar{e}_n(v).$$

□

Theorem 2.52 (Structure Theorem for Compact Operators)**.** *A linear operator T on $\mathcal{H}$ is compact if and only if there exist a sequence (perhaps finite) of pairs (e_i, f_i) of vectors and a decreasing sequence $\{\lambda_i\}$ of positive numbers such that the vectors $\{e_i\}$ are orthonormal, the vectors $\{f_j\}$ are orthonormal,* $\lim \lambda_i = 0$ *when the sequence is infinite, and*

$$T = \sum \lambda_i (e_i \otimes \bar{f}_i).$$

Moreover, $||T|| = \lambda_1$.

Proof. Assume T is compact. We know $P = T^*T$ is positive and compact. Thus we have an orthonormal sequence f_n of vectors and a decreasing sequence λ_n^2 of eigenvalues for P such that $\lambda_1^2 = ||P||$ and $P = \sum \lambda_n^2 f_n \otimes \bar{f}_n$. Define $e_n = \frac{1}{\lambda_n} Tf_n$. Note:

$$\begin{aligned}
(e_m, e_n) &= (\frac{1}{\lambda_m} Tf_m, \frac{1}{\lambda_n} Tf_n) \\
&= \frac{1}{\lambda_m \lambda_n} (T^*Tf_m, f_n) \\
&= \frac{1}{\lambda_m \lambda_n} (Pf_m, f_n) \\
&= \frac{\lambda_m^2}{\lambda_m \lambda_n} (f_m, f_n) \\
&= \delta_{m,n}.
\end{aligned}$$

So the sequence $\{e_n\}$ is orthonormal.

Note P is zero on the vectors perpendicular to the eigenvectors f_n. Thus T^*T is zero on these vectors. But $T^*Tv = 0$ if and only if $Tv = 0$ for $(Tv, Tv) = (T^*Tv, v)$. Hence $T(v - \sum(v, f_n)f_n) = 0$. Consequently,

$$\begin{aligned}
Tv &= \sum (v, f_n) T f_n \\
&= \sum (v, f_n) \lambda_n e_n \\
&= \sum \lambda_n (e_n \otimes \bar{f}_n) v.
\end{aligned}$$

Now assume T has form $\sum \lambda_n (e_n \otimes \bar{f}_n) v$. Note $Tf_1 = \lambda_1 e_1$ implies $||T|| \geqslant \lambda_1$. Now

$$\begin{aligned}
||Tv||^2 &= \sum \lambda_n^2 |(v, f_n)| \\
&\leqslant \lambda_1^2 \sum |(v, f_n)|^2 \\
&\leqslant \lambda_1^2 ||v||^2
\end{aligned}$$

by Bessel's inequality. So $||T|| = \lambda_1$.

Consequently if $T = \sum \lambda_n (e_n \otimes \bar{f}_n)$ and $T_N = \sum_{k=1}^{N} \lambda_k (e_n \otimes \bar{f}_n)$, then $||T - T_N|| = \lambda_{N+1} \to 0$ as $N \to \infty$.

But T_N is compact for T_N has finite rank. By Theorem 2.46, T is compact. □

Exercise Set 2.2

1. Let Ω be an open subset of $\mathbb{R}^n$ and let $F = (f_1, f_2, \ldots, f_m)$ be a function from Ω into $\mathbb{R}^m$. Suppose F is differentiable at a point x in Ω. Show the partial derivatives $D_i f_j(x)$ exist and the linear transformation D is defined by

$$D(a_1, a_2, \ldots, a_n) = (\sum_{i=1}^{n} a_i D_i f_1(x), \sum_{i=1}^{n} a_i D_i f_2(x), \ldots, \sum_{i=1}^{n} a_i D_i f_m(x)).$$

2. Let $F : \Omega \to \mathbb{R}^m$ be a function on an open subset Ω of $\mathbb{R}^n$. Show F is continuous at each point where it is differentiable.

3. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ where $D_1 f(0,0)$ and $D_2 f(0,0)$ exist but f is not differentiable.

4. Show that $F = (f_1, f_2, \ldots, f_m)$ mapping an open set $\Omega \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ is differentiable at a point x in Ω if and only if each f_i is differentiable at x.

5. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Assume $f : \Omega \to \mathbb{R}$ is a function such that the partial derivatives $D_i f$ exist and are continuous on Ω. Show f is continuously differentiable on Ω.

6. Let $\alpha \in \mathbb{N}_0^n$ and suppose $f_1, f_2, \ldots, f_m$ are C^∞ functions on $\mathbb{R}^n$. Show

$$D^\alpha (f_1 f_2 \cdots f_m) = \sum_{\beta_1 + \beta_2 + \cdots \beta_m \leqslant \alpha} \binom{\alpha}{\beta_1,\ \beta_2,\ \cdots,\ \beta_m} D^{\beta_1} f_1 D^{\beta_2} f_2 \cdots D^{\beta_m} f_m$$

where

$$\binom{\alpha}{\beta_1,\ \beta_2,\ \cdots,\ \beta_m} = \frac{\alpha!}{\beta_1!\beta_2!\cdots\beta_m!}.$$

7. In $\mathbb{R}^n$, one has coordinates

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\ &\cdots \\ x_{n-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\ x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1} \end{aligned}$$

where $0 \leqslant \theta_j \leqslant \pi$ for $j = 1, \cdots, n-2$ and $0 \leqslant \theta_{n-1} \leqslant 2\pi$. Show in terms of these coordinates, one has

$$dx_1 dx_2 \cdots dx_n = r^{n-1}\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} dr d\theta_1 \cdots d\theta_{n-1}.$$

8. Let $B_n = \{x \in \mathbb{R}^n \mid |x| \leqslant 1\}$. Show

$$\lambda_n(B_n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

(**Hint:** Give an inductive argument and use the Beta function $B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}\,dt = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$.)

9. Show that $\sigma(S^{n-1})$ is the usual surface measure of S^{n-1}; i.e.,

$$\sigma(S^{n-1}) = \lim_{h\to 0+}\frac{1}{h}\lambda((1+h)B_n - B_n)).$$

10. Show the measure σ is invariant under rotations.

11. Let $\alpha > 0$ and $1 \leqslant p < \infty$. Show that the Chebyshev's inequality

$$\lambda_n\left(\{x \in \mathbb{R}^n \mid |f(x)| > \alpha\}\right) \leqslant \left(\frac{|f|_p}{\alpha}\right)^p$$

holds.

12. Show Corollary 2.26 follows from Proposition 2.25.

13. Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then $\ker(T^*) = T(\mathcal{H})^\perp$.

14. Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. Show $\mathcal{H} = \ker T \oplus T^*(\mathcal{K})$ is an orthogonal decomposition of $\mathcal{H}$.

15. Suppose $\{e_\alpha\}_{\alpha\in A}$ is an orthonormal basis of $\mathcal{H}$ and w_α for $\alpha \in A$ are vectors in a Hilbert space $\mathcal{K}$. Show if $\sum_{\alpha\in A}||w_\alpha||^2 < \infty$, the sums $T = \sum_\alpha w_\alpha \otimes \bar{e}_\alpha$ and $S = \sum e_\alpha \otimes \bar{w}_\alpha$ converge in operator norm and define Hilbert-Schmidt operators T and S with $T^* = S$ and $Te_\alpha = w_\alpha$ for all $\alpha \in A$. In particular, T is a Hilbert-Schmidt operator if and only if $T = \sum Te_\alpha \otimes \bar{e}_\alpha$ where $\sum ||Te_\alpha||_{\mathcal{K}}^2 < \infty$.

16. Let K and L be Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ with L^2 kernels R and S. Show $\int R(x,z)S(z,y)\,dz$ is an L^2 kernel for KL.

17. Let (X,μ) and (Y,ν) be σ-finite measure spaces. Suppose T is a Hilbert-Schmidt operator from $L^2(Y,\nu)$ into $L^2(X,\mu)$ with kernel K. Show T^* is Hilbert-Schmidt and has kernel K^* where $K^*(x,y) = \overline{K(y,x)}$.

18. Let (X,μ) and (Y,ν) be σ-finite measure spaces. Let $e_\alpha(\cdot)$, $\alpha \in A$ and $f_\beta(\cdot)$, $\beta \in B$ be complete orthonormal bases of $L^2(X)$ and $L^2(Y)$, respectively. Show the functions $e_\alpha \times f_\beta$ defined by $e_\alpha \times f_\beta(x,y) = e_\alpha(x)f_\beta(y)$ form a complete orthonormal basis of $L^2(X \times Y, \mu \times \nu)$.

19. Let v and w be vectors in a Hilbert space $\mathcal{H}$ and let A and B be bounded linear operators on $\mathcal{H}$. Show:

(a) $||v \otimes \bar{w}|| = ||v||\,||w|| = ||v \otimes \bar{w}||_2$

(b) $(v \otimes \bar{w})^* = w \otimes \bar{v}$

(c) $A(v \otimes \bar{w})B^* = Av \otimes \overline{Bw}$.

20. Recall if f, $h \in L^2(\mathbb{R}^n)$, then $f \times h$ is the rank one operator given by

$$(f \otimes h)(g)(x) = \left[\int g(y)h(y)\,dy\right] f(x).$$

Let $\{h_j\}$ be an orthonormal basis. Show T is a Hilbert-Schmidt on $L^2(\mathbb{R}^n)$ if and only if $T = \sum_{j,k} \lambda_{j,k} h_j \otimes h_k$ where $\sum |\lambda_{j,k}|^2 < \infty$.

21. Show T is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$ if and only if T has form $T = \sum f_j \otimes h_j$ where f_j, $h_j \in L^2(\mathbb{R}^n)$, the sequence $\{h_j\}_{j=1}^\infty$ is orthogonal, and $\sum ||f_j||^2||h_j||^2 < \infty$. In this case $||T||_2^2 = \sum ||f_j||^2||h_j||^2$.

22. Find a bounded linear transformation T that cannot be written in form $\sum v_i \otimes \bar{w}_i$ where this series converges in the norm topology.

23. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the Banach space of bounded linear operators on $\mathcal{H}$. For each $v \in \mathcal{H}$, define a seminorm $|\cdot|_v$ on $\mathcal{B}(\mathcal{H})$ by

$$|A|_v = ||Av||_{\mathcal{H}}.$$

The locally convex vector space topology defined on $\mathcal{B}(\mathcal{H})$ by the seminorms $|\cdot|_v$ is called the strong operator topology on $\mathcal{B}(\mathcal{H})$.

(a) Show the strong operator topology on $\mathcal{B}(\mathcal{H})$ is weaker than the norm topology on $\mathcal{B}(\mathcal{H})$.

(b) Show if T is a bounded linear operator on $\mathcal{H}$ and $\{e_\alpha\}_{\alpha\in A}$ is a complete orthonormal basis, then

$$T = \sum_\alpha Te_\alpha \otimes \bar{e}_\alpha$$

in the strong operator topology.

24. Show the set of trace class operators forms a linear subspace of $B(\mathcal{H})$.

25. Show T^* is trace class if T is trace class and $\mathrm{Tr}(T^*) = \overline{\mathrm{Tr}(T)}$.

26. Give another proof of Theorem 2.46 by showing that if T_n is compact and converges in norm to T, then the closure of the set $\{Tv \mid ||v|| = 1\}$ is totally bounded. (Recall a subset of a metric space is compact if and only if it is complete and totally bounded.)

27. Let $\mathcal{H}$ be a Hilbert space. Show the adjoint of any finite rank operator on $\mathcal{H}$ is a finite rank operator.

28. Let $\mathcal{H}$ be a Hilbert space. Show the finite rank operators on $\mathcal{H}$ form a norm dense subspace of the space of compact operators.

29. Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Show:

(a) The operator T is a Hilbert-Schmidt operator if and only if T has form

$$T = \sum \lambda_{i,j} e_i \otimes \bar{f}_j$$

where $\sum |\lambda_{i,j}|^2 < \infty$ and the sequences $\{e_i\}$ and $\{f_j\}$ are orthonormal.

(b) The operator T is a trace class operator if and only if T has form

$$T = \sum \lambda_{i,j} e_i \otimes \bar{f}_j$$

where $\sum |\lambda_{i,j}| < \infty$ and the sequences $\{e_i\}$ and $\{f_i\}$ are orthonormal.

30. Let F be a measurable subset of $\mathbb{R}^n$. Let E be the set of x in F such that there is a $\delta > 0$ with $\lambda(B_\delta(x) \cap F) = 0$. Show that E has measure 0. (**Hint:** Take a countable base for the topology of $\mathbb{R}^n$.)

5. The Schwartz Space

Two important vector spaces of functions on $\mathbb{R}^n$ are the space $C_c^\infty(\mathbb{R}^n)$ of C^∞ complex valued functions with compact support and the space of C^∞ complex valued functions on $\mathbb{R}^n$ which along with their derivatives vanish rapidly at ∞. When these spaces are topologized with their 'Schwartz topologies', they are known as Schwartz spaces. In order to see the vector space $C_c^\infty(\mathbb{R}^n)$ is not trivial, one can use the following Lemma.

Lemma 2.53. *Let $0 < r < R$. There exists a smooth nonnegative function φ on $\mathbb{R}^n$ with such that $0 \leqslant \varphi(x) \leqslant 1$, $\varphi(x) = 1$ if and only if $|x| \leqslant r$, and $\varphi(x) = 0$ if and only if $|x| \geqslant R$.*

Proof. Exercise 2.3.1 shows the function $f(t) = 0$ if $t \leqslant 0$ and $f(t) = e^{-1/t}$ if $t > 0$ is C^∞ on $\mathbb{R}$. Define $g(t) = f(R^2 - t)f(t - r^2)$. Then g is C^∞, $g \geqslant 0$, and $g(t) = 0$ zero if and only if $t \leqslant r^2$ or $t \geqslant R^2$. Next define G by $G(t) := \int_t^\infty g(s)\,ds / \int_{-\infty}^\infty g(s)\,ds$. Then G is smooth, non-negative, $G(t) = 0$ if and only if $t \geqslant R^2$, and $G(t) = 1$ if and only if $t \leqslant r^2$. The function $\varphi(x) := G(|x|^2)$ satisfies the conditions in the Lemma. $\square$

Corollary 2.54. *Suppose $K \subseteq U$ where K is a compact subset of $\mathbb{R}^n$ and U is open. Then there is a $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leqslant \phi \leqslant 1$, $\phi(x) = 1$ for $x \in K$, and $\operatorname{supp} \phi \subseteq U$.*

Proof. For each $p \in U$ choose $r(p) > 0$ such that if $|x - p| \leqslant 2r(p)$, then $x \in U$. Since K is compact, we can find a finite sequence $p_1, p_2, \ldots, p_m$ such that if $r_k = r(p_k)$, then

$$K \subseteq \bigcup_{k=1}^m \{x \mid |x - p_k| < r_k\}.$$

Now for each k, choose a C^∞ function ϕ_k such that $0 \leqslant \phi_k \leqslant 1$, $\phi_k(x) = 0$ if $|x - p_k| \geqslant 2r_k$, and $\phi_k(x) = 1$ if $|x - p_k| \leqslant r_k$. Set

$$\phi(x) = 1 - (1 - \phi_1)(1 - \phi_2) \cdots (1 - \phi_k).$$

Clearly $0 \leqslant \phi \leqslant 1$ and $\phi \in C^\infty$. Also if $|x - p_k| < r_k$, $\phi(x) = 1$ and we see $\phi = 1$ on K. Furthermore, if x is not in the compact subset $\cup_{k=1}^m \{x \mid |x - p_k| \leqslant 2r_k\}$ of U, then $\phi(x) = 0$ and we see ϕ has compact support inside U. $\square$

The space of rapidly decreasing smooth functions on $\mathbb{R}^n$ consists of the smooth complex valued functions f on $\mathbb{R}^n$ satisfying

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |D^\alpha f(x)| < \infty$$

for all $N \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$. Moreover, this space with seminorms $|\cdot|_{N,\alpha}$ given by

$$|f|_{N,\alpha} = \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |D^\alpha f(x)|$$

is the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^n$ and is denoted by both $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}_n$. It contains $C_c^\infty(\mathbb{R}^n)$ as a vector subspace.

We note many other norms and seminorms are continuous in this topology. Indeed,

$$\begin{aligned}
|f|_p^p &= \int |f(x)|^p\,dx \\
&= \int |f(x)|^p(1+|x|^2)^{Np}(1+|x|^2)^{-Np}\,dx \\
&\leqslant |f|_{N,0}^p \int (1+|x|^2)^{-Np}\,dx \\
&\leqslant C_{N,p}^p |f|_{N,0}^p
\end{aligned}$$

where in view of Remark 2.27 one has $C_{N,p}^p = \int (1+|x|^2)^{-Np}\,dx < \infty$ when $Np > \frac{n}{2}$. Hence

$$|f|_p \leqslant C_{N,p}|f|_{N,0} \tag{2.9}$$

and we see convergence in $\mathcal{S}(\mathbb{R}^n)$ implies convergence in every L^p space.

Proposition 2.55. *The space $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$. Moreover, it is dense in $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$.*

Proof. By Lemma 2.53, there is a function $\phi \in C_c^\infty(\mathbb{R}^n)$ with $0 \leqslant \phi \leqslant 1$, $\phi(x) = 1$ if $|x| \leqslant 1$ and $\phi(x) = 0$ if $|x| \geqslant 2$.

For each $\beta \in \mathbb{N}_0^n$, let $M_\beta = \max_{x\in\mathbb{R}^n} |D^\beta\phi(x)|$. Set $\phi_k(x) = \phi(\frac{x}{k})$. Note $\phi_k(x) = 1$ if $|x| \leqslant k$ and $\phi_k(x) = 0$ if $|x| \geqslant 2k$. Also note

$$|D^\beta\phi_k(x)| = \frac{1}{k^{|\beta|}}\left|D^\beta\phi(\frac{x}{k})\right| \leqslant M_\beta$$

for all β and x. Let $f \in \mathcal{S}(\mathbb{R}^n)$. We claim $\phi_k f \to f$ in $\mathcal{S}(\mathbb{R}^n)$. Indeed, by Leibniz's rule,

$$\begin{aligned}
|\phi_k f - f|_{N,\alpha} &= \sup_{x\in\mathbb{R}^n}(1+|x|^2)^N|D^\alpha(\phi_k f)(x) - D^\alpha f(x)| \\
&= \sup_{|x|\geqslant k}(1+|x|^2)^N|D^\alpha(\phi_k f)(x) - D^\alpha f(x)| \\
&\leqslant \sup_{|x|\geqslant k}(1+|x|^2)^N|D^\alpha f(x)| + \sup_{|x|\geqslant k}(1+|x|^2)^N|D^\alpha(\phi_k f)(x)| \\
&\leqslant \sup_{|x|\geqslant k}(1+|x|^2)^N|D^\alpha f(x)| + \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\sup_{|x|\geqslant k}(1+|x|^2)^N|D^\beta\phi_k(x)D^{\alpha-\beta}f(x)| \\
&\leqslant \sup_{|x|\geqslant k}(1+|x|^2)^N|D^\alpha f(x)| + \sum_{\beta\leqslant\alpha}M_\beta\binom{\alpha}{\beta}\sup_{|x|\geqslant k}(1+|x|^2)^N|D^{\alpha-\beta}f(x)|.
\end{aligned}$$

Thus $|\phi_k f - f|_{N,\alpha} \to 0$ as $k \to \infty$ if $\sup_{|x|\geqslant k}(1+|x|^2)^N|D^\gamma f(x)| \to 0$ for each $\gamma \in \mathbb{N}_0^n$. But if $|x| \geqslant k$,

$$\begin{aligned}(1+|x|^2)^N|D^\gamma f(x)| &\leqslant \frac{1}{1+|x|^2}(1+|x|^2)^{N+1}|D^\gamma f(x)| \\ &\leqslant \frac{1}{1+|x|^2}|f|_{N,\gamma} \\ &\leqslant \frac{1}{1+k^2}|f|_{N,\gamma}.\end{aligned}$$

To see the second statement, we recall that the linear span of characteristic functions $\chi_{\prod[a_j,b_j]}$ of rectangles are dense in $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$. It therefore suffices to show one can approximate in L^p the function $\chi_{\prod[a_j,b_j]}$ by a C^∞ function of compact support. Now again using Lemma 2.53, we can choose a function $\phi \in C_c^\infty(\mathbb{R})$ with $0 \leqslant \phi \leqslant 1$ and $\phi(x) = 1$ if and only if $-1 \leqslant x \leqslant 1$. Define $\phi_k(x)$ by

$$\begin{aligned}\phi_1(x_1, x_2, \ldots, x_n) &= \prod_{j=1}^n \phi\left(\frac{2}{b_j - a_j}(x_j - a_j) - 1\right) \\ \phi_k(x_1, x_2, \ldots, x_n) &= \phi_1^k.\end{aligned} \tag{2.10}$$

Note ϕ_k converge pointwise to $\chi_{\prod[a_j,b_j]}$ and $0 \leqslant \phi_k \leqslant \phi_1$. This implies $\phi_k \to \chi_{\prod[a_j,b_j]}$ in L^p for $1 \leqslant p < \infty$. □

Since the locally convex topology on $\mathcal{S}(\mathbb{R}^n)$ is defined by the countable collection of seminorms $|\cdot|_{N,\alpha}$ and these seminorms are separating, Exercises 2.1.1 and 2.1.10 show the space $\mathcal{S}(\mathbb{R}^n)$ is metrizable.

Proposition 2.56. *The locally convex metrizable topological space $\mathcal{S}(\mathbb{R}^n)$ is complete.*

Proof. Let $\{f_k\}$ be a Cauchy sequence. Since $|D^\alpha f_k - D^\alpha f_l|_\infty = |f_k - f_l|_{0,\alpha}$, we see $D^\alpha f_k$ are uniformly Cauchy for all α. Hence for each α, $D^\alpha f_k$ converges uniformly to a continuous function f_α. This implies f_0 is C^∞ and $D^\alpha f_0 = f_\alpha$ for all α. Moreover, $(1+|x|^2)^N|D^\alpha f_k(x) - D^\alpha f_l(x)| \leqslant |f_k - f_l|_{N,\alpha}$ and thus if $\epsilon > 0$ and K is chosen so that one has $|f_k - f_l|_{N,\alpha} \leqslant \epsilon$ when $k, l \geqslant K$, then letting $l \to \infty$ gives

$$(1+|x|^2)^N|D^\alpha f_k(x) - D^\alpha f(x)| \leqslant \epsilon \quad \forall x \in \mathbb{R}^n \text{ for } k \geqslant K.$$

Consequently $|f_k - f_0|_{N,\alpha} \leqslant \epsilon$ for $k \geqslant K$. So $|f_k - f_0|_{N,\alpha} \to 0$ as $k \to \infty$ and $|f_0|_{N,\alpha} \leqslant |f_0 - f_K|_{N,\alpha} + |f_K|_{N,\alpha} < \infty$ for all N and α. Thus $f_0 \in \mathcal{S}(\mathbb{R}^n)$ and f_k converges to f_0 in $\mathcal{S}(\mathbb{R}^n)$. □

There are several linear transformations of the space of complex valued functions f on $\mathbb{R}^n$. The following occur frequently and have special interest.

(11)

	Translation	$\lambda(y)f(x) = f(x-y)$
(11)	**Dilation**	$\delta(a)f(x) = a^{-n/2}f(a^{-1}x)$ and
	Multiplication	For $g : \mathbb{R}^n \to \mathbb{C}$, $M_g f(x) = g(x)f(x)$

Moreover, there are three idempotent operators on the complex valued functions:

	Conjugation	$\bar{f}(x) := \overline{f(x)}$
(12)	**Check**	$\check{f}(x) := f(-x)$
	Adjoint	$f^*(x) := \overline{f(-x)}$

The linear transformations $\lambda(y)$ and $\delta(a)$ will map $\mathcal{S}(\mathbb{R}^n)$ into itself as will each of the three idempotent operations. To insure the multiplication operator M_g maps $\mathcal{S}(\mathbb{R}^n)$ into itself one needs to put restrictions on the function $g(x)$.

A complex valued function g on $\mathbb{R}^n$ is said to be **tempered** or to grow polynomially if there is a constant K and an integer N such that $|g(x)| \leqslant K(1+|x|^2)^N$ for all x. A smooth function g is said to be **infinitely tempered** if each of its derivatives $D^\alpha g$ is tempered. We denote the vector space of infinitely tempered smooth functions by $\mathcal{S}'_\infty(\mathbb{R}^n)$.

For each $\alpha \in \mathbb{N}_0^n$, the polynomial function x^α is defined by

$$x^\alpha(x_1, \ldots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

A polynomial $p(x)$ of degree m is a function of form

$$\sum_{|\alpha| \leqslant m} c_\alpha x^\alpha$$

where $c_\alpha \neq 0$ for some α with $|\alpha| = m$. We note

$$\frac{|x^\alpha|}{(1+|x|^2)^{|\alpha|/2}} \leqslant 1 \text{ for all } x.$$

This implies every polynomial function $p(x)$ is in $\mathcal{S}'_\infty(\mathbb{R}^n)$.

Moreover, every C^∞ function $g(x)$ which satisfies $|D^\alpha g|_\infty < \infty$ for all α is infinitely tempered.

Proposition 2.57. *Let $g \in \mathcal{S}'_\infty(\mathbb{R}^n)$. Then M_g maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$ and is continuous.*

Proof. Clearly $gf \in C^\infty$ if $f \in \mathcal{S}(\mathbb{R}^n)$. For each $\beta \in \mathbb{N}_0^n$, choose $N_\beta \in \mathbb{N}$ and $K_\beta \geqslant 0$ so that for $x \in \mathbb{R}^n$, one has

$$|D^\beta g(x)| \leqslant K_\beta (1 + |x|^2)^{N_\beta}.$$

Then using Leibniz's rule,

$$\begin{aligned} |gf|_{N,\alpha} &= \sup_{x\in\mathbb{R}^n} (1 + |x|^2)^N |D^\alpha (gf)(x)| \\ &\leqslant \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \sup_{x\in\mathbb{R}^n} (1 + |x|^2)^N |D^\beta g(x)|\,|D^{\alpha-\beta} f(x)| \\ &\leqslant \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \sup_{x\in\mathbb{R}^n} (1 + |x|^2)^{-N_\beta} |D^\beta g(x)| (1 + |x|^2)^{N+N_\beta} |D^{\alpha-\beta} f(x)| \\ &\leqslant \sum_{\beta\leqslant\alpha} K_\beta \binom{\alpha}{\beta} |f|_{N+N_\beta,\alpha-\beta}. \end{aligned}$$

This implies $gf \in \mathcal{S}(\mathbb{R}^n)$ and the map $M_g : f \mapsto gf$ is continuous at 0. Since M_g is a linear transformation, M_g is continuous everywhere. □

Proposition 2.58. *For $\alpha \in \mathbb{N}_0^n$, $y \in \mathbb{R}^n$, and $a > 0$, the linear transformations D^α, $\lambda(y)$, and $\delta(a)$ map $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$ and are continuous in the Schwartz topology. Moreover, the three idempotent operations $\phi \mapsto \bar{\phi}$, $\phi \mapsto \check{\phi}$, and $\phi \mapsto \phi^*$ are also continuous.*

Proof. Note each of these are linear or conjugate linear transformations. It thus suffices to show continuity at 0. That D^α is continuous at 0 follows from the equality

$$|D^\alpha f|_{N,\beta} = |f|_{N,\alpha+\beta}.$$

To see $\lambda(y)$ is continuous at 0, note

$$\begin{aligned} |\lambda(y) f|_{N,\beta} &= \sup_{x\in\mathbb{R}^n} (1 + |x|^2)^N |D^\beta f(x - y)| \\ &= \sup_{x\in\mathbb{R}^n} (1 + |x + y|^2)^N |D^\beta f(x)| \\ &= |M_g D^\beta f|_{0,0} \end{aligned}$$

where $g(x) = (1 + |x + y|^2)^N$ is a polynomial function and thus is in $\mathcal{S}'_\infty(\mathbb{R}^n)$. By Proposition 2.57 and the already established statement that $f \mapsto D^\beta f$ is continuous, the mapping $f \mapsto M_g D^\beta f$ is continuous. Hence, if $f \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, then $|M_g D^\beta f|_{0,0} \to 0$, and we see $\lambda(y)$ is continuous at 0.

Now let $a > 0$. The chain rule implies

$$\begin{aligned}
|\delta(a)f|_{N,\beta} &= \sup(1+|x|^2)^N |D^\beta \delta(a) f(x)| \\
&= a^{-n/2} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N |a^{-|\beta|}(D^\beta f)(a^{-1}x)| \\
&= a^{-n/2-|\beta|} \sup_{x\in\mathbb{R}^n} (1+|ax|^2)^N |D^\beta f(x)| \\
&= a^{-n/2-|\beta|} |M_g D^\beta f|_{0,0}
\end{aligned}$$

where now $g(x) = (1+|ax|^2)^N$ is in $\mathcal{S}'_\infty(\mathbb{R}^n)$. Again, by Proposition 2.57, $\delta(a)$ is continuous at the zero function. Finally, if S is one of the three idempotent operators, note $|S(\phi)|_{N,\alpha} = |\phi|_{N,\alpha}$ and thus S is continuous at 0. □

Lemma 2.59. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then the mapping $x \mapsto \lambda(x)\phi$ is a continuous mapping of $\mathbb{R}$ into $\mathcal{S}(\mathbb{R}^n)$.*

Proof. We need only show $|\lambda(x)\phi - \lambda(x_0)\phi|_{N,\alpha} \to 0$ as $x \to x_0$. Now by the Mean Value Theorem, there is a y^* of form $y^* = y - (tx + (1-t)x_0)$ where $0 < t < 1$ with

$$\begin{aligned}
(1+|y|^2)^N |D^\alpha\phi(y-x) - D^\alpha\phi(y-x_0)| &= (1+|y|^2)^N \Big|\sum_{k=1}^n (x_k - x_{0,k})\partial_k D^\alpha \phi(y^*)\Big| \\
&\leqslant (1+|y|^2)^N \sum_{k=1}^n |x_k - x_{0,k}|\,|\partial_k D^\alpha\phi(y^*)| \\
&\leqslant \left(\frac{1+|y|^2}{1+|y^*|^2}\right)^N \sum_{k=1}^n |x_k - x_{0,k}|\,(1+|y^*|^2)^N |\partial_k D^\alpha \phi(y^*)| \\
&\leqslant \left(\frac{1+|y|^2}{1+|y^*|^2}\right)^N \sum_{k=1}^n |x_k - x_{0,k}|\,|\phi|_{N,\alpha+e_k}.
\end{aligned}$$

Now using Exercise 2.3.4, we have

$$\begin{aligned}
&\frac{1+|y|^2}{1+|y^*|^2} = \frac{1+|y|^2}{1+|y-(tx+(1-t)x_0)|^2} \\
&\quad = \frac{1+|y|^2}{(1+|y-(tx+(1-t)x_0)|^2)(1+|tx+(1-t)x_0|^2)} \big(1+|tx+(1-t)x_0|^2\big) \\
&\quad \leqslant 2(1+|tx+(1-t)x_0|^2) \\
&\quad \leqslant 2(1+2|x|^2+2|x_0|^2).
\end{aligned}$$

Putting these together we see

$$|\lambda(x)\phi - \lambda(x_0)\phi|_{N,\alpha} \leqslant 2^N(1+2|x|^2+2|x_0|^2)^N \sum_{k=1}^n |x_k - x_{0,k}|\,|\phi|_{N,\alpha+e_k}.$$

So $x \mapsto \lambda(x)\phi$ is continuous. □

$\mathcal{S}(\mathbb{R}^n)$ is a topological vector space. It is also closed under pointwise multiplication. The following proposition establishes that with this multiplication $\mathcal{S}(\mathbb{R}^n)$ is a *topological algebra*.

Proposition 2.60. *The mapping $(f, g) \mapsto f \cdot g$ is a continuous bilinear mapping from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Clearly multiplication is bilinear. We show it is continuous. Let f_0 and g_0 be in $\mathcal{S}(\mathbb{R}^n)$ and let N and α be given. Suppose $\epsilon > 0$. Then

$$\begin{aligned} |fg - f_0g_0|_{N,\alpha} &\leqslant |fg - f_0g|_{N,\alpha} + |f_0g - f_0g_0|_{N,\alpha} \\ &= |(f - f_0)g|_{N,\alpha} + |M_{f_0}(g - g_0)|_{N,\alpha}. \end{aligned}$$

Note $\mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{S}'_\infty(\mathbb{R}^n)$. Thus by Proposition 2.57, there is a neighborhood V of g_0 in $\mathcal{S}(\mathbb{R}^n)$ such that $|M_{f_0}(g - g_0)|_{N,\alpha} < \frac{\epsilon}{2}$ if $g \in V$. Hence we need only show that we can make $|(f - f_0)g|_{N,\alpha} < \frac{\epsilon}{2}$ for g in some neighborhood V' of g_0 and f in some neighborhood U of f_0. But by Leibniz's rule,

$$(1 + |x|^2)^N D^\alpha((f - f_0)g)(x) = (1 + |x|^2)^N \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta(f - f_0)(x)\, D^{\alpha-\beta}g(x).$$

Hence $|(1 + |x|^2)^N D^\alpha((f - f_0)g)(x)| \leqslant M \sum_{\beta\leqslant\alpha} |f - f_0|_{0,\beta} |g|_{N,\alpha-\beta}$ where $M = \max_{\beta\leqslant\alpha} \binom{\alpha}{\beta}$. We restrict g to lie in the neighborhood V' of g_0 consisting of those g which satisfy $|g - g_0|_{N,\gamma} < 1$ for all $\gamma \leqslant \alpha$. Take K to be the largest of the numbers $1 + |g_0|_{N,\gamma}$ for $\gamma \leqslant \alpha$. Since $|g|_{N,\gamma} \leqslant 1 + |g_0|_{N,\gamma}$ for $\gamma \leqslant \alpha$, we see $|(1 + |x|^2)^N D^\alpha((f - f_0)g)(x)| \leqslant KM \sum_{\beta\leqslant\alpha} |f - f_0|_{0,\beta}$. Take U to be the open neighborhood of f_0 consisting of the functions f satisfying $\sum_{\beta\leqslant\alpha} |f - f_0|_{0,\beta} < \frac{\epsilon}{2KM}$. Then for $(f, g) \in U \times (V \cap V')$ one has

$$|fg - f_0g_0|_{N,\alpha} < \epsilon$$

and consequently pointwise multiplication is continuous. □

Corollary 2.61. *Let $g \in \mathcal{S}'_\infty(\mathbb{R}^n)$. Then*

$$|f|_{g,\alpha} := \sup_{x\in\mathbb{R}^n} |g(x)D^\alpha f(x)| \text{ and } |f|_{\alpha,g} = \sup_{x\in\mathbb{R}^n} |D^\alpha(gf)(x)|$$

define continuous seminorms on $\mathcal{S}(\mathbb{R}^n)$.

Proof. That these are seminorms is clear. That they are continuous follows from the fact that $|f|_{g,\alpha} = |M_g D^\alpha f|_{0,0}$ and $|f|_{\alpha,g} = |D^\alpha(M_g f)|_{0,0}$. □

In particular, one could have defined the Schwartz topology on $\mathcal{S}(\mathbb{R}^n)$ by using all the seminorms $|\cdot|_{p,\alpha}$ where p is a polynomial function. In many situations, it is more convenient to use these seminorms.

We finish this section with some basic results on Schwartz space.

Proposition 2.62. *Let A be an invertible linear transformation of $\mathbb{R}^n$. Then $f \mapsto f \circ A$ is a linear homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Clearly $f \circ A$ is C^∞ by the chain rule. Moreover,

$$\begin{aligned}\frac{\partial}{\partial x_j}(f \circ A)(x_1, \ldots, x_n) &= \frac{\partial}{\partial x_j} f\Big(\sum x_l a_{l,1}, \ldots, \sum x_l a_{l,n}\Big) \\ &= \sum_k (D^{e_k} f)(xA) a_{j,k}.\end{aligned}$$

Rewriting we see

$$D^{e_j}(f \circ A) = (D_1 f \circ A, \ldots, D_n f \circ A) \cdot A_j$$

where A_j is the jth row of A.

Repeating one sees

$$D^\alpha(f \circ A) = \sum_{|\beta| = |\alpha|} P_\beta(A) D^\beta f \circ A$$

where $P_\beta(A)$ is a homogeneous polynomial of degree $|\alpha|$ in the entries of the matrix for A.

From this we see

$$\begin{aligned}|p(x) D^\alpha(f \circ A)(x)| &\leqslant \sum_{|\beta|=|\alpha|} |P_\beta(A)| |p(x) D^\beta f(Ax)| \\ &\leqslant \sum_{|\beta|=|\alpha|} |P_\beta(A)| \, |p \circ A^{-1}(Ax) D^\beta f(Ax)| \\ &\leqslant \sum_{|\beta|=|\alpha|} |P_\beta(A)| \, ||f||_{p \circ A^{-1}, \beta}.\end{aligned}$$

This implies $f \circ A$ is Schwartz and $f \mapsto f \circ A$ is continuous at 0. By linearity, $f \mapsto f \circ A$ is continuous at every f. Since $f \mapsto f \circ A^{-1}$ is the inverse, we have $f \mapsto f \circ A$ is a homeomorphism. □

Exercise Set 2.3

1. Show the function f defined by

$$f(t) = \begin{cases} 0 & \text{if } t \leqslant 0 \\ e^{-\frac{1}{t}} & \text{if } t > 0 \end{cases}$$

is C^∞ on $\mathbb{R}$.

2. Let $p(x)$ be a polynomial function and let $\lambda > 0$. Show the function $x \mapsto p(x) e^{-\lambda |x|^2}$ is in $\mathcal{S}(\mathbb{R}^n)$.

3. Show that

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2}\, dx = 1\,.$$

(Hint: Do the case $n = 2$ by using polar coordinates. Then use Fubini's Theorem to do the other cases.)

4. Show $\frac{1+|x+y|^2}{(1+|x|^2)(1+|y|^2)} \leqslant 2$.

5. Show that the following are equivalent:

- $f \in \mathcal{S}\,(\mathbb{R}^n)$;
- For all $\alpha, \beta \in \mathbb{N}_0^n$, $|f|_{\alpha,\beta} := \sup \left|D^\alpha(x^\beta f)(x)\right| < \infty$;
- For all $\alpha, \beta \in \mathbb{N}_0^n$, $|f|'_{\alpha,\beta} := \sup \left|x^\alpha D^\beta f(x)\right| < \infty$;
- For all polynomials p and all $\alpha \in \mathbb{N}_0^n$,
$$|f|_{p,\alpha} = \sup |p(x) D^\alpha f(x)| < \infty;$$
- For all polynomials p and all $\alpha \in \mathbb{N}_0^n$,
$$|f|_{\alpha,p} = \sup |D^\alpha(p(x)f(x))| < \infty;$$
- For all $N \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$, $|f|'_{N,\alpha} = \sup(1+|x|)^N\, |D^\alpha f(x)| < \infty$.

Furthermore show each family of seminorms $\{|\cdot|_{N,\alpha}\}$, $\{|\cdot|_{\alpha,\beta}\}$, $\left\{|\cdot|'_{\alpha,\beta}\right\}$, $\{|\cdot|_{p,\alpha}\}$, $\{|\cdot|_{\alpha,p}\}$, and $\left\{|\cdot|'_{N,\alpha}\right\}$ defines the same topology on $\mathcal{S}(\mathbb{R}^n)$.

6. Let g be a tempered measurable complex valued function on $\mathbb{R}^n$. Show that the map

$$\mathcal{S}(\mathbb{R}^n) \ni f \mapsto T_g(f) := \int_{\mathbb{R}^n} f(x)g(x)\, dx \in \mathbb{C}$$

is a well defined continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$.

7. Let A be a linear transformation of $\mathbb{R}^n$ given by

$$A(x_1, \ldots, x_n) = (\sum a_{1,j}x_j, \sum a_{2,j}x_2, \ldots, \sum a_{n,j}x_j).$$

Suppose $f \in C^\infty(\mathbb{R}^n)$. Show

$$\left(\frac{\partial}{\partial x_k}\right)^m (f \circ A) = \sum_{|\beta|=m} (a_{1,k}, a_{2,k}, \ldots, a_{n,k})^\beta (D^\beta f) \circ A$$

8. Let $\Phi = (\phi_1, \ldots, \phi_n)$ be C^∞ mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$ with C^∞ inverse $\Psi = (\psi_1, \ldots, \psi_n)$. Assume all the ϕ_i and ψ_j are in $\mathcal{S}'_\infty(\mathbb{R}^n)$; i.e., all their derivatives have polynomial growth. Show

$$f \mapsto f \circ \Phi$$

is a linear homeomorphism of $\mathcal{S}_n(\mathbb{R}^n)$.

9. Suppose $\phi \in \mathcal{S}(\mathbb{R})$ and $\int_{-\infty}^{\infty} \phi(t)\,dt = 0$. Show

$$F(x) = \int_{-\infty}^{x} \phi(t)\,dt$$

is in $\mathcal{S}(\mathbb{R})$. **Hint:** Consider $\int_x^\infty \phi(t)\,dt$.

10. Let f be Schwartz on $\mathbb{R}^m \times \mathbb{R}^n$. Show $y \mapsto f(x,y) = f_x(y)$ is Schwartz on $\mathbb{R}^n$ for each $x \in \mathbb{R}^m$.

11. Show $x \mapsto f_x$ where $f_x(y) = f(x,y)$ is continuous from $\mathbb{R}^m$ into $\mathcal{S}(\mathbb{R}^n)$ for each $f \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$.

12. Let I be a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$ and suppose $f \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$. Define $I(f)(x) = I(f_x)$ where $f_x(y) = f(x,y)$. Show I is a continuous linear transformation of $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^m)$.

6. Topologies on Spaces of Smooth Compactly Supported Functions

Let Ω be an open subset of $\mathbb{R}^n$ and let $C^\infty(\Omega)$ be the set of all C^∞ complex valued functions on Ω. For each compact subset K of Ω, define a seminorms $|\cdot|_{K,\alpha}$ on $C^\infty(\Omega)$ by

$$|\phi|_{K,\alpha} = \max_{x \in K} |D^\alpha \phi(x)|.$$

The space $C^\infty(\Omega)$ equipped with the locally convex topological vector space on $C^\infty(\Omega)$ defined by these seminorms is called $\mathcal{E}(\Omega)$. These seminorms make $\mathcal{E}(\Omega)$ a separated locally convex topological vector space. To simplify notation, the space $\mathcal{E}(\mathbb{R}^n)$ is denoted by $\mathcal{E}_n$.

For each compact subset K of Ω, set $\mathcal{D}_K(\Omega)$ to be the subspace of $C^\infty(\Omega)$ consisting of those f with $\operatorname{supp} f \subseteq K$. We give $\mathcal{D}_K(\Omega)$ the relative topology of $\mathcal{E}(\Omega)$ on $\mathcal{D}_K(\Omega)$. It is the locally convex topology defined by the restrictions of the seminorms $|\cdot|_{K,\alpha}$ to $\mathcal{D}_K(\Omega)$. Exercise 2.1.10 shows this space is metrizable.

Lemma 2.63. *The mapping $\phi \mapsto \phi|_\Omega$ is a continuous mapping from $\mathcal{S}_n$ into $\mathcal{E}(\Omega)$.*

Proof. Let $T(\phi) = \phi|_\Omega$. Then

$$|T\phi|_{K,\alpha} = \max_{x \in K} |D^\alpha \phi(x)| \leqslant |\phi|_{0,\alpha}.$$

Consequently T is continuous. $\square$

For each open subset Ω of $\mathbb{R}^n$, $C_c^\infty(\Omega)$ will denote the vector space of all complex valued C^∞ functions on Ω with compact support in Ω. We note we may identify this space with the space of C^∞ functions on $\mathbb{R}^n$ which have compact support in the open set Ω.

Definition 2.64. *The Schwartz topology on $C_c^\infty(\Omega)$ is the inductive limit topology of the topological vector subspaces $\mathcal{D}_K(\Omega)$ where K is a compact subset of Ω. The space $C_c^\infty(\Omega)$ with the Schwartz topology is called $\mathcal{D}(\Omega)$.*

Note $\mathcal{D}_{K_1}(\Omega) \cup \mathcal{D}_{K_2}(\Omega) \subseteq \mathcal{D}_{K_1 \cup K_2}(\Omega)$ and $\cup_{K \subseteq \Omega} \mathcal{D}_K(\Omega) = \mathcal{D}(\Omega)$. Hence by Proposition 2.16, a seminorm $|\cdot|$ on $\mathcal{D}(\Omega)$ is continuous in the Schwartz topology on $\mathcal{D}(\Omega)$ if and only if its restriction to each subspace $\mathcal{D}_K(\Omega)$ is continuous.

The space $\mathcal{D}(\mathbb{R}^n)$ is denoted by $\mathcal{D}_n$ and is the space of all C^∞ complex valued functions with compact support with its Schwartz topology.

One could also take $C_c^\infty(\Omega)$ to be the space of all C^∞ functions on $\mathbb{R}^n$ with compact support in Ω. In this situation, we can take the relative topology of $\mathcal{S}_n$ on $C_c^\infty(\Omega)$. The next result shows the relative topology is weaker than the Schwartz topology. Exercise 2.4.9 shows the relative topology is strictly weaker.

Proposition 2.65. *The relative topology of $\mathcal{S}_n$ on $C_c^\infty(\Omega)$ is weaker than the Schwartz topology.*

Proof. Give $C_c^\infty(\Omega)$ the relative topology of $\mathcal{S}_n$. Let $I : \mathcal{D}(\Omega) \to C_c^\infty(\Omega)$ be the identity map. Since I is a linear transformation, Proposition 2.15 shows that I is continuous if and only if $I|_{\mathcal{D}_K(\Omega)}$ is continuous for each compact subset K of Ω. But

$$|f|_{K,\alpha} = |D^\alpha f|_\infty \leqslant |f|_{0,\alpha}.$$

Thus $I|_{D_K(\Omega)}$ is continuous at 0. Hence I restricted to each $\mathcal{D}_K(\Omega)$ is continuous. So I is continuous. This gives $G = I^{-1}(G)$ is open in $\mathcal{D}(\Omega)$ for every open subset G in $C_c^\infty(\Omega)$. □

Lemma 2.66. *The relative topology of $\mathcal{D}(\Omega)$ on $\mathcal{D}_K(\Omega)$ is the topology of $\mathcal{D}_K(\Omega)$.*

Proof. Recall that the inductive limit topology on $\mathcal{D}(\Omega)$ is the strongest topology such that the relative topology on each $\mathcal{D}_K(\Omega)$ is weaker than the topology of $\mathcal{D}_K(\Omega)$. Hence, we need only show every open set in $\mathcal{D}_K(\Omega)$ is open in the relative topology.

Let U be open in $\mathcal{D}_K(\Omega)$ and let $f_0 \in U$. Choose a finite set F of $\alpha's$ and an $\epsilon > 0$ such that $f \in U$ if $|f - f_0|_{K,\alpha} < \epsilon$ for $\alpha \in F$. The seminorms $|\cdot|_\alpha$ on $\mathcal{D}(\Omega)$ defined by

$$|f|_\alpha = \max_x |D^\alpha f(x)|$$

are continuous on $\mathcal{D}(\Omega)$ for their restrictions to each $\mathcal{D}_{K'}(\Omega)$ are continuous. Let $G = \{f \in \mathcal{D}(\Omega) \mid |f - f_0|_\alpha < \epsilon \text{ for } \alpha \in F\}$. Then G is open in $\mathcal{D}(\Omega)$ and $f_0 \in G \cap D_K(\Omega) \subseteq U$. Thus each $f_0 \in U$ is interior in U in the relative topology. Hence U is open in the relative topology. □

Theorem 2.67. *The inclusion mappings $\mathcal{D}(\mathbb{R}^n) \overset{\iota}{\hookrightarrow} \mathcal{S}(\mathbb{R}^n) \overset{\kappa}{\hookrightarrow} \mathcal{E}(\mathbb{R}^n)$ are continuous. Furthermore $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R}^n)$.*

Proof. We first note ι is continuous for by Proposition 2.65, the relative topology of $\mathcal{S}_n$ on $C_c^\infty(\mathbb{R}^n)$ is weaker than the Schwartz topology on $\mathcal{D}_n$. Next we show $\mathcal{D}_n$ is dense in $\mathcal{S}_n$.

Let $f \in \mathcal{S}_n$ and suppose φ is C^∞ and $0 \leqslant \varphi \leqslant 1$ and $\varphi(x) = 1$ if and only if $|x| \leqslant 1$ and $\varphi(x) = 0$ if $|x| \geqslant 2$. Then $\varphi(x/n)f(x) \in \mathcal{D}_n$. We claim it converges to f in $\mathcal{S}_n$. Indeed $|p(x)D^\alpha(\varphi(x/n)f(x) - f(x))| = 0$ if $|x| \leqslant n$ while it is $|p(x)D^\alpha f(x)|$ for $|x| \geqslant 2n$. Between n and $2n$ it is $|p(x)\sum\binom{\alpha}{\beta}D^\beta((\varphi(x/n)-1)D^{\alpha-\beta}f(x)| = |p(x)\sum\binom{\alpha}{\beta}n^{-|\beta|}(D^\beta\varphi)(x/n)D^{\alpha-\beta}f(x)|$. We thus see there is an M independent of n such that

$$|p(x)\sum\binom{\alpha}{\beta}D^\beta((\varphi(x/n)-1)D^{\alpha-\beta}f(x)| \leqslant M\sum_{\beta\leqslant\alpha}|p(x)D^{\alpha-\beta}f(x)|$$

for $n \leqslant x \leqslant 2n$. But note $\sup_{x\geqslant|n|}|p(x)D^\gamma f(x)| \to 0$ for every p and γ as $n \to \infty$. Thus $|\varphi(x/n)f(x) - f(x)|_{p,\alpha} \to 0$ as $n \to \infty$.

That the mapping κ is continuous following from Lemma 2.63. Let $K_j = \overline{B_j(0)}$, $j \in \mathbb{N}$. Let $\varphi_j \in \mathcal{D}(\mathbb{R}^n)$ be such that $0 \leqslant \varphi_j \leqslant 1$, and $\varphi|_{K_j} = 1$. Let $\varphi \in \mathcal{E}(\mathbb{R}^n)$. Then $\varphi_j\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi_j\varphi \to \varphi$. $\square$

Proposition 2.68. *Let $1 \leqslant p < \infty$. Then $\mathcal{D}(\Omega) \subseteq L^p(\Omega)$ and the inclusion mapping $\mathcal{D}(\Omega) \ni f \mapsto f \in L^P(\Omega)$ is continuous and has dense range.*

Proof. By Proposition 2.15, it suffices to show $f \mapsto f$ from $C_K^\infty(\Omega) \to L^p(\Omega)$ is continuous. Since this is a linear transformation, we need only show continuity at 0 . But $|f|_p^p = \int_K |f(x)|^p\, d\lambda_n(x) \leqslant \int_K |f|_{K,\mathbf{0}}^p\, d\lambda_n(x) \leqslant \lambda_n(K)\,|f|_{K,\mathbf{0}}^p$. Thus

$$|f|_p \leqslant \lambda_n(K)^{\frac{1}{p}}|f|_{K,\mathbf{0}}$$

and continuity at 0 is established.

To see that the range is dense, the argument in the second part of the proof of Proposition 2.55 can be applied to the situation when Ω is a proper open subset of $\mathbb{R}^n$. $\square$

One can show that Proposition 2.58 continues to hold on $\mathcal{D}(\mathbb{R}^n)$.

Proposition 2.69. *For $\alpha \in \mathbb{N}_0^n$, $y \in \mathbb{R}^n$, and $a > 0$, the linear transformations D^α, $\lambda(y)$, and $\delta(a)$ map $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}(\mathbb{R}^n)$ and are continuous. Moreover, the three idempotent operations $\phi \mapsto \bar{\phi}$, $\phi \mapsto \check{\phi}$, and $\phi \mapsto \phi^*$ are homeomorphisms.*

6.1. Convergence in Schwartz Spaces. Recall subset B of a locally convex topological vector space X is **bounded** if for each open subset U of X containing 0, there is an $R > 0$ such that $B \subseteq \lambda U$ for all $\lambda > R$. By Exercise 2.1.13, a subset of X is bounded if and only if each continuous seminorm is bounded on X.

Proposition 2.70. *Let Ω be an open subset of $\mathbb{R}^n$. Then a subset B is a bounded subset of $\mathcal{D}(\Omega)$ if and only if there is a compact subset K of Ω such that B is a bounded subset of $\mathcal{D}_K(\Omega)$.*

Proof. Suppose B is a bounded subset of $\mathcal{D}(\Omega)$. We first show there is a compact subset K of Ω with $B \subseteq \mathcal{D}_K(\Omega)$. Assume no such K exists. By Exercise 2.4.4, there is a sequence K_i of compact subsets of Ω such that $K_i \subseteq K_{i+1}^\circ \subseteq K_{i+1}$ and $\cup K_i = \Omega$. Since $B \nsubseteq \mathcal{D}_{K_i}(\Omega)$, we can choose $\phi_i \in B$, $\phi_i \notin D_{K_i}(\Omega)$. In particular, there exists $x_i \notin K_i$ such that $\phi_i(x_i) \neq 0$. Define U by

$$U = \{\varphi \in \mathcal{D}(\Omega) \mid |\varphi(x_j)| < \frac{1}{j}|\varphi_j(x_j)| \text{ for all } j\}\,.$$

Note U is balanced and convex. If $K \subset \Omega$, then

$$\#\{j \in \mathbb{N} \mid x_j \in K\} < \infty$$

and hence $U \cap \mathcal{D}_K(\Omega)$ is open. Corollary 2.17 implies U is open in $\mathcal{D}(\Omega)$. In particular, there exists a $\lambda > 0$ such that $B \subseteq \lambda U$. But, if $j > \lambda$, then $\varphi_j \notin \lambda U$, which contradicts $\varphi_j \in B$. Hence, there exists a K such that $B \subset \mathcal{D}_K(\Omega)$.

Finally we note that a subset B of $\mathcal{D}_K(\Omega)$ is bounded in $\mathcal{D}_K(\Omega)$ if and only if it is bounded in $\mathcal{D}(\Omega)$. Indeed, by Lemma 2.66, the topology on $\mathcal{D}_K(\Omega)$ is the relative topology from $\mathcal{D}(\Omega)$. Thus every open neighborhood V of 0 in $\mathcal{D}_K(\Omega)$ has form $G \cap D_K(\Omega)$ where G is an open neighborhood of 0 in $\mathcal{D}(\Omega)$. In particular, there is an $R > 0$ such that $B \subseteq \lambda V$ for $\lambda > R$ if and only if such an R exists with $B \subseteq \lambda G$ for $\lambda > R$. □

Proposition 2.71. *The space $\mathcal{D}_K(\Omega)$ is a Fréchet space.*

Proof. We have already noted the separated locally convex topological vector space $\mathcal{D}_K(\Omega)$ is metrizable for its topology is defined by countably many seminorms; i.e. see Exercise 2.1.10.

We have to prove that $\mathcal{D}_K(\Omega)$ is complete. Let $\{f_k\}_k$ be a Cauchy sequence in $\mathcal{D}_K(\Omega)$. Then for each α, the sequence $\{D^\alpha f_k\}$ converges uniformly to a continuous function g_α on $C(\Omega)$. Let $f = g_0$. It follows by Exercise 2.4.13 that f is in $C^\infty(\Omega)$ and $D^\alpha f_k$ converges uniformly to $D^\alpha f$. Clearly $\operatorname{supp}(f) \subset K$. Hence $f_k \to f$ in the topology of $\mathcal{D}_K(\Omega)$. □

Corollary 2.72. *Let Ω be a nonempty open subset of $\mathbb{R}^n$. Then a sequence $\{\phi_k\}_{k=1}^{\infty}$ is Cauchy in $\mathcal{D}(\Omega)$ if and only if there is a compact subset K of Ω such that it is Cauchy in $\mathcal{D}_K(\Omega)$. Moreover, the space $\mathcal{D}(\Omega)$ is complete.*

Proof. We first note that if K is a compact subset of Ω, then since $\mathcal{D}_K(\Omega)$ has the relative topology of $\mathcal{D}(\Omega)$, a sequence in $\mathcal{D}_K(\Omega)$ is Cauchy if and only if it is Cauchy in $\mathcal{D}(\Omega)$.

Now let $\{\phi_k\}_k$ be a Cauchy sequence in $\mathcal{D}(\Omega)$. By Exercise 2.1.14, this sequence is bounded. Proposition 2.70 then implies there exists a compact set $K \subset \Omega$, such that $\phi_k \in \mathcal{D}_K(\Omega)$ for all k. Consequently, the sequence ϕ_k is Cauchy in $\mathcal{D}_K(\Omega)$.

By Proposition 2.71, there exists a $\phi \in \mathcal{D}_K(\Omega)$ such that $\phi_k \to \phi$ in $\mathcal{D}_K(\Omega)$. But then, as $\mathcal{D}_K(\Omega)$ carries the relative topology, one has $\phi_k \to \phi$ in $\mathcal{D}(\Omega)$. $\square$

Proposition 2.73. *Let V be a locally convex topological vector space and Ω be a nonempty open subset of $\mathbb{R}^n$. Then a linear transformation $T : \mathcal{D}(\Omega) \to V$ is continuous if and only if $T(\phi_k) \to T(\phi)$ for every convergent sequence $\phi_k \to \phi$ in $\mathcal{D}(\Omega)$.*

Proof. By Proposition 2.15, T is continuous if and only if $T|_{\mathcal{D}_K(\Omega)}$ is continuous for each compact subset K of Ω. Since each $\mathcal{D}_K(\Omega)$ is a metric space and $\phi_k \to \phi$ in $\mathcal{D}(\Omega)$ if and only if there is a K with $\phi_k \to \phi$ in $\mathcal{D}_K(\Omega)$, T is continuous if and only if $T(\phi_k) \to T(\phi)$ whenever $\phi_k \to \phi$. $\square$

Exercise Set 2.4

1. Show if Ω is a nonempty open subset of $\mathbb{R}^n$, then the inclusion mapping from $\mathcal{D}(\Omega)$ into $\mathcal{E}(\Omega)$ is continuous.

2. Show if Ω_1 and Ω_2 are nonempty open subsets of $\mathbb{R}^n$ and $\Omega_1 \subseteq \Omega_2$, then the restriction mapping from $\mathcal{E}(\Omega_2)$ into $\mathcal{E}(\Omega_1)$ is continuous.

3. Give a proof for Proposition 2.69.

4. Let Ω be an open subset of $\mathbb{R}^n$.

(a) Show there is a countable increasing sequence $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ of compact subsets of Ω such that $\Omega = \cup K_i$ and $K_i \subseteq K_{i+1}^{\circ}$ for all i. Conclude $\mathcal{D}(\Omega) = \cup \mathcal{D}_{K_i}(\Omega)$.

(b) Show the inductive limit topology defined by the subspaces $\mathcal{D}_{K_i}(\Omega)$ is the same as the inductive limit topology defined by all the subspaces $\mathcal{D}_K(\Omega)$ where K is a compact subset of Ω.

5. Prove that $\mathcal{E}(\Omega)$ is complete.

6. Let Ω be an open subset of $\mathbb{R}^n$ and suppose K is a nonempty compact subset.

(a) Show if B is a bounded subset of $\mathcal{D}_K(\Omega)$, then the collection of functions inside B is equicontinuous.

(b) Show every bounded subset of $\mathcal{D}_K(\Omega)$ has compact closure. Conclude that the space $\mathcal{D}_K(\Omega)$ has the Heine-Borel property; i.e., a subset of $\mathcal{D}_K(\Omega)$ is compact if and only if it is closed and bounded.

7. Show $C_c^\infty(\mathbb{R})$ with the relative topology from $\mathcal{S}_1$ is not complete in the sense that there is a sequence $\{f_n\}$ in $C_c^\infty(\mathbb{R})$ such that for each open neighborhood V of 0 in $\mathcal{S}_1$, there is a $K \in \mathbb{N}$ with $f_m - f_n \in V$ for $m, n \geqslant K$, but $\{f_n\}$ does not have a limit in $\mathcal{D}_1$. (Hint: Use Lemma 2.53 to find a ϕ which is C^∞, is nonzero, and vanishes for $x \notin [0,1]$. Then take $f_n = \phi + \frac{1}{2}\lambda_1\phi + \frac{1}{4}\lambda_2\phi + \cdots \frac{1}{2^{n-1}}\lambda_n\phi$.)

8. Let Ω_1 and Ω_2 be nonempty open subsets of $\mathbb{R}^n$ with $\Omega_1 \subseteq \Omega_2$. We can redefine $\mathcal{D}(\Omega)$ to be all C^∞ functions on $\mathbb{R}^n$ which have compact support inside Ω.

(a) Show the inclusion mapping of $\mathcal{D}(\Omega_1)$ into $\mathcal{D}(\Omega_2)$ is continuous.

(b) Show the relative topology of $\mathcal{D}(\mathbb{R})$ on $\mathcal{D}(-1,1)$ is strictly weaker than the topology of $\mathcal{D}(-1,1)$.

9. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Give $C_c^\infty(\Omega)$ the relative topology of $\mathcal{E}(\Omega)$. Show this topology on $C_c^\infty(\Omega)$ is strictly weaker than the relative topology of $\mathcal{S}(\mathbb{R}^n)$ on $C_c^\infty(\Omega)$.

10. Show there is no countable family of seminorms on $\mathcal{D}_n$ which defines the Schwartz topology on $\mathcal{D}_n$, and as a consequence show the topology on $\mathcal{D}_n$ is not first countable and hence cannot be metrizable.

11. A seminorm $||\cdot||$ on $\mathcal{D}(\Omega)$ is said to be admissible if its restriction to every $\mathcal{D}_K(\Omega)$ where $K \subseteq \Omega$ and K is compact is continuous on $\mathcal{D}_K(\Omega)$.

(a) Show the mapping that assigns to each admissible seminorm to the open set $\{f \in \mathcal{D}(\Omega) \mid ||f|| < 1\}$ is a one-to-correspondence between all admissible seminorms $||\cdot||$ on $\mathcal{D}(\Omega)$ and all balanced convex sets U contained in Ω such that $U \cap \mathcal{D}_K(\Omega)$ is open in $\mathcal{D}_K(\Omega)$ for all compact $K \subseteq \Omega$.

(b) Show a seminorm on $\mathcal{D}(\Omega)$ is continuous if and only if it is admissible.

12. Let Ω be an open subset of $\mathbb{R}^n$. For each compact subset K of Ω, let $C_K(\Omega)$ be the subset of $C_c(\Omega)$ consisting of those $\phi \in C_c(\Omega)$ with supp $\phi \subseteq K$. Define a norm on this space by $|\phi|_K = \max_{x\in K}|\phi(x)|$. Give $C_c(\Omega)$ the inductive limit topology obtained from the subspaces $C_K(\Omega)$. Show $C_c(\Omega)$ is complete.

13. Let Ω be an open subset of $\mathbb{R}^n$. Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of C^∞ functions on Ω such that for each multiindex α, the sequence $\{D^\alpha f_k\}_{k=1}^{\infty}$ is uniformly Cauchy on Ω. Show there is a C^∞ function f on Ω such that $D^\alpha f_k \to D^\alpha f$ uniformly for all multiindices α.

14. Show a subset of $\mathcal{D}(\Omega)$ is compact if and only if it is closed and bounded. **Hint:** Suppose F is a closed and bounded subset of $\mathcal{D}_n$. Then $F \subseteq \mathcal{D}_K(\Omega)$ for some compact subset K, and since this is in a metric space (the relative topology of $\mathcal{D}_(\Omega)$ is the topology on $\mathcal{D}_K$), you need only show sequential compactness; use the boundedness of the seminorms defining the topology on $\mathcal{D}_K$ to show one can apply Ascoli's Theorem.

15. In Lemma 2.59, we showed $x \mapsto \lambda(x)\phi$ is a continuous mapping from $\mathbb{R}$ into $\mathcal{S}(\mathbb{R}^n)$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$. Suppose $f \in \mathcal{D}(\mathbb{R}^n)$. Show $x \mapsto \lambda(x)f$ is a continuous mapping of $\mathbb{R}$ into $\mathcal{D}(\mathbb{R}^n)$.

16. Let Ω_1 and Ω_2 be nonempty subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Let $\Phi \in \mathcal{D}(\Omega_1 \times \Omega_2)$. For $x \in \Omega_1$, define Φ_x by $\Phi_x(y) = \Phi(x, y)$. Show $x \mapsto \Phi_x$ is continuous mapping from $\mathcal{D}(\Omega_1)$ into $\mathcal{D}(\Omega_2)$.

17. Let $\Omega \subset \mathbb{R}^n$ be open and non-empty. For $\alpha \in \mathbb{N}_0^n$, let $\mathcal{C}_c^\alpha(\Omega)$ be the space of compactly supported functions such that $D^\beta f \in \mathcal{C}(\Omega)$ for all $\beta \leqslant \alpha$. For each compact set $K \subset \Omega$, let $\mathcal{C}_K^\alpha(\Omega) = \{f \in \mathcal{C}^\alpha(\Omega) \mid \mathrm{supp}(f) \subset K\}$. The seminorms $|\cdot|_{K,\beta}$, $\beta \leqslant \alpha$ define a topology on $C_K^\alpha(\Omega)$. Denote the corresponding topological vector space by $\mathcal{D}_K^\alpha(\Omega)$. Let $\mathcal{D}^\alpha(\Omega)$ have the inductive limit topology of the subspaces $\mathcal{D}_K^\alpha(\Omega)$ where K is a compact subset of Ω. Show the following:

(a) $\mathcal{D}_K^\alpha(\Omega)$ and $\mathcal{D}^\alpha(\Omega)$ are complete.

(b) The inclusion maps $\mathcal{D}_K(\Omega) \hookrightarrow \mathcal{D}_K^\alpha(\Omega)$ and $\mathcal{D}(\Omega) \hookrightarrow \mathcal{D}^\alpha(\Omega)$ are continuous.

18. Show that $\mathcal{D}_K(\Omega)$ is closed subspace of $\mathcal{D}(\Omega)$ for all compact subsets $K \subset \Omega$. **Hint:** Let $f \notin \mathcal{D}_K$ and choose x_0 with $f(x_0) \neq 0$. Let $U = \{\phi \in \mathcal{D}_n : |\phi(x_0)| < |f(x_0)|\}$; show U is open in $\mathcal{D}_n$ and $f + U$ misses $\mathcal{D}_K$.

19. Let K be a compact subset of an open subset Ω of $\mathbb{R}^n$. Show if $\mathcal{F}$ is a closed subset of $\mathcal{D}_K(\Omega)$ and $\alpha \in \mathbb{N}_0^n$, then $D^\alpha\mathcal{F}$ is a closed subset of $\mathcal{D}_K(\Omega)$.

20. Let $1 \leqslant p < \infty$, and $K \subset L^p(\mathbb{R}^n)$ be a closed and bounded subset. Then K is compact in $L^p(\mathbb{R}^n)$ if and only if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ and a compact subset $L \subset \mathbb{R}^n$, $L^o \neq \varnothing$, such that for each

$f \in K$ and $y \in \mathbb{R}^n$ with $|y| < \delta$ we have

$$\left(\int |f(x+y) - f(x)|^p \, d\mu(x) \right)^{1/p} < \epsilon$$

and

$$\left(\int_{L^c} |f(x)|^p \, d\mu \right)^{1/p} < \epsilon \, .$$

7. Convolution on $\mathbb{R}^n$

The space of C^∞ functions on $\mathbb{R}^n$ which are L^p form a dense subspace of $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$. Thus for $f \in L^p(\mathbb{R}^n)$ and for any $n \in \mathbb{N}$ there exists a smooth function f_n such that $|f - f_n|_p < \frac{1}{n}$. One way to obtain such a function is to convolve f with a Schwartz function. Formally convolution is defined by

$$f * h(x) = \int_{\mathbb{R}^n} f(y) h(x-y) \, dy.$$

One needs to determine when and in what sense this integral converges. We begin this section with some of the central facts concerning convolution.

Recall we have defined translation by y by $\lambda(y)f(x) = f(x-y)$. Since Lebesgue measure is translation invariant, the operators $\lambda(y)$ for $y \in \mathbb{R}^n$ are isometries of $L^p(\mathbb{R}^n)$.

Lemma 2.74. *Let* $1 \leqslant p < \infty$. *The map* $\lambda : \mathbb{R}^n \to B(L^p(\mathbb{R}^n))$ *is a strongly continuous homomorphism; i.e., if* $f \in L^p(\mathbb{R}^m)$, *then*

$$\lim_{y \to y_0} \lambda(y) f = \lambda(y_0) f$$

for all $y_0 \in \mathbb{R}^n$.

Proof. Let $f \in L^p(\mathbb{R}^n)$. Suppose $\{y_k\}_{k=1}^\infty$ is a sequence in $\mathbb{R}^n$ converging to y_0. Let $\epsilon > 0$. Choose a continuous function g with compact support K satisfying $|g - f|_p < \frac{\epsilon}{3}$. Note $\lambda(y_n)g$ converges pointwise to $\lambda(y_0)g$ and $|\lambda(y_n)g - \lambda(y_0)g|_p^p \leqslant 2^p |g|_\infty^p 1_{K'}$ where K' is the compact set $K + \{y_0, y_1, y_2, \ldots\}$. By the Lebesgue Dominated Convergence Theorem, $|\lambda(y_k)g - \lambda(y_0)g|_p^p \to 0$ as $k \to \infty$. Thus there is an $N \in \mathbb{N}$ so that $|\lambda(y_k)g - \lambda(y_0)g|_p < \frac{\epsilon}{3}$ if $k \geqslant N$. Hence for $k \geqslant N$,

$$\begin{aligned} |\lambda(y_k)f - \lambda(y_0)f|_p &\leqslant |\lambda(y_k)f - \lambda(y_k)g|_p + |\lambda(y_k)g - \lambda(y)g| + |\lambda(y_0)g - \lambda(y_0)f|_p \\ &< 2|f-g|_p + \frac{\epsilon}{3} < \epsilon. \end{aligned}$$

So $y \mapsto \lambda(y)f$ is continuous at y_0 for each $f \in L^p(\mathbb{R}^n)$. $\square$

Assume that f and g are functions on $\mathbb{R}^n$ such that the function $y \mapsto f(y)g(x-y)$ is integrable for almost all x. Then the function

$$f * g(x) := \int f(y)g(x-y)\,dy$$

is defined a.e. and is called *the convolution* of f and g.

Lemma 2.75. *Suppose f, g, and h are complex valued measurable functions on $\mathbb{R}^n$. Then the following hold:*

(a) *$f * g(x)$ exists if and only if $g * f(x)$ and then*

$$f * g(x) = g * f(x).$$

(b) *Suppose $f * g(x)$ and $f * h(x)$ exist. Then $f * (c_1 g + c_2 h)(x)$ exists for any complex numbers c_1 and c_2. Moreover,*

$$f * (c_1 g + c_2 h)(x) = c_1 f * g(x) + c_2 f * h(x).$$

Proof. For (a) note if $f(y)g(x-y)$ is integrable in y, then by making the change in variables $z = x - y$, the function $z \mapsto f(x-z)g(z)$ is integrable in z and

$$\int f(y)g(x-y)\,dy = \int g(z)f(x-z)\,dz.$$

In particular, $f * g(x)$ exists if and only if $g * f(x)$ exists and then

$$f * g(x) = g * f(x).$$

Finally (b) follows immediately from the linearity of the integral and the fact that $y \mapsto f(y)(c_1 g(x-y) + c_2 h(x-y))$ is integrable if $y \mapsto f(y)g(x-y)$ and $y \mapsto f(y)h(x-y)$ are integrable. $\square$

Lemma 2.76. *Let $1 \leqslant p \leqslant q \leqslant \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then the following hold:*

(a) *$f * g(x)$ exists for all x and $|f * g(x)| \leqslant |f|_p |g|_q$.*

(b) *$f * g$ is continuous.*

Proof. Note (a) follows from the Hölder inequality because $y \mapsto g(x-y)$ is in $L^q(\mathbb{R}^n)$ and has the same norm as g.

For (b), we recall $\check{f}(y) = f(-y)$. Since $f * g(x) = g * f(x)$, we have $\int g(y)f(x-y)\,dy = \int \check{f}(y-x)g(y)\,dy$. Thus $f * g(x) = \int \lambda(x)\check{f}(y)\,g(y)\,dy$. Hence

$$\begin{aligned} |f * g(x) - f * g(x_0)| &\leqslant \int |\lambda(x)\check{f}(y) - \lambda(x_0)\check{f}(y)|\,|g(y)|\,dy \\ &\leqslant |\lambda(x)\check{f} - \lambda(x_0)\check{f}|_p |g|_q. \end{aligned}$$

Since $\check{f} \in L^q(\mathbb{R}^n)$ and $1 \leqslant p < \infty$, Lemma 2.74 implies $f * g(x) \to f * g(x_0)$ as $x \to x_0$. $\square$

Lemma 2.77. *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ where $1 \leqslant p \leqslant \infty$. Then $f * g \in L^p(\mathbb{R}^n)$ and*

$$|f * g|_p \leqslant |f|_1 |g|_p.$$

In particular, the mapping $g \mapsto f * g$ is a bounded linear operator on $L^p(\mathbb{R}^n)$.

Proof. We note $f * g(x) = \int f(x-y)g(y)\,dy$. Define $K(x,y) = f(x-y)$. Then $|K_x|_1 = |f|_1$ and $|K^y|_1 = |f|_1$ for all x and y. By Lemma 2.28, $(Tg)(x) = \int K(x,y)g(y)\,dy$ exists a.e. x, $Tg \in L^p(\mathbb{R}^n)$, and the linear operator T defined by the kernel K has norm at most $|f|_1$. □

One can obtain Lemma 2.77 using more direct arguments. See Exercise 2.5.1 and Exercise 2.5.2.

A $*$ **algebra** $\mathcal{A}$ is an algebra over the complex numbers having a mapping $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(a+b)^* = a^*+b^*$, $(ca)^* = \bar{c}a^*$, and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$ and $c \in \mathbb{C}$. The element a^* is called the **adjoint** of a.

A **Banach $*$ algebra** is a $*$ algebra $\mathcal{B}$ with a complete norm $||\cdot||$ having properties $||a^*|| = ||a||$ and $||ab|| \leqslant ||a||\,||b||$ for all $a, b \in \mathcal{B}$.

The involution $f \mapsto f^*$ on complex valued functions on $\mathbb{R}^n$ is defined by

$$f^*(x) = \overline{f(-x)}. \tag{A}$$

One has $(f+g)^* = f^* + g^*$, $(f^*)^* = f$, $(cf)^* = \bar{c}f^*$ and $|f^*|_p = |f|_p$ if f is measurable.

Proposition 2.78. *Under convolution and adjoint operation* (A)*, the algebra $L^1(\mathbb{R}^n)$ is a commutative Banach $*$ algebra.*

Proof. By Lemmas 2.75 and 2.77, $f * g = g * f$ and $|f * g|_1 \leqslant |f|_1|g|_1$ for all f, $g \in L^1(\mathbb{R}^n)$. We have noted $|f^*|_1 = |f|_1$. To finish we need to check convolution is associative and $(f * g)^* = g^* * f^*$ for $f, g \in L^1(\mathbb{R}^n)$.

For associativity, note if $f, g, h \in L^1(\mathbb{R}^n)$, then by the changes of variables $x \mapsto x + y$ followed by $y \mapsto y + z$, one sees

$$\begin{aligned}\int\int\int |f(z)g(y-z)h(x-y)|\,dx\,dy\,dz &= \int\int\int |f(z)g(y)h(x)|\,dx\,dy\,dz \\ &= |f|_1|g|_1|h|_1 \\ &< \infty.\end{aligned}$$

Fubini's Theorem implies $(y,z) \mapsto f(z)g(y-z)h(x-y)$ is integrable for a.e. x and for a.e. x,

$$\begin{aligned}
(f*g)*h(x) &= \int (f*g)(y)h(x-y)\,dy \\
&= \int \left(\int f(z)g(y-z)\,dz \right) h(x-y)\,dy \\
&= \int\int f(z)g(y-z)h(x-y)\,dy\,dz \\
&= \int\int f(z)g(y)h(x-(y+z))\,dy\,dz \\
&= \int f(z) \int g(y)h(x-y-z)\,dy\,dz \\
&= \int f(z)g*h(x-z)\,dz \\
&= f*(g*h)(x).
\end{aligned}$$

Thus convolution is associative. Finally translation invariance of Lebesgue measure implies

$$\begin{aligned}
(f*g)^*(x) &= \overline{f*g(-x)} \\
&= \overline{\int f(y)g(-x-y)\,dy} \\
&= \overline{\int f(y-x)g(-y)\,dy} \\
&= \int \overline{g(-y)}\,\overline{f(y-x)}\,dy \\
&= \int g^*(y)f^*(x-y)\,dy \\
&= g^* * f^*(x) \text{ a.e. } x.
\end{aligned}$$

□

The next two lemmas deals with the regularity properties of $f*g$ in case one of the function is in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Lemma 2.79. *Let* $1 \leqslant p \leqslant \infty$. *Let* $f \in L^p(\mathbb{R}^n)$ *and* $g \in \mathcal{S}(\mathbb{R}^n)$. *Then* $f*g$ *is smooth and*

$$p(D)(f*g) = f*(p(D)g)$$

for any polynomial p.

Proof. By Lemma 2.76, if $h \in \mathcal{S}(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$ where $1 \leqslant p < \infty$, then $f*h(x)$ exists for all x and is continuous in x. It follows that it is enough to show that $D_j(f * g)(x)$ exists and equals $f * D_j g(x)$. As $D_j g \in \mathcal{S}(\mathbb{R}^n)$ the general statement follows by iteration.

Let $0 < |h| < 1$. Then

$$\frac{f * g(x + he_j) - f * g(x)}{h} = \int \frac{f(y)\,(g(x + he_j - y) - g(x - y))}{h}\,dy.$$

Now by the Mean Value Theorem there is an h^* depending on x, y, and h with $|h^*| < |h| < 1$ such that

$$g(x + he_j - y) - g(x - y)) = h\,D_j g(x + h^* e_j - y).$$

Hence

$$\frac{f * g(x + he_j) - f * g(x)}{h} = \int f(y) D_j g(x + h^* e_j - y)\,dy.$$

Take q with $\frac{1}{p} + \frac{1}{q} = 1$. Choose $M > 0$ and $K > 0$ such that

$$(1 + |y|^2)^{-M} \in L^q(\mathbb{R}^n) \text{ and } |D_j g(z)| \leqslant \frac{K}{(1 + |z|^2)^M} \text{ for all } z.$$

Note using the triangle inequality that

$$1 + |y - x - h^* e_j + (x + h^* e_j)|^2 \leqslant 2(1 + |y - x - h^* e_j|^2)(1 + |x + h^* e_j|^2).$$

From this one sees:

$$\frac{1}{1 + |x + h^* e_j - y|^2} \leqslant \frac{2(1 + |x + h^* e_j|^2)}{1 + |y|^2}.$$

Thus

$$\begin{aligned}
|f(y) D_j g(x + h^* e_j - y)| &\leqslant |f(y)|\,\frac{K}{(1 + |x + h^* e_j - y|^2)^M} \\
&\leqslant 2^M |f(y)|\,\frac{K(1 + |x + h^* e_j|^2)^M}{(1 + |y|^2)^M} \\
&\leqslant 2^M K(1 + (|x| + 1))^2)^M |f(y)|\,(1 + |y|^2)^{-M}
\end{aligned}$$

which by Hölder's inequality is integrable in y. The Lebesgue Dominated Convergence Theorem then implies

$$\begin{aligned}
\lim_{h \to 0} \frac{f * g(x + he_j) - f * g(x)}{h} &= \int \lim_{h \to 0} f(y) D_j g(x + h^* e_j - y)\,dy \\
&= \int f(y) D_j g(x - y)\,dy \\
&= f * D_j g(x).
\end{aligned}$$

□

Lemma 2.80. *If $f, g \in \mathcal{S}(\mathbb{R}^n)$ then $f * g \in \mathcal{S}(\mathbb{R}^n)$ and the bilinear map*

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (f, g) \mapsto f * g \in \mathcal{S}(\mathbb{R}^n)$$

is continuous.

Proof. Using Lemma 2.79, we see $f * g$ is C^∞ and $D^\alpha(f * g) = D^\alpha f * g$. Moreover, using Exercise 2.3.4, we see

$$\begin{aligned}(1+|r|^2)^N|D^\alpha(f*g)(r)| &\leqslant \int \frac{(1+|r|^2)^N}{(1+|r-s|^2)^N}(1+|r-s|^2)^N|D^\alpha f(r-s)|\,|g(s)|\,d_ns \\ &\leqslant ||f||_{N,\alpha}\int \frac{(1+|r|^2)^N}{(1+|r-s|^2)^N}\frac{(1+|s|^2)^{N+k}|g(s)|}{(1+|s|^2)^N}\frac{1}{(1+|s|^2)^k}\,d_ns \\ &\leqslant 2^N||f||_{N,\alpha}||g||_{N+k,0}\int \frac{1}{(1+|s|^2)^k}\,dx.\end{aligned}$$

Hence $||f * g||_{N,\alpha} < C||f||_{N,\alpha}||g||_{N+k,0}$ for some constant C. This shows $f * g$ is Schwartz and implies convolution is continuous. ☐

Exercise Set 2.5

1. Suppose f and g are in $L^1(\mathbb{R}^n)$. Use Fubini's Theorem to show $f * g$ is in $L^1(\mathbb{R}^n)$ and then

$$|f * g|_1 \leqslant |f|_1|g|_1.$$

2. Suppose f is in $L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ where $1 < p \leqslant \infty$. Choose q with $\frac{1}{p} + \frac{1}{q} = 1$. Use the fact that the dual of $L^q(\mathbb{R}^n)$ is $L^p(\mathbb{R}^n)$ to show $f * g(x)$ exists a.e. and $|f * g|_p \leqslant |f|_1|g|_p$.

3. Let

$$f(x) = \begin{cases} 1 & 0 \leqslant x < 1 \\ 0 & \text{otherwise.}\end{cases}$$

For $N \in \mathbb{N}$, define f_N by $f_1 = f$ and $f_{N+1} = f_N * f$.

(a) Evaluate the functions f_2 and f_3.

(b) Show that $\mathrm{Supp}(f_N) = [0, N]$ and $f_N(x) > 0$ for $x \in (0, N)$.

(c) For $N \geqslant 2$ show that $f \in C^{N-2}(\mathbb{R})$.

(d) If $f \in C(\mathbb{R})$, show

$$\int_{-\infty}^{\infty} f(x)f_N(x)\,dx = \int_0^1 \ldots \int_0^1 f(x_1 + \ldots + x_N)\,dx_1\ldots dx_N$$

(e) Show $\sum_{k=-\infty}^{\infty} f_N(x-k) = 1$ for all $x \in \mathbb{R}$.

(f) Verify each function f_N is symmetric with respect to the center of $[0, N]$, i.e.,

$$f_N(\frac{N}{2} + x) = f_N(\frac{N}{2} - x)\,.$$

(g) Show the following difference equation holds:

$$f_{N+1}(x) = \frac{x}{N} f_N(x) + \frac{N+1-x}{N} f_{N-1}(x-1)\,.$$

4. Let f and g be even functions. Show $f * g$ is an even function. That is show if x is in the domain of $f * g$, then $-x$ is in the domain of $f * g$ and then

$$f * g(x) = f * g(-x).$$

5. Let $f(x) = e^{-|x|}$ for $x \in \mathbb{R}$. Show $f * f(x) = (1 + |x|)\, e^{-|x|}$.

6. Let $f(x) = e^{-x}\,\chi_{[0,\infty)}(x)$. Set $f_1 = f$ and $f_{N+1} = f * f_N$ for $N > 1$. Evaluate the function f_N for all N.

7. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Suppose that $g \in C_c^\infty(\mathbb{R}^n)$. Show $f * g(x)$ exists everywhere and $f * g \in C^\infty(\mathbb{R}^n)$.

8. Suppose $f(x) = 0$ for $x \notin E$ and $g(x) = 0$ for $x \notin F$. Show $f * g(x) = 0$ for any $x \notin E + F$. In particular, if f and g have compact supports $\operatorname{supp} f$ and $\operatorname{supp} g$ and $f * g$ is defined everywhere, then $f * g$ has compact support contained in $\operatorname{supp} f + \operatorname{supp} g$.

9. Suppose $f * g(x)$ is defined a.e. x. Show $\lambda(a)f * \lambda(b)g(x)$ is defined a.e. and

$$\lambda(a)f * \lambda(b)g(x) = \lambda(a+b)(f * g)(x)$$

for a.e. x.

10. Recall $\delta(a)f(x) = a^{-n/2} f(a^{-1}x)$ for $a > 0$ and f a function on $\mathbb{R}^n$. Show

$$\delta(a)f * \delta(a)g(x) = (f * g)(a^{-1}x) = a^{n/2}\delta(a)(f * g)(x).$$

11. Show $\chi_{[0,1]} * \chi_{[a,b]}$ is the function defined by:

$$\chi_{[0,1]}*\chi_{[a,b]}(x)(x) = \begin{cases} 0 & \text{if } x \leqslant a \\ x-a & \text{if } a \leqslant x \leqslant \min\{a+1, b\} \\ \min\{1, b-a\} & \text{if } \min\{a+1,b\} \leqslant x \leqslant \max\{a+1, b\} \\ b+1-x & \text{if } \max\{a+1,b\} \leqslant x \leqslant b+1 \\ 0 & \text{if } b+1 \leqslant x. \end{cases}$$

12. Let $c > 0$. Show $\chi_{[0,c]} * \chi_{[a,b]}$ is the function defined by

$$\chi_{[0,c]} * \chi_{[a,b]}(x) = \begin{cases} 0 & \text{if } x \leqslant a \\ x-a & \text{if } a \leqslant x \leqslant \min\{a+c, b\} \\ \min\{c, b-a\} & \text{if } \min\{a+c,b\} \leqslant x \leqslant \max\{a+c, b\} \\ b+c-x & \text{if } \max\{a+c,b\} \leqslant x \leqslant b+c \\ 0 & \text{if} b+c \leqslant x. \end{cases}$$

13. Let $l \leqslant m$. Show the convolution of $\chi_{[2^l,2^{l+1}]}$ with $\chi_{[2^m,2^{m+1}]}$ is given by:

$$\chi_{[2^l,2^{l+1}]} * \chi_{[2^m,2^{m+1}]}(x) = \begin{cases} 0 & \text{if } x \leqslant 2^m \\ x - 2^m & \text{if } 2^m \leqslant x \leqslant 2^m + 2^l \\ 2^l & \text{if } 2^m + 2^l \leqslant x \leqslant 2^{m+1} \\ 2^{m+1} + 2^l - x & \text{if } 2^{m+1} \leqslant x \leqslant 2^{m+1} + 2^l \\ 0 & \text{if } 2^{m+1} + 2^l \leqslant x. \end{cases}$$

14. Let $f = \sum_{k=1}^{\infty} 2^{k/2}\chi_{[2^{-k},2^{-k+1}]}$.

(a) Show $f \in L^1(\mathbb{R})$ but $f \notin L^2(\mathbb{R})$.

(b) Show $f * f$ is not continuous.

15. Give an example of $f, h \in L^2(\mathbb{R})$ where $f * h(x) = \int f(y)h(x-y)\, d_1y$ is not an L^2 function.

16. Show the Hilbert space of Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$ is an example of Hilbert algebra. That is show the space $\mathcal{B}_2(\mathcal{H},\mathcal{H})$ is a Banach $*$ algebra with multiplication given by composition and the adjoint given by operator adjoint and then show one has the following additional properties:

(a) For each $A \in \mathcal{B}_2(\mathcal{H},\mathcal{H})$, the linear transformation L_A of $\mathcal{B}_2(\mathcal{H},\mathcal{H})$ given by $L_A(B) = AB$ is bounded and has norm $||A||$.

(b) $L_A^* = L_{A^*}$

(c) The linear span of the set $\{AB \mid A, B \in \mathcal{B}_2(\mathcal{H},\mathcal{H})\}$ is dense in $\mathcal{B}_2(\mathcal{H},\mathcal{H})$.

8. Regularization and Approximate Identity

Notice that by Lemmas 2.77 and 2.79, the convolution of a Schwartz function g and any L^p function f is a smooth function in L^p. Since the space of Schwartz functions contains $C_c^\infty(\mathbb{R}^n)$ and any rapidly decreasing function such as the "Gaussian" $g(x) = \exp(-\pi|x|^2)$, there is a variety of choices for the function g. We shall see that if $g \geqslant 0$, $\int g(x)\, dx = 1$, and the integral of g is small off a small diameter ball about the origin, then the convolution $f * g$ is not only smooth but is also close to f in the space L^p. In fact for g, we can take compactly supported functions similar to those constructed at the beginning of Section 5, i.e., suppose $\varphi \geqslant 0$, $\varphi(x) = 1$ if $|x| < \frac{\delta}{2}$ and $\varphi(x) = 0$ if $|x| \geqslant \delta$. Then take $g = \frac{1}{M}\phi$ where $M = \int \phi(x)\, dx$. We note if $|g|_1 = 1$, then $|f * g|_p \leqslant |f|_p$ for $1 \leqslant p < \infty$.

To organize the process we instead start with any Schwartz function $h \geqslant 0$ satisfying $|h|_1 = \int h(x)\, dx = 1$ and define h_t for $t > 0$ by

$$h_t(x) = t^{-n} h(t^{-1}x).$$

The function h_t is still Schwartz and since

$$\int h_t(x)\, dx = \int t^{-n} h(t^{-1}x)\, dx = \int h(x)\, dx = 1,$$

one still has $|h_t|_1 = 1$. As $t \to 0+$, the function h_t localizes more about the origin. In this section we use this to show

$$\lim_{t \to 0+} h_t * f = f \quad \text{in} \quad L^p(\mathbb{R}^n).$$

When $h \geqslant 0$, $h \in C_c^\infty(\mathbb{R}^n)$, and $\int h\, dx = 1$, the family of functions $\{h_t\}_{t>0}$ is sometimes called a *mollifier*. The convolution $f * h_t$ is then called a *regularization* or *mollification* of the function f. The significance of convolution is that the regularization of $f \in L^p$, $1 \leqslant p < \infty$, approximates f; thus f can be approximated by smooth functions in a very controlled way.

The functions h_t are also used in another manner. We note the Banach algebra $L^1(\mathbb{R}^n)$ has no identity; i.e., there is no function g in $L^1(\mathbb{R}^n)$ such that $g * f = f$ for all $f \in L^1(\mathbb{R}^n)$. However, for any f, one has $h_t * f$ is almost f for t near 0. The family of functions $\{h_t\}$ is then an example of an *approximate identity* for the Banach algebra $L^1(\mathbb{R}^n)$, namely

$$\lim_{t \to 0+} |h_t * f - f|_1 = 0$$

for all $f \in L^1(\mathbb{R}^n)$.

To show the h_t's form an approximate identity and can be used as mollifiers, we start with a simple lemma.

Lemma 2.81. *Let $g \in L^1(\mathbb{R}^n)$ and suppose $\delta, \epsilon > 0$. As usual for $t > 0$, set $g_t(x) = t^{-n} g(t^{-1}x)$. Then there exists a $T > 0$ such that*

$$\int_{|x| \geqslant \delta} |g_t(x)|\, dx < \epsilon$$

for all $0 < t \leqslant T$.

Proof. Choose $R > 0$ such that

$$\int_{|x| \geqslant R} |g(x)|\, dx < \epsilon\,.$$

Set $T = \delta/R$. Then for $t \leqslant T$ we have $\delta t^{-1} \geqslant \delta T^{-1} = R$ and hence

$$\begin{aligned}\int_{|x|\geqslant\delta} |g_t(x)|\, dx &= t^{-n} \int_{|x|\geqslant\delta} \left|g(t^{-1}x)\right|\, dx \\ &= \int_{|y|\geqslant\delta t^{-1}} |g(y)|\, dy \qquad (y = t^{-1}x) \\ &\leqslant \int_{|y|\geqslant R} |g(y)|\, dy \\ &< \epsilon\end{aligned}$$

□

Theorem 2.82. *Suppose $h \geqslant 0$, $|h|_1 = 1$, and $1 \leqslant p < \infty$. For $t > 0$, set $h_t(x) = t^{-n}h(x/t)$. Then*

$$\lim_{t\to 0+} |f * h_t - f|_p = 0$$

for each $f \in L^p(\mathbb{R}^n)$. In particular, the family $\{h_t\}_{t>0}$ is an approximate identity in $L^1(\mathbb{R}^n)$.

Proof. We have already noted that the change of variables $x = ty$ shows

$$\int h_t(x)\, dx = \int t^{-n}h(t^{-1}x)\, dx = \int h(y)\, dy = 1.$$

Consequently, $|h_t|_1 = 1$ for $t > 0$. Assume f is in $L^p(\mathbb{R}^n)$. By Lemma 2.74, we can choose $\delta > 0$ such that

$$|f - \lambda(y)f|_p < \epsilon/2 \qquad \text{for } |y| < \delta\,.$$

Let $T > 0$ be such that for $t < T$ we have

$$\int_{|x|\geqslant\delta} h_t\, d\mu < \frac{\epsilon}{2(2|f|_p + 1)}\,.$$

Then using $\int h_t(y)\, dy = 1$, we have

$$\begin{aligned}|f - h_t * f|_p &= \left(\int \left|f(x) - \int h_t(y)f(x-y)\, dy\right|^p\, dx\right)^{1/p} \\ &= \left(\int \left|\int h_t(y)(f(x) - f(x-y))\, dy\right|^p\, dx\right)^{1/p}.\end{aligned}$$

We next apply Lemma 2.29 and obtain:

$$\begin{aligned}|f - h_t * f|_p &\leqslant \int \left(\int |h_t(y)(f(x) - f(x-y)\,)|^p \; dx \right)^{1/p} dy \\ &= \int h_t(y) \left(\int |f(x) - f(x-y)|^p \; dx \right)^{1/p} dy \\ &= \int_{|y|<\delta} h_t(y) \left(\int |f(x) - f(x-y)|^p \; dx \right)^{1/p} dy \\ &+ \int_{|y|\geqslant\delta} h_t(y) \left(\int |f(x) - f(x-y)|^p \; dx \right)^{1/p} dy\,.\end{aligned}$$

Note for any t, the first integral satisfies

$$\begin{aligned}\int_{|y|<\delta} h_t(y) \left(\int |f(x) - f(x-y)|^p \; dx \right)^{1/p} dy &= \int_{|y|<\delta} h_t(y)\, |f - \lambda(y) f|_p \, dy \\ &< \epsilon/2 \int h_t(y)\, dy \\ &= \epsilon/2\end{aligned}$$

while for $0 < t < T$, the second integral can be estimated by

$$\begin{aligned}&\int_{|y|\geqslant\delta} h_t(y) \left(\int |f(x) - f(x-y)|^p \, dx \right)^{1/p} dy \\ &\qquad \leqslant \int_{|y|\geqslant\delta} h_t(y)(|f|_p + |\lambda(y) f|_p)\, dy = 2|f|_p \int_{|y|\geqslant\delta} h_t(y)\, dy \\ &\qquad\qquad < \frac{2|f|_p \epsilon}{2(2|f|_p + 1)} < \epsilon/2\,.\end{aligned}$$

Hence $|f - h_t * f|_p < \epsilon$ if $0 < t < T$. □

Lemma 2.83. *Assume $h \in \mathcal{S}(\mathbb{R}^n)$, $h \geqslant 0$, and $\int h(x)\, dx = 1$.*

(a) *If f is a bounded continuous function, then*

$$\lim_{t \to 0+} f * h_t(x) = f\,(x)$$

for each x in $\mathbb{R}^n$. Moreover, the convergence is uniform on compact subsets of $\mathbb{R}^n$.

(b) *If $f \in \mathcal{S}(\mathbb{R}^n)$, then $h_t * f$ converges to f in the Schwartz topology as $t \to 0+$. Thus the family $\{h_t\}_{t>0}$ forms an "approximate identity" in the algebra $\mathcal{S}(\mathbb{R}^n)$.*

(c) *If, in addition, h has compact support and $f \in \mathcal{D}(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$, then $h_t * f \to f$ in $\mathcal{D}(\Omega)$ as $t \to 0+$.*

Proof. Assume f is bounded and continuous, and K is a compact subset of $\mathbb{R}^n$. Choose $\delta > 0$ such that $|f(x) - f(x-y)| < \frac{\epsilon}{2}$ for $|y| < \delta$ and $x \in K$. Choose $T > 0$ such that $\int_{|y| \geqslant \delta} h_t \, d\mu < \frac{\epsilon}{2(2||f||_\infty + 1)}$ for $0 < t < T$. Then for $x \in K$ and $0 < t < T$ one has:

$$\begin{aligned} |f(x) - h_t * f(x)| &\leqslant \int h_t(y)|f(x) - f(x-y)| \, dy \\ &= \int_{|y|<\delta} h_t(y)|f(x) - f(x-y)| \, dy \\ &+ \int_{|y| \geqslant \delta} h_t(y)|f(x) - f(x-y)| \, dy \\ &< \frac{\epsilon}{2} + 2|f|_\infty \int_{|y| \geqslant \delta} h_t(y) \, dy \\ &< \epsilon \, . \end{aligned}$$

This shows (a). For (b) we show

$$|h_t * f - f|_{N,\alpha} \to 0 \text{ as } t \to 0.$$

But this is $|D^\alpha(h_t * f) - D^\alpha f|_{N,0}$ which by Lemma 2.79 is $|h_t * D^\alpha f - D^\alpha f|_{N,0}$. Since $D^\alpha f \in \mathcal{S}(\mathbb{R}^n)$, it suffices to show

$$|h_t * f - f|_{N,0} \to 0$$

for each N. But

$$|h_t * f - f|_{N,0} = \sup_x (1 + |x|^2)^N |h_t * f(x) - f(x)|.$$

For $0 < t < 1$ and a fixed L with $L > N$, we have

$$\begin{aligned} (1+|x|^2)^N |h_t * f(x)| &= t^{-n}(1 + |x|^2)^N \left| \int h(t^{-1}y) f(x-y) \, dy \right| \\ &\leqslant (1 + |x|^2)^N \int h(y) \, |f(x - ty)| \, dy \\ &= \int h(y) \frac{(1 + |x|^2)^N}{(1 - |x - ty|^2)^L} (1 - |x - ty|^2)^L \, |f(x - ty)| \, dy \\ &= \frac{|f|_{L,0}}{(1 + |x|^2)^{L-N}} \int h(y) \frac{(1 + |x|^2)^L}{(1 + |x - ty|^2)^L} \, dy \\ &\leqslant \frac{|f|_{L,0}}{(1 + |x|^2)^{L-N}} \int h(y) \frac{(1 + |x|^2)^L}{(1 + |x - ty|^2)^L} \left(\frac{1 + |y|^2}{1 + |ty|^2} \right)^L \, dy \\ &\leqslant \frac{|f|_{L,0}}{(1 + |x|^2)^{L-N}} \int (1 + |y|^2)^L h(y) \frac{(1 + |x|^2)^L}{(1 + |x - ty|^2)^L (1 + |ty|^2)^L} \, dy. \end{aligned}$$

Now note $\frac{1+|x|^2}{(1+|x-ty|^2)(1+|ty|^2)} \leqslant 4$ for

$$\begin{aligned}1+|x|^2 &\leqslant (1+|x-ty+ty|^2)\\ &\leqslant (1+(|x-ty|+|ty|)^2)\\ &\leqslant 1+4|x-ty|^2+4|ty|^2\\ &\leqslant 4(1+|x-ty|^2)(1+|ty|^2).\end{aligned}$$

Hence for $1<t<1$,

$$(1+|x|^2)^N|h_t * f(x)| \leqslant \frac{4^L|f|_{L,0}}{(1+|x|^2)^{L-N}}\int (1+|y|^2)^L h(y)\,dy$$

Now $(1+|y|^2)^L h(y)$ is Schwartz and thus is an L^1 function. Consequently we can choose $R_1>0$ such that

$$(|+|x|^2)^N|h_t * f(x)| \leqslant \frac{\epsilon}{4} \text{ for all } |x|>R_1 \text{ and all } 0<t<1.$$

Next since $(1+|x|^2)^N f \in \mathcal{S}(\mathbb{R}^n)$, we can choose $R \geqslant R_1$ such that

$$(1+|x|^2)^N|f(x)| \leqslant \frac{\epsilon}{4} \text{ for all } |x|>R.$$

By part (a), there is a T with $0<T<1$, such that $0<t<T$ implies

$$|h_t * f(x) - f(x)| < \frac{\epsilon}{2(1+R^2)^N} \text{ for all } x \text{ with } |x| \leqslant R.$$

Hence if $0<t<T$, we have

$$\begin{aligned}|h_t * f - f|_{N,0} &= \sup_x (1+|x|^2)^N|h_t * f(x) - f(x)|\\ &\leqslant \max_{|x|\leqslant R}(1+R^2)^N|h_t * f(x)-f(x)|\\ &+ \sup_{|x|>R}(1+|x|^2)^N(|h_t * f(x)|+|f(x)|)\\ &< \frac{\epsilon}{2} + \sup_{|x|>R}(1+|x|^2)^N|h_t * f(x)| + \sup_{|x|>R_1}(1+|x|^2)^N|f(x)|\\ &\leqslant \frac{\epsilon}{2}+\frac{\epsilon}{4}+\frac{\epsilon}{4} = \epsilon.\end{aligned}$$

For (c), note if $r>0$ with $B_r(0) \supseteq \operatorname{supp}(h)$, then $\operatorname{supp}(h_t) \subseteq B_{rt}(0)$ for $t>0$. Since $\operatorname{supp}(f)$ is a compact subset of Ω, there is a $\delta>0$ so that $K := \operatorname{supp}(f)+\overline{B_{r\delta}(0)} \subseteq \Omega$. Thus if $0<t<\delta$, $\operatorname{supp}(f)+\operatorname{supp}(h_t) \subseteq K \subseteq \Omega$. Thus by Exercise 2.5.8, $h_t * f$ has support in the compact set K for $0<t<\delta$. Since $\mathcal{D}(\Omega)$ has the inductive limit topology of the topological subspaces $\mathcal{D}_K(\Omega)$ where K is a compact subset of Ω, we need only show $h_t * f \to f$ in $\mathcal{D}_K(\Omega)$ where $0<t<\delta$ and $t \to 0+$. Since the seminorms $|\cdot|_\alpha$ for $\alpha \in \mathbb{N}_0^n$ define the topology on $\mathcal{D}_K(\Omega)$, it suffices to show $|h_t * f - f|_\alpha \to 0$ as $t \to 0+$. Again using Lemma 2.79, since $D^\alpha f$ again has compact support in supp(f), it suffices to show $|h_t * f - f|_0 \to 0$ as $t \to 0+$. Now part (a) shows $|h_t * f - f|_0 = \max_{x\in K} |h_t * f(x) - f(x)| \to 0$ as $t \to 0+$. $\square$

Example: Let $g(x) = e^{-\pi|x|^2}$. Then $|g|_1 = 1$ according to Exercise 2.3.3. The function $g_t(x)$ given by

$$g_t(x) = t^{-n}e^{-\pi|x|^2/t^2}$$

is localized more and more around 0 as t nears 0.

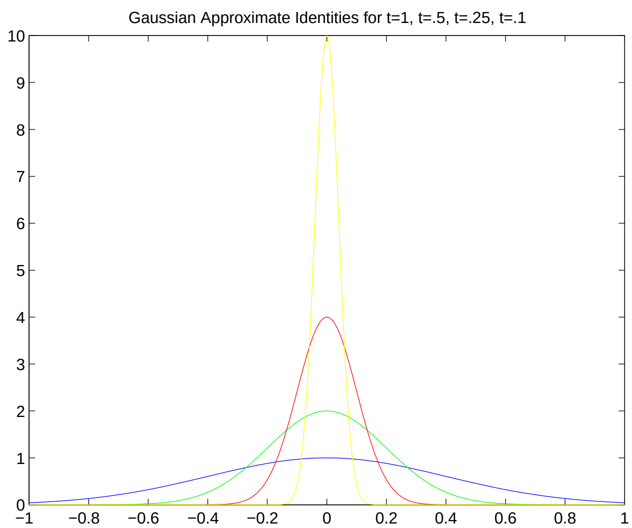

Figure 1. Gaussian Approximate Identities

The following Lemma's proof depends on some complex analysis. Namely, an entire function on $\mathbb{C}^n$ and its derivatives are uniform limits of a power series on compact subsets of $\mathbb{C}^n$. This result can be found in [**19**, Chapter1].

Lemma 2.84. *Suppose f is a function in $\mathcal{S}(\mathbb{R}^n)$ and K is a compact subset of $\mathbb{R}^n$. Let $m \in \mathbb{N}_0$. Then there is a sequence $P_k(x)$ of polynomials*

$$P_k(x) = \sum_{|\alpha|\leqslant N_k} c_{k,\alpha}x^\alpha$$

such that

$$\max_{x\in K} |D^\gamma(P_k - f)(x)| \to 0$$

as $k \to \infty$ for all $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leqslant m$.

Proof. By Lemma 2.83, $g_t(x) = \frac{1}{t^n}e^{-|x|^2/t^2}$ has the property that $g_t * f \to f$ in $\mathcal{S}$ as $t \to 0+$. Thus if $k > 0$, there is a $t_k > 0$ with

$$|g_{t_k} * f - f|_{0,\alpha} < \frac{1}{2^{k+1}}$$

if $|\alpha| \leqslant m$.

Define $F_k(z) = g_{t_k} * f(z) = \int g_{t_k}(z-y)f(y)\,dy$ where

$$g_t(z) = \frac{1}{t^n} e^{-\frac{1}{t^2}(z_1^2+z_2^2+\cdots+z_n^2)}.$$

F_k is an entire function on $\mathbb{C}^n$ and thus has a global power series

$$F_k(z) = \sum_\beta c_\beta z^\beta$$

where

$$c_\beta = \frac{D_z^\beta F_k(0)}{\beta!}.$$

Moreover, this power series and its derivatives convergence uniformly on compact sets to F_k and its derivatives, respectively. Hence there is a polynomial $P_k(z)$ such that

$$|D_z^\alpha P_k(z) - D_z^\alpha F_k(z)| < \frac{1}{2^{k+1}}$$

if $|\alpha| \leqslant m$ and $z \in K$. But this implies

$$|D^\alpha P_k(x) - D^\alpha F_k(x)| < \frac{1}{2^{k+1}}$$

if $|\alpha| \leqslant m$ and $x \in K$. Consequently, if $x \in K$, then

$$|D^\gamma P_k(x) - D^\gamma f(x)| \leqslant |D^\gamma P_k(x) - D^\gamma F_k(x)| + |F_k - f|_{0,\gamma} < \frac{1}{2^k}$$

for $|\gamma| \leqslant m$. □

The second statement in the following theorem is reproved using the Hermite functions in Chapter 4. Indeed, see Proposition 4.73.

Theorem 2.85. *The mapping $(f, h) \mapsto f \otimes h$ defined by*

$$f \otimes h(x,y) = f(x)h(y)$$

is a bilinear and continuous mapping of $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^{2n})$. Moreover, the linear span of the set of functions $f \otimes h$ is dense in $\mathcal{S}(\mathbb{R}^{2n})$.

Proof. First note

$$\begin{aligned}(1+|(x,y)|^2)^N |D^{\alpha,\beta}(f\otimes h)(x,y)| &= (1+|x|^2+|y|^2)^N |D^\alpha f(x)\, D^\beta h(y)| \\ &= \sum_{r+s+t=N} \frac{N!}{r!s!t!} |x|^{2s}|y|^{2t}|D^\alpha f(x)|\,|D^\beta h(y)| \\ &\leqslant \sum_{r+s+t=N} \frac{N!}{r!s!t!}(1+|x|^2)^s(1+|y|^2)^t |D^\alpha f(x)|\,|D^\beta h(y)| \\ &= \sum_{r+s+t=N}\frac{N!}{r!s!t!}|f|_{s,\alpha}|h|_{t,\beta}.\end{aligned}$$

So

$$|f \otimes h|_{N,(\alpha,\beta)} \leqslant C|f|_{N,\alpha}|h|_{N,\beta}$$

where $C = 3^N$. Also

$$\begin{aligned}|f \otimes h - f_0 \otimes h_0|_{N,(\alpha,\beta)} &\leqslant |f \otimes h - f \otimes h_0|_{N,(\alpha,\beta)} + |f \otimes h_0 - f_0 \otimes h_0|_{N,(\alpha,\beta)} \\ &\leqslant C|f|_{N,\alpha}|h - h_0|_{N,\beta} + C|f - f_0|_{N,\alpha}|h_0|_{N,\beta} \\ &\leqslant C(|f_0|_{N,\alpha} + |f - f_0|_{N,\alpha})|h - h_0|_{N,\beta} + C|f - f_0|_{N,\alpha}|h_0|_{N,\beta}.\end{aligned}$$

This implies continuity at (f_0, h_0).

By Proposition 2.55, we know the functions $F(x, y)$ in $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ with compact support are dense. Hence it suffices to show we can approximate such F closely by linear combinations of tensors $f \otimes h$. Let $K_1 \times K_2$ be a product of compact sets with the property that the interiors of K_1 and K_2 contain the projections of the support of F on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Using Corollary 2.54, one can find Schwartz functions $\phi_1(x)$ and $\phi_2(y)$ such that ϕ_j has support in K_j, $0 \leqslant \phi_j \leqslant 1$ and $\phi_j = 1$ on a open set U containing the appropriate projection of the support of F onto either $\mathbb{R}^m$ or $\mathbb{R}^n$. Fix p in $\mathbb{N}$. Lemma 2.84 implies there is a sequence $P_k(x, y)$ of polynomials such that if

$$|P_k - F|_{p, K_1 \times K_2} = \max_{|(\alpha,\beta)| \leqslant p} \max_{(x,y) \in K_1 \times K_2,} |D^{(\alpha,\beta)} P_k(x, y) - D^{(\alpha,\beta)} F(x, y)|,$$

then $|P_k - F|_{p, K_1 \times K_2} \to 0$ as $k \to \infty$. Set $F_k(x, y) = \phi_1(x)P_k(x, y)\phi_2(y)$ and $F_0(x, y) = \phi(x)F(x, y)\phi(y)$. Note $F_0 = F$ and

$$\begin{aligned}D^{(\alpha,\beta)}(F_k(x, y) - F(x, y)) &= D^{(\alpha,\beta)}(F_k - F_0)(x, y) \\ &= D^{(\alpha,\beta)}(\phi_1 \otimes \phi_2)(P_k - F)(x, y) \\ &= \sum_{\gamma \leqslant \alpha} \sum_{\delta \leqslant \beta} \binom{\alpha}{\gamma} \binom{\beta}{\delta} (D^{\alpha-\gamma}\phi_1 \otimes D^{\beta-\delta}\phi_2) D^{(\gamma,\delta)}(P_k - F)(x, y).\end{aligned}$$

Let $M = \max_{|(\gamma,\delta)| \leqslant p} |D^{(\gamma,\delta)}(\phi_1 \otimes \phi_2)|_\infty$. Since all $D^{\alpha-\gamma}\phi_1 \otimes D^{\beta-\delta}\phi_2$ vanish off $K_1 \times K_2$ and

$$1 = \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} = \sum_{\delta \leqslant \beta} \binom{\beta}{\delta},$$

we have

$$\left| D^{(\alpha,\beta)}(F_k(x, y) - F(x, y)) \right| \leqslant M|P_k - F|_{p, K_1 \times K_2}$$

for all (x, y) and all (α, β) with $|(\alpha, \beta)| \leqslant p$. Thus for any p and any $\epsilon > 0$, we can find an F_k with

$$|F_k - F|_{K_1 \times K_2, (\alpha,\beta)} \leqslant \epsilon$$

whenever $|(\alpha, \beta)| \leqslant p$. Since F_k and F have support in $K_1 \times K_2$, we conclude every $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ neighborhood of F contains a function of form F_k. But

F_k is a sum of functions of form $c_{\alpha,\beta} x^\alpha \phi_1(x) y^\beta \phi_2(y)$ and thus is a linear combination of tensors $f_\alpha \otimes g_\beta$ where $f_\alpha \in \mathcal{S}(\mathbb{R}^m)$ and $g_\beta \in \mathcal{S}(\mathbb{R}^n)$. $\square$

A minor modification of this proof gives the following Theorem.

Theorem 2.86. *Let Ω_1 and Ω_2 be nonempty open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Then the mapping $(f, g) \mapsto f \otimes g$ from $\mathcal{D}(\Omega_1) \times \mathcal{D}(\Omega_2)$ into $\mathcal{D}(\Omega_1 \times \Omega_2)$ is a continuous bilinear mapping whose range spans a dense linear subspace of $\mathcal{D}(\Omega_1 \times \Omega_2)$.*

Exercise Set 2.6

1. Using Lemma 2.79, Exercise 2.5.8, and Theorem 2.82, show $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$. This gives an alternate proof to the second part of Proposition 2.55.

2. Let $f \in L^p(\mathbb{R}^n)$. For $t > 0$ define $u(t,x)$ by

$$u(t,x) = f * h_t(x)$$

where $h(x) = e^{-\pi|x|^2}$ and $h_t(x) = t^{-n/2}h(x/\sqrt{t})$. Show that $u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{C}$ is smooth, satisfies the heat equation

$$\partial_t u = \frac{1}{4\pi}\Delta u,$$

and

$$\lim_{t\to 0} u(t,\cdot) = f \text{ in } L^p(\mathbb{R}^n).$$

3. Let Ω be an open subset of $\mathbb{R}^n$. Since $L^p(\Omega)$ is a subspace of $L^p(\mathbb{R}^n)$, $h * f$ is defined for $h \in L^1(\mathbb{R}^n)$. Suppose $h \in C_c^\infty(\mathbb{R}^n)$ is nonnegative and has integral one. Let $f \in \mathcal{D}(\Omega)$. Show $h_t * f \to f$ in $\mathcal{D}(\Omega)$ as $t \to 0+$.

4. By following the proof of Theorem 2.85, check the validity of Theorem 2.86.

5. Let M_n be the vector space of all complex Borel measures on $\mathbb{R}^n$. For each measure $\mu \in M_n$, let $|\mu|$ be the variation measure of μ. Then M_n with norm $||\mu|| = |\mu|(\mathbb{R}^n)$ is a Banach space. By the Riesz representation theorem, M_n is the dual space of the Banach space $C_0(\mathbb{R}^n)$ consisting of continuous functions vanishing at ∞. For μ and $\nu \in M_n$ define $\mu^* \in M_n$ and $\mu * \nu \in M_n$ by $\mu^*(f) = \overline{\int \overline{f(-x)}\, d\mu(x)}$ and $\mu * \nu(f) = \int f(x+y)\, d\mu(x)d\nu(y)$.

(a) Show M_n is a commutative Banach $*$ algebra.

(b) For $g \in L^1(\mathbb{R}^n)$, define $\lambda_g \in M_n$ by $\lambda_g(f) = \int f(x)g(x)\, dx$. Show

$$\lambda_g * \lambda_h = \lambda_{g*h} \text{ for } g, h \in L^1(\mathbb{R}^n).$$

(c) For $f \in L^1(\mathbb{R}^n)$, show $(\lambda_f)^* = \lambda_{f^*}$.

(d) For $f \in L^1(\mathbb{R}^n)$ and $\mu \in M_n$, define $f * \mu(x) = \int f(x-y)\, d\mu(y)$. Show $f * \mu \in L^1(\mathbb{R}^n)$ and

$$\lambda_f * \mu = \lambda_{f*\mu}.$$

(e) Show $f \mapsto \lambda_f$ is an isometric $*$ homomorphism of $L^1(\mathbb{R}^n)$ under convolution onto a closed ideal in M_n.

(f) Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\mu \in M_n$. Show $f * \mu$ is a smooth function.

Chapter 3

The Fourier Transform on $\mathbb{R}^n$

In this chapter we present the Fourier transform on $\mathbb{R}^n$. This transform permeates mathematics and its applications. It is used to attack problems ranging from partial differential equations to analytic number theory to stochastic processes to filtering and noise reduction. It also is fundamental in its importance in the representation theory of nonabelian groups and in providing the basic means in the analysis of functions which are invariant under noncommuting transformation groups. In this chapter, we view the Fourier transform as a transform on function spaces. By selectively altering the spaces on which this transform acts, one extends the reach of the transform to more general functions (distributions) or alters its topological behavior by restricting to the space of Schwartz functions.

1. The Fourier Transform on $L^1(\mathbb{R}^n)$

The easiest space on which to define the Fourier transform is on $L^1(\mathbb{R}^n)$. Indeed for $\omega \in \mathbb{R}^n$, the function e_ω defined on $\mathbb{R}^n$ by

$$e_\omega(y) = e^{2\pi i \omega \cdot y}$$

where $\omega \cdot y = \sum_{j=1}^n \omega_j y_j$ is in $L^\infty(\mathbb{R})$, the dual space of L^1. Thus

$$\hat{f}(\omega) = \mathcal{F}f(\omega) := \int_{\mathbb{R}^n} f(y)\overline{e_\omega(y)}\, dy = \int_{\mathbb{R}^n} f(y) e^{-2\pi i \omega \cdot y}\, dy$$

exists for each $\omega \in \mathbb{R}^n$. The function $\hat{f}$ is called the *Fourier transform* of the L^1 function f and it is defined at each point $\omega \in \mathbb{R}^n$.

Example 3.1. *Let $a_j < b_j$, $j = 1, \dots, n$, and let $f = \chi_I$ where $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then*

$$\begin{aligned}\widehat{f}(\omega) &= \int \chi_I(x) e^{-2\pi i \sum x_j \omega_j}\, dx \\ &= \prod_{j=1}^{n} \int_{a_j}^{b_j} e^{-2\pi i x_j \omega_j}\, dx_j \,.\end{aligned}$$

Assume for the moment that $n = 1$ and $a = a_1 < b = b_1$ and $\omega = \omega_1 \neq 0$. Then

$$\begin{aligned}\int_a^b e^{-2\pi i x \omega}\, dx &= \frac{e^{-2\pi i a \omega} - e^{-2\pi i b \omega}}{2\pi i \omega} \\ &= \frac{\sin\left((b-a)\pi\omega\right)}{\pi\omega} e^{-i\pi(a+b)\omega}.\end{aligned}$$

For $\omega = 0$ the integral is simply $b - a$. Consequently in general one has

$$\begin{aligned}\hat{f}(\omega) &= \prod_{j=1}^{n} \frac{e^{-2\pi i a_j \omega_j} - e^{-2\pi i b_j \omega_j}}{2\pi i \omega_j} \\ &= e^{-\pi i \sum_{j=1}^{n} (a_j + b_j)\omega_j} \prod_{j=1}^{n} \frac{\sin\left((b_j - a_j)\pi\omega_j\right)}{\pi\omega_j} \,.\end{aligned}$$

In the case $b_j > 0$ and $a_j = -b_j$ the transformed function is given by

$$\widehat{f}(\omega) = \pi^{-n} \prod_{j=1}^{n} \frac{\sin(2\pi b_j \omega_j)}{\omega_j} \,.$$

Note that $\widehat{f}$ is continuous, and Exercise 3.1.2 implies $\widehat{f}$ is not in $L^1(\mathbb{R}^n)$. It is in $L^2(\mathbb{R}^2)$. In particular this example shows that the Fourier transform ***does not map*** *$L^1(\mathbb{R}^n)$* ***into*** *$L^1(\mathbb{R}^n)$ in general.*

Example 3.2. *Let $g(x) = e^{-\pi|x|^2}$, the Gaussian. Let $h(\omega) = \widehat{g}(\omega)$. We notice that*

$$\begin{aligned}\overline{h}(\omega) &= \overline{\int g(x) e^{-2\pi i x \cdot \omega}\, dx} \\ &= \int g(x) e^{2\pi i x \cdot \omega}\, dx \\ &= \int g(-x) e^{-2\pi i x \cdot \omega}\, dx \\ &= h(\omega)\end{aligned}$$

because $g(x) = g(-x)$. Hence h is real and

$$h(\omega) = \int g(x) \cos(2\pi x \cdot \omega)\, dx \,.$$

We will calculate h in two different ways.

1) Assume first that $n = 1$. We notice that $\omega \mapsto g(x)e^{-2\pi i x\omega}$ is differentiable with derivatives uniformly bounded by

$$\left|D_\omega^\alpha g(x)e^{-2\pi i x\omega}\right| \leqslant (2\pi)^{|\alpha|}\,|x^\alpha g(x)|$$

which is integrable as a function of x. Hence $h(\omega)$ is differentiable and

$$h'(\omega) = -2\pi i \int x e^{-\pi x^2} e^{-2\pi i x\omega}\, dx\,.$$

Now $-2\pi x e^{-\pi x^2} = D(e^{-\pi x^2})$. Integrating by parts and using $g(x)$ has limit zero at $\pm\infty$ one has

$$\begin{aligned} h'(\omega) &= -i \int e^{-\pi x^2} D_x(e^{-2\pi i x\omega})\, dx \\ &= -2\pi\omega \int e^{-\pi x^2} e^{-2\pi i x\omega}\, dx \\ &= -2\pi\omega h(\omega)\,. \end{aligned}$$

It follows that h is a solution to the differential equation

$$h'(\omega) = -2\pi\omega h(\omega)\,.$$

Hence

$$h(\omega) = Ae^{-\pi\omega^2}$$

for some $A > 0$. Furthermore

$$h(0) = \int e^{-\pi x^2}\, dx = 1\,;$$

(cf. Exercise 2.3.3). Hence $h(\omega) = e^{-\pi\omega^2} = g(\omega)$. For general n, Fubini's Theorem implies

$$h(\omega) = \prod_{j=1}^n h(\omega_j) = \prod_{j=1}^n g(\omega_j) = e^{-\pi|\omega|^2}\,.$$

2) The second way to evaluate the integral uses complex integration. Again it suffices to do the case $n = 1$. Completing the square gives

$$\begin{aligned} h(\omega) &= \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\omega}\, dx \\ &= \int_{-\infty}^{\infty} e^{-\pi\left(x^2+2ix\omega\right)}\, dx \\ &= e^{-\pi\omega^2} \int_{-\infty}^{\infty} e^{-\pi\left(x^2+2ix\omega+(i\omega)^2\right)}\, dx \\ &= e^{-\pi\omega^2} \int_{-\infty}^{\infty} e^{-\pi(x+i\omega)^2}\, dx \\ &= e^{-\pi\omega^2} \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx \\ &= e^{-\pi\omega^2} \end{aligned}$$

Here we have used Cauchy's integral formula to see $\int_{-\infty}^{\infty} e^{-\pi(x+i\omega)^2}\, dx = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx$.

The Fourier transform has several interesting properties relative to differentiation, multiplication, and translation. Recall if p is the polynomial function given by $p(x) = \sum_{|\alpha|\leqslant k} a_\alpha x^\alpha$ with $a_\alpha \in \mathbb{C}$, then $p(D)$ is the differential operator with constant coefficients defined by

$$p(D) = \sum_{|\alpha|\leqslant k} a_\alpha D^\alpha.$$

Moreover, recall for $y \in \mathbb{R}^n$ and $a > 0$, and a complex valued function g on $\mathbb{R}^n$, we have linear transformations:

Translation	$\lambda(y)f(x) = f(x-y)$
Dilation	$\delta(a)f(x) = a^{-n/2} f(a^{-1}x)$ and
Multiplication	$M_g f(x) = g(x)f(x)$

on the space of complex valued functions on $\mathbb{R}^n$ (cf. p.86 of Chapter 2). Define $\tau(y)$ by $\tau(y) = M_{e_{-y}}$. Thus

$$\tau(y)f(x) = e^{-2\pi i x\cdot y} f(x).$$

The maps $\lambda(y)$ and $\tau(y)$ are isometries on $L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant \infty$ while $\delta(a)$ is an isometry only on $L^2(\mathbb{R}^n)$. It, however, is bounded on every other L^p space; in particular, its norm on L^1 is $a^{-n/2}$.

We next give three idempotent operations on the space of complex valued functions. These were used in few instances in Chapters 1 and 2. They are conjugation, check, and adjoint. Conjugation and adjoint are conjugate linear transformations while check is linear. All three are isometries on each $L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant \infty$.

Conjugation	$\overline{f}(x) := \overline{f(x)}$
Check	$f^\vee(x) := f(-x)$ and
Adjoint	$f^* = \overline{f^\vee}$

In the next lemma we use the following two properties of the functions e_ω.

$$e_\omega(x+y) = e_\omega(x)e_\omega(y) \tag{3.1}$$

and

$$\overline{e_\omega(x)} = e_{-\omega}(x) = e_\omega(-x)\,. \tag{3.2}$$

Lemma 3.3 (Basic Properties)**.** *Let $f, g \in L^1(\mathbb{R}^n)$.*

(a) $\widehat{f} \in C(\mathbb{R}^n)$ *and* $|\widehat{f}|_\infty \leqslant |f|_1$.

(b) $\widehat{f+g} = \widehat{f} + \widehat{g}$ *and* $\widehat{cf} = c\widehat{f}$ *and for* $c \in \mathbb{C}$.

(c) $\widehat{\lambda(y)f} = \tau(y)\hat{f}$.

(d) $\widehat{\tau(y)f} = \lambda(y)\hat{f}$.

(e) *If* $a > 0$, *then* $\widehat{\delta(a)f} = \delta(1/a)\hat{f}$.

(f) $\hat{\overline{f}}(\omega) = \overline{\hat{f}(-\omega)}$, *i.e.*, $\hat{\overline{f}} = (\hat{f})^*$.

(g) $\widehat{f^\vee} = \hat{f}^\vee$.

(h) $\widehat{f^*} = \overline{\hat{f}}$.

(i) $\widehat{f * g} = \hat{f}\hat{g}$.

Proof. (a) Since $|e_\omega(x)| = 1$, we have

$$\left|\widehat{f}(\omega)\right| = \left|\int f(x)e_{-\omega}(x)\,dx\right| \leqslant \int |f(x)|\,dx \leqslant |f|_1\,.$$

Hence $|\widehat{f}|_\infty \leqslant |f|_1$. Moreover,

$$\begin{aligned}\left|\widehat{f}(\omega_1) - \widehat{f}(\omega_2)\right| &= \left|\int f(x)(e^{-2\pi i\omega_1\cdot x} - e^{-2\pi i\omega_2\cdot x})\,dx\right| \\ &\leqslant \int |f(x)|\left|e^{-2\pi i(\omega_1-\omega_2)\cdot x} - 1\right|\,dx.\end{aligned}$$

As

$$\left|e^{-2\pi i(\omega_1-\omega_2)\cdot x} - 1\right| \leqslant 2,$$

the Lebesgue Dominated Convergence Theorem implies

$$\lim_{\omega_1 \to w_2} \left| \widehat{f}(\omega_1) - \widehat{f}(\omega_2) \right| = 0\,.$$

Hence $\widehat{f}$ is continuous. Note (b) is a direct consequence of the linearity of integration. For (c) we note:

$$\begin{aligned}
\widehat{\lambda(y)f}(\omega) &= \int \lambda(y) f(x) e_{-\omega}(x)\, dx \\
&= \int f(x-y) e_{-\omega}(x)\, dx \\
&= \int f(x) e_{-\omega}(x+y)\; du \\
&= e_{-\omega}(y) \int f(x) e_{-\omega}(x)\; du \\
&= [\tau(y)\widehat{f}](\omega)\,.
\end{aligned}$$

(d) follows by reversing the calculations in the proof of (c).

For (e) we have:

$$\begin{aligned}
\widehat{\delta(a)f}(\omega) &= a^{n/2} \int f(ax) e_{-\omega}(x)\, dx \\
&= a^{-n/2} \int f(x) e_{-\omega}(x/a)\; dx \\
&= a^{-n/2} \int f(x) e_{-\omega/a}(x)\; dx \\
&= \left[\delta(1/a)\widehat{f}\right](\omega)\,.
\end{aligned}$$

To see (f) note

$$\begin{aligned}
\widehat{\bar{f}}(\omega) &= \int \bar{f}(u) e_{-\omega}(x)\, dx \\
&= \overline{\int f(x) e_{\omega}(x)\, dx} \\
&= \overline{\widehat{f}(-\omega)} \\
&= \widehat{f}^*(\omega)\,.
\end{aligned}$$

For (g) we see

$$\begin{aligned}\widehat{f^\vee}(\omega) &= \int f(-x)e_{-\omega}(x)\,dx\\ &= \int f(x)e_{-\omega}(-x)\ dx\\ &= \int f(x)e_{\omega}(x)\ dx\\ &= \widehat{f}(-\omega)\\ &= \widehat{f}^\vee(\omega).\end{aligned}$$

while (h) follows from (e) and (f) and $f^* = \overline{f^\vee}$.

To see (i), note if $f, g \in L^1(\mathbb{R}^n)$, then $(w, y) \mapsto f(w)g(y-w)e^{-2\pi i x\cdot y}$ is integrable for

$$\int\int |f(w)g(y-w)|\,dy\,dw = \iint |f(w)g(y)|\,dy\,dw = |f|_1|g|_1 < \infty.$$

Hence by Fubini's Theorem,

$$\begin{aligned}\widehat{f*g}(x) &= \int f*g(y)e^{-2\pi i x\cdot y}\,dy\\ &= \iint f(w)g(y-w)e^{-2\pi i x\cdot y}\,dw\,dy\\ &= \iint f(w)g(y)e^{-2\pi i x\cdot(y+w)}\,dy\,dw\\ &= \int f(w)e^{-2\pi i x\cdot w}\,dw\int g(y)e^{-2\pi i x\cdot y}\,dy\\ &= \widehat{f}(x)\widehat{g}(x).\end{aligned}$$

□

Exercise Set 3.1

1. Find the Fourier transform of the following functions:

(a) $\chi_{[-1/2,1/2]}$;

(b) $e^{-|x|}$;

(c) $e^{-|3x-2|}$;

(d) $e^{-\left(ax^2+by+c\right)}$

(e) $\chi_{[0,\infty,0]}e^{-x}$;

(f) xe^{-x^2};

(g) $\chi_{[0,1]}\sin(2\pi x)$;

(h) $\frac{1}{(x+i)^n}$, $n \geqslant 2$;

2. Let $\operatorname{sinc}(\omega) = \frac{\sin\omega}{\omega}$ for $\omega \neq 0$. Show sinc is not in $L^1(\mathbb{R})$.

3. By Exercise 2.6.5, the space M_n of complex Borel measures on $\mathbb{R}^n$ is a Banach $*$ algebra under convolution. Moreover if $f \in L^1(\mathbb{R}^n)$, then λ_f defined by $\lambda_f(E) = \int_E f(x)\,dx$ is in M_n. For μ in M_n, define

$$\hat{\mu}(\omega) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\omega}\,d\mu(x)\,.$$

(a) Show that $\hat{\mu}$ is continuous.

(b) Show if f is in $L^1(\mathbb{R}^n)$, then $\widehat{\lambda_f} = \hat{f}$.

(c) Express the function $\widehat{\mu^*}$ in terms of $\hat{\mu}$.

(d) Show that $\widehat{\mu * \nu} = \hat{\mu}\hat{\nu}$ for $\mu, \nu \in M_n$.

(e) Show $\mu = 0$ if $\hat{\mu} = 0$..

2. The Fourier Transform on $\mathcal{S}(\mathbb{R}^n)$

The Fourier transform has particularly nice behavior on the space $\mathcal{S}(\mathbb{R}^n)$ of Schwartz functions. This is true in part because functions in $\mathcal{S}(\mathbb{R}^n)$ are smooth with all derivatives integrable; and if P is a polynomial, then $Pf \in L^p(\mathbb{R}^n)$ for all $p \geqslant 1$, so one need not worry about integrability conditions. In this section we show that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a topological isomorphism of algebras whose inverse is given by $\mathcal{F}^{-1}f = \mathcal{F}(f)^\vee$.

Theorem 3.4. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and p be a polynomial. Then for each $\omega \in \mathbb{R}^n$, the following hold:*

(a) $\widehat{p(D)f}(\omega) = p(2\pi i\omega)\hat{f}(\omega)$;

(b) $\hat{f}$ *is smooth and* $p(D)\hat{f}(\omega) = (p(-2\pi i\cdot)f(\cdot))^\wedge(\omega)$.

Proof. Linearity implies it suffices to show (a) and (b) for the polynomials x^α. For (a), integrating by parts $|\alpha|$ times yields

$$\begin{aligned}\int D_x^\alpha f(x)e^{-2\pi i x\cdot\omega}\,dx =& (-1)^{|\alpha|}\int f(x)D_x^\alpha(e^{-2\pi i x\cdot\omega})\,dx\\ =&(-1)^{|\alpha|}\int(-2\pi i)^{|\alpha|}\omega^\alpha f(x)e^{-2\pi i x\cdot\omega}\,dx\\ =&(2\pi i\omega)^\alpha\hat{f}(\omega).\end{aligned}$$

For (b) we let $1 \leqslant j \leqslant n$. Then

$$\begin{aligned}\frac{\mathcal{F}(f)(\omega+te_j)-\mathcal{F}(f)(\omega)}{t} &= \int f(x)\frac{e^{-2\pi i(\omega+te_j)\cdot x}-e^{-2\pi i\omega\cdot x}}{t}\,dx\\ &= \int f(x)\,e^{-2\pi i x\cdot\omega}\frac{e^{-2\pi i t x_j}-1}{t}\,dx\,.\end{aligned}$$

But

$$\left|f(x)\,e^{-2\pi i x\cdot\omega}\frac{e^{-2\pi i t x_j}-1}{t}\right| \leqslant |2\pi x_j f(x)|$$

for all $t \neq 0$ and $x \mapsto x_j f(x)$ is integrable. By the Lebesgue Dominated Convergence Theorem, one sees $D_j(\mathcal{F}(f))(\omega) = \lim_{t\to 0}\frac{\mathcal{F}(f)(\omega+te_j)-\mathcal{F}(f)(\omega)}{t}$ exists and

$$D_j(\mathcal{F}(f))(\omega) = \int(-2\pi i x_j)f(x)\,e^{-ix\cdot\omega}\,dx\,.$$

As $x \mapsto (-2\pi i)^{|\alpha|}x^\alpha f(x)$ is in $\mathcal{S}(\mathbb{R}^n)$ for any α, it follows by iteration that $D^\alpha\hat{f}(\omega)$ exists and is the Fourier transform of $x \mapsto (-2\pi i x)^\alpha f(x)$. Since $x^\alpha f(x)$ is Schwartz and thus L^1, (a) of Lemma 3.3 shows $D^\alpha\mathcal{F}(f)$ is continuous for all α. Hence $\mathcal{F}(f)$ is smooth. $\square$

The Laplacian Δ on $\mathbb{R}^n$ is the second order differential operator defined by:

$$\Delta = \partial_1^2 + \ldots + \partial_n^2 \tag{3.3}$$

Note $\Delta = p(D)$ where $p(x) = |x|^2$. Since $(1+|\omega|^2)^N = ((1-p(2\pi i\omega)/(4\pi^2))^N$, (a) implies

$$(1+|\omega|^2)^N\mathcal{F}(f)(\omega) = \mathcal{F}\left((1-\frac{\Delta}{4\pi^2})^N f\right)(\omega).$$

Hence, also using (b), we have:

$$\left(1+|\omega|^2\right)^N D^\alpha\hat{f}(\omega) = i^\alpha(2\pi)^{|\alpha|-2N}\mathcal{F}\left((4\pi^2-\Delta_y)^N y^\alpha f(y)\right)(\omega)\,. \tag{3.4}$$

Corollary 3.5. *The Fourier transform is a continuous linear transformation from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Since $\mathcal{F}$ is linear, it suffices to show $\mathcal{F}$ is continuous at 0. Let $T_{N,\alpha}$ be the linear transformation of $\mathcal{S}(\mathbb{R}^n)$ defined by $T_{N,\alpha}(f) = (4\pi^2-\Delta_y)^N y^\alpha f(y)$. Using (a) of Lemma 3.3 one has

$$\begin{aligned}|\mathcal{F}(f)|_{N,\alpha} &= \sup_\omega\left(1+|\omega|^2\right)^N\left|D^\alpha\hat{f}(\omega)\right|\\ &= (2\pi)^{|\alpha|-2N}\sup_\omega\left|\mathcal{F}\left((4\pi^2-\Delta_y)^N y^\alpha f(y)\right)(\omega)\right| \quad = (2\pi)^{|\alpha|-2N}\sup_\omega|\mathcal{F}(T_{N,\alpha}(f))\\ &\leqslant (2\pi)^{|\alpha|-2N}|T_{N,\alpha}(f)|_1.\end{aligned}$$

By inequality 2.9 of Chapter 2, $|T_{N,\alpha}f|_1 \leqslant C_{N,0}\,|T_{N,\alpha}f|_{N,0}$ for $N > n/2$. Since Propositions 2.57 and 2.58 imply $T_{N,\alpha}$ is continuous, we see for each $\epsilon > 0$, there is a neighborhood V of 0 in $\mathcal{S}(\mathbb{R}^n)$ such that $|T_{N,\alpha}f|_1 < \frac{\epsilon}{(2\pi)^{|\alpha|-2N}}$ for $f \in V$. Thus $|\mathcal{F}(f)|_{N,\alpha} < \epsilon$ for $f \in V$. □

Lemma 3.6. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $\widehat{f * g} = \hat{f}\hat{g}$.*

Proof. By Lemma 2.80 we know if $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $f * g \in \mathcal{S}(\mathbb{R}^n)$ so $\widehat{f * g}$ is well defined. Using Fubini's Theorem, we see:

$$\begin{aligned}
\widehat{f * g}(\omega) &= \int f * g(x)\, e_{-\omega}(x)\, dx \\
&= \int \left[\int f(y)g(x-y)\ \ dy\right] e_{-\omega}(x)\, dx \\
&= \int \left[\int f(y)g(x-y)\, e_{-\omega}(x)\, dx\right] dy \\
&= \int f(y) \left[\int g(x)e_{-\omega}(x+y)\ dx\right] dy \\
&= \int f(y)e_{-\omega}(y)\ \left[\int g(x)e_{-\omega}(x)\, dx\right] dy \\
&= \widehat{f}(\omega)\widehat{g}(\omega)\,.
\end{aligned}$$

□

Lemma 3.7. *If $f, g \in L^1(\mathbb{R}^n)$, then the functions*

$$\omega \mapsto \widehat{f}(\omega)g(\omega),\ \ \omega \mapsto f(\omega)\widehat{g}(\omega)$$

are integrable and

$$\int \widehat{f}(\omega)g(\omega)\, d\omega = \int f(x)\hat{g}(x)\, dx\,.$$

Proof. First note $\hat{f}g$ and $f\hat{g}$ are integrable for by (a), their absolute values are bounded by the L^1 functions $|f|_1|g|$ and $|g|_1|f|$.

Using Fubini's Theorem and the integrability of the function

$$x, y \mapsto f(x)e^{-2\pi i x\cdot y)}g(y) \text{ on } \mathbb{R}^n \times \mathbb{R}^n,$$

we see:

$$\begin{aligned}
\int \widehat{f}(\omega)g(\omega)\, d\omega &= \int \left(\int f(x)e^{-2\pi i x\cdot\omega}\, dx\right)\, g(\omega)\, d\omega \\
&= \int f(x) \left(\int g(\omega)e^{-2\pi i x\cdot\omega}\, d\omega\right)\, dx \\
&= \int f(x)\widehat{g}(x)\, dx.
\end{aligned}$$

□

Corollary 3.8. *The Fourier transform is one-to-one on $L^1(\mathbb{R}^n)$.*

Proof. Let $f \in L^1(\mathbb{R}^n)$ and assume $\hat{f} = 0$. Then

$$0 = \int \hat{f}(\omega) h(\omega)\, d\omega = \int f(x)\hat{h}(x)\, dx$$

for every Schwartz function h. Since the Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$, we see $\int f(x)\phi(x)\, dx = 0$ for all Schwartz functions ϕ. Using (2.10), we see $\int_Q f(x)\, dx = 0$ for all rectangles Q. Thus $f(x) = 0$ for a.e. x. □

Proposition 3.9. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{F}^2(f)(\omega) = f(-\omega)$.*

Proof. Let $f, g \in \mathcal{S}_n$. By (e) of Lemma 3.3 and Lemma 2,

$$\int \hat{f}(x)\delta(a)h(x)\, dx = \int f(x)\delta(\frac{1}{a})\hat{h}(x)\, dx.$$

Thus

$$\int \hat{f}(x)\frac{1}{\sqrt{a^n}}h(\frac{x}{a})\, dx = \int f(x)\sqrt{a^n}\hat{h}(ax)\, dx.$$

This gives

$$\int \hat{f}(x)h(\frac{x}{a})\, dx = a^n \int f(x)\hat{h}(ax)\, dx = \int f(\frac{x}{a})\hat{h}(x)\, dx.$$

Letting $a \to \infty$ and using the Lebesgue Dominated Convergence Theorem, we see

$$\mathcal{F}(\hat{f})(0)h(0) = \int \hat{f}(x)h(0)\, dx = \int f(0)\hat{h}(x)\, dx = f(0)\mathcal{F}(\hat{h})(0).$$

Take $h(x) = e^{-\pi|x|^2}$. Example 3.2 shows

$$\hat{h}(\omega) = h(\omega).$$

Hence $h(0) = \mathcal{F}(\hat{h})(0) = 1$ and thus

$$\mathcal{F}^2(f)(0) = f(0).$$

But now (c) of Lemma 3.3 gives:

$$\begin{aligned} f(-y) &= [\lambda(y)f](0) \\ &= \mathcal{F}\left[(\lambda(y)f)^\wedge\right](0) \\ &= \mathcal{F}\left[\hat{f}e_{-y}\right](0) \\ &= \int \hat{f}(\omega)e_{-y}(\omega)\, d\omega \\ &= \mathcal{F}^2 f(y)\,. \end{aligned}$$

□

Theorem 3.10. *$\mathcal{F}$ is a topological algebraic isomorphism of $\mathcal{S}(\mathbb{R}^n)$ with inverse $^\vee \circ \mathcal{F}$.*

Proof. By Corollary 3.5, we know by now that $\mathcal{F}$ is a continuous linear transformation of $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. By Proposition 3.9 we have

$$\mathcal{F}^2 f = f^\vee$$

or $(^\vee \circ \mathcal{F}) \circ \mathcal{F} = \mathrm{id}$. Hence $^\vee \circ \mathcal{F}$ is a left-inverse for $\mathcal{F}$. By part (g) of Lemma 3.3, $^\vee \circ \mathcal{F} = \mathcal{F} \circ {}^\vee$; it thus follows that $^\vee \circ \mathcal{F}$ is also a right inverse. The theorem follows now since by Proposition 2.58, $^\vee : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a topological isomorphism. □

Let f be a Schwartz function on $\mathbb{R}^m \times \mathbb{R}^n$. Then the **partial Fourier transform** $\mathcal{F}_1 f$ is defined by

$$\mathcal{F}_1 f(\omega, y) = \int_{\mathbb{R}^m} f(x, y) e^{-2\pi i x \cdot \omega}\, dx$$

for $\omega \in \mathbb{R}^m$. It is easy to check this function is smooth and by Exercise 3.2.1 one has

$$p(D)\mathcal{F}_1 f(\omega, y) = \mathcal{F}_1(p(-2\pi i \cdot) f)(\omega, y) \text{ and} \tag{3.5}$$

$$\mathcal{F}_1(p(D) f)(\omega, y) = p(2\pi i \omega)\mathcal{F}_1 f(\omega, y) \tag{3.6}$$

for any polynomial p in x.

Theorem 3.11. *The partial Fourier transform $\mathcal{F}_1$ is a linear homeomorphism of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ onto $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$.*

Proof. We show continuity. Let $f_k \to 0$ in Schwartz space. Then $(-ix)^\alpha D_y^\beta f_k \to 0$ in Schwartz space.

Using Equations 3.5 and 3.6 we have

$$(1 + |(w, y)|^2)^N |D^{(\alpha,\beta)}(\mathcal{F}_1 f)(\omega, y)| = (1 + |\omega|^2 + |y|^2)^N |\mathcal{F}_1((-2\pi i x)^\alpha D^\beta f)(\omega, y)|$$

$$= |\mathcal{F}_1((1 + \frac{1}{4\pi^2}\left\{\left(-i\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(-i\frac{\partial}{\partial x_n}\right)^2\right\} + |y|^2)(-2\pi i x)^\alpha D^\beta f)(w, y)|$$

which is bounded in (ω, y). Thus $\mathcal{F}_1 f$ is Schwartz. Hence it suffices to show if $h_k \to 0$ in Schwartz space,

$$|\mathcal{F}_1(h_k)|_\infty \to 0.$$

But

$$|\mathcal{F}_1 h_k(\omega, y)| \leqslant \int |h_k(x, y)|(1 + |x|^2 + |y|^2)^M (1 + |x|^2 + |y|^2)^{-M}\, d_n x|$$

$$\leqslant |h_k|_{M,0} \int \frac{1}{(1 + |x|^2)^M}\, d_n x \to 0$$

as $k \to \infty$ for $M > n/2$. □

Example 3.12. ***(Continuation of Example 3.2)*** *Again we take* $h(x) = e^{-\pi|x|^2}$ *and define functions* h_t *by*

$$h_t(x) = t^{-n/2}h(x/\sqrt{t}) = t^{-n/2}e^{-\pi|x|^2/t}\,.$$

Then by Theorem 2.82, $\{h_t\}_{t>0}$ *is an approximate identity on* $L^p(\mathbb{R}^n)$ *for* $1 \leqslant p < \infty$*; i.e.,* $\lim_{t\to 0+}|h_t * f - f|_p = 0$ *for each* $f \in L^p$*. Moreover,*

$$\mathcal{F}h_t(x) = t^{-n/4}\mathcal{F}(\delta(\sqrt{t})h)(x) = t^{-n/4}\left(\delta\left(\frac{1}{\sqrt{t}}\right)\hat{h}\right)(x) = \hat{h}(\sqrt{t}x) = e^{-\pi t|x|^2}.$$

Hence if we define linear operators $H_t : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *by*

$$H_t(f)(x) := h_t * f(x)\,,$$

then by Lemma 2.77

$$||H_t|| \leqslant 1\,.$$

Thus the fact that $\{h_t\}$ *is an approximate identity on* L^p *is equivalent to* $H_t \to id$ *in the strong operator topology as* $t \to 0+$ *(the strong operator topology is defined in Exercise 2.2.23). Moreover, since* $\mathcal{F}(h_t)(\omega) = e^{-\pi t|\omega|^2}$*, one has:*

$$\begin{aligned}\widehat{h_s * h_t}(\omega) &= e^{-s|\omega|^2}e^{-t|\omega|^2}\\ &= e^{-(s+t)|\omega|^2}\\ &= \widehat{h_{t+s}}(\omega)\,.\end{aligned}$$

As the Fourier transform is injective it follows that

$$h_s * h_t = h_{s+t}$$

which implies that

$$H_{s+t} = H_sH_t\,.$$

Thus the family $\{H_t\}_{t>0}$ *is a semigroup of bounded operators on* $L^p(\mathbb{R}^n)$ *converging strongly to the identity as* $t \to 0+$*.*

One of the consequences of the last theorem is a simple proof of the *Riemann-Lebesgue Lemma*:

Theorem 3.13 (Riemann-Lebesgue Lemma)**.** *Let* $f \in L^1(\mathbb{R}^n)$*. Then* $\hat{f} \in C_0(\mathbb{R}^n)$*; that is* $\hat{f}$ *is continuous and*

$$\lim_{|\omega|\to\infty}\hat{f}(\omega) = 0\,.$$

Proof. By Lemma 3.3, we know that $\hat{f}$ is continuous and bounded. Thus we only have to show that $\lim_{|\omega|\to\infty}\hat{f}(\omega) = 0$. Let $\epsilon > 0$. Using Proposition 2.55, we can choose a g in $\mathcal{S}(\mathbb{R}^n)$ such that

$$|f - g|_1 < \epsilon/2\,.$$

As $\hat{g} \in \mathcal{S}(\mathbb{R}^n)$ there is a $R > 0$ such that $|\hat{g}(\omega)| < \epsilon/2$ for all $|\omega| \geqslant R$. But then for $|\omega| \geqslant R$ we have

$$\begin{aligned} |\hat{f}(\omega)| &\leqslant \left|\hat{f}(\omega) - \hat{g}(\omega)\right| + |\hat{g}(\omega)| \\ &\leqslant |f - g|_1 + |\hat{g}(\omega)| \\ &< \epsilon\,. \end{aligned}$$

□

3. Inversion for L^1 Functions

In this section we show that for an L^1 function f, one can obtain $f(x)$ by the inversion formula if the Fourier transform of f is L^1 or we can obtain $f(x)$ from an improper Riemann integral when f is well behaved near x.

Theorem 3.14. *Let $f \in L^1(\mathbb{R}^n)$ and suppose $\hat{f} \in L^1(\mathbb{R}^n)$. Then f is continuous and $\hat{\hat{f}}(-x) = f(x)$ for all x.*

Proof. Using Lemma 3.7 twice, we see if $\phi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\begin{aligned} \int f(x)\hat{\hat{\phi}}(x)\,dx &= \int \hat{\phi}(y)\hat{f}(y)\,dy \\ &= \int \phi(x)\hat{\hat{f}}(x)\,dx. \end{aligned}$$

But by Proposition 3.9, $\hat{\hat{\phi}}(x) = \phi(-x)$. Thus

$$\int f(-x)\phi(x)\,dx = \int \phi(x)\hat{\hat{f}}(x)\,dx$$

for all Schwartz functions ϕ. Moreover, by Lemma 2.53, there is a sequence of Schwartz function ϕ_n so that $\phi_n \to \chi_B$ a.e. where B is closed ball in $\mathbb{R}^n$ and the supports of all ϕ_n are contained in B', a ball containing B. Thus

$$\int_B (\hat{\hat{f}}(x) - f(-x))\,dx = 0$$

for all closed balls B. This implies $\hat{\hat{f}}(x) = f(-x)$ for a.e. x. But by Lemma 3.3, $\hat{\hat{f}}$ is continuous. Thus f is continuous. □

Next we attempt to reconstruct $f(x)$ as an improper Riemann integral. Formally, Fourier inversion on $\mathbb{R}$ is given by $f(x) = \int \hat{f}(t)e^{2\pi itx}\,dt$ but in what sense can one interpret this integral?

For a Schwartz function f the answer is clear. One has

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i t x}\, dt \\ &= \lim_{N\to\infty} \int_{-N}^{N} \int f(y) e^{-2\pi i y t}\, dy\, e^{2\pi i t x}\, dt \\ &= \lim_{N\to\infty} \int \int_{-N}^{N} f(y) e^{2\pi i t(x-y)}\, dt\, dy \\ &= \lim_{N\to\infty} \int f(y) \int_{-N}^{N} e^{2\pi i t(x-y)}\, dt\, dy \\ &= \lim_{N\to\infty} D_N * f(x) \end{aligned}$$

where

$$D_N(x) = \int_{-N}^{N} e^{2\pi i t x}\, dt.$$

Now

$$D_N(x) = \frac{e^{2\pi i t x}}{2\pi i x}\Big|_{t=-N}^{t=N} = \frac{e^{2\pi i N x} - e^{-2\pi i N x}}{2\pi i x} = \frac{\sin 2\pi N x}{\pi x}. \tag{3.7}$$

The function D_N is called the **Dirichlet kernel**. See Exercise 3.2.11 for some further properties.

Lemma 3.15. *The improper Riemann integral*

$$\lim_{N\to\infty} \int_{-N}^{N} \frac{\sin t}{t}\, dt = 2 \int_{0}^{\infty} \frac{\sin t}{t}\, dt = \alpha$$

exists.

We shall see shortly that $\alpha = \pi$.

Proof.

$$\int_0^\infty \frac{\sin t}{t}\, dt = \sum_{n=1}^{\infty} \int_{(n-1)\pi}^{n\pi} \frac{\sin t}{t}\, dt$$

is an alternating series with terms going to 0. □

This lemma along with a change of variables gives the following improper integral.

$$\int_0^\infty D_N(x)\, dx = \frac{\alpha}{2\pi}$$

Indeed,

$$\begin{aligned}
\int_0^\infty D_N(x)\,dx &= \int_0^\infty \frac{\sin(2\pi Nx)}{\pi x}\,dx \\
&= \int_0^\infty \frac{\sin x}{\pi(x/2\pi N)}\,d(x/2\pi N) \\
&= \int_0^\infty \frac{\sin x}{\pi x}\,dx \\
&= \frac{\alpha}{2\pi}.
\end{aligned}$$

The Riemann-Lebesgue Lemma implies

$$\lim_{a\to\infty}\int f(x)\sin ax\,dx = 0$$

for $f \in L^1(\mathbb{R})$. These provide the central points in the proof of the following theorem.

Theorem 3.16. *Suppose $f \in L^1(\mathbb{R})$ and $f(x+)$ and $f(x-)$ exist and there is an $\alpha > 0$ and a $\delta > 0$ such that*

$$|f(x+h) - f(x+)| < Mh^\alpha$$

and

$$|f(x-h) - f(x-)| < Mh^\alpha$$

for $0 < h < \delta$. Then

$$\int_{-N}^N \hat{f}(y)e^{2\pi ixy}\,dy = D_N * f(x) \to \frac{1}{2}(f(x+) + f(x-))$$

as $N \to \infty$.

Proof.

$$\begin{aligned}
D_N * f(x) &= \int_{-\infty}^\infty D_N(t)f(x-t)\,dt \\
&= \int_0^\infty D_N(t)f(x-t)\,dt + \int_{-\infty}^0 D_N(t)f(x-t)\,dt \\
&= \int_0^\infty D_N(t)f(x-t)\,dt + \int \chi_{[-\infty,0]}(-t)D_N(-t)f(x+t)\,dt \\
&= \int_0^\infty D_N(t)(f(x+t) + f(x-t))\,dt.
\end{aligned}$$

We now show

$$\lim_{N\to\infty}\int_0^\infty D_N(t)f(x+t)\,dt = \frac{\alpha}{2\pi}f(x+)$$

and by a similar argument

$$\lim_{N\to\infty}\int_0^\infty D_N(t)f(x-t)\,dt = \frac{\alpha}{2\pi}f(x-).$$

Once this has been done, we need only verify $\alpha = \pi$. Note we have

$$\begin{aligned}
&\left|\int_0^\infty D_N(t)f(x+t)\,dt - \frac{\alpha}{2\pi}f(x+)\right| = \left|\int_0^\infty D_N(t)(f(x+t)-f(x+))\,dt\right| \\
&\leqslant \left|\int_0^b \frac{\sin 2\pi Nt}{\pi t}(f(x+t)-f(x+))\,dt\right| + \left|\int_b^\infty \frac{\sin 2\pi Nt}{\pi t}(f(x+t)-f(x+))\,dt\right| \\
&\leqslant \left|\int_0^b \frac{\sin 2\pi Nt}{\pi t}(f(x+t)-f(x+))\,dt\right| + \left|\int_b^\infty \frac{\sin 2\pi Nt}{\pi t}f(x+t)\,dt\right| \\
&\quad + \left|\int_b^\infty \frac{\sin 2\pi Nt}{\pi t}f(x+)\,dt\right|
\end{aligned}$$

Let $\epsilon > 0$. Choose $b > 0$ such that $|f(x+)\int_b^\infty \frac{\sin 2\pi Nt}{\pi t}\,dt| < \frac{\epsilon}{4}$. Next pick $b > \delta > 0$ so that $|f(x+t) - f(x+)| < Mt^\alpha$ if $0 < t < \delta$. Then if $\psi(t) = \frac{f(x+t)-f(x+)}{\pi t}$, we have

$$\int_0^b \frac{\sin 2\pi Nt}{\pi t}(f(x+t)-f(x+))\,dt = \int \chi_{(0,\delta)}(t)\psi(t)\sin 2\pi Nt\,dt + \int \chi_{(\delta,b)}(t)\psi(t)\sin 2\pi Nt\,dt.$$

Hence

$$\begin{aligned}
&\left|\int_0^\infty D_N(t)f(x+t)\,dt - \frac{\alpha}{2\pi}f(x+)\right| \leqslant |\int \chi_{(0,\delta)}(t)\,\psi(t)\sin 2\pi Nt\,dt| \\
&\quad + \left|\int \chi_{(\delta,b)}(t)\,\psi(t)\sin 2\pi Nt\,dt\right| + \left|\int_b^\infty \frac{f(x+t)}{\pi t}\sin 2\pi Nt\,dt\right| + \frac{\epsilon}{4}.
\end{aligned}$$

Now note $\chi_{(0,\delta)}(t)|\psi(t)| \leqslant \chi_{(0,\delta)}(t)\frac{Mt^\alpha}{\pi t} = \frac{M}{\pi}\chi_{(0,\delta)}(t)t^{\alpha-1}$, $\chi_{(\delta,b)}(t)\psi(t)$, and $x_{(b,\infty)}(t)\frac{f(x+t)}{t}$ are all integrable in t. Hence the Lebesgue Lemma implies we can find an N_0 such that for $N \geqslant N_0$ we have

$$|\int \chi_{(0,\delta)}(t)\,\psi(t)\sin 2\pi Nt\,dt| + \left|\int \chi_{(\delta,b)}(t)\,\psi(t)\sin 2\pi Nt\,dt\right| + \left|\int_b^\infty \frac{f(x+t)}{\pi t}\sin 2\pi Nt\,dt\right| < \frac{3\epsilon}{4}.$$

Thus for $N \geqslant N_0$, $\left|\int_0^\infty D_N(t)f(x+t)\,dt - \frac{\alpha}{2\pi}f(x+)\right| < \epsilon$.

To finish, we note if f is an always positive Schwartz function, then

$$f(x) = \frac{1}{2}(f(x+)+f(x-)) = \lim_{N\to\infty} D_N * f(x) = \frac{\alpha}{2\pi}(f(x+)+f(x-)).$$

This implies $\alpha = \pi$. □

Corollary 3.17. *Suppose $f \in L^1(\mathbb{R})$ and the limits $f(x+)$, $f(x-)$, $f'(x+) := \lim\limits_{h\to 0+}\frac{f(x+h)-f(x+)}{h}$, and $f'(x-) := \lim\limits_{h\to 0-}\frac{f(x+h)-f(x-)}{h}$ exist. Then*

$$\frac{1}{2}(f(x+)+f(x-)) = \lim_{N\to\infty}\int_{-N}^N \hat{f}(y)e^{2\pi ixy}\,dy.$$

Exercise Set 3.2

1. For $f \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$, show the partial Fourier transform $\mathcal{F}_1 f$ given by

$$\mathcal{F}_1 f(\omega, y) = \int f(x,y) e^{-2\pi i x\cdot\omega}\, dx$$

satisfies

$$\begin{aligned} p(D)\mathcal{F}_1 f(\omega, y) &= \mathcal{F}_1(p(-2\pi i\cdot) f)(\omega, y) \text{ and} \\ F_1(p(D)f)(\omega, y) &= p(2\pi i\omega)\mathcal{F}_1 f(\omega, y) \end{aligned}$$

for any polynomial p in x.

2. Show the partial Fourier transform $\mathcal{F}_1$ satisfies properties corresponding to properties (a) to (g) in Exercise 3.2.3.

3. Using

$$\int \left| \int f(x) e^{-2\pi i x\cdot\omega}\, dx \right|^2 d\omega = \int |f(x)|^2\, dx$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, show

$$\int | \int f(x) e^{-aix\cdot\omega} \sqrt{\left(\frac{a}{2\pi}\right)^n}\, dx|^2 \sqrt{\left(\frac{a}{2\pi}\right)^n} d\omega = \int |f(x)|^2 \sqrt{\left(\frac{a}{2\pi}\right)^n}\, dx$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ and $a > 0$. Now define

$$\mathcal{F}_a(f)(\omega) = \int f(x) e^{-iax\cdot\omega}\, d_a x$$

where $d_a x$ is the measure $\sqrt{\left(\frac{a}{2\pi}\right)^n}\, dx$. Show $\mathcal{F}_a$ is a linear homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ that satisfies:

(a) Show $\mathcal{F}_a^2 f = \check{f}$ for $f \in \mathcal{S}(\mathbb{R}^n)$.

(b) Show $\mathcal{F}_a$ extends to a unitary transformation of the Hilbert space $L^2(\mathbb{R}^n, d_a x)$.

(c) Show $\mathcal{F}_a(\lambda(y)f)(\omega) = e^{iay\omega}\mathcal{F}_a(f)(\omega)$ for $f \in \mathcal{S}(\mathbb{R}^n)$

(d) Show $\mathcal{F}_a(\delta(b)f) = \delta(\frac{1}{b})\mathcal{F}_a(f)$ for $f \in \mathcal{S}(\mathbb{R}^n)$

(e) Show $\mathcal{F}_a(f^*) = \overline{\mathcal{F}_a(f)}$ for $f \in \mathcal{S}(\mathbb{R}^n)$

(f) Show $\mathcal{F}_a(p(D)f)(\omega) = p(ai\omega)\mathcal{F}_a f(\omega)$ for $f \in \mathcal{S}(\mathbb{R}^n)$

(g) Show $\mathcal{F}_a(pf)(\omega) = p(\frac{iD}{a})\mathcal{F}_a(f)(\omega)$ for polynomials p and $f \in \mathcal{S}(\mathbb{R}^n)$.

The important cases are $a = 1$ and $a = 2\pi$.

4. Let f be Schwartz on $\mathbb{R}^m \times \mathbb{R}^n$. Show $y \mapsto f(x,y) = f_x(y)$ is Schwartz on $\mathbb{R}^n$ for each $x \in \mathbb{R}^m$.

5. Show $x \mapsto f_x$ is continuous from $\mathbb{R}^m$ into $\mathcal{S}(\mathbb{R}^n)$ for each $f \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$.

6. Let $f \in L^1(\mathbb{R}^n)$ be such that

$$\int_{\mathbb{R}^n} |x|^k \, |f(x)| \, dx < \infty.$$

Then $D^\alpha \widehat{f}(\omega)$ exists for all α with $|\alpha| \leqslant k$ and

$$D^\alpha \hat{f}(\omega) = (-2\pi i)^{|\alpha|} \int x^\alpha f(x) \, e^{-2\pi i x \cdot \omega} \, dx.$$

7. Suppose f is C^∞ and has support inside $\{x \mid |x| \leqslant R\}$. Define $\hat{f}(z) = \int f(x) e^{-2\pi i z \cdot x} \, dx$. Show $\hat{f}(z)$ is a holomorphic function and show for each $N \in \mathbb{N}$, there is a constant C_N such that

$$|\hat{f}(z)| \leqslant C_N (1 + |z|^2)^{-N} e^{2\pi R |\mathrm{Im}(z)|}$$

for all $z \in \mathbb{C}^n$.

8. Let $f \in L^1(\mathbb{R})$ and let $h_t(y) = \frac{1}{t} e^{-\pi y^2/t^2}$ be the heat kernel. Show if

$$\lim_{r \to 0+} \frac{1}{2r} \int_{-r}^{r} (f(x-y) - f(x)) \, dy = 0,$$

then

$$\lim_{t \to 0+} h_t * f(x) = f(x).$$

Hint: Consider $\int_{-T}^{T} h_t'(y) F(y) \, dy$ where $F(y) = \int_{-y}^{y} (f(x-z) - f(x)) \, dz$.

9. Let $f \in L^1(\mathbb{R}^n)$. Show that

$$\lim_{t \to 0} H_t(f)(x) = f(x)$$

for all $x \in \mathbb{R}^n$ such that

$$\lim_{r \to 0} \frac{1}{r^n} \int_{|x-y| \leqslant r} (f(y) - f(x)) \, dy = 0.$$

10. Let $f \in L^1(\mathbb{R}^n)$ and assume that $\hat{f} \in L^1(\mathbb{R}^n)$. Use H_t to show that

$$f(x) = \int \hat{f}(\omega) e^{2\pi i x \cdot \omega} \, d\omega$$

for almost all $x \in \mathbb{R}^n$. In particular f is continuous almost everywhere. Note this gives another proof of Theorem 3.14.

11. Recall from (3.7) that the Dirichlet kernel is given by

$$D_N(u) = \int_{-N}^{N} e^{2\pi i u x} \, dx = \frac{\sin 2\pi N x}{\pi x} \quad \text{for } N > 0.$$

Let $f \in L^2(\mathbb{R})$.

(a) Show $D_N * f(x)$ is defined for all x and gives a continuous function.

(b) Show $D_N * f(x) = \int_0^\infty D_N(t)(f(x+t) + f(x-t)) \, dt$.

(c) Show $D_N * f$ is in $L^2(\mathbb{R})$ and $D_N * f \to f$ in $L^2(\mathbb{R})$ as $N \to \infty$.

(d) Show $D_N(t) = \mathcal{F}(\chi_{[-N,N]})$, this function is sometimes called the sinc_N function.

12. Suppose f is a real valued integrable function on $\mathbb{R}$ and suppose $f(x+)$, $f(x-)$, $f'(x+)$, and $f'(x-)$ exist. Define

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos px\,dx \text{ and } B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin px\,dx.$$

Show

$$\frac{1}{2}(f(x+) + f(x-)) = \lim_{N\to\infty}\int_0^N A(p)\cos px + B(p)\sin px\,dp.$$

4. The Fourier Transform on $L^2(\mathbb{R}^n)$ and its Spectral Decomposition

In the previous section we showed that the Fourier transform is a topological isomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto itself. We will show that this transform has a natural extension of order 4 to the Hilbert space $L^2(\mathbb{R}^n)$ onto itself. Then we will look at its spectral decomposition relative to a particularly well behaved basis of eigenfunctions. Indeed, the Hermite operator $4\pi x^2 - \frac{d^2}{dx^2}$ commutes with the Fourier transform and has discrete spectrum with eigenspaces of dimension one. The orthonormal basis formed by these eigenfunctions are eigenvectors for the Fourier transform and thus give a spectral resolution of the Fourier transform. Moreover, as we shall see in the next chapter, an L^2 function is Schwartz if and only if its Fourier coefficients relative to the Hermite basis are rapidly decreasing. This will give another characterization of $\mathcal{S}(\mathbb{R}^n)$.

Recall (see Exercise 3.3.1) if U is a densely defined linearly transformation from a Hilbert space $\mathcal{H}$ having dense range in Hilbert space $\mathcal{H}'$ and $(Uu, Uv)' = (u, v)$ for all $u, v \in \mathrm{Dom}(U)$, then U extends uniquely to a unitary isomorphism from $\mathcal{H}_1$ onto $\mathcal{H}_2$.

Lemma 3.18. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$(\hat{f}, \hat{g})_2 = (f, g)_2\,.$$

Proof. Recall $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. Using Lemma 3.7 followed by (f) of Lemma 3.3 and the inversion formula $\hat{\hat{h}} = \check{h}$ for Schwartz functions h, we see

$$\begin{aligned}(\widehat{f}, \widehat{g})_2 &= \int \widehat{f}(x)\overline{\widehat{g}(x)}\, dx \\ &= \int f(x)\, \mathcal{F}(\overline{\widehat{g}})(x)\, dx \\ &= \int f(x) \left(\widehat{\widehat{g}}\right)^* (x)\, dx \\ &= \int f(x)\, (\check{g})^* (x)\, dx \\ &= \int f(x)\overline{g(x)}\, dx \\ &= (f, g)_2.\end{aligned}$$

□

Theorem 3.19 (Plancherel)**.** *The mapping $\mathcal{F}$ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ extends to a unitary isometry of L^2 onto L^2. We denote this extension again by $\mathcal{F}$ or $f \mapsto \hat{f}$. It satisfies $\hat{\hat{f}} = \check{f}$ for $f \in L^2$.*

Proof. The first statement follows the density of $\mathcal{S}(\mathbb{R}^n)$ in L^2, Theorem 3.10, Lemma 3.18 and Exercise 3.3.1 The second is a consequence of Proposition 3.9. □

Remark 3.20. Notice that if $f \in L^2(\mathbb{R}^n)$, then the Fourier transform of f is **not necessarily** given by an integral of the form

$$\int f(x)e^{-2\pi i x\cdot\omega}\, dx$$

as we do **not know if this integral exists**. In fact the Fourier transform of f is the L^2 limit of any sequence $\{\hat{f}_k\}_{k=1}^{\infty}$ where $f_k \in \mathcal{S}(\mathbb{R}^n)$ converges in L^2 to f.

We have seen that the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary isomorphism of order 4, i.e., $\mathcal{F}^4 = \mathrm{id}$. It follows that the only possible eigenvalues of $\mathcal{F}$ are $1, i, -1$, and $-i$. Furthermore since $\mathcal{F}^2(f)(x) = f(-x)$, $\mathcal{F}^2(f) = f$ if and only if f is even and $\mathcal{F}^2(f) = -f$ if and only if f is odd. Therefore eigenfunctions with eigenvalues ± 1 have linear span the even functions in L^2 and the eigenfunctions corresponding to $\pm i$ have linear span the odd functions. Let A be the set of all n-tuples of nonnegative integers. We will next construct a multi-indexed sequence of polynomials H_α, $\alpha \in A$, such that the functions $h_\alpha(x) = H_\alpha(x)e^{-\pi|x|^2}$, $\alpha \in A$, are an orthogonal basis for $L^2(\mathbb{R}^n)$ consisting of eigenfunctions for the Fourier transform. It is also clear since H_α are polynomial functions that the h_α

are in $\mathcal{S}(\mathbb{R}^n)$. The functions h_α are not just eigenfunctions for the Fourier transform but are also in fact eigenfunctions for an unbounded symmetric operator H commuting with the Fourier transform. This operator H is a slight renormalization of the usual Hermite operator

$$|x|^2 - \Delta^2.$$

We will call the polynomials H_α *the Hermite polynomials* and the functions h_α the *Hermite functions* although because of our normalization these differ slightly from the standard definitions.

The Hermite polynomials H_α can be defined in terms of the following *generating function*:

$$e^{-2\pi(|t|^2-2t\cdot x)} = e^{2\pi|x|^2}e^{-2\pi|x-t|^2} = \sum_{\alpha\in A} H_\alpha(x)\frac{t^\alpha}{\alpha!}\,.$$

This series converges for all x and t and defines each function H_α uniquely. Furthermore,

$$H_\alpha(x) = D_t^\alpha e^{-2\pi(|t|^2-2t\cdot x)}|_{t=0} = e^{2\pi|x|^2}D_t^\alpha e^{-2\pi|x-t|^2}|_{t=0}\,,$$

where the subscript t indicates a differentiation with respect to the variable t. As

$$D_t^\alpha e^{-2\pi|x-t|^2}|_{t=0} = (-1)^{|\alpha|}D_x^\alpha e^{-2\pi|x|^2},$$

we obtain the lemma:

Lemma 3.21. *Let $\alpha \in A$. Then*

$$H_\alpha(x) = (-1)^{|\alpha|}e^{2\pi|x|^2}D^\alpha e^{-2\pi|x|^2}\,.$$

The polynomial $H_\alpha(x)$ is called *the Hermite polynomial* of order α while the function

$$h_\alpha(x) = H_\alpha(x)e^{-\pi|x|^2} \tag{3.8}$$

is the corresponding *Hermite function*. The Hermite functions are in $\mathcal{S}(\mathbb{R}^n)$.

Remark 3.22. Usually the Hermite polynomials are defined by

$$(-1)^{|\alpha|}e^{|x|^2/2}D^\alpha e^{-|x|^2/2},$$

a difference which comes from the **normalization** of the Fourier transform. We refer to Exercise 3.3.6 where the different normalizatons are compared.

Lemma 3.23. $h_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j)$.

Proof. This follows from $D^\alpha = D^{\alpha_1}\cdots D^{\alpha_n}$ and $e^{-\pi|x|^2} = \prod e^{-\pi x_j^2}$. □

Lemma 3.24. *The Hermite polynomial H_α has form*

$$H_\alpha(x) = (4\pi)^{|\alpha|}x^\alpha + \sum_{\beta<\alpha} c_{\alpha,\beta}x^\beta.$$

Proof. This is clear if $|\alpha| = 0$ and for $\alpha + e_i$, we have

$$\begin{aligned}
H_{\alpha+e_i}(x) &= (-1)^{|\alpha|+1} e^{2\pi|x|^2} D_{e_i} D^\alpha e^{-2\pi|x|^2} \\
&= -e^{2\pi|x|^2} D_{e_i} \left(H_\alpha(x) e^{-2\pi|x|^2} \right) \\
&= -e^{2\pi|x|^2} \left((D_{e_i} H_\alpha(x)) e^{-2\pi|x|^2} + H_\alpha(x) D_{e_i} e^{-2\pi|x|^2} \right) \\
&= -e^{2\pi|x|^2} \left(\sum_{\beta<\alpha} c'_{\alpha,\beta} x^\beta \, e^{-2\pi|x|^2} + H_\alpha(x)(-4\pi x_i) e^{-2\pi|x|^2} \right) \\
&= (4\pi)^{|\alpha+e_i|} x^\alpha + \sum_{\beta<\alpha+e_i} c_{\alpha+e_i,\beta} x^\beta.
\end{aligned}$$

□

Corollary 3.25. *Every polynomial on $\mathbb{R}^n$ can be written as a linear combination of the Hermite polynomials.*

Proof. Induction shows each x^α is in the linear span of the $H_\beta(x)$ where $\beta \leqslant \alpha$. □

Lemma 3.26. *Let $n = 1$. Then for $k \geqslant 0$*

$$H_{2k}(0) = \frac{(-2\pi)^k (2k)!}{k!} \quad \textit{and} \quad H_{2k+1}(0) = 0.$$

Proof. Setting $x = 0$ in the generating formula gives

$$\begin{aligned}
e^{-2\pi t^2} &= \sum_{n=o}^{\infty} \frac{(-2\pi t^2)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(-2\pi)^n (2n)!}{n!} \frac{t^{2n}}{(2n)!} \\
&= \sum_{k=0}^{\infty} H_k(0) \frac{t^k}{k!}\,.
\end{aligned}$$

This obviously yields the lemma. □

For $n = 1$ the first few Hermite polynomials are given by:

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 4\pi x \\
H_2(x) &= 16\pi^2 x^2 - 4\pi \\
H_3(x) &= 64\pi^3 x^3 - 48\pi^2 x
\end{aligned}$$

Our next aim is to show the Hermite polynomials can be obtained naturally by recursion. If E and F are two differential operators, then their *commutator* $[E, F]$ is given by

$$[E, F] = EF - FE\,.$$

If E and F have constant coefficients, then $[E, F] = 0$, for $\partial_i \partial_j = \partial_j \partial_i$. Furthermore, note one always has

$$[E, F] = -[F, E]\,.$$

If at least one of the two differential operators has variable coefficients, then they may not commute. This can be seen easily using

$$\partial_j(x_j f) = f + x_j \partial_j f\,.$$

This behavior can be formulated more generally in terms of multiplication operators. We recall a multiplication operator has form $M_g(f)(x) = g(x)f(x)$ where g is a function. The g used in the following lemma is assumed to be smooth.

Lemma 3.27. *Let $\Delta = D_1^2 + \ldots + D_n^2$ be the Laplacian on $\mathbb{R}^n$. Then:*

(a) $[D^\alpha, D^\beta] = [M_g, M_h] = 0$

(b) $[D_j, M_g] = M_{D_j g}$

(c) $[D^\alpha, M_{x_j}] = \alpha_j D^{\alpha - e_j}$

(d) $[\Delta, M_g] = M_{\Delta g} + 2\sum M_{D_j g} D_j$.

In particular,

(e) $\left[\Delta, M_{x_j}\right] = 2D_j$

(f) $\left[D_j, M_{x_j^2}\right] = 2x_j$.

Proof. Note (a) is immediate and (b) follows by a simple calculation using the product rule.

For (c), note from Leibniz's Rule, Lemma 2.21, we have

$$\begin{aligned} D^\alpha(x_i f) &= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta x_i \cdot D^{\alpha-\beta} f \\ &= x_i D^\alpha f + \alpha_i D^{\alpha - e_i} f \end{aligned}$$

for $D^\beta x_i = 0$ for $\beta \neq e_i$ or $\beta \neq 0$. Thus

$$\begin{aligned} [D^\alpha, M_{x_i}]f &= (D^\alpha(x_i f) - x_i D^\alpha f) \\ &= x_i D^\alpha f + \alpha_i D^{\alpha - e_i} f - x_i D^\alpha f \\ &= \alpha_i D^{\alpha - e_i} f. \end{aligned}$$

For (d), we first calculate $[D_j^2, M_g]$.

$$\begin{aligned}
[D_j^2, M_g]f &= D_j^2(gf) - gD_j^2 f \\
&= D_j((D_j g)f + gD_j f) - gD_j^2 f \\
&= (D_j^2 g)f + (D_j g)(D_j f) + (D_j g)(D_j f) + gD_j^2 f - gD_j^2 f \\
&= (D_j^2 g)f + 2(D_j g)(D_j f).
\end{aligned}$$

Thus $[D_j^2, M_g] = M_{D_j^2 g} + 2M_{D_j g}D_j$. Consequently,

$$[\Delta, M_g] = \sum M_{D_j^2 g} + 2\sum M_{D_j g}D_j = M_{\Delta g} + 2\sum_{j=1}^{n} M_{D_j g}D_j.$$

Finally (e) follows from (d) and (f) follows from (b). $\square$

Lemma 3.28. *Let $\alpha \in A$. Then the following hold:*

(a) $\partial_j H_\alpha(x) = 4\pi\alpha_j H_{\alpha - e_j}(x)$ *;*

(b) $H_{\alpha + e_j}(x) = (4\pi x_j - D_j)H_\alpha(x)$ *. In particular the Hermite polynomials are defined by the recursion formula*

$$\begin{aligned}
H_0(x) &= 1 \\
H_{\alpha+e_j}(x) &= 4\pi x_j H_\alpha(x) - D_j H_\alpha(x);
\end{aligned}$$

(c) $(2\pi x_j - D_j)h_\alpha(x) = h_{\alpha+e_j}(x)$.

(d) $(2\pi x_j + D_j)h_\alpha(x) = 4\pi\alpha_j h_{\alpha-e_j}(x)$.

Proof. For (a) using the Leibniz Rule, one has:

$$\begin{aligned}
\partial_j H_\alpha(x) &= \partial_{x,j} D_t^\alpha e^{-2\pi(|t|^2 - 2t\cdot x)}|_{t=0} \\
&= D_t^\alpha \left(4\pi t_j e^{-2\pi|t|^2 + 4\pi t\cdot x}\right)|_{t=0} \\
&= 4\pi\alpha_j D_t^{\alpha - e_j} e^{-\pi|t|^2 + 2\pi t\cdot x}|_{t=0} \\
&= 4\pi\alpha_j H_{\alpha - e_j}(x).
\end{aligned}$$

Using the product formula for differentiation,

$$\begin{aligned}
D_j H_\alpha(x) &= (-1)^{|\alpha|}(D_j e^{2\pi|x|^2} D^\alpha e^{-2\pi|x|^2}) \\
&= (-1)^{|\alpha|}\left(4\pi x_j e^{\pi|x|^2} D^\alpha e^{-\pi|x|^2} + e^{\pi|x|^2} D_j D^\alpha e^{-\pi|x|^2}\right) \\
&= 4\pi x_j H_\alpha(x) - H_{\alpha+e_j}(x).
\end{aligned}$$

This gives (b).

Now (c) follows from the product rule and (b). Indeed,

$$\begin{aligned}(2\pi x_j - D_j)h_\alpha(x) &= (2\pi x_j - D_j)e^{-\pi|x|^2}H_\alpha(x)\\ &= 4\pi x_j e^{-\pi|x|^2}H_\alpha(x) - e^{-\pi|x|^2}D_jH_\alpha(x)\\ &= e^{-\pi|x|^2}(4\pi x_j - D_j)H_\alpha(x)\\ &= e^{-\pi|x|^2}H_{\alpha+e_j}(x)\\ &= h_{\alpha+e_j}(x).\end{aligned}$$

Finally for (d),

$$\begin{aligned}(2\pi x_j + D_j)h_\alpha(x) &= (2\pi x_j + D_j)e^{-\pi|x|^2}H_\alpha(x)\\ &= e^{-\pi|x|^2}D_jH_\alpha(x)\\ &= e^{-\pi|x|^2}(4\pi\alpha_j H_{\alpha-e_j}(x))\\ &= 4\pi\alpha_j h_{\alpha-e_j}(x).\end{aligned}$$

□

Lemma 3.29. *Let $E_j = 2\pi x_j - D_j$. Then $\mathcal{F}\circ E_j = -iE_j\circ\mathcal{F}$.*

Proof. This follows from Theorem 3.4. □

Proposition 3.30. *Let $\alpha\in A$. Then $\mathcal{F}(h_\alpha) = (-i)^{|\alpha|}h_\alpha$.*

Proof. Assume first that $\alpha = 0$. Then $h_\alpha(x) = e^{-\pi|x|^2}$ and according to Example 3.2 we have $\mathcal{F}(e^{-\pi|x|^2}) = e^{-\pi|\omega|^2}$. Hence the Theorem holds for $|\alpha| = 0$.

Assume that the claim holds for α. To do induction, we show it holds for $\alpha + e_j$. But by the last Lemma and (c) of Lemma 3.28, we have

$$\begin{aligned}F(h_{\alpha+e_j}) &= F((2\pi x_j - D_j)h_\alpha)\\ &= F\circ E_j(h_\alpha)\\ &= -iE_j\circ F(h_\alpha)\\ &= -iE_j((-i)^{|\alpha|}h_\alpha)\\ &= (-i)^{|\alpha+e_j|}h_{\alpha+e_j}.\end{aligned}$$

□

We now define the *Hermite differential operator* H by

$$H = 4\pi^2 M_{|x|^2} - \Delta \tag{3.9}$$

The Hermite differential operator maps $\mathcal{S}(\mathbb{R}^n)$ into itself. Hence H is a densely defined differential operator on $L^2(\mathbb{R}^n)$. Notice that

$$H = \sum_{j=1}^{n} H_j$$

where H_j is the Hermite operator in "one dimension" $H_j = 4\pi^2 x_j^2 - D_j^2$. The Hermite operator is special because it commutes with the Fourier transform.

Lemma 3.31. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ then*

$$\mathcal{F}(Hf) = H(\mathcal{F}f)\,.$$

Proof. By Theorem 3.4 we have $\mathcal{F}(pf) = p(iD/2\pi)\mathcal{F}f$ and $\mathcal{F}(p(D)f) = p(2\pi i\omega)\mathcal{F}f(\omega)$ for any polynomial p and $\omega \in \mathbb{R}^n$. Let $p(x) = (2\pi)^2|x|^2$ and $q(x) = |x|^2$. Then

$$p(\frac{i}{2\pi}D) = \frac{-4\pi^2}{4\pi^2}\Delta = -\Delta = -q(D)$$

and

$$q(2\pi i\omega) = -p(\omega).$$

Hence

$$\begin{aligned}
\mathcal{F}(Hf)(\omega) &= \mathcal{F}((p - q(D))f)(\omega) \\
&= p(\frac{i}{2\pi}D)\hat{f}(\omega) - q(2\pi i\omega)\hat{f}(\omega) \\
&= H\hat{f}(\omega)\,.
\end{aligned}$$

$\square$

Lemma 3.32. *The Hermite operator is symmetric and positive on $\mathcal{S}(\mathbb{R}^n)$. Thus $(Hf, g)_2 = (f, Hg)_2$ and $(Hf, f)_2 > 0$ when $f \neq 0$.*

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and let $p(x) = 4\pi^2\,|x|^2$ be as in the proof of the last lemma. Using that p is real valued and integrating by parts gives

$$\begin{aligned}
(Hf, g) &= (pf - \Delta f, g) \\
&= \int (p(x)f(x) - \Delta f(x))\overline{g(x)}\,dx \\
&= (f, pg) - \int (\Delta f)(x)\overline{g(x)}\;dx \\
&= (f, pg) - \int f(x)\overline{\Delta g(x)}\,dx \\
&= (f, pg - \Delta g) \\
&= (f, Hg).
\end{aligned}$$

For positivity let $\varphi(x) = 2\pi|x|$. Then $\varphi f \in L^2(\mathbb{R}^n)$ and $\varphi f \neq 0$ unless $f = 0$. Hence

$$\begin{aligned}(Hf, f) &= 4\pi^2 \int |x|^2 f(x)\overline{f(x)}\,dx - \int \sum_{j=1}^{n} [D_j^2 f](x)\overline{f(x)}\,dx \\ &= |\varphi f|_2^2 + \int \sum_{j=1}^{n} D_j f(x)\overline{D_j f(x)}\,dx \\ &= |\varphi f|_2^2 + \sum_{j=1}^{n} |D_j f|_2^2 > 0\end{aligned}$$

if $f \neq 0$. □

Lemma 3.33. *For $j = 1, \ldots, n$, $[H, 2\pi x_j - D_j] = 4\pi(2\pi x_j - D_j)$. Thus:*

$$H(2\pi x_j - D_j) = (2\pi x_j - D_j)(H + 4\pi)\,.$$

Proof. Set $p(x) = 4\pi^2|x|^2$. Lemma 3.27 implies

$$\begin{aligned}[H, 2\pi M_{x_j} - D_j] &= [M_p, 2\pi M_{x_j}] - [M_p, D_j] - [\Delta, 2\pi M_{x_j}] + [\Delta, D_j] \\ &= M_{D_j p} - 2\sum_{k=1}^{n} M_{D_k 2\pi x_j} D_k \\ &= 8\pi^2 x_j - 4\pi D_j \\ &= 4\pi\,(2\pi x_j - D_j)\,.\end{aligned}$$

□

Lemma 3.34. *The space of functions of the form $x \mapsto p(x)e^{-\pi|x|^2}$, where p is a polynomial, is dense in $L^2(\mathbb{R}^n)$.*

Proof. By Exercise 3.3.8 the span of functions with form $x \mapsto \prod_{j=1}^n f_j(x_j)$, $f_j \in L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. Thus we can assume $n = 1$.

We have

$$\int p(x)e^{-\pi x^2} f(x)\,dx = 0$$

for all polynomial functions $p(x)$. Note $e^{-\pi(x+ia)^2} = e^{\pi a^2}e^{-2\pi i a x}e^{-\pi x^2}$. Set

$$p_n(x) = e^{\pi a^2}\sum_{k=1}^{n} \frac{(-2\pi i a)^k}{k!} x^k.$$

For each n we have:

$$\int p_n(x)e^{-\pi x^2} f(x)\,dx = 0.$$

Now

$$
\begin{aligned}
|p_n(x)e^{-\pi x^2} f(x)| &\leqslant e^{\pi a^2} \left(\sum_{k=0}^{n} \frac{(2\pi |ax|)^k}{k!} \right) e^{-\pi x^2} |f(x)| \\
&\leqslant e^{\pi a^2} e^{2\pi |ax|} e^{-\pi x^2} |f(x)|.
\end{aligned}
$$

This later function is integrable for $x \mapsto e^{2\pi|ax|}e^{-\pi x^2}$ is an L^2 function for each a. Thus by the Lebesgue Dominated Convergence Theorem,

$$\int e^{-\pi(x+ia)^2} f(x)\, dx = 0$$

for each $a \in \mathbb{R}$. This implies

$$\int e^{-2\pi iax} e^{-\pi x^2} f(x)\, dx = 0$$

for all a. Consequently, the Fourier transform of the function $x \mapsto e^{-\pi x^2} f(x)$ is zero. Since the Fourier transform is one-to-one (on L^1 or L^2), we have $f(x) = 0$ a.e. x. $\square$

Proposition 3.35. *The Hermite functions h_α, $\alpha \in A$, are eigenfunctions for the Hermite operator $H = 4\pi^2|x|^2 - \Delta$ and span a dense subset of $L^2(\mathbb{R}^n)$. Moreover, the eigenvalue for h_α is $2\pi(2\,|\alpha| + n)$.*

Proof. By Corollary 3.25, every polynomial function on $\mathbb{R}^n$ is a linear combination of the Hermite polynomials $H_\alpha(x)$. Consequently, every function of form $p(x)e^{-\pi x^2}$ is a linear combination of Hermite functions $h_\alpha(x)$. By the previous lemma, we see the Hermite functions span a dense subspace of $L^2(\mathbb{R}^n)$.

Next note $H(h_0)(x) = (4\pi^2\,|x|^2 - \Delta)(e^{-\pi|x|^2}) = 2\pi n e^{-\pi|x|^2} = 2\pi n\, h_0(x)$.

Assume that $H(h_\alpha) = 2\pi(2\,|\alpha| + n)h_\alpha$ for all α with $|\alpha| = k$. Suppose $|\beta| = k + 1$. Then there exists a j such that $\alpha = \beta - e_j \in A$. Obviously $|\alpha| = k$. From Lemmas 3.28 and 3.33 we see

$$
\begin{aligned}
H(h_\beta) &= H(2\pi x_j - D_j)h_\alpha \\
&= (2\pi x_j - D_j)(H + 4\pi)h_\alpha \\
&= (2\pi x_j - D_j)(2\pi(2\,|\alpha| + n) + 4\pi)h_\alpha \\
&= 2\pi(2\,|\alpha + e_j| + n)h_\beta.
\end{aligned}
$$

$\square$

Theorem 3.36. *The functions*

$$e_\alpha = \frac{2^{n/4}}{\sqrt{(4\pi)^{|\alpha|}\alpha!}} h_\alpha,\ \alpha \in A$$

form a complete orthonormal basis of $L^2(\mathbb{R}^n)$ consisting of eigenfunctions of the Fourier transform with eigenvalue $i^{-|\alpha|}$.

Proof. We begin by determining the inner products of the Hermite functions. First recall that $h_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j)$. Hence

$$(h_\alpha, h_\beta) = \prod_{j=1}^n (h_{\alpha_j}, h_{\beta_j}).$$

Thus the inner product of h_α and h_β in $L^2(\mathbb{R}^n)$ is determined by the inner products of the functions h_k and h_l in $L^2(\mathbb{R})$.

Thus we assume $n = 1$. We have $H = 4\pi^2 x^2 - \frac{d^2}{dx^2}$ and $Hh_k = 2\pi(2k + 1)h_k$. Moreover by Lemma 3.32,

$$(Hh_k, h_l) = (h_k, Hh_l).$$

Consequently,

$$2\pi(2k + 1)(h_k, h_l) = 2\pi(2l + 1)(h_k, h_l).$$

Thus when $k \neq l$, we see the inner product (h_k, h_l) is 0. By Lemma 3.28, $(2\pi x - D)h_k = h_{k+1}$. Consequently,

$$\begin{aligned}(h_{k+1}, h_{k+1}) &= ((2\pi x - D)h_k, (2\pi x - D)h_k) \\ &= ((2\pi x + D)(2\pi x - D)h_k, h_k) \\ &= ((4\pi^2 x^2 - D^2 + 2\pi)h_k, h_k) \\ &= ((H + 2\pi)h_k, h_k).\end{aligned}$$

Using $Hh_k = 2\pi(2k + 1)h_k$ from Proposition 3.35, we have

$$(h_{k+1}, h_{k+1}) = 2\pi(2k + 2)(h_k, h_k) = 4\pi(k + 1)(h_k, h_k).$$

But $(h_0, h_0) = \int e^{-2\pi x^2}\, dx = \frac{1}{\sqrt{2}} \int e^{-\pi x^2}\, dx = \frac{1}{\sqrt{2}}$. These together imply

$$(h_k, h_k) = \frac{(4\pi)^k k!}{\sqrt{2}}.$$

Putting this together in the general case gives:

$$(h_\alpha, h_\beta)_2 = \frac{(4\pi)^{|\alpha|} \alpha_1! \cdots \alpha_n!}{\sqrt{2^n}} \delta_{\alpha,\beta}.$$

Thus the vectors

$$e_\alpha = \frac{2^{n/4}}{\sqrt{(4\pi)^{|\alpha|}\alpha!}} h_\alpha,\ \alpha \in A \tag{3.10}$$

form an orthonormal set in $L^2(\mathbb{R}^n)$. Proposition 3.35 gives they are complete. Moreover, by Theorem 3.30, e_α is an eigenfunction of the Fourier transform with eigenvalue $(-i)^{|\alpha|}$. □

Corollary 3.37. *For $j = 1, \ldots, n$,*

$$(2\pi x_j - D_j)e_\alpha(x) = \sqrt{4\pi(\alpha_j + 1)}\, e_{\alpha+e_j}(x) \text{ and}$$
$$(2\pi x_j + D_j)e_\alpha(x) = \sqrt{4\pi\alpha_j}\, e_{\alpha-e_j}(x).$$

Proof. Using (c) and (d) of Lemma 3.28, we have:

$$\begin{aligned}(2\pi x_j - D_j)e_\alpha &= (2\pi x_j - D_j)\frac{2^{n/4}}{\sqrt{(4\pi)^{|\alpha|}\alpha!}}h_\alpha \\ &= \frac{2^{n/4}}{\sqrt{(4\pi)^{|\alpha|}\alpha!}}h_{\alpha+e_j} \\ &= \frac{2^{n/4}\sqrt{4\pi(\alpha_j+1)}}{\sqrt{(4\pi)^{|\alpha+e_j|}(\alpha+e_j)!}}h_{\alpha+e_j} \\ &= \sqrt{4\pi(\alpha_j+1)}\, e_{\alpha+e_j}\end{aligned}$$

and

$$\begin{aligned}(2\pi x_j + D_j)e_\alpha &= (2\pi x_j + D_j)\frac{2^{n/4}}{\sqrt{(4\pi)^{|\alpha|}\alpha!}}h_\alpha \\ &= \frac{2^{n/4}4\pi\alpha_j}{\sqrt{(4\pi)^{|\alpha|}\alpha!}}h_{\alpha-e_j} \\ &= \frac{2^{n/4}\sqrt{4\pi\alpha_j}}{\sqrt{(4\pi)^{|\alpha-e_j|}(\alpha-e_j)!}}h_{\alpha-e_j} \\ &= \sqrt{4\pi\alpha_j}\, e_{\alpha-e_j}\end{aligned}$$

□

We thus obtain:

$$D_j e_\alpha(x) = \sqrt{\pi\alpha_j}\, e_{\alpha-e_j}(x) + \sqrt{\pi(\alpha_j+1)}\, e_{\alpha+e_j}(x) \tag{3.11}$$

$$x_j e_\alpha(x) = \sqrt{\frac{\alpha_j}{4\pi}}\, e_{\alpha-e_j}(x) + \sqrt{\frac{\alpha_j+1}{4\pi}}\, e_{\alpha+e_j}(x). \tag{3.12}$$

Exercise Set 3.3

1. Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces. Let U be a linear transformation from a dense linear subspace D of $\mathcal{H}$ onto a dense linear subspace D' of $\mathcal{H}'$. Suppose

$$(Uv, Uw)' = (u, w)$$

for all $u, w \in D$.

(a) Show if $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in D, then $\{Uu_n\}_{n=1}^{\infty}$ is a Cauchy sequence in D'.

(b) Show $\tilde{U}$ defined on $\mathcal{H}$ by $\tilde{U}v = \lim Uv_n$ if $v = \lim_n v_n$ where $v_n \in D$ is a unitary isomorphism of $\mathcal{H}$ onto $\mathcal{H}'$.

2. Let $h \in L^1(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$. Show $\mathcal{F}(f * h) = \hat{h}\hat{f}$.

3. Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$ satisfying $T^4 = I$. Show $\mathcal{H}$ is the direct sum of four orthogonal eigenspaces for T.

4. The essential support ess-supp(f) of a measurable complex valued function f on $\mathbb{R}^n$ is the complement of the union of all open subsets U of $\mathbb{R}^n$ on which $f = 0$ a.e. For $T > 0$, let

$$L^2_T(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \mid \text{ess-supp}(\hat{f}) \subset [-T, T]^n \right\}.$$

the space of T-band limited functions.

(a) Show that the operator defined on the dense subspace $\mathcal{S}(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$ by

$$h \mapsto \mathrm{sinc}_T * h$$

where

$$\mathrm{sinc}_T(x) = \prod_{j=1}^{n} \frac{\sin(2\pi T x_j)}{\pi x_j}$$

extends to the orthogonal projection P_T of $L^2(\mathbb{R}^n)$ onto $L^2_T(\mathbb{R}^n)$.

(b) Show that

$$f = \lim_{T \mapsto \infty} P_T(f)$$

for $f \in L^2(\mathbb{R}^n)$.

5. Consider the differential operators $D_t^\alpha = \left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial t_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial t_n}\right)^{\alpha_n}$ and $D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. For f in $C^\infty(\mathbb{R}^n)$, show

$$D_t^\alpha f(x - t)|_{t=0} = (-1)^{|\alpha|} D_x^\alpha f(x).$$

6. In many books the Hermite polynomials are defined by using the generating function $e^{t\cdot x - t\cdot t/2}$. In this case, the Hermite polynomials $\tilde{H}_\alpha(x)$ satisfy:

$$e^{t\cdot x - t\cdot t/2} = \sum_\alpha \tilde{H}_\alpha(x) \frac{t^\alpha}{\alpha!}.$$

Show that

$$(4\pi)^{|\alpha|/2} \tilde{H}_\alpha(\sqrt{4\pi}x) = H_\alpha(x).$$

7. Let $0 \leqslant \psi(t) \leqslant 1$ be a smooth function with support in $(1/2, \infty)$ such that $\psi(t) = 1$ for $t \geqslant 1$. Let $p = \left[\frac{n}{2}\right] + 1$. Let

$$f(x) := \frac{1}{|x|^p}\psi(|x|) .$$

Show the following:

(a) $f \in L^2(\mathbb{R}^n)$.

(b) $Hf(x) = \frac{1}{|x|^{p-2}}\psi\,(|x|) - \Delta f(x)$ is not in $L^2(\mathbb{R}^n)$.

8. Show that the linear span of the functions

$$(x_1, \ldots, x_n) \mapsto f_1(x_1) \cdot \ldots \cdot f_n(x_n)$$

where $f_1, f_, \ldots, f_n$ are in $L^2(\mathbb{R})$ is a dense linear subspace of $L^2(\mathbb{R}^n)$.

Chapter 4

Further Topics and Applications

1. Holomorphic Functions on $\mathbb{C}^n$

In the last chapters we determined the image of the two spaces, $L^2(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, under the Fourier transform. In both cases the Fourier transform was an isomorphism. In the next section we will determine the image of the space of compactly supported functions. This will be a certain space of *holomorphic* functions on $\mathbb{C}^n$, so we need to review the most basic facts from complex analysis. We present the material mostly without proofs.

Definition 4.1. *Let $\varnothing \neq U$ be an open set in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be a function. We say that f is complex differentiable or holomorphic at $z \in U$ if the limit*

$$f'(z) := \lim_{w\to z} \frac{f(w) - f(z)}{w - z} \tag{4.1}$$

exists. The function f is differentiable on U if it is differentiable at every point $z \in U$.

A function on $\mathbb{C}$ can also be viewed as a function on $\mathbb{R}^2 \to \mathbb{R}^2$ by

$$(x, y) \mapsto (u(x, y), v(x, y))$$

where $u(x, y) = \mathrm{Re} f(x + iy)$ and $v(x, y) = \mathrm{Im} f(x + iy)$. Therefore if $f'(z)$ exists, the partial derivatives

$$D_x f(z) = \frac{\partial f}{\partial x}(z) = \lim_{h\to 0} \frac{f(z + h) - f(z)}{h} = D_x u(x, y) + iD_x v(x, y) \tag{4.2}$$

and

$$(4.3) \quad D_y f(z) = \frac{\partial f}{\partial y}(z) = \lim_{h\to 0} \frac{f(z+ih) - f(z)}{h} = D_y u(x,y) + iD_y v(x,y)$$

are well defined. Assume that $f : U \to \mathbb{C}$ is holomorphic at $z_0 = x_0 + iy_0$. Then, because the limit in (4.1) is independent of how we take the limit $w \to z_0$,

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \\ &= D_x f(z_0) \\ &= D_x u(x_0,y_0) + iD_x v(x_0,y_0) \\ &= \lim_{h\to 0} \frac{f(z_0+hi) - f(z_0)}{ih} \\ &= \frac{1}{i}\frac{\partial f}{\partial y}(z_0) \\ &= -iD_y u(x_0,y_0) + D_y v(x_0,y_0) \end{aligned}$$

Thus

$$\begin{aligned} D_x u(x_0,y_0) &= D_y v(x_0,y_0) \\ D_y u(x_0,y_0) &= -D_x v(x_0,y_0)\,. \end{aligned}$$

This system of equation is called the *(real) Cauchy-Riemann equations*. This motivates also the following: Define the first order differential operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ by

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

Then a holomorphic function satisfies the differential equation

$$(4.4) \qquad \frac{\partial f}{\partial \bar{z}} = 0$$

which is called the (complex) *Cauchy-Riemann equation.*

Theorem 4.2. *Let $U \subset \mathbb{C}$ be open, $U \neq \varnothing$, and let $f : U \to \mathbb{C}$. Then f is holomorphic at a point $z_0 = x_0 + iy_0 \in U$ if and only if f is differentiable as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ at the point (x_0, y_0) and f satisfies the Cauchy-Riemann equation at z_0. In this case we have*

$$\begin{aligned} f'(z_0) &= D_x u(x_0,y_0) + iD_x v(x_0,y_0) \\ &= D_y v(x_0,y_0) - iD_y v(x_0,y_0)\,. \end{aligned}$$

where $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$.

Let $I = [a, b]$ be a closed interval in $\mathbb{R}$ and let $\gamma : I \to U$ be a piecewise-continuously differentiable path in $\mathbb{C}$. Define the *integral of f over γ* by

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\gamma'(t)\, dt\,.$$

We will from now on always assume that γ is piecewise-continuously differentiable path. We set $C(\gamma) := \gamma([a, b])$. We say that γ is a *parametrization* of the curve C if $C = C(\gamma)$. We say that γ is *simple* if γ is injective as a function from $[a, b)$ to $\mathbb{C}$, i.e., $\gamma(t) = \gamma(s)$ implies that $s = t$. Notice, that we allow $\gamma(a) = \gamma(b)$. We say that γ is *closed* if $\gamma(a) = \gamma(b)$. We say that the curve C is *simple* (*closed*) if there exists a simple (closed) parametrization γ of C. If C is a simple and closed curve then we say that C is a *Jordan curve*. We can travel the same parameterized curve $C = C(\gamma)$ in two directions, $t \mapsto \gamma(t)$ and $t \mapsto \gamma((b + a) - s)$, so that we have two different *orientations*. A Jordan curve divides the complex region into 3 parts, the curve itself, an open bounded connected interior known as the *interior of C*, and an unbounded open set called the *exterior of C*. A parametrization γ of a Jordan curve C is *positively oriented* if $i\gamma'(t)$ points to the interior region determined by the curve.

Example 4.3. *Let C be a circle with radius $R > 0$ and center z_0. Thus $\gamma(t) = R\, e^{2\pi i t} + z_0$, $t \in [0, 1]$. Then γ is injective and closed. Thus C is a Jordan curve. Furthermore, since $i\gamma'(t) = -2\pi R\, e^{2\pi i t}$, $i\gamma'(t)$ points to the interior of the circle. So γ is positively oriented. Let $f(z) = A(z - z_0)^n$, $n \in \mathbb{N}_0$. Then $f(\gamma(t)) = A\, R^n\, e^{2\pi i n t}$. Thus*

$$\int_\gamma f(z)\, dz = 2\pi A R^{n+1} i \int_0^1 e^{2\pi i(n+1)t}\, dt = 0\,.$$

Furthermore

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz &= \frac{1}{2\pi i}\int_0^1 \frac{A R^n e^{2\pi i n t}}{R\, e^{2\pi i t}} 2\pi i R\, e^{2\pi i t}\, dt \\ &= A R^n \int_0^1 e^{2\pi i n t}\, dt \\ &= \begin{cases} A & \text{if } \; n = 0 \\ 0 & \text{if } \; n > 0 \end{cases} \\ &= f(z_0)\,.\end{aligned}$$

If $\gamma(t)$ and $\eta(t)$ are two continuous paths defined on $[a, b]$ then we say that γ and η are *homotopic* if there exists a continuous function $F : [a, b] \times [0, 1] \to U$ such that $F(s, 0) = \gamma(t)$ and $F(s, 1) = \eta(s)$ for all $t \in [0, 1]$.

Theorem 4.4 (Independence of the path)**.** *Let f be a complex differentiable function on a non-empty open subset U of $\mathbb{C}$. Let $\gamma, \eta : [a,b] \to U$ be two homotopic curves. Then $\int_\gamma f(z)\,dz = \int_\eta f(z)\,dz$.*

Corollary 4.5. *Suppose that $f : \mathbb{C}^n \to \mathbb{C}$ is complex differentiable and that $\gamma : I \to \mathbb{C}^n$ is a closed curve. Then*

$$\int_I f(\gamma(t))\,dt = 0\,.$$

Proof. The curve γ is homotopic to the constant curve $\eta(t) = \gamma(t_0)$ where $t_0 \in I$ is fixed. The claim follows then from the fact that

$$\int_\eta f(z)\,dz = 0\,.$$

$\square$

Theorem 4.6 (Cauchy's integral formula)**.** *Let f be a complex differentiable function on a non-empty simply connected open subset U of $\mathbb{C}$. Let $\gamma : [a,b] \to U$ be a positively oriented Jordan curve. Assume that z_0 is in the interior of γ. Then*

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}dz\,.$$

The Cauchy integral formula has several important consequences. The first follows from the fact that the function inside the integral is infinitely differentiable as a function of z_0. Thus we obtain the following theorem:

Theorem 4.7. *Let $\varnothing \neq U$ be an open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be complex differentiable. Then f is infinitely differentiable and for each $n \in \mathbb{N}_0$,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

where γ is any positively oriented Jordan curve in U surrounding z_0 such that the interior of γ is contained in U.

Example 4.8. *Let γ be the circle $\gamma(t) = z_0 + R\,e^{it}$, $t \in [0, 2\pi]$ as in Example 4.3. Then $\gamma'(t) = iR\,e^{it}$ and $\gamma(t) - z_0 = R\,e^{it}$, Hence the Cauchy integral formula becomes*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + R\,e^{it})\,dt\,.$$

Let $F(s,t) := z_0 + Rt\,e^{is}$. Then F is continuous and $F(s,0) = z_0$, $F(s,1) = z_0 + R\,e^{is} = \gamma(s)$. Hence γ and the constant curve $s \mapsto z_0$ are homotopic. As the path integral is independent of the path it follows that

$$\int_\gamma f(z)\,dz = \int_{t \mapsto z_0} f(z)\,dz = 0\,.$$

We notice that the same argument holds for ***any*** *curve that is homotopic to the constant curve $t \mapsto z_0$. Hence the integral of a holomorphic function of a curve homotopic to a point is always zero.*

The second application of Cauchy's integral formula is that complex differentiable functions are actually analytic. The function f is called *analytic* or *holomorphic* on U if for each $z_0 \in U$ there exists an open ball $B_R(z_0) \subset U$ such that f has an power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{4.5}$$

on $B_R(z)$. The maximal radius $\rho > 0$ such that the power series in (4.5) converges is called *the convergence radius* of the power series. The convergence radius can be evaluated as

$$\frac{1}{\rho} = \overline{\lim} \sqrt[k]{|a_k|}\,.$$

Assume that $a_k \neq 0$ for all k. Then we can also evaluate ρ by

$$\frac{1}{\rho} = \overline{\lim} \frac{|a_{k+1}|}{|a_k|}\,. \tag{4.6}$$

Example 4.9. *Let $a_k = \frac{1}{k!}$. Then $|a_{k+1}/a_k| = 1/(k+1) \to 0$ for $k \to \infty$. Hence $\rho = \infty$ and the series $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ converges for all $z \in \mathbb{C}$. The limit is the exponential function*

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}\,.$$

Example 4.10. *Let $a_{2k} = \frac{(-1)^k}{(2k)!}$ and $a_{2k+1} = 0$. Then $a_k = 0$ if k is odd. To use (4.6) define $b_k = a_{2k} = (-1)^k/(2k)!$, $k \in \mathbb{Z}_0$. Then $b_k \neq 0$ and*

$$\frac{|b_{k+1}|}{|b_k|} = \frac{1}{(2k+2)(2k+1)} \to 0 \quad \text{as} \quad k \to \infty\,.$$

Hence $\sum_{k=0}^{\infty} b_k z^k$ exists for all $z \in \mathbb{C}$. It follows that

$$\sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} b_k (z^2)^k$$

exists for all $z \in \mathbb{C}$. Furthermore

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} = \cos(z)\,.$$

We note that every analytic function is complex differentiable and that

$$a_k = \frac{f^{(k)}(z_0)}{k!}\,.$$

Furthermore

$$f^{(n)}(z) = \sum_{k=n}^{\infty} a_k k(k-1)\cdots(k-n+1)(z-z_0)^{k-n}$$
$$= \sum_{k=0}^{\infty} (k+1)\cdots(k+n) a_{k+n}(z-z_0)^k .$$

Theorem 4.11. *Let $R > 0$ and assume that f is complex differentiable on $B_R(z_0)$. Then f is analytic on $B_R(z_0)$.*

The last few theorems together give the following equivalent description of holomorphic functions on open subsets of $\mathbb{C}$:

Theorem 4.12. *Let $\varnothing \neq U \subset \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$. Then the following are equivalent:*

(a) *f is complex differentiable on U;*

(b) *f is holomorphic on U;*

(c) *Let γ be a positively oriented Jordan curve contained in U. Then*

$$f(z) = \int_\gamma \frac{f(w)}{w-z}\, dw$$

for all z in the interior of γ.

Let us now consider functions defined on open subsets of $\mathbb{C}^n$. In the remaining part of this section, U will be for a non-empty open subset of $\mathbb{C}^n$. Let $f : U \to \mathbb{C}^m$. Recall that f is differentiable at a point $z \in U$ if there exists a $\mathbb{R}$-linear map $Df(z) : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ such that

$$f(w) = f(z) + Df(z)(w-z) + o(|z-w|) .$$

Definition 4.13. *The function $f : U \to \mathbb{C}$ is called:*

(a) *Complex differentiable at $z \in U$ if there exists a complex linear map $Df(z) : \mathbb{C}^n \to \mathbb{C}$ such that*

$$f(w) = f(z) + Df(z)(w-z) + o(|w-z|) .$$

(b) *Complex differentiable on U if it is complex differentiable at every point of U.*

(c) *Analytic or holomorphic on U if for each $z \in U$ there exists positive numbers $r_j > 0$ such that the poly disc*

$$P(z,r) = \{w \in \mathbb{C}^n \mid \forall j\, :\, |w_j - z_j| < r_j\} \subset U$$

and on $P(z,r)$ we have

$$f(w) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (w-z)^\alpha$$

with $a_\alpha \in \mathbb{C}$.

Most of the results that we will need follows from the one-dimensional case because of the following lemma. Let $r \in \mathbb{R}^n$, we will write $r > 0$ if $r_j > 0$ for all j.

Lemma 4.14. *Let $z \in \mathbb{C}^n$ and $r \in \mathbb{R}^n$, $r > 0$. Suppose that $f : P(z, r) \to \mathbb{C}$ is continuous. Then f is complex differentiable if and only if for all $1 \leqslant j \leqslant n$ the function*

$$P(z_j, r_j) \ni w \mapsto f(z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_n) \in \mathbb{C}$$

is complex differentiable.

Let $r, s \in \mathbb{R}^n$, $r, s > 0$ such that $s_j < r_j$ for $j \leqslant n$. Let $z \in \mathbb{C}^n$ and define $\gamma_j(\theta) = s_j e^{2\pi i\theta} + z_j$ and let $\gamma(\theta_1, \ldots, \theta_n) := (\gamma_1(\theta_1), \ldots, \gamma_n(\theta_n))$. Finally let f be a continuous function on the image $I \subset P(z, r)$ of γ. Then

$$\begin{aligned} \mathrm{ch}(f)(z) &:= \left(\frac{1}{2\pi i}\right)^n \int_I \frac{f(w)}{(w_1 - z_1)\cdots(w_n - z_n)}\, dw_1 \ldots dw_n \\ &= \int_{[0,2\pi]^n} f(\gamma_1(\theta_1), \ldots, \gamma_n(\theta_n))\, d\theta_1 \ldots d\theta_n \end{aligned}$$

is called the *Cauchy integral of f over I*. We notice that by Cauchy's Integral formula and Lemma 4.14 it follows that $f(z) = \mathrm{ch}(f)(z)$ if f is holomorphic. Furthermore the integral is independent of the choice of path γ_j as long as γ_j is homotopic to the circle. Similar to Theorem 4.12 we have for more than one variables:

Theorem 4.15. *Let $\varnothing \neq U \subset \mathbb{C}^n$ be open and let $f : U \to \mathbb{C}$ be continuous. Then the following are equivalent:*

(a) *f is complex differentiable on U.*
(b) *f is holomorphic on U.*
(c) $f = \mathrm{ch}(f)$

We will also need the following simple lemma for the proof of the Paley-Wiener theorem.

Lemma 4.16. *Suppose that $K \subset \mathbb{R}^n$ is compact and $\varnothing \neq U \subset \mathbb{C}^n$ open. Let μ be a finite measure on K and suppose that $f : K \times U \to \mathbb{C}$ is measurable and bounded on compact sets. Then if $U \ni z \to f(x, z) \in \mathbb{C}$ is holomorphic for each $x \in \mathbb{K}$, then the function*

$$F(z) := \int_K f(x, z)\, d\mu(x)$$

is holomorphic.

Proof. We can assume that $\mu(K) > 0$ and that $U = P(z_0, r)$ with $r > 0$. The function F is continuous by Lebesgue dominated convergence theorem. We now calculate

$$\begin{aligned}\operatorname{ch}(F)(z) &= \int_{[0,2\pi]^n} F(\gamma_1(\theta_1), \ldots, \gamma_n(\theta_1))\, d\theta_1 \ldots d\theta_n \\ &= \int_{[0,2\pi]^n} \int_K f(x, \gamma_1(\theta_1), \ldots, \gamma_n(\theta_1))\, d\mu(x)\ d\theta_1 \ldots d\theta_n\,.\end{aligned}$$

As $L := [0, 2\pi]^n \times K$ is compact and $f(\cdot, \gamma(\cdot))$ is bounded on L it follows that $f(\cdot, \gamma(\cdot))$ is integrable on L with respect to $\mu \times (d\theta_1 \times \cdots \times d\theta_n)$. By Fubini's Theorem and Theorem 4.15 we get

$$\begin{aligned}\operatorname{ch}(F)(z) &= \int_{[0,2\pi]^n} \int_K f(x, \gamma_1(\theta_1), \ldots, \gamma_n(\theta_1))\, d\mu(x)\ d\theta_1 \ldots d\theta_n \\ &= \int_K \int_{[0,2\pi]^n} f(x, \gamma_1(\theta_1), \ldots, \gamma_n(\theta_1))\ d\theta_1 \ldots d\theta_n\, d\mu(x) \\ &= \int_K f(x, z)\, d\mu(x) \\ &= F(z)\,.\end{aligned}$$

Hence by Theorem 4.15, it follows that F is holomorphic. □

2. The Paley-Wiener Theorem

We begin by defining the *Paley-Wiener spaces*. Let $R > 0$ and let $\mathrm{PW}_R(\mathbb{C}^n)$ be the space of holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ satisfying

$$\pi_{R,N}(F) := \sup_{z\in\mathbb{C}^n} (1 + |z|^2)^N e^{-R|\mathrm{Im}(z)|} |F(z)| < \infty$$

for all $N \in \mathbb{N}$. We notice that the family of seminorms $\{\pi_{R,N}\}_{N\in\mathbb{N}}$ defines a locally convex Hausdorff topology on $\mathrm{PW}_R(\mathbb{C}^n)$. Next we let $\mathrm{PW}(\mathbb{C}^n) = \cup_{R>0}\mathrm{PW}_R(\mathbb{C}^n)$. We give $\mathrm{PW}(\mathbb{C}^n)$ the inductive limit topology. Recall that for a compact set $K \subset \mathbb{R}^n$, $\mathcal{D}_K(\mathbb{R}^n)$ denotes the space of smooth functions on $\mathbb{R}^n$ with support in K. For a positive number $R > 0$, we let $\mathcal{D}_R(\mathbb{R}^n) = \mathcal{D}_{B_R(0)}(\mathbb{R}^n)$. Thus $\mathcal{D}_R(\mathbb{R}^n)$ is the space of smooth functions with support contained in a closed ball of radius R centered at zero. We start with the following simple Lemma.

Lemma 4.17. *Let $n \geqslant 2$, $R > 0$, and $f \in \mathrm{PW}_R(\mathbb{C}^n)$. Let $z_0 \in \mathbb{C}$. Then the function $F : \mathbb{C}^{n-1} \to \mathbb{C}$,*

$$F(z) := \int_{\mathbb{R}} f(z, z_0 + x)\, dx$$

is in $\mathrm{PW}_R(\mathbb{C}^{n-1})$ and the map $\mathrm{PW}_R(\mathbb{C}^n) \to \mathrm{PW}_R(\mathbb{C}^{n-1})$, $f \mapsto F$, is continuous.

Proof. We have $|f(z, z_0+x)(1+|z|^2+|z_0+x|^2)^{N+1}e^{-R|\mathrm{Im}(z,z_0)|}| \leqslant \pi_{N+1,R}(f)$ for all $z \in \mathbb{C}^{n-1}$ and $x \in \mathbb{R}$. Hence

$$(1 + |z_0 + x|^2)|f(z, z_0 + x)(1 + |z|^2)^N \leqslant \pi_{N+1,R}(f).$$

This gives

$$(1 + |z|^2)^N \int_{\mathbb{R}} |f(z, z_0 + x)|\, dx \leqslant \pi_{N+1,R}(f) \int_{\mathbb{R}} \frac{1}{1 + |z_0 + x|^2}\, dx$$

and consequently

$$\pi_{N,R}(F) \leqslant C_{z_0} \pi_{N+1,R}(f)$$

where $C_{z_0} = \int_{\mathbb{R}} \frac{1}{1+|z_0+x|^2}\, dx < \infty$. This implies $F \in \mathrm{PW}_R(\mathbb{C}^{n-1})$ and $f \mapsto F$ is continuous at 0. Since $f \to F$ is a linear transformation, we see $f \mapsto F$ is continuous. □

Theorem 4.18. *Let* $f \in \mathrm{PW}(\mathbb{C}^n)$ *and let* $\sigma \in \mathbb{R}^n$*. Then*

$$\int_{\mathbb{R}^n} f(x)\, d\lambda = \int_{\mathbb{R}^n} f(x + i\sigma)\, d\lambda\,.$$

Proof. We may assume that $f \in \mathrm{PW}_R(\mathbb{C}^n)$ for some $R > 0$. Let $T > 0$ and let $\sigma = (\sigma_1, \ldots, \sigma_n)$. Consider the curves $\gamma_1(t) = t$, $-T \leqslant t \leqslant T$, $\gamma_2(t) = T + it\sigma_n$, $0 \leqslant t \leqslant 1$, $\gamma_3(t) = -t + i\sigma_n$, $-T \leqslant t \leqslant T$, and $\gamma_4(t) = -T + (1 - t)i\sigma_n$, $0 \leqslant t \leqslant 1$. Finally let $\gamma(t)$ be the sum of those four curves:

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } -T \leqslant t \leqslant T \\ \gamma_2(t - T) & \text{if } T \leqslant t \leqslant T + 1 \\ \gamma_3(t - (2T + 1)) & \text{if } T + 1 \leqslant t \leqslant 3T + 1 \\ \gamma_4(t - (3T + 1)) & \text{if } 3T + 1 \leqslant t \leqslant 3T + 2 \end{cases}$$

Then for each $z \in \mathbb{C}^{n-1}$

$$\int_\gamma f(z, w)dw = 0$$

by Corollary 4.5. Thus

$$\int_{-T}^{T} f(z,t)dt + i\int_0^1 f(z, T + it\sigma_n)\, dt - \int_{-T}^{T} f(z, t + i\sigma_n)\, dt - i\int_0^1 f(z, -T + it\sigma_n)\, dt = 0\,.$$

Consider now the limit $T \to \infty$. Using that

$$\begin{aligned} |f(z, \pm T + it\sigma_n)| &\leqslant \pi_{R,1}(f)(1 + |z|^2 + T^2 + \sigma_n^2 t^2)^{-1} e^{R\sqrt{|\mathrm{Im}(z)| + t|\sigma_n|}} \\ &\leqslant C' T^{-2}, \end{aligned}$$

we see

$$\lim_{T\to\infty} \int_0^1 f(z, T + it\sigma_n)\, dt = \int_0^1 f(z, -T + it\sigma_n)\, dt = 0\,.$$

Thus

$$\int_{-\infty}^{\infty} f(z, t)\, dt = \int_{-\infty}^{\infty} f(z, t + i\sigma_n)\, dt$$

By Lemma 4.17, we know that the function

$$F(z) = \int_{-\infty}^{\infty} f(z, t_n)\, dt_n = \int_{-\infty}^{\infty} f(z, t_n + i\sigma_n)\, dt_n$$

is in $\mathrm{PW}_R(\mathbb{C}^{n-1})$. By iterating this argument, we see:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z, t_{n-1}, t_n)\, dt_n dt_{n-1} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z, t_{n-1} + i\sigma_{n-1}, t_n + i\sigma_n)\, dt_n dt_{n-1} \\ &\vdots \\ \int_{\mathbb{R}^n} f(t)\, dt &= \int_{\mathbb{R}^n} f(t + i\sigma)\, dt.\end{aligned}$$

$\square$

Theorem 4.19. *Let $F \in \mathrm{PW}_R(\mathbb{C}^n)$ and let $\alpha \in \mathbb{N}_0^n$. Then $D^\alpha F \in \mathrm{PW}_R(\mathbb{C}^n)$ and the mapping $F \mapsto D^\alpha F$ is continuous.*

Proof. It suffices to show $\frac{\partial}{\partial z_1} F \in \mathrm{PW}_R(\mathbb{C}^n)$, $F \mapsto \frac{\partial}{\partial z_1} F$ is continuous, and then argue inductively.

We write $z = (z_1, z')$ where $z_1 \in \mathbb{C}$ and $z' \in \mathbb{C}^{n-1}$. Consider the curve C: $\gamma(t) = (z_1 + e^{it}, z')$ for $0 \leqslant t \leqslant 2\pi$. Then

$$\frac{\partial}{\partial z_1} F(z_1, z') = \frac{1}{2\pi i} \int_C \frac{F(w_1, z')}{(w_1 - z_1)^2}\, dw_1.$$

This gives

$$\begin{aligned}(1 + |z|^2)^N \left|\frac{\partial}{\partial z_1} F(z)\right| e^{-R|\mathrm{Im} z|} &= \frac{1}{2\pi} \left| \int_C \frac{(1 + |z|^2)^N (1 + |(w_1, z')|^2)^N F(w_1, z') e^{-R|\mathrm{Im} z|}}{(1 + |(w_1, z')|^2)^N |w_1 - z_1|^2} \right| d|w_1| \\ &\leqslant \frac{1}{2\pi} \int_C \frac{(1 + |z|^2)^N \pi_{R,N}(F)}{(1 + |(w_1, z')|^2)^N} |dw_1|.\end{aligned}$$

Now on C, $w_1 = z_1 + e^{it}$ and thus

$$\begin{aligned}
\frac{(1+|z|^2)^N}{(1+|(w_1,z')|^2)^N} &= \frac{(1+|z_1+e^{it}-e^{it}|^2+|z'|^2)^N}{(1+|w_1|^2+|z'|^2)^{N+1}} \\
&= \frac{(1+|w_1-e^{it}|^2+|z'|^2)^N}{(1+|w_1|^2+|z'|^2)^N} \\
&\leqslant \frac{(1+|w_1|^2+2|w_1|+1+|z'|^2)^N}{(1+|w_1|^2+|z'|^2)^N} \\
&= \left(\frac{1+|w_1|^2+|z'|^2}{(1+|w_1|^2+|z'|^2)} + \frac{2|w_1|+1}{1+|w_1|^2+|z'|^2}\right)^N \\
&\leqslant \left(1+\frac{2|w_1|+1}{1+|w_1|^2}\right)^N \\
&= \left(\frac{|w_1|^2+2|w_1|+2}{|w_1|^2+1}\right)^N \\
&\leqslant 5^N.
\end{aligned}$$

Hence

$$\begin{aligned}
(1+|z|^2)^N \left|\frac{\partial}{\partial z_1}F(z)\right| e^{-R|\mathrm{Im} z|} &\leqslant \frac{1}{2\pi}\int_C \frac{(1+|z|^2)^N \pi_{R,N}(F)}{(1+|(w_1,z')|^2)^N}\,|dw_1| \\
&\leqslant \frac{5^N}{2\pi}\pi_{R,N}(F)\int_C |dw_1| \\
&= 5^N \pi_{R,N}(F).
\end{aligned}$$

This gives $\frac{\partial}{\partial z_1}F \in \mathrm{PW}_R(\mathbb{C}^n)$ and $\pi_{R,N}(\frac{\partial}{\partial z_1}F) \leqslant 5^N \pi_{R,N}(F)$. Thus $F \mapsto \frac{\partial}{\partial z_1}F$ is continuous. $\square$

Corollary 4.20. *Let $F \in \mathrm{PW}_R(\mathbb{C}^n)$. Then $F|_{\mathbb{R}^n}$ is a Schwartz function. Moreover, $F \mapsto F|_{\mathbb{R}^n}$ is a continuous linear transformation from $\mathrm{PW}_R(\mathbb{C}^n)$ into $\mathcal{S}(\mathbb{R}^n)$.*

Proof. The above proof implies

$$(1+|x|^2)^N |D^\alpha F(x)| \leqslant 5^{N|\alpha|}\pi_{R,N}(F).$$

Thus if $f = F|_{\mathbb{R}^n}$, then

$$|f|_{N,\alpha} \leqslant 5^{N|\alpha|}\pi_{R,N}(F).$$

$\square$

Recall $\mathcal{D}(\mathbb{R}^n)$ denotes the space of complex C^∞ functions with compact support.

Theorem 4.21 (The Paley-Wiener Theorem)**.** *For $f \in \mathcal{D}(\mathbb{R}^n)$, we extend the domain of definition of the Fourier transform $\mathcal{F}(f)$ to $\mathbb{C}^n$ by*

$$\mathcal{F}(f)(z) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot z}\, dx.$$

Let $R > 0$. Then $\mathcal{F}$ is a topological isomorphism of $\mathcal{D}_R(\mathbb{R}^n)$ onto $\mathrm{PW}_{2\pi R}(\mathbb{C}^n)$.

Consequently, the Fourier transform is a topological isomorphism of $\mathcal{D}(\mathbb{R}^n)$ onto $\mathrm{PW}(\mathbb{C}^n)$.

Proof. By Lemma 4.16, F defined by $F(z) = \int_{B_R(0)} f(y) e^{2\pi i z \cdot y}\, dy$ is a holomorphic function on $\mathbb{C}^n$.

Let $z = (z_1, z_2, \ldots, z_n)$ where $z_j = s_j + it_j$. Then on $B_R(0)$ we have the estimate

$$\left|e^{-2\pi i z \cdot y}\right| \leqslant e^{2\pi \sum t_j y_j} \leqslant e^{2\pi \sum |t_j| |y_j|} \leqslant \exp\left(2\pi \sqrt{\sum_{j=1}^n t_j^2} \sqrt{\sum_{j=1}^n y_j^2}\right) \leqslant e^{2\pi R |\mathrm{Im}(z)|}.$$

Hence

$$|F(z)| \leqslant e^{2\pi R|\mathrm{Im}(z)|} |f|_1 \leqslant \mathrm{vol}(B_R(0)) |f|_\infty\, e^{2\pi R |\mathrm{Im}(z)|}.$$

Integration by parts gives

$$\begin{aligned}(2\pi i z)^\alpha F(z) &= \int_{B_R(0)} (-1)^{|\alpha|} f(y)\, D_y^\alpha e^{-2\pi i z \cdot y}\, dy \\ &= \int_{B_R(0)} D^\alpha f(y)\, e^{-2\pi i z \cdot y}\, dy\,.\end{aligned}$$

Thus $|z^\alpha|\, |F(z)| \leqslant (2\pi)^{-|\alpha|} \mathrm{vol}(B_R(0)) |D^\alpha f|_\infty e^{2\pi R |\mathrm{Im}(z)|}$.

Now by expanding $(1 + |z|^2)^N = \sum_{|\alpha| \leqslant 2N} a_\alpha\, |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n}$ we see that

$$\left(1 + |z|^2\right)^N e^{-2\pi R |\mathrm{Im}(z)|}\, |F(z)| \leqslant \mathrm{vol}(B_R(0)) \sum a_\alpha (2\pi)^{-|\alpha|}\, |D_1^{\alpha_1} \cdots D_n^{\alpha_n} f|_\infty\,.$$

This implies $\mathcal{F} : \mathcal{D}_R(\mathbb{R}^n) \to \mathrm{PW}_{2\pi R}(\mathbb{C}^n)$ is continuous.

Let now $F \in \mathrm{PW}_{2\pi R}(\mathbb{C}^n)$. By Corollary 4.20, F is Schwartz on $\mathbb{R}^n$. Thus f given by

$$f(x) = \mathcal{F}^{-1}(f)(x) = \int_{\mathbb{R}^n} F(y)^{2\pi i y \cdot x}\, dy.$$

is a Schwartz function.

Now let $F \in \mathrm{PW}_{2\pi R}(\mathbb{C}^n)$ and $x \in \mathbb{R}^n$ with $|x| > R$. We first note $H(z) = F(z)e^{2\pi i z\cdot x}$ is in $\mathrm{PW}_{2\pi(R+|x|)}(\mathbb{C}^n)$. Indeed,

$$\begin{aligned}|(1+|z|^2)^N|H(z)|e^{-2\pi(R+|x|)|\mathrm{Im}(z)|} &= (1+|z|^2)^N|F(z)|e^{-2\pi R|\mathrm{Im} z|} \\ &\quad \times e^{-2\pi \mathrm{Im} z\cdot x}e^{-2\pi|x|\,|\mathrm{Im}(z)|} \\ &\leqslant \pi_{N,2\pi R}(F)e^{2\pi|\mathrm{Im} z||x|}e^{-2\pi|x|\,|\mathrm{Im} z|} \\ &\leqslant \pi_{N,2\pi R}(F).\end{aligned}$$

Hence by Theorem 4.18, if $z = a + ib$ where $a, b \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} H(y)\,dx = \int_{\mathbb{R}^n} H(y+ib)\,dx = \int_{\mathbb{R}^n} H(y+a+ib)\,dx.$$

Consequently,

$$f(x) = \int F(y)e^{2\pi i y\cdot x}\,dy = \int F(y+z)e^{2\pi i(y+z)\cdot x}\,dy$$

for any $z \in \mathbb{C}^n$.

Now take $z = \alpha i x$ where $\alpha > 0$. Then

$$\begin{aligned}f(x) &= \int F(y+i\alpha x)e^{2\pi i(y+\alpha i x)\cdot x}\,dy \\ &= e^{-2\alpha\pi|x|^2}\int F(y+i\alpha x)e^{2\pi i y\cdot x}\,dy\end{aligned}$$

and we see

$$\begin{aligned}|f(x)| &\leqslant e^{-2\alpha\pi|x|^2}\int \pi_{N,2\pi R}(F)(1+|y|^2)^{-N}e^{2\pi R\alpha|x|} \\ &= e^{2\pi\alpha|x|(R-|x|)}\pi_{N,2\pi R}(F)\int(1+|y|^2)^{-N}\,dy.\end{aligned}$$

Take N so that $\int(1+|y|^2)^{-N}\,dy < \infty$ and let $\alpha \to \infty$. We obtain $f(x) = 0$ for $|x| > R$. Hence $\mathrm{supp}\,(f) \subset \overline{B_R(0)}$, or $f \in \mathcal{D}_R(\mathbb{R}^n)$.

Let $\alpha \in \mathbb{N}_0^n$. Choose $N > 0$ such that $y \mapsto (1+|y|^2)^{-N}\,|y_1^{\alpha_1}\cdots y_n^{\alpha_n}|$ is integrable. Then

$$\begin{aligned}|D^\alpha f(x)| &\leqslant (2\pi)^{|\alpha|}\int |y_1^{\alpha_1}\cdots y_n^{\alpha_n}F(y)|\,dy \\ &\leqslant (2\pi)^{|\alpha|}\int |y_1^{\alpha_1}\cdots y_n^{\alpha_n}|\,(1+|y|^2)^{-N}\,dy\,\sup_{t\in\mathbb{R}^n}\left(1+|t|^2\right)^{-N}|F(t)| \\ &\leqslant C\pi_{N,R}(F)\,.\end{aligned}$$

Thus the Fourier transform is a topological isomorphism.

The last statement follows from the fact that $\mathcal{D}(\mathbb{R}^n)$ and $\mathrm{PW}(\mathbb{C}^n)$ are inductive limits of $\mathcal{D}_R(\mathbb{R}^n)$ and $\mathrm{PW}_R(\mathbb{C}^n)$, respectively. □

Remark 4.22. The discussion in the proof concerning $H(z)$ establishes the following:

If $F \in \mathrm{PW}_R(\mathbb{C}^n)$ and $a \in \mathbb{R}^n$, then H defined by

$$H(z) = e^{ia\cdot z}F(z)$$

is in $\mathrm{PW}_{R+|a|}(\mathbb{C}^n)$.

Exercise Set 4.1

1. Show, using the definition, that the function $z \mapsto \bar{z}$, i.e., $x+iy \mapsto x-iy$, is not holomorphic.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map corresponding to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to the standard basis e_1, e_2. Show T is complex linear if and only if $a = d$ and $b = -c$. Use that to show that if $f : \mathbb{C} \to \mathbb{C}$ is differentiable, then f is complex differentiable if and only if $Df(z)$ is complex linear. Derive from this that f is complex differentiable if and only if f satisfies the Cauchy-Riemann differential equation.

3. Find the points at which the following functions are complex differentiable

(a) $f(z) = x^2 + 2ixy$;
(b) $f(z) = 2xy + i(x + \frac{2}{3}y^3)$;
(c) $f(z) = x^2 + y^2 = z\bar{z}$;

4. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(x+iy) = \begin{cases} 0 & \text{if} \quad z = 0 \\ \frac{(1+i)x^3-(1-i)y^3}{x^2+y^2} & \text{if} \quad z \neq 0 \end{cases}.$$

Show f solves to Cauchy-Riemann equation at $z = 0$ and f is not differentiable at zero. Why does this not contradict Theorem 4.2?.

5. Let f be a holomorphic function on an open subset $U \subset \mathbb{C}$. Write $f = u + iv$ as usual. Show $\Delta u = \Delta v = 0$.

6. Let f be a holomorphic function on an open subset of $\mathbb{C}$. Show the function $z \mapsto \overline{f(\bar{z})}$ is holomorphic.

7. Find the radius of convergence for the following functions:

(a) $f(z) = \sum_{n=0}^{\infty} nz^n$;
(b) $f(z) = \sum_{n=0}^{\infty} n!z^n$;
(c) $f(z) = \sum_{n=0}^{\infty} e^n z^n$;
(d) $f(z) = \sum_{n=0}^{\infty} e^{n!} z^n$.

8. Let $\langle\cdot,\cdot\rangle$ be a positive definite Hermitian form on $\mathbb{C}^n$. Thus $z, w \mapsto \langle z, w\rangle$ is complex linear in the first variable, $\langle z, w\rangle = \overline{\langle w, z\rangle}$ and $\langle z, z\rangle > 0$ for all $z \neq 0$. Show $(x, y) := \mathrm{Re}\langle x, y\rangle$ defines a inner product on $\mathbb{R}^{2n} = \mathbb{C}^n$ as a real vector space. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Show $e_1, \ldots, e_n, ie_1, \ldots, ie_n$ is an orthonormal basis of $\mathbb{R}^{2n}$.

9. Define $J : \mathbb{C}^n \to \mathbb{C}^n$ by $x \mapsto ix$. Show the matrix expression of J with the basis in problem 1.8 is given by

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix.

10. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be $\mathbb{R}$-linear. Show the following three conditions are equivalent:

(a) T is complex linear;

(b) $TJ = JT$;

(c) The matrix representation of T with respect to the basis in problem 1.8 is given by

$$T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

where $A, B \in M_n(\mathbb{R})$.

11. Prove Lemma 4.17.

12. Prove the second part of the Paley-Wiener theorem.

13. For $a \in \mathbb{R}^n$, define $\tau(a)f(z) = f(z - a)$ and $e_a(z) = e^{ia \cdot z}$. Furthermore, let $p(z)$ be a complex polynomial.

(a) Show $\tau(a)$ is a linear homeomorphism of $\mathrm{PW}(\mathbb{C}^n)$.

(b) Show $f \mapsto e_a f$ is a linear homeomorphism of $\mathrm{PW}(\mathbb{C}^n)$.

(c) Show $f \mapsto pf$ is a continuous linear transformation of each space $\mathrm{PW}_R(\mathbb{C}^n)$.

14. Show $\mathrm{PW}_R(\mathbb{C}^n)$ is a Fréchet space.

15. Show if B is a bounded subset of $\mathrm{PW}(\mathbb{C}^n)$, then there is an $R > 0$ such that $B \subseteq \mathrm{PW}_R(\mathbb{C}^n)$.

16. Show for each $R > 0$, the space $\mathrm{PW}_R(\mathbb{C}^n)$ has the Heine-Borel property.

17. Show the space $\mathrm{PW}(\mathbb{C}^n)$ has the Heine-Borel property.

18. Show the space $\mathrm{PW}(\mathbb{C}^n)$ is complete.

19. Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and such that on $\mathbb{R}^+$ we have $|f(t)| \leqslant Ce^{t\beta}$ for some $\beta \in \mathbb{R}$ and $C > 0$. Let

$$\mathbb{C}_\beta := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\} .$$

Show the **Laplace transform** of f:

$$\mathcal{L}(f)(z) := \int_0^\infty e^{-zt} f(t)\, dt$$

is defined for $z \in \mathbb{C}_\beta$ and that $\mathcal{L}(f) : \mathbb{C}_\beta \to \mathbb{C}$ is holomorphic. Furthermore show that for all $x > \beta$ the following inversion formula holds for $t > 0$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{L}(f)(x + iy) e^{(x+iy)t}\, dy .$$

(Hint: Relate the Laplace transform to the Fourier transform.)

20. (Hardy spaces) Let

$$L^2_+(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) \mid \mathrm{supp}\,(\hat{f}) \subset [0, \infty) \right\}$$

and

$$L^2_-(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}) \mid \mathrm{supp}\,(\hat{f}) \subset (-\infty, 0] \right\} .$$

Let $\mathcal{H}(\mathbb{C}_+)$ be the space of all holomorphic functions on $\mathbb{C}_+$ such that $f(\cdot + iy) \in L^2(\mathbb{R})$ for all $y > 0$, i.e.,

$$\int_{-\infty}^\infty |f(x + iy)|^2\, dx < \infty \qquad \forall y > 0$$

and such that the L^2-limit

$$\beta_+(f) := \lim_{y \to 0} f(\cdot + iy)$$

exists. Similarly we define $\mathcal{H}(\mathbb{C}_-)$, $\mathbb{C}_- := \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$, to be the space of all holomorphic functions on $\mathbb{C}_-$ such that $f(\cdot - iy) \in L^2(\mathbb{R})$ for all $y > 0$, i.e.,

$$\int_{-\infty}^\infty |f(x - iy)|^2\, dx < \infty \qquad \forall y > 0$$

and such that the L^2-limit

$$\beta_-(f) := \lim_{y \to 0} f(\cdot - iy)$$

exists. Show the following:

(a) Let $f \in L^2_+(\mathbb{R})$. Then the f extends to a holomorphic function $F_+(z)$ on $\mathbb{C}_+$. It is given by

$$F_+(z) = \int \hat{f}(x) e^{2\pi izx}\, dx .$$

Furthermore $F_+ \in \mathcal{H}(\mathbb{C}_+)$ and $\beta_+(F_+) = f$.

(b) Let $f \in L^2_-(\mathbb{R})$. Then f extends to a function $F_- \in \mathcal{H}(\mathbb{C}_-)$ such that $\beta_-(F_-) = f$.

(c) If $f \in L^2(\mathbb{R})$ then there exists $F \in \mathcal{H}(\mathbb{C}_+)$ and $G \in \mathcal{H}(\mathbb{C}_-)$ such that

$$f = \beta_+(F) - \beta_-(G) .$$

Thus any L^2-function on $\mathbb{R}$ is the difference of a boundary value of a holomorphic function on the upper half plane and a holomorphic function on the lower half plane.

21. Let X be a Hausdorff topological space, and let $\mathcal{H} \subset \mathcal{C}(X)$ be a Hilbert space. Assume that the maps

$$\mathrm{ev}_x : \mathcal{H} \to \mathbb{C} , \quad f \mapsto f(x)$$

are continuous for all $x \in X$. Show there exists a function $K : X \times X \to \mathbb{C}$ such that the following holds:

(a) If $K_y : X \to \mathbb{C}$ is the function $K_y(x) = K(x, y)$, then $K_y \in \mathcal{H}$ for all y. In particular K_y is continuous.

(b) We have $f(y) = (f \mid K_y)$ for all $y \in X$ and all $f \in \mathcal{H}$.

(c) $K(x, y) = \overline{K(y, x)}$ for all $x, y \in X$. In particular $y \mapsto K(x, y)$ is continuous for all $x \in X$.

(d) Let $x, y \in X$, then $(K_y \mid K_x) = K(x, y)$. In particular $||K_x||^2 = K(x, x)$ for all $x \in X$.

(e) The linear span

$$\mathcal{H}_0 := \left\{ \sum_{\text{finite}} c_j K_{x_j} \in \mathcal{H} \mid c_j \in \mathbb{C},\, x_j \in X \right\}$$

is dense in $\mathcal{H}$.

(f) The function K is positive definite in the sense, that for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in X$, and $c_1, \ldots, c_k \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{k} c_i \bar{c}_j K(x_j, x_i) \geqslant 0 .$$

The function K is called a *reproducing kernel for the Hilbert space* $\mathcal{H}$.

22. (The Fock space and the Segal-Bargmann transform) This exercise is a set of problems dealing with the Fock space $\mathcal{F}(\mathbb{C}^n)$ of holomorphic functions on $\mathbb{C}^n$ and the Segal-Bargmann transform $B : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n)$.

(a) Let $\mathcal{F}(\mathbb{C}^n)$ be the space of holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ such that

$$||F||^2 := \pi^{-n} \int |F(x + iy)|^2 \, e^{-|z|^2} \, dxdy < \infty$$

where $z = x + iy$. For $F, G \in \mathcal{F}(\mathbb{C}^n)$, let

$$(F \mid G) = \pi^{-n} \int F(x+iy)\, \overline{G(x+iy)}\, e^{-|z|^2}\, dxdy.$$

Show $\mathcal{F}(\mathbb{C}^n)$ is a Hilbert space.

(b) Show every polynomial p is in $\mathcal{F}(\mathbb{C}^n)$ and the vector space $\mathbb{C}[z_1, \ldots, z_n]$ of polynomials is dense in $\mathcal{F}(\mathbb{C}^n)$.

(c) Evaluate the norm of the monomials $p_\alpha(z) = z^\alpha, \alpha \in \mathbb{N}_0^n$.

(d) Show $\left\{ ||p_\alpha||^{-1}\, p_\alpha \mid \alpha \in \mathbb{N}_0^n \right\}$ is an orthonormal basis for $\mathcal{F}(\mathbb{C}^n)$.

(e) Define $R : \mathcal{F}(\mathbb{C}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ by

$$RF(x) = e^{-|x|^2/2} F(x)\,.$$

Show R is injective and that $Rp \in L^2(\mathbb{R}^n)$ for every polynomial p.

(f) Show $\mathrm{Im}(R)$ is dense in $L^2(\mathbb{R}^n)$. Hence $R : \mathcal{F}(\mathbb{C}^n) \to L^2(\mathbb{R}^n)$ is densely defined. (Hint: Use the Hermite polynomials.)

(g) Show $\mathcal{F}(\mathbb{C}^n) \to \mathbb{C}$, $F \mapsto F(z)$, is continuous for each $z \in \mathbb{C}^n$.

(h) Show the reproducing kernel for $\mathcal{F}(\mathbb{C}^n)$ is given by

$$K(z, w) = e^{z \cdot \overline{w}}$$

where $u \cdot v = \sum_{j=1}^n u_j v_j$.

(i) Show R is closed.

(j) As R is closed and densely defined the map $R^* : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n)$ is well defined. Show that

$$R^* f(z) = e^{z \cdot z/2} f * H(z)$$

where $H(z) = e^{-\frac{1}{2} z \cdot z}$. In particularly $RR^* f(z) = f * H(z)$. (Hint: Use that $R^* f(z) = (R^* f \mid K_z)$ where as usually $K_z(w) = K(w, z)$.)

(k) Using the heat semigroup Show $\sqrt{RR^*}(f)(x) = (2\pi)^{-n/2} f * L(x)$ where $L(z) = H(\sqrt{2} z)$.

(l) Polarizing R we can write $R = B\sqrt{RR^*}$ where $B : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{C}^n)$ is an unitary isomorphism. Show B is given by

$$Bf(z) = \left(\frac{2}{\pi}\right)^{n/4} \int f(x) e^{-z \cdot x + 2x \cdot z - \frac{1}{2} x \cdot z}\, dx\,.$$

The map B is the so-called **Segal-Bargmann transform**.

3. Applications: The wave equation

It is well known that waves behaves differently in even and odd dimension. In odd space dimensions the solution vanishes in finite time, depending on the support of the initial data, whereas this is not necessarily the case in even dimensions. Mathematically this can be formulated by saying that the solution $u(x,t)$ of the Cauchy problem

$$\begin{aligned} \Delta_x u(x,t) &= c^{-2} u_{tt}(x,t) \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned} \tag{4.7}$$

when $f, g \in \mathcal{D}_R(\mathbb{R}^n)$ and $c > 0$ satisfies the *Huygens' principle* when $n = 2k+1$ and not necessarily when $n = 2k$. Notice that if the function $u(x,t)$ satisfies (4.7) for $c = 1$, then the function

$$v(x,t) = u(x, t/c)$$

satisfies (4.7). Thus we can assume that $c = 1$. We can the state the Huygens' principle by saying that:

$$u(x,t) = 0 \quad \text{if} \quad |t| \geqslant R + |x| \,.$$

We start by taking the Fourier transform $\mathcal{F}_x$ in the x-variable and write $\hat{u}(\omega, t) = \mathcal{F}_x u(x,t)$. This results in the Cauchy problem

$$\frac{d^2}{dt^2}\hat{u}(\lambda,t) = -(2\pi)^2 |\lambda|^2 \hat{u}(\lambda,t), \quad \hat{u}(\lambda,0) = \hat{f}(\lambda), \quad \frac{d}{dt}\hat{u}(\lambda,0) = \hat{g}(\lambda)\,.$$

For each fixed λ this Cauchy problem has a unique solution

$$\hat{u}(\lambda,t) = \hat{g}(\lambda)\cos(2\pi|\lambda|t) + \hat{f}(\lambda)\frac{\sin(2\pi\,|\lambda|\,t)}{2\pi\,|\lambda|}\,.$$

Taking the inverse Fourier transform and using Corollary 2.26, we obtain the solution

$$\begin{aligned} u(x,t) &= \int \left(\hat{g}(\lambda)\cos(2\pi\,|\lambda|\,t) + \hat{f}(\lambda)\frac{\sin(2\pi\,|\lambda|\,t)}{2\pi\,|\lambda|}\right) e^{2\pi i\lambda\cdot x}\,d\lambda \\ &= \int_0^\infty \int_{S^{n-1}} \left(\hat{g}(r\omega)\cos(2\pi rt) + \hat{f}(r\omega)\frac{\sin(2\pi rt)}{2\pi r}\right) e^{2\pi i r\omega\cdot x} r^{n-1}\,d\sigma(\omega)\,dr\,. \end{aligned}$$

Now assume that $n = 2k+1$ is odd. To simplify the argument, we consider each part in the inner integral separately.

First, for $r > 0$, set

$$G(x,r) = r^{n-1}\int_{S^{n-1}} \hat{g}(r\omega) e^{2\pi i r\omega\cdot x}\,d\sigma(\omega).$$

Since S^{n-1} is compact and $\hat{g} \in \mathrm{PW}_{2\pi R}(\mathbb{C}^n)$, we can define $G(x,z)$ for $z \in \mathbb{C}$ by

$$G(x,z) = z^{n-1} \int_{S^{n-1}} \hat{g}(z\omega) e^{2\pi i z\omega\cdot x}\, d\sigma(\omega).$$

Note for each $x \in \mathbb{R}^n$, the function $G(x,z)$ is holomorphic in z. Moreover by Theorem 4.21, $\hat{g} \in \mathrm{PW}_{2\pi R}(\mathbb{C}^n)$ and thus by Remark 4.22, H defined by

$$H(\xi) = \hat{g}(\xi) e^{2\pi i \xi\cdot x}$$

is in $\mathrm{PW}_{2\pi(R+|x|)}(\mathbb{C})$. Thus given N, there is a $C > 0$ with

$$|H(\xi)| \leqslant C(1+|\xi|^2)^{-N} e^{(2\pi R+|x|)|\mathrm{Im}(\xi)|}.$$

Since $\omega \in S^{n-1} \subseteq \mathbb{R}^n$, we see

(a) $|G(x,a+ib)| \leqslant C\sigma(S^{n-1})(1+(a^2+b^2))^{-N} e^{2\pi(R+|x|)|b|}(a^2+b^2)^{(n-1)/2}$

and consequently $G(x,\cdot)$ is in $\mathrm{PW}(\mathbb{C})$ for each $x \in \mathbb{R}^n$.

As $n-1$ is even and the surface measure $d\sigma$ on S^{n-1} is invariant under the inversion $\omega \mapsto -\omega$, it follows that $G(z) = G(-z)$, i.e. G is even. Now by Remark 4.22, $z \mapsto G(x,z)e^{2\pi i z t}$ is in $\mathrm{PW}(\mathbb{C})$. Using all this and Theorem 4.18, we see if u_1 is defined by $u_1(x,t) = \int_0^\infty G(x,r)\cos(2\pi r t)\, dr$, then

$$\begin{aligned} u_1(x,t) &= \int_0^\infty G(x,t) \frac{e^{2\pi i r t} + e^{-2\pi i r t}}{2}\, dr \\ &= \frac{1}{2}\int_0^\infty G(x,t) e^{2\pi i r t}\, dr + \frac{1}{2}\int_0^\infty G(x,-t) e^{-2\pi i r t}\, dr \\ &= \frac{1}{2}\int_{-\infty}^\infty G(x,r) e^{2\pi i r t}\, dr \\ &= \frac{e^{-2\pi b t}}{2}\int_{-\infty}^\infty G(x,r+ib) e^{2\pi i r t}\, dr\,. \end{aligned}$$

By (a) if we take $N > \frac{n-1}{2} + 1 = k+1$ and $C' = C\sigma(S^{n-1})$, then

$$\begin{aligned} |u_1(x,t)| &\leqslant e^{2\pi(R+|x|-t)b} C' \int_{-\infty}^\infty \frac{(r^2+b^2)^{(n-1)/2}}{(1+r^2+b^2)^N}\, dr \\ &\leqslant e^{2\pi(R+|x|-t)b} C' \int_{-\infty}^\infty \frac{1}{1+r^2}\, dr \\ &\leqslant e^{2\pi(R+|x|-t)b} C' \pi\,. \end{aligned}$$

If $t > R+|x|$ then by letting $b \to \infty$, we see

$$u_1(x,t) = 0\,.$$

If $t < 0$ and $|t| > R+|x|$ the same conclusion holds by letting $b \to -\infty$.

For the second integral we use a similar argument where for $x \in \mathbb{R}^n$ and $r,\, t \in \mathbb{R}$ we define $H(x,r)$ and $u_2(x,t)$ by

$$H(x,r) = r^{n-2} \int_{S^{n-1}} \hat{f}(r\omega)\, e^{2\pi i r\omega\cdot x}\, d\sigma(\omega)$$

and

$$u_2(x,t) = \int_0^\infty H(x,r) \sin(2\pi r t)\, dr\,.$$

Again since $n \geqslant 3$, we note that $r \mapsto H(x,r)$ extends to a holomorphic function on $\mathbb{C}$ such that

$$\begin{aligned}|H(x,a+ib)| &\leqslant C(1+a^2+b^2)^{-N} e^{2\pi(R+|x|)|b|}(a^2+b^2)^{(n-1)/2} \\ &\leqslant C' e^{2\pi(R+|x|)|b|}(1+r^2)^{-1}\,.\end{aligned}$$

Furthermore, since $n-2$ is odd, $H(x,r)$ is odd as a function of r. As $r \mapsto \sin(2\pi r t)$ is odd, we see

$$\begin{aligned}u_2(x,t) &= \frac{1}{2i}\int_0^\infty H(x,r)\left(e^{2\pi i r t} - e^{-2\pi i r t}\right)\, dr \\ &= \frac{1}{2i}\int_0^\infty H(x,r)e^{2\pi i r t} + H(x,-r)e^{-2\pi r t}\, dr \\ &= \frac{1}{2i}\int_{-\infty}^\infty H(x,r)e^{2\pi i r t}\, dr\,.\end{aligned}$$

and just as before it follows that for $t > R + |x|$ we have

$$u_2(x,t) = 0\,.$$

Since $u(x,t) = u_1(x,t) + u_2(x,t)$, one sees $u(x,t) = 0$ for $|t| \geqslant R + |x|$.

The case $n = 1$ is handled differently. We define

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(u)\, du\,.$$

Then

$$u_x(x,t) = \frac{1}{2}(f'(x+t) + f'(x-t)) + \frac{1}{2}\left(g(x+t) - g(x-t)\right)$$

and

$$u_{xx}(x,t) = \frac{1}{2}(f''(x+t) + f''(x-t)) + \frac{1}{2}\left(g'(x+t) - g'(x-t)\right)\,.$$

Similarly

$$u_t(x,t) = \frac{1}{2}(f'(x+t) - f'(x-t)) + \frac{1}{2}\left(g(x+t) + g(x-t)\right)$$

and hence

$$u_{tt}(x,t) = \frac{1}{2}(f''(x+t) + f''(x-t)) + \frac{1}{2}\left(g'(x+t) - g'(x-t)\right)\,.$$

It follows that $u_{xx}(x,t) = u_{tt}(x,t)$. Furthermore

$$u(x,0) = f(x)$$

and

$$\begin{aligned} u_t(x,0) &= \frac{1}{2}(f'(x+t) - f'(x-t)) + \frac{1}{2}\left(g(x+t) + g(x-t)\right)\Big|_{t=0} \\ &= g(x)\,. \end{aligned}$$

Hence $u(x,t)$ is the solution to the Cauchy problem (4.7). But if $t > |x| + R$ then

$$f(x+t) + f(x-t) = 0$$

and

$$\int_{x-t}^{x+t} g(u)\,du = \int_{-\infty}^{\infty} g(u)\,du\,.$$

Hence $u(x,t) = 0$ for $t > |x| + R$ if and only if $\int_{-\infty}^{\infty} g(u)\,du = 0$. This condition can be formulated in the following way. Let

$$h(x) = \int_{-\infty}^{x} g(u)\,du\,.$$

Then $h(x) = 0$ if $|x| \geqslant R$, h is smooth, and $h'(x) = g(x)$. This is equivalent to $g \in \frac{d}{dx}\mathcal{D}_R(\mathbb{R})$ and we have:

Lemma 4.23. *Suppose that $n = 1$. Then the solution $u(x,t)$ to the Cauchy problem (4.7) satisfies the Huygens' principle if and only if $g \in \frac{d}{dx}\mathcal{D}_R(\mathbb{R})$.*

Exercise Set 4.2

1. Use the Fourier transform to solve the Cauchy problem

$$\frac{\partial u}{\partial x}(x,t) = \frac{\partial u}{\partial t}(x,t), \quad u(x,0) = f(x)$$

for some $f \in \mathcal{S}(\mathbb{R}^n)$.

4. Distributions

In general we can not talk about the derivative of a generic L^p-function. The function $\chi_{[0,1]}$ is differentiable almost everywhere, and the derivative where it exists is zero. But what about the jump points 0 and 1? The theory of distributions – or generalized functions, as they are sometimes called – allows us to differentiate *any* locally integrable function as many times as we please! The result is then, in general, *not* a function, but a *distribution*.

Let us start by reviewing the dual of a topological vector space. Let V, W be complex, locally convex, Hausdorff topological vector spaces. $\mathrm{Hom}(V,W)$

will denote the space of *continuous* linear transformations $T : V \to W$. We make $\mathrm{Hom}(V, W)$ into a vector space by defining

$$(\lambda T + S)(v) := \lambda T(v) + S(v)$$

for $T, S \in \mathrm{Hom}(V, W)$ and $\lambda \in \mathbb{C}$. Then the dual V' is defined by

$$V' = \mathrm{Hom}(V, \mathbb{C})\,.$$

For $\nu \in V'$ and $u \in V$ we write

$$\nu(u) = \langle u, \nu \rangle$$

to underline the duality between the two spaces V and V'. The space V' separates points in V, i.e.; if $u \neq 0$ then there exists a $\nu \in V'$ such that $\langle u, \nu \rangle \neq 0$.

Now if V is a topological vector space, then the weak $*$ topology on V' is the locally convex Hausdorff topology on V' given by the semi-norms $|\cdot|'_v$ for $v \in V$ where

$$|\nu|'_v = |\langle v, \nu \rangle| = |\nu(v)| \tag{4.8}$$

This is the *weakest locally convex topology* that makes all the linear functionals

$$V' \ni \nu \mapsto \langle u, \nu \rangle \in \mathbb{C}$$

for $u \in V$ continuous.

Let Ω be a nonempty subset of $\mathbb{R}^n$. In Definition 2.64, we defined the topological vector space $D(\Omega)$. It is the inductive limit topology on $C_c^\infty(\Omega)$ of the subspaces $D_K(\Omega)$. The locally convex topology on $D_K(\Omega)$ is defined using the semi-norms $|\cdot|_{K,\alpha}$ where

$$|\phi|_{K,\alpha} = \max_{x \in K} |D^\alpha \phi(x)|.$$

Definition 4.24. *Let $\varnothing \neq \Omega \subseteq \mathbb{R}^n$ be an open set. The elements of $\mathcal{D}(\Omega)'$ are called* ***distributions*** *or* ***generalized functions***. *The elements of $\mathcal{D}(\Omega)$ are called* ***test functions***.

If T is a distribution and φ a test function, then we introduce the following notation for the value $T(\varphi)$ of T at the point φ:

$$T(\varphi) = \langle \varphi, T \rangle = \int \varphi(x)\, dT(x)\,.$$

By Propositions 2.15 and 2.73 we have the following criteria for deciding whether a linear functional T on $\mathcal{D}(\Omega)$ is a distribution:

(9) *T is a distribution if and only if the restriction of T to $\mathcal{D}_K(\Omega)$ is continuous for every compact set $K \subset \Omega$.*

$$(10) \qquad \begin{array}{l} T \textit{ is a distribution if and only if } T \textit{ is sequentially contin-} \\ \textit{uous; i.e. if } \varphi_j \to \varphi \textit{ in } \mathcal{D}(\Omega), \textit{ then } T(\varphi_j) \to T(\varphi). \end{array}$$

Example 4.25 (The δ-distributions)**.** *Let $x \in \mathbb{R}^n$. Define $\delta_x : \mathcal{D} \to \mathbb{C}$ by*

$$\langle \varphi, \delta_x \rangle := \varphi(x) .$$

As $|\langle \varphi, \delta_x \rangle| \leqslant |\varphi|_{K,0}$ for all compact sets K containing x it follows that δ_x is continuous. When $x = 0$ we use the notation δ for δ_0.

Example 4.26 (Locally integrable functions)**.** *Let $\Omega \neq \varnothing$ be open. A measurable function $f : \Omega \to \mathbb{C}$ is called locally integrable if*

$$\int_K |f(x)|\, dx < \infty$$

for all compact sets $K \subset \Omega$. Let $\mathcal{L}_{loc}(\Omega)$ be the space of locally integrable functions and let

$$L_{loc}(\Omega) := \mathcal{L}_{loc}(\Omega) / \{ f \in \mathcal{L}_{loc}(\Omega) \mid f(y) = 0 \ a.e.\} .$$

Note that $L^p(\Omega) \subset L_{loc}(\Omega)$ for all $1 \leqslant p \leqslant \infty$. For $f \in L_{loc}(\Omega)$ define $T_f : \mathcal{D}(\Omega) \to \mathbb{C}$ by

$$\langle \varphi, T_f \rangle = \int \varphi(x) f(x)\, dx .$$

Let K be a compact set containing supp (φ)*. Then*

$$|\langle \varphi, T_f \rangle| \leqslant \left(\int_K |f\,(x)|\, dx \right) |\varphi|_\infty = C_K |\varphi|_{K,0}$$

where $C_K = \int_K |f\,(x)|\, dx < \infty$. Hence T_f is continuous, and thus is a distribution. We say that a distribution T is a locally integrable function if there exists a $f \in L_{loc}(\Omega)$ such that $T = T_f$. In that case Exercise 4.3.4 shows f is unique and we simply write $T = f$. Thus $T = f$ if and only if

$$\langle \varphi, T \rangle = \int \varphi(x) f(x)\, dx$$

for all test functions ϕ. We say that T is an L^p-function, a smooth function, etc. if $T = f$ with $f \in L^p$, f smooth, etc. Let $H(x) = \chi_{[0,\infty)}$. Then H and the corresponding distribution

$$\mathcal{D}(\mathbb{R}) \ni \varphi \mapsto \langle \varphi, H \rangle = \int_0^\infty \varphi(x)\, dx \in \mathbb{C}$$

is called the Heaviside distribution.

Theorem 4.27. *Let $\{T_n\}$ be a sequence of distributions on Ω such that $\{T_n(\varphi)\}$ converges for every test function φ in $\mathcal{D}(\Omega)$. Then $T : \mathcal{D}(\Omega) \to \mathbb{C}$ defined by*

$$T(\varphi) = \lim_{n\to\infty} T_n(\varphi)$$

is a distribution.

Proof. This follows from Theorem 2.18, the Uniform Boundedness Principle. $\square$

Recall that we have

$$\mathcal{D}(\mathbb{R}^n) \overset{\iota}{\hookrightarrow} \mathcal{S}(\mathbb{R}^n) \overset{\kappa}{\hookrightarrow} \mathcal{E}(\mathbb{R}^n)\,. \tag{4.11}$$

Here ι and κ are the canonical inclusion maps. Both maps are continuous with dense image, c.f. Theorem 2.67. We will use this sequence to clarify the connection between elements in the dual of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{E}(\Omega)$ and distributions. For that we first recall the some facts on the *transpose* of a linear transformation. Let V and W be topological vector spaces and suppose $T : V \to W$ is a continuous linear transformation. Then the transpose T^t is the linear transformation from W' to V' defined by

$$\langle v, T^t(\nu)\rangle = \langle T(v), \nu\rangle$$

for $\nu \in W'$ and $v \in V$. Since T and ν are continuous, we see $T^t(\nu)$ is a composition of continuous transformations and thus is continuous. Moreover, the linear map $T^t : W' \to V'$ is continuous in the weak $*$ topologies since

$$|T^t(\nu)|_v = |\langle v, T^t(\nu)\rangle| = |\langle T(v), \nu\rangle| = |\nu|_{T(v)}.$$

We will also need the following Lemma:

Lemma 4.28. *Assume that $T(V) \subseteq W$ is dense in W. Then $T^t : W' \to V'$ is injective.*

Proof. Assume $T^t(\nu) = 0$. Then $\langle v, T^t(\nu)\rangle = 0$ for all $v \in V$. This says $\langle T(v), \nu\rangle = 0$ for all $v \in V$. Thus ν vanishes on a dense subset of W and we see $\nu = 0$. $\square$

As a result of this general discussion, we see (4.11) gives the following sequence of injective continuous maps:

$$\mathcal{E}(\mathbb{R}^n)' \overset{\kappa^t}{\hookrightarrow} \mathcal{S}(\mathbb{R}^n)' \overset{\iota^t}{\hookrightarrow} \mathcal{D}(\mathbb{R}^n)'\,.$$

Similarly we have $\mathcal{E}(\Omega)' \hookrightarrow \mathcal{D}(\Omega)'$ for any open set $\Omega \neq \varnothing$. Notice, that for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $T \in \mathcal{E}(\mathbb{R}^n)'$ we have:

$$\langle (\kappa \circ \iota)^t(T), \varphi\rangle = \langle T, (\kappa \circ \iota)(\varphi)\rangle = \langle T, \varphi\rangle\,.$$

Thus $(\kappa \circ \iota)^t(T) = T|_{\mathcal{D}(\mathbb{R}^n)}$, or the inclusion $\mathcal{E}(\mathbb{R}^n)' \hookrightarrow \mathcal{D}(\mathbb{R}^n)'$ is simply the restriction map. The same holds for the other inclusions. Thus we can view both $\mathcal{S}(\mathbb{R}^n)'$ and $\mathcal{E}(\mathbb{R}^n)'$ as subsets of $\mathcal{D}(\mathbb{R}^n)'$. Namely, they are the distributions on $\mathbb{R}^n$ which have extensions as continuous linear functionals to the larger spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R}^n)$.

Let $U \subseteq \Omega \subseteq \mathbb{R}^n$ be open. We say that a distribution $T \in \mathcal{D}(\Omega)'$ vanishes on U if $\langle f, T\rangle = 0$ for all $f \in \mathcal{D}(\Omega)$ with $\mathrm{supp}\,(f) \subset U$. Let U_T be the union of all open sets U such that T vanishes on U. Then U_T is open and hence $\Omega \backslash U_T$ is closed in Ω. Define the support of T by:

$$\operatorname{supp}(T) = \Omega \backslash U_T\,. \tag{4.12}$$

Lemma 4.29. *Suppose $\phi \in \mathcal{D}(\Omega)$ and $\operatorname{supp}\phi \subseteq U_1 \cup U_2 \cup \cdots \cup U_N$ where $U_1, U_2, \ldots, U_N$ are open subsets of Ω. Then there are $\phi_1, \phi_2, \ldots, \phi_N \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\phi_i \subseteq U_i$ and $\phi = \phi_1 + \phi_2 + \cdots + \phi_N$.*

Proof. Let $K = \operatorname{supp}\phi$. Then for each $p \in K$, choose a closed ball B_p of radius $r_p > 0$ and center p with $B_p \subseteq U_i$ for some i. Since K is compact, we can choose a finite sequence B_{p_k} for $k = 1, 2, \ldots, m$ of such balls that cover K. Set K_i to be the union of the sets $B_{p_k} \cap K$ where $B_{p_k} \subseteq U_i$. Then K_i is a compact subset of U_i and $K = \cup_{i=1}^{N} K_i$. By Corollary 2.54, there is a $\psi_i \in C_c^\infty(U_i)$ with $0 \leqslant \psi_i \leqslant 1$ and $\psi_i = 1$ on K_i.

Set $\phi_i = \frac{\phi\psi_i}{\sum \psi_i}$ where we define $\frac{0}{0}$ to be 0. Note ϕ_i is C^∞ on the open set where $\sum \psi_i > 0$ and thus is C^∞ on an open set containing the support K of ϕ. Moreover, each ϕ_i is 0 on the open set $\Omega - K$. Thus each ϕ_i is in $C_c^\infty(U_i)$ and clearly $\sum \phi_i = \phi$.

□

Proposition 4.30. *Let T be a distribution on nonempty open subset Ω. Then T vanishes on $\Omega \backslash \operatorname{supp} T$.*

Proof. Let ϕ have support K where $K \subseteq \Omega \backslash \operatorname{supp} T = U_T$. Since U_T is the union of all open sets $U \subseteq \Omega$ where T vanishes on U, we see these sets U form an open cover of K. Since K is compact there are amongst these sets a finite collection $U_1, U_2, \ldots, U_m$ which cover K. By Lemma 4.29, there exists $\phi_i \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\phi_i \subseteq U_i$ and

$$\phi_1 + \phi_2 + \cdots + \phi_m = \phi.$$

But since T vanishes on each U_i, we see $T(\phi) = \sum_{i=1}^{m} T(\phi_i) = 0$. So T vanishes on U_T. □

Lemma 4.31. *Let $T \in \mathcal{D}'(\Omega)$. Then $T \in \mathcal{E}(\Omega)'$ if and only if $\operatorname{supp}(T)$ is compact in Ω.*

Proof. Let $T \in \mathcal{E}(\Omega)'$. Since T is continuous there is a compact set K, a finite subset $F \subseteq \mathbb{N}_0^n$, and a $M > 0$ such that if $|f|_{K,\alpha} \leqslant \frac{1}{M}$ for $\alpha \in F$, then $|T(f)| \leqslant 1$. This implies $|T(f)| \leqslant M \max\{|f|_{K,\alpha} \mid \alpha \in F\}$ for $f \in \mathcal{E}(\Omega)$. Consequently, if f has compact support in the complement of K in Ω, $T(f) = 0$; and we see T vanishes off K and thus has compact support.

Conversely, if T is a distribution with compact support K, we can by Corollary 2.54 and Exercise 4.3.1 find a $\psi \in C_c^\infty(\Omega)$ with $\psi = 1$ on an open set $V \subseteq \Omega$ containing K. Extend T to all of $\mathcal{E}(\Omega)$ by

$$T^e(f) = T(\psi f).$$

T^e is continuous on $\mathcal{E}(\Omega)$ for the mapping $f \mapsto f\psi$ is a continuous linear map from $\mathcal{E}(\Omega)$ into $\mathcal{D}_K(\Omega)$ and by the definition of inductive limit topology, the inclusion mapping from $\mathcal{D}_K(\Omega)$ into $\mathcal{D}(\Omega)$ is continuous. Moreover $T^e = T$ on $\mathcal{D}(\Omega)$. Indeed, if $f \in \mathcal{D}(\Omega)$, then $f - \psi f$ is 0 off V and hence has support in $\Omega \backslash V \subseteq \Omega \backslash K$. Since T vanishes on $\Omega \backslash K$, we see $T(f - \psi f) = 0$. Hence $T^e(f) = T(f)$. $\square$

Definition 4.32. *A distribution T on $\mathbb{R}^n$ is **a tempered distribution** if T has a continuous extension to $\mathcal{S}(\mathbb{R}^n)$; since this extension is unique, we also call it T.*

Since a distribution T on a nonempty open subset Ω of $\mathbb{R}^n$ has compact support if and only if it is the restriction of a unique continuous linear functional on $\mathcal{E}(\Omega)$, we identify T with this unique extension and again call the extension T.

Definition 4.33. *Let Ω be a nonempty open subset of $\mathbb{R}^n$ and suppose $f : \Omega \to \mathbb{C}$ is a measurable function. Then the* **support** *of f is the complement in Ω of the union of all open subset V of Ω for which $f\chi_V$ is 0 a.e.*

Note a point x in Ω is in the support of f if and only if there is an open neighborhood V of x such that $f = 0$ a.e. on V. It is an easy exercise to show if f is continuous, then the support of f is the closure in Ω of the set $\{x \in \Omega \mid f(x) \neq 0\}$. Moreover, two measurable functions equal almost everywhere have the same support.

Remark 4.34. Let $f \in L_{\mathrm{loc}}(\Omega)$. Then $\operatorname{supp} f = \operatorname{supp} T_f$.

Indeed, T_f vanishes on an open subset V of Ω if and only if $T_f(\phi) = 0$ for all test functions ϕ with $\operatorname{supp} \phi \subseteq V$ if and only if $\int_V \phi(x) f(x)\, dx = 0$ for all such ϕ. The conclusion will then follow if one shows $\int_V \phi(x)\, f(x)\, dx = 0$ for all $\phi \in C_c^\infty(\Omega)$ with $\operatorname{supp} \phi \subseteq V$ if and only if $f = 0$ a.e. on V. But this we leave as an exercise.

Example 4.35. *Suppose $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ is is dominated by a polynomial function $p(x)$. Hence $|f(x)| \leqslant p(x)$ a.e. x. Note if $P(x)$ is the polynomial $(1 + |x|^2)p(x)$ and ϕ is a Schwartz function, we have $|\phi|_{P,0} = \max |(1 + |x|^2)^n p(x)\phi(x)| < \infty$. Thus T_f defined by*

$$T_f(\phi) = \int f(x)\phi(x)\, dx \text{ for } \phi \in \mathcal{S}(\mathbb{R}^n)$$

satisfies

$$\begin{aligned}|\langle \phi, T_f\rangle| &\leqslant \int |f(x)\,\phi(x)|\,dx\\ &\leqslant \int |p(x)\,\phi(x)|\,dx\\ &= |\phi|_{P,0}\int \frac{1}{(1+|x|^2)^n}\,dx.\end{aligned}$$

By Corollary 2.61, $|\cdot|_{P,0}$ is a continuous seminorm. Thus T_f on $\mathcal{D}(\mathbb{R}^n)$ has a continuous extension to $\mathcal{S}(\mathbb{R}^n)$ and hence is a tempered distribution .

Example 4.36. *Let Ω be a nonempty open subset of $\mathbb{R}^n$ and let μ be a Borel measure on Ω. Then every $f \in \mathcal{D}(\Omega)$ is integrable if and only if $\mu(K) < \infty$ for all compact subsets K of Ω. For open subsets Ω of $\mathbb{R}^n$, one has $\mu(K) < \infty$ for all compact subsets K of Ω if and only if μ is a Radon measure on Ω. For a discussion of Radon measures see the material at the beginning of Section 1 in Chapter 6.*

*Suppose μ is a Radon measure on Ω. Note if $\phi \in \mathcal{D}_K(\Omega)$ where K is a compact subset of Ω, then $|\langle \phi, T_\mu\rangle| \leqslant \int |\phi|\ d\mu \leqslant |\phi|_\infty \mu(\mathrm{supp}\phi) \leqslant |\phi|_{K,0}\mu(K)$. Thus by (4), T_μ is distribution on Ω. In the case when $\Omega = \mathbb{R}^n$ and T_μ has a continuous extension to $\mathcal{S}(\mathbb{R}^n)$, we say μ is a **tempered measure**. Moreover, by Exercise 4.3.5, the distribution T_μ on Ω has a continuous extension to $\mathcal{E}(\Omega)$ if and only if μ has compact support; that is there is a compact subset K of Ω such that $\mu(\Omega\backslash K) = 0$.*

For example if μ is a measure on $\mathbb{R}^n$ such that $\int (1+|x|^2)^r\,d\mu(x) < \infty$ is finite for some $r \in \mathbb{R}$, then μ is a tempered measure. Indeed, we have for any nonnegative integer $m \geqslant -r$,

$$\begin{aligned}|\langle \phi, T_\mu\rangle| &\leqslant \int |\phi(x)|\,d\mu(x)\\ &= \int (1+|x|^2)^m |\phi(x)|\,(1+|x|^2)^{-m}\,d\mu(x)\\ &\leqslant |\phi|_{m,0}\int (1+|x|^2)^{-m}\,d\mu(x)\\ &\leqslant |\phi|_{m,0}\int (1+|x|^2)^{r}\,d\mu(x).\end{aligned}$$

Example 4.37. *Assume that f is a measurable function such that for some $r \in \mathbb{R}$ and $1 \leqslant p < \infty$, we have*

$$\int |f(x)|^p (1+|x|^2)^r\,dx < \infty.$$

Then T_f is a tempered distribution. To see this we first consider $T_{|f|}$. Note $T_{|f|}(\phi) = \int |f(x)|\, \phi(x)\, dx$ and so $T_{|f|}$ is T_μ where μ is the measure defined by

$$\mu(E) = \int_E |f(x)|\, dx.$$

We thus know $T_{|f|}$ is tempered if the measure μ is tempered. Choose q with $\frac{1}{p} + \frac{1}{q} = 1$ and set $g_s(x) = (1 + |x|^2)^s$. Then

$$\begin{aligned}
\int (1 + |x|^2)^s\, d\mu(x) &= \int (1 + |x|^2)^s |f(x)|\, dx \\
&= \int (1 + |x|^2)^s |f(x)| (1 + |x|^2)^{r/p}\, dx \\
&\leqslant |g_s|_q \left(\int |f(x)|^p (1 + |x|^2)^r\, dx \right)^{1/p} \\
&< \infty
\end{aligned}$$

if s is chosen so that $|g_s|_q < \infty$. Moreover, as seen in the last part of Example 4.36, we have $\int |\phi(x)\, f(x)|\, dx = \int |\phi(x)|\, d\mu(x) \leqslant C|\phi|_{m,0}$ where $C = \int (1 + |x|^2)^s\, d\mu(x) = \int (1 + |x|^2)^s\, |f(x)|\, dx < \infty$ and m is any nonnegative integer with $m \geqslant -s$. From $|T_f(\phi)| = |\int f(x)\phi(x)\, dx| \leqslant \int |f(x)\phi(x)|\, dx \leqslant C|\phi|_{m,0}$, it follows that T_f is a tempered distribution.

Exercise Set 4.3

1. Show if Ω is a nonempty open subset of $\mathbb{R}^n$ and K is a compact subset of Ω, then there is an open set V with compact closure with $K \subseteq V \subseteq \bar{V} \subseteq \Omega$.

2. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Show the inclusion mapping from $\mathcal{D}(\Omega)$ into $\mathcal{D}(\mathbb{R}^n)$ is continuous and thus the restriction of every distribution to $\mathcal{D}(\Omega)$ is a distribution on Ω.

3. Let Ω be a nonempty subset of $\mathbb{R}^n$ and suppose $f : \Omega \to \mathbb{C}$ is a continuous function. Show the support of f as a measurable function is $\overline{\{x \in \Omega \mid f(x) \neq 0\}} \cap \Omega$.

4. Let V be an open subset of $\mathbb{R}^n$ and suppose $f \in L_{\text{loc}}(V)$. Show $\int \phi f\, dx = 0$ for all $\phi \in C_c^\infty(V)$ if and only if $f = 0$ a.e. on V.

5. Let μ be a Borel measure on Ω, a nonempty open subset of $\mathbb{R}^n$. Then the support of the measure μ (denoted by $\operatorname{supp} \mu$) is the complement of the union of all open subsets of Ω which have measure 0. Show if $\mu(K) < \infty$ for all compact subsets K of Ω, then the distribution T_μ on Ω has compact support if and only if $\operatorname{supp} \mu$ is a compact subset of Ω.

6. Let μ be a measure on an open subset Ω of $\mathbb{R}^n$. Show $\operatorname{supp} \mu = \{x \in \Omega \mid \mu(N) > 0 \text{ for every neighborhood } N \text{ of } x\}$

7. Let μ be a complex Borel measure on a nonempty open set Ω of $\mathbb{R}^n$. Define T_μ by

$$T_\mu(\phi) = \int \phi \, d\mu$$

for $\phi \in \mathcal{D}(\Omega)$. The support of the measure μ is the complement of the union of all open subsets U of Ω such that $\mu(E) = 0$ for all Borel subsets E of U. Show the support of the distribution T_μ is the support of the measure μ.

8. Prove Lemma 4.34.

9. Show $\operatorname{supp}(\delta_x) = \{x\}$.

10. Let $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geqslant 0$ and let $p(x)$ be a nonnegative polynomial. For $s \geqslant 0$ define $P^s : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$ by

$$P^s(\varphi) = \int_{\mathbb{R}^n} \varphi(x) p(x)^s \, ds \text{ for } \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Show P^s is a distribution, and that

$$\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 0\} \ni s \to P^s(\varphi) \in \mathbb{C}$$

is a holomorphic function in s for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

11. Suppose that $n = 1$ and let $p(x) = x^2$. Show the following:

(a) $\left(\frac{d}{dx}\right)^2 p(x)^s = 2s(2s-1)p(x)^{s-1}$.

(b) If $\operatorname{Re}(s) > 0$, then

$$P^s(f) = \frac{2^{-2}}{(s+1)(s+1/2)} \int f''(x) p^{s+1}(x) \, dx \, .$$

(c) The distribution P^s can be extended to $\{s \in \mathbb{C} \setminus \{1/2\} \mid \operatorname{Re}(s) > -1\}$ such that $P^s(f)$ is meromorphic by defining

$$P^s(\varphi) = \frac{2^{-2}}{(s+1)(s+1/2)} \int \varphi''(x) p^{s+1}(x) \, dx \, .$$

(d) P^s can be extended to all $s \in \mathbb{C}$ where $s \notin -\frac{1}{2}\mathbb{N}$.

(e) If $\varphi \in \mathcal{D}(\mathbb{R})$, then

$$\mathbb{C} \ni s \mapsto \frac{1}{\Gamma(2\lambda + 1)} P^s(\varphi) \in \mathbb{C}$$

is holomorphic.

12. Let $p(x) = |x|^2$. Find a differential operator D and a polynomial $\beta(s)$ such that

$$Dp(x)^s = \beta(s) p(x)^{s-1} \, .$$

Use this to extend the distribution P^s to almost all $s \in \mathbb{C}$.

13. Define a function $x_+ : \mathbb{R} \to [0, \infty]$ by $x_+ = x$ if $x \geqslant 0$ and $x_+ = 0$ if $x \leqslant 0$. Thus $x_+ = xH(x)$ where $H(x)$ is the Heaviside function. The function x_+^s is locally integrable for $\mathrm{Re}(s) > -1$ and hence defines a distribution which we denote by the same letter:

$$\langle x_+^s, \varphi \rangle = \int_0^\infty x^s \varphi(x)\, dx\,.$$

Show $s \mapsto x_+^s$ extends to $\{s \in \mathbb{C} \mid s \neq -1, -2, -3, \ldots\} = \mathbb{C} \setminus -\mathbb{N}$ such that

$$s \mapsto \langle x_+^s, \varphi \rangle$$

is holomorphic on the above set and

$$s \mapsto \frac{1}{\Gamma(\lambda + 1)} \langle x^s, \varphi \rangle$$

is holomorphic on $\mathbb{C}$.

14. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be linear and $\det F \neq 0$. If S is a distribution define $S \circ F$ by

$$S \circ F(\varphi) := |\det F|^{-1} \int \varphi(F^{-1}(x))\, dS(x)\,.$$

Show $S \circ F$ is a distribution. If $S = f$ is a locally integrable function then $S \circ F = f \circ F$.

15. **(Definition of the Principal value)** A function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called **homogeneous of degree** μ if for all $\lambda > 0$

$$f(\lambda x) = \lambda^\mu f(x)\,.$$

Similarly a distribution T is called **homogeneous of degree** μ if for all test functions φ we have

$$\int \lambda^{-n} \varphi(\lambda^{-1} x)\, dS(x) = \lambda^\mu \int \varphi(x)\, dS(x).$$

(a) Show the δ-distribution is homogeneous and find its degree.

(b) Assume that $f \in \mathcal{C}(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree $-n$ and that $\int_{S^{n-1}} f(\omega)\, d\sigma(\omega) = 0$. Define $\mathrm{PV}(f) : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$ by

$$\langle \mathrm{PV}(f), \varphi \rangle := \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus \overline{B_\epsilon(0)}} f(x)\varphi(x)\, dx\,.$$

Show $\mathrm{PV}(f)$ is a distribution. This distribution is called the **principal value** of f.

(c) Show for any $\epsilon > 0$

$$\langle \mathrm{PV}(f), \varphi \rangle = \int_{|x| \leqslant \epsilon} f(x)(\varphi(x) - \varphi(0))\, dx + \int_{|x| \geqslant \epsilon} f(x)\varphi(x)\, dx\,.$$

(d) Show $\mathrm{PV}(f)$ is homogeneous of degree $-n$.

16. Define distributions

$$\frac{1}{x \pm i0} = \lim_{y \to 0^\pm} \frac{1}{x + iy}$$

where the limit is taken in $\mathcal{D}'(\mathbb{R})$. Show

$$\delta_0 = \frac{1}{2\pi i}\left(\frac{-1}{x+i0} + \frac{1}{x-i0}\right).$$

Thus δ_0 is a *difference of boundary values* of a holomorphic function in the upper half plane and a holomorphic function in the lower half plane.

5. Differentiation of Distributions

Next we show how the notion of *differentiation* and some other simple operations on functions can be defined on distributions.

As we have seen every locally integrable function and hence every differentiable function f can be viewed as a distribution. The definition of the derivative of T_f if distributions are to be viewed as extensions of functions must satisfy

$$(T_f)' = T_{f'}$$

i.e., the derivative of f as a distribution is the same as the derivative of f. The motivation of *how these definitions are made* is that the extended operation should agree with the previous definition if the distribution is a function. Thus for a test function φ on $\mathbb{R}$ one has

$$\langle T_{f'}, \varphi\rangle = \int \varphi(x) f'(x)\, dx = -\int \varphi'(x) f(x)\, dx = -\langle T_f, \varphi'\rangle\,.$$

More generally, in $\mathbb{R}^n$, one has to define

$$D^\alpha T(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi)$$

for any multiindex $\alpha \in \mathbb{N}_0^n$. In fact every *differential operator* has an extension as an operator on distributions. Namely:

Definition 4.38. *Let $\Omega \neq \varnothing$ be open. A linear map $D = \sum_{|\alpha| \leqslant N} a_\alpha D^\alpha : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$, $a_\alpha \in \mathcal{E}(\Omega)$, is called a differential operator of order N if there exists an α with $|\alpha| = N$ and $a_\alpha \neq 0$. Denote by* $\operatorname{ord}(D)$ *the order of the differential operator D.*

Lemma 4.39. *Let D be a differential operator and $\varnothing \neq \Omega \subseteq \mathrm{R}^n$ be open. Then $D : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$, $\varphi \mapsto D\varphi$, is continuous.*

Proof. Let $K \subset \Omega$ be compact, and $\alpha \in \mathbb{N}_0^n$. Write $D = \sum_\beta a_\beta D^\beta$. Then

$$|D\varphi|_{K,\alpha} \leqslant \sum_\beta (\sup_{x \in K} |a_\beta(x)|) |\varphi|_{K,\alpha+\beta}\,.$$

Using fact (4) and Lemma 2.66, we see D is continuous from $\mathcal{D}(\Omega)$ to $\mathcal{D}(\Omega)$. $\square$

Example 4.40. *If $a : \Omega \to \mathbb{C}$ is a smooth function, then the map*

$$\mathcal{D}(\Omega) \to \mathcal{D}(\Omega) \text{ given by } \varphi \mapsto a\varphi$$

is a differential operator of order zero. The Laplacian $\Delta = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\right)^2$ is a differential operator of degree two.

Remark 4.41. If D is a differential operator and $\varphi \in \mathcal{D}(\Omega)$. Then it follows from the definition that $\operatorname{supp}(D\varphi) \subset \operatorname{supp}(\varphi)$ and $D : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is continuous. See Exercise 4.4.23 to see that essentially one can use this fact to give an alternative definition of a differential operator.

Let $D = \sum_\alpha a_\alpha \partial^\alpha$ be a differential operator of order N. Define a differential operator $D^\# : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$, *the transpose of* D, by

$$\int \varphi(x) D\psi(x)\, dx = \int D^\# \varphi(x)\, \psi(x)\, dx\,. \tag{4.13}$$

for all test functions ψ and φ. Thus, by integration by parts we calculate by using Leibnitz's rule:

$$\begin{aligned}\int \varphi(x) \sum_\alpha a_\alpha D^\alpha \psi(x)\, dx &= \sum_\alpha (-1)^{|\alpha|} \int D^\alpha (a_\alpha \varphi)(x) \psi(x)\, dx \\ &= \sum_\alpha \sum_{\beta \leqslant \alpha} (-1)^{|\alpha|} \binom{\alpha}{\beta} \int \left(D^{\alpha-\beta} a_\alpha(x) D^\beta \varphi(x)\right) \psi(x)\, dx.\end{aligned}$$

Thus

$$\left(\sum_\alpha a_\alpha D^\alpha\right)^\# = \sum_\alpha \sum_{\beta \leqslant \alpha} (-1)^{|\alpha|} \binom{\alpha}{\beta} D^{\alpha-\beta} a_\alpha D^\beta\,. \tag{4.14}$$

In particular it follows that $D^\#$ is a differential operator and $\operatorname{ord}(D^\#) = \operatorname{ord}(D)$.

Moreover, $(D^\#)^\# = D$. Indeed, if ϕ is in $\mathcal{D}(\Omega)$, then

$$\begin{aligned}\int (D^\#)^\# \phi\,(x)\, \psi(x)\, dx &= \int \phi(x)\, D^\# \psi(x)\, dx \\ &= \int D\phi(x)\psi(x)\, dx\end{aligned}$$

for all $\psi \in \mathcal{D}(\Omega)$ and so $(D^\#)^\# \phi = D\phi$.

If T is a distribution then we define $DT : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ by

$$\langle \varphi, DT \rangle := \langle D^\# \varphi, T \rangle\,, \quad f \in \mathcal{D}(\Omega)\,. \tag{4.15}$$

Lemma 4.42. *Let $T \in \mathcal{D}(\Omega)'$ and let D be a differential operator. Then DT is a distribution.*

Proof. The map $\mathcal{D}(\Omega) \ni \varphi \mapsto D^{\#}\varphi \in \mathbb{C}$ is continuous by Lemma 4.39. Hence the composition

$$\mathcal{D}(\Omega) \ni \varphi \mapsto D\varphi \mapsto T(D^{\#}\varphi) \in \mathbb{C}$$

is continuous. $\square$

Remark 4.43. Let $D : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ be a differential operator. By dualizing, we get a continuous linear map $D^t : \mathcal{D}(\Omega)' \to \mathcal{D}(\Omega)'$ given by $\langle \varphi, D^t T\rangle = \langle D\varphi, T\rangle$. Note $D^{\#}T = D^t T$ for

$$\langle \phi, D^{\#}T\rangle = \langle (D^{\#})^{\#}\phi, T\rangle = \langle D\phi, T\rangle = \langle \phi, D^t T\rangle.$$

Example 4.44. *Assume that $n = 1$ and let $H = \chi_{[0,\infty)}$ be the Heaviside function. Then*

$$\langle \varphi, H\rangle = \int_0^\infty \varphi(x)\, dx\,.$$

We now calculate

$$\langle \varphi, \frac{dH}{dx}\rangle = \langle -\varphi', H\rangle = -\int_0^\infty \varphi'(x)\, dx = \varphi(0)\,.$$

Thus $\frac{dH}{dx} = \delta$.

Example 4.45. *Assume that $n = 1$. Let $f(x) = |x|$. Then*

$$\begin{aligned}\int \varphi(x) \frac{d}{dx}\,|x|\; dx &= -\int \varphi'(x)|x|\, dx \\ &= -\int_0^\infty \varphi'(x)x\, dx + \int_{-\infty}^0 \varphi'(x)x\, dx \\ &= \int_0^\infty \varphi(x)\, dx - \int_{-\infty}^0 \varphi(x)\, dx \\ &= \int_{-\infty}^\infty \varphi(x)\operatorname{sign}(x)\, dx\,.\end{aligned}$$

Thus

$$\frac{d|x|}{dx} = \operatorname{sign}(x)\,.$$

Let $f \in L^1_{\mathrm{loc}}(\Omega)$. Then $D^\alpha f$ is a distribution for every multiindex $\alpha \in \mathbb{N}_0^n$, but as the example of the Heaviside function shows, $D^\alpha f$ is not necessarily a function (see Exercise 4.4.6). But in case there exists a locally integrable function g_α such that

$$D^\alpha T_f = T_{g_\alpha}$$

then g_α is unique (up to set of measure zero) and is called the *weak or distributional derivative of f*. We simply write $D^\alpha f = g_\alpha$. Thus by the definition, $D^\alpha f$ exists weakly if and only if there exists a locally integrable function g_α such that for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ we have

$$(-1)^{|\alpha|} \int D^\alpha \varphi(x) f(x)\, dx = \int \varphi(x) g_\alpha(x)\, dx\,.$$

In (5) of Section 5 of Chapter 2, we defined translation $\lambda(y)$, dilation $\delta(a)$, and multiplication M_g by g on any complex valued function f on $\mathbb{R}^n$; namely $\lambda(y)f(x) = f(x-y)$, $\delta(a)f(x) = a^{-n/2}f(a^{-1}x)$, and $M_g f = gf$. There we also defined three idempotent operations: conjugation, check, and adjoint. One can do these same operations on distributions. Namely, let T be a distribution on $\mathbb{R}^n$ and suppose $y \in \mathbb{R}^n$, $a > 0$, and $g \in \mathcal{E}(\mathbb{R}^n)$. Then $\lambda(y)T$, $\delta(a)T$, gT, $\bar{T}$, $\check{T}$, and T^* are defined on functions $\varphi \in \mathcal{D}(\mathbb{R}^n)$ by:

(16)	**Translation**	$\lambda(y)T(\varphi) = T(\lambda(-y)\varphi)$
	Dilation	$\delta(a)T(\varphi) = T(\delta(a^{-1})\varphi)$
	Multiplication	$gT(\varphi) = T(g\varphi)$
	Conjugation	$\bar{T}(\varphi) = T(\bar{\varphi})$
	Check	$\check{T}(\varphi) = T(\check{\varphi})$ and
	Adjoint	$T^*(\varphi) = T(\varphi^*)$

Multiplication can be done on any set. Thus if Ω is a nonempty open subset of $\mathbb{R}^n$ and $g \in \mathcal{E}(\Omega)$ and T is a distribution on Ω, then gT is defined by $\langle \phi, gT\rangle = \langle g\phi, T\rangle$ for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

Exercise 4.4.1 shows that $\lambda(y)$, $\delta(a)$, and M_g where $M_g\phi = g\phi$ are continuous on $\mathcal{D}(\mathbb{R}^n)$ and thus $\lambda(y)T$, $\delta(a)T$, and gT are distributions. Furthermore, as seen in Exercise 4.4.2, if $T = T_f$ where f is a locally integrable function, then $\lambda(y)T_f = T_{\lambda(y)f}$, $\delta(a)T_f = T_{\delta(a)f}$, and $gT_f = T_{gf}$. Similarly, if T is a distribution, then $\bar{T}$, $\check{T}$, and T^* are distributions and one has $\bar{T}_f = T_{\bar{f}}$, $\check{T}_f = T_{\check{f}}$, and $(T_f)^* = T_{f^*}$.

Remark 4.46. In the more specialized case when T is a tempered distribution, one knows by Proposition 2.58 that translation, dilation, differentiation, and the three idempotent operations are continuous on $\mathcal{S}(\mathbb{R}^n)$ and consequently $\lambda(y)T$, $\delta(a)T$, $D^\alpha T$, $\bar{T}$, $\check{T}$, and T^* are tempered distributions. Moreover, the continuity of M_g given in Proposition 2.57 shows gT is a tempered distribution when $g \in \mathcal{S}'_\infty(\mathbb{R}^n)$. In particular, e_yT is tempered when $e_y(x) = e^{2\pi i x\cdot y}$.

Exercise Set 4.4

1. Let $y \in \mathbb{R}^n$ and $a > 0$. Show $\lambda(y)$ and $\delta(a)$ are continuous linear homeomorphisms of $\mathcal{D}(\mathbb{R}^n)$. Also show if Ω is an open subset of $\mathbb{R}^n$ and $g \in \mathcal{E}(\Omega)$, then the map from $\mathcal{D}(\Omega)$ to $\mathcal{D}(\Omega)$ given by $\phi \mapsto g\phi$ is a continuous linear transformation.

2. Show if $y \in \mathbb{R}^n$ and $a > 0$, then $\lambda(y)T_f = T_{\lambda(y)f}$ and $\delta(a)T_f = T_{\delta(a)f}$ for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Also show $gT_f = T_{gf}$ whenever $g \in \mathcal{E}(\Omega)$ and $f \in L_{\text{loc}}(\Omega)$.

3. Let $f : \Omega \to \mathbb{C}$ be m-times continuously differentiable. Show $D^\alpha f$ is the weak derivative of f for all α with $|\alpha| \leqslant m$.

4. Suppose that the sequence of distributions $\{T_n\}$ converges to $T \in \mathcal{D}'(\mathbb{R}^n)$. Show $D^\alpha T_n \to D^\alpha T$ for all $\alpha \in \mathbb{N}_0^n$.

5. Suppose that $n = 1$. Evaluate the distributional derivative of the following functions and distributions:

(a) $f(x) = \text{sign}(x)$.

(b) $f(x) = x\,|x|$.

(c) $f(x) = xH(x)$

(d) $f(x) = H(x+1)H(1-x)$.

(e) δ_a, $a \in \mathbb{R}$.

(f) $x\delta_a$, $a \in \mathbb{R}$.

(g) $f(x) = x\log(x)H(x)$.

6. Show there is no locally integrable function f such that $T_f = \delta$.

7. Show every L^p function, and every measurable function whose absolute value is majorized by a polynomial is a tempered distribution. If f is such a function the corresponding distribution is defined by

$$T_f(\phi) = \int f(x)\,\phi(x)\,dx.$$

8. Let $L(x) = \log(x)\chi_{(0,\infty)}(x)$.

(a) Show L is locally integrable on $\mathbb{R}$, and hence that L defines a distribution on $\mathbb{R}$.

(b) Show

$$L(\varphi) = \int_0^\infty \frac{1}{x}\varphi(x)\,dx$$

if $\text{supp}\,(\varphi) \subset (0,\infty)$.

(c) Show $\varphi \mapsto \int_0^\infty \frac{1}{x}\varphi(x)\,dx$ does not define a distribution on $\mathbb{R}$.

(d) Determine the distribution L'.

9. Suppose that the distribution S is homogeneous of degree λ. Then $D^\alpha S$ is homogeneous of degree $\lambda - |\alpha|$. (See Exercise 4.3.15 for the definition of homogeneous.)

10. Antiderivatives of distributions: The following steps show how to find an antiderivative of a distribution T.

(a) Show if $h \in \mathcal{D}(\mathbb{R})$ and $\int h(x)\, dx = 1$, then the mapping $\phi \mapsto \Phi$ defined by

$$\Phi(x) = \int_{-\infty}^{x} \left(\phi(t) - (\int \phi) h(t) \right) \, dt$$

is a continuous linear mapping of $\mathcal{D}(\mathbb{R})$ into $\mathcal{D}(\mathbb{R})$; show this mapping is also continuous from $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$.

(b) Let T be a distribution on $\mathbb{R}$. Define S_h by

$$\langle \phi, S_h \rangle = -\langle \Phi, T \rangle.$$

Show

$$S_h' = T.$$

(c) Show if S_1 and S_2 are distributions on $\mathbb{R}$ with $S_1' = S_2'$, then $S_1 = S_2 + c$ where c is a constant, i.e., show there is a constant c such that

$$S_1(\phi) = S_2(\phi) + T_c(\phi) = S_2(\phi) + c \int \phi(x)\, dx.$$

(d) Show if T is tempered, then the distribution S_h is tempered.

(e) Show if F is absolutely continuous with derivative f and $T = T_f$, then $S_h = F + c$ for some constant c.

11. Find the antiderivatives of the following distributions:

(a) $T = 1$.

(b) δ_a.

(c) $\operatorname{sign}(x)$.

12. Let Ω be a connected open subset of $\mathbb{R}^n$. Show if T is a distribution on Ω and $D^\alpha T = 0$ for all α with $|\alpha| = 1$, then $T = c = T_c$ for some constant c. (**Hint:** Show if T is a distribution on the interval $(a, b) \subseteq \mathbb{R}$ with $\frac{d}{dx} T = 0$, then $T = c$. Then use Theorem 2.86).

13. Let f be a locally integrable function on $\mathbb{R}^n$. Show the support of the distribution T_f is the complement of the union of all open subsets U of $\mathbb{R}^n$ such that $f|_U$ is zero almost everywhere.

14. Let $f : \Omega \to \mathbb{C}$ be m-times continuously differentiable. Show $D^\alpha f$ is the weak derivative of f for all α with $|\alpha| \leqslant m$.

15. Let D be a differential operator of order N. Show the expression $D = \sum_{|\alpha|\leqslant N} a_\alpha \partial^\alpha$ is unique. (Hint: Apply D to the polynomials $x \mapsto x^\gamma$.)

16. Let Ω be an open subset of $\mathbb{R}^n$ and suppose $p(x) = \sum_{|\alpha|\leqslant N} a_\alpha x^\alpha$ is a polynomial. Then $L = p(D) = \sum a_\alpha D^\alpha$ is a differential operator with constant coefficients. Show if T is a distribution on Ω, then Definition (4.15) agrees with the natural definition of $p(D)T$ as $\sum a_\alpha D^\alpha T$.

17. Let T be a distribution on a nonempty open subset Ω of $\mathbb{R}^n$. The distribution T is said to have finite order if there is a $N \in \mathbb{N}_0$ such that for each compact subset K of Ω and each $\epsilon > 0$, there is a $\delta_K > 0$ such that if $\phi \in \mathcal{D}(K)$ and $\max_{x\in K} |D^\alpha \phi(x)| < \delta_K$ for $|\alpha| \leqslant N$, then $|T(\phi)| < \epsilon$. The smallest N for which this is true is called the **order** of T. If T has order 0, then T is said to be a Radon distribution on Ω.

(a) Show if the order of T is 0 and $T(\phi) \geqslant 0$ if $\phi \geqslant 0$, then there is a Radon measure μ on Ω such that

$$T(\phi) = \int \phi(x)\, d\mu(x)$$

for $\phi \in \mathcal{D}(\Omega)$.

(b) Show if T is a Radon distribution on Ω, then there are Radon measures μ_1, μ_2, μ_3, and μ_4 such that

$$T(\phi) = \int \phi\, d\mu_1 - \int \phi\, d\mu_2 + i \int \phi\, d\mu_3 - i \int \phi\, d\mu_4.$$

(Hint: See Rudin, Real and Complex Analysis on complex measures and the dual of $C_c(\Omega)$.)

18. Show if T is a distribution on Ω with compact support, then T has finite order.

19. Show if T is a Radon distribution on Ω with compact support, then there is a complex measure μ on Ω with $T(\phi) = \int \phi\, d\mu$ for $\phi \in \mathcal{D}(\Omega)$.

20. Show if T is a distribution on Ω with compact support, then there exists an N and complex Borel measures μ_α on Ω such that

$$T = \sum_{|\alpha|\leqslant N} D^\alpha T_{\mu_\alpha}.$$

Hint: Let N be the order of T and U be an open subset of Ω with $\operatorname{supp} T \subseteq U \subseteq \bar{U} \subseteq \Omega$ and $\bar{U}$ compact. Show the mapping $\{D^\alpha \phi|_U\}_{|\alpha|\leqslant N} \mapsto T(\phi)$ is continuous and linear on relative topology of the product topology of $\prod_{|\alpha|\leqslant N} C_c(U)$. Then use the Hahn-Banach Theorem.

21. Show the following version of Taylor's formula. Suppose f is C^∞. Then

$$f(x) = \sum_{|\alpha|\leqslant N} \frac{x^\alpha}{\alpha!} D^\alpha(f)(0) + (N+1) \sum_{|\alpha|=N+1} \frac{x^\alpha}{\alpha!} \int_0^1 (1-t)^N D^\alpha f(tx)\, dt.$$

22. Show if T is a distribution on Ω and T has support $\{x_0\}$ where $x_0 \in \Omega$, then T has form

$$T(\phi) = \sum_{|\alpha|\leqslant N} c_\alpha D^\alpha \phi(x_0)$$

for some finite N. (**Hint:** Exercise 4.4.21 may be useful.)

23. Let Ω be an open connected subset of $\mathbb{R}^n$ and suppose $D : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is a continuous linear mapping with the property that $D(\operatorname{supp}\phi) \subseteq \operatorname{supp}\phi$. Show there exist $a_\alpha \in \mathcal{E}(\Omega)$ such that for each $p \in \Omega$, $\{\alpha \mid a_\alpha(p) \neq 0\}$ is finite and

$$D(\phi) = \sum_\alpha a_\alpha D^\alpha \phi.$$

24. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Show there is a nondifferential operator on $\mathcal{D}(\Omega)$ whose restriction to each $\mathcal{D}_K(\Omega)$ is a differential operator.

6. The Fourier Transform of Tempered Distributions

As we have seen, the Fourier transform of functions works well for integrable, square integrable, compactly supported smooth, and Schwartz functions. In the cases of square integrable functions, compactly supported smooth functions, and Schwartz functions, the Fourier transform provides a topological isomorphism between well known spaces. We shall use the topological isomorphism given in Theorem 3.10 and the fact that

$$\int \mathcal{F}(\phi)(y)\psi(y)\, dy = \int \phi(y)\mathcal{F}(\psi)(y)\, dy \tag{4.17}$$

for Schwartz functions ϕ and ψ to Show $\mathcal{F}$ also extends to the space $\mathcal{S}(\mathbb{R}^n)'$ of tempered distributions. We remark that Equation (4.17) is a simple consequence of Lemma 3.7.

Note since $\phi \mapsto \hat{\phi}$ is a topological isomorphism of $\mathcal{S}(\mathbb{R}^n)$, one sees if T is a tempered distribution, then S defined by

$$S(\phi) = T(\hat{\phi})$$

is also a tempered distribution. We also note that if $T = T_\phi$ where $\phi \in \mathcal{S}(\mathbb{R}^n)$, then $S = T_{\hat{\phi}}$. Indeed,

$$\begin{aligned} S(\psi) &= T_\phi(\hat{\psi}) \\ &= \int \hat{\psi}(x)\,\phi(x)\,dx \\ &= \int \psi(x)\hat{\phi}(x)\,dx \\ &= T_{\hat{\phi}}(\psi) \end{aligned}$$

for $\psi \in \mathcal{S}(\mathbb{R}^n)$. This observation is the basis for the following definition.

Definition 4.47. *Let T be a tempered distribution. Define $\mathcal{F}(T) = \hat{T}(\varphi) = T(\hat{\varphi})$. Then $\hat{T}$ is the Fourier transform of T. The notation $\mathcal{F}(T)$ is also used for $\hat{T}$.*

If $T = f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a locally integrable function, then we will also write $\hat{f}$ for $\widehat{T_f}$. We have seen this is consistent when f is a Schwartz function. By Exercise 4.5.1 one also has:

$$\hat{T}_f = T_{\hat{f}} \text{ if } f \in L^1(\mathbb{R}^n) \text{ or } f \in L^2(\mathbb{R}^n) \tag{4.18}$$

Recall from (4.8) the seminorms $|\cdot|'_\phi$ where $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $|T|'_\phi = |T(\phi)|$ make the space of tempered distributions into a locally convex topological vector space. This topology is the weak $*$ topology, and it is the smallest topology which makes the functions $T \mapsto T(\varphi)$ continuous for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Recall $e_y(x) = e^{2\pi i x\cdot y}$. In Remark 4.46, we noted if T is a tempered distribution, then $D^\alpha T$, $\lambda(y)T$, e_yT, $\delta(a)T$, $\bar{T}$, $\check{T}$, and T^* are tempered distributions. Recall that $\lambda(y)T(\varphi) = T(\lambda(-y)\varphi) = \int \varphi(x+y)dT(x)$, $e_yT(\varphi) = T(e_y\varphi) = \int e^{2\pi i yx}\varphi(x)dT(x)$, and $\delta(a)T(\varphi) = T(\delta(a^{-1})\varphi) = a^{n/2}\int \varphi(ax)\,dT(x)$ for $a > 0$.

Theorem 4.48. *The Fourier transform is a topological isomorphism of $\mathcal{S}(\mathbb{R}^n)'$. Furthermore*

(a) $(\mathcal{F})^2(T) = \check{T}$ *and* $\mathcal{F}^4 = I$;

(b) $\widehat{\lambda(y)T} = e_y\widehat{T}$;

(c) $\widehat{e_yT} = \lambda(y)\widehat{T}$;

(d) $\widehat{\delta(a)T} = \delta(a)\widehat{T}$;

(e) $\widehat{\overline{T}} = \widehat{T}^*$;

(f) $\widehat{T^*} = \overline{\widehat{T}}$;

(g) $\widehat{\check{T}} = \check{\widehat{T}}$;

(h) $\mathcal{F}(p(\frac{D}{2\pi i})T) = p\widehat{T}$.

(i) $\widehat{pT} = p(-\frac{D}{2\pi i})\widehat{T}$;

Proof. Claim $\mathcal{F}$ is continuous. It suffices to show $T \mapsto \hat{T}(\varphi)$ is continuous for each φ. But this is $T \mapsto T(\hat{\varphi})$ which is continuous.

Let ϕ be a Schwartz function. To see (a), note Proposition 3.9 implies $\mathcal{F}^2T(\phi) = T(\mathcal{F}^2\phi) = T(\check{\phi}) = \check{T}(\phi)$ and thus $\mathcal{F}^4T(\phi) = \check{T}(\mathcal{F}^2\phi) = \check{T}(\check{\phi}) = T(\check{\check{\phi}}) = T(\phi)$.

From Lemma 3.3, we have $\lambda(y)\hat{\phi} = \mathcal{F}(\tau(y)\phi) = \widehat{e_{-y}\phi}$ and $e_y\hat{\phi} = \widehat{\lambda(-y)\phi}$ and thus

$$\mathcal{F}(\lambda(y)T)(\phi) = \lambda(y)T(\hat{\phi}) = T(\lambda(-y)\hat{\phi}) = T(\mathcal{F}(e_y\phi)) = \hat{T}(e_y\phi) = e_y\hat{T}(\phi) \text{ and}$$
$$\mathcal{F}(e_yT)(\phi) = e_yT(\hat{\phi}) = T(e_y\hat{\phi}) = T(\mathcal{F}(\lambda(-y)\phi)) = \hat{T}(\lambda(-y)\phi) = \lambda(y)\hat{T}(\phi).$$

This gives (b) and (c).

For (d), suppose one has $a > 0$. Then since $\mathcal{F}(\delta(a)\phi) = \delta(a^{-1})\mathcal{F}(\phi)$, we see

$$\begin{aligned}\mathcal{F}(\delta(a)T)(\phi) &= \delta(a)T(\hat{\phi}) = T(\delta(a)\hat{\phi}) = T(\mathcal{F}(\delta(a^{-1})\phi)) \\ &= \hat{T}(\delta(a^{-1})\phi) = \delta(a)\hat{T}(\phi).\end{aligned}$$

Next, to see (e) and (f), note by Lemma 3.3 that $\mathcal{F}(\phi^*) = \bar{\hat{\phi}}$ and $\mathcal{F}(\bar{\phi}) = \mathcal{F}(\phi)^*$ and thus

$$\mathcal{F}(\bar{T})(\phi) = \bar{T}(\hat{\phi}) = T(\bar{\hat{\phi}}) = T(\mathcal{F}(\phi^*)) = \hat{T}(\phi^*) = (\hat{T})^*(\phi)$$

and

$$\mathcal{F}(T^*)(\phi) = T^*(\hat{\phi}) = T((\mathcal{F}\phi)^*) = T(\mathcal{F}(\bar{\phi})) = \hat{T}(\bar{\phi}).$$

Again using Lemma 3.3, we know $\mathcal{F}(\check{\phi}) = \check{\hat{\phi}}$ and so

$$\mathcal{F}(\check{T})(\phi) = \check{T}(\hat{\phi}) = T(\check{\hat{\phi}}) = T(\mathcal{F}(\check{\phi})) = \mathcal{F}(T)(\check{\phi}) = \check{\hat{T}}(\phi).$$

Finally to see $\mathcal{F}(p(\frac{D}{2\pi i})T) = p\hat{T}$ and $\mathcal{F}(pT) = p(\frac{-D}{2\pi i})\mathcal{F}(T)$, we use Theorem 3.4 which gives

$$\mathcal{F}(p\phi) = p(-\frac{D}{2\pi i})\mathcal{F}(\phi) \text{ and}$$
$$\mathcal{F}(p(D)\phi)(\omega) = p(2\pi i\omega)\hat{\phi}(\omega)$$

for polynomials $p(x) = \sum_{|\alpha|\leqslant N} a_\alpha x^\alpha$.

Hence

$$\begin{aligned}
\mathcal{F}(p(\frac{D}{2\pi i})T)(\phi) &= p(\frac{D}{2\pi i})T(\hat{\phi}) \\
&= \sum_{|\alpha|\leqslant N} \frac{a_\alpha}{(2\pi i)^{|\alpha|}} D^\alpha T(\hat{\phi}) \\
&= \sum_{|\alpha|\leqslant N} (-1)^{|\alpha|} \frac{a_\alpha}{(2\pi i)^{|\alpha|}} T(D^\alpha \hat{\phi}) \\
&= T(p(-\frac{D}{2\pi i})\hat{\phi}) \\
&= T(\mathcal{F}(p\,\phi)) \\
&= \hat{T}(p\phi) \\
&= p\hat{T}(\phi)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}(pT)(\phi) &= pT(\hat{\phi}) \\
&= T(p\hat{\phi}) \\
&= T(\mathcal{F}(p(\frac{D}{2\pi i})\phi) \\
&= \hat{T}(p(\frac{D}{2\pi i})\phi) \\
&= \hat{T}(\sum_{|\alpha|\leqslant N} \frac{D^\alpha}{(2\pi i)^{|\alpha|}}\phi) \\
&= \sum_{|\alpha|\leqslant N} \left(\frac{-D}{2\pi i}\right)^\alpha \hat{T}(\phi) \\
&= p(\frac{-D}{2\pi i})\hat{T}(\phi).
\end{aligned}$$

□

Exercise Set 4.5

1. Suppose $f \in L^1(\mathbb{R}^n)$ or $f \in L^2(\mathbb{R}^n)$. Show

$$\mathcal{F}(T_f) = T_{\hat{f}}.$$

2. Let $f \in \mathcal{S}_(\mathbb{R}^n)$. Show $\lambda(x)(T_f) = T_{\lambda(x)f}$, and $gT_f = T_{gf}$ if g is a tempered C^∞ function.

3. Let $x \in \mathbb{R}^n$. Determine the following Fourier transforms:

(a) $\mathcal{F}(e_x)$

(b) $\mathcal{F}(\epsilon_x)$ where ϵ_x is the point mass measure at x

(c) Explain why one writes $\mathcal{F}(\epsilon_x) = e_{-x}$ and $\mathcal{F}(e_x) = \epsilon_x$.

4. Let $p(x)$ be a polynomial on $\mathbb{R}^n$ and for $x \in \mathbb{R}^n$, let ϵ_x be the point mass measure at x. Determine the following:

(a) $\mathcal{F}(p\epsilon_x)$

(b) $\mathcal{F}(p(D)\epsilon_x)$

5. Assume that $n = 1$. Evaluate the Fourier transform of the following distributions:

(a) $f(x) = |x|$.

(b) $f(x) = x\,|x|$.

(c) $f(x) = xH(x)$

(d) $f(x) = \cos(\lambda x)$ for some $\lambda \in \mathbb{R}$.

(e) $f(x) = x\sin(\lambda x)$ for some $\lambda \in \mathbb{R}$.

6. Let f and g be Schwartz functions. Show using the Fourier transform that if $p(x)$ is a polynomial function, then $p(D)(f * g) = (p(D)f) * g = f * p(D)g$.

7. Let T be a distribution on $\mathbb{R}^n$ and let ϕ be a test function. Show if $p(x)$ is a polynomial function, then $p(D)(T * \phi) = (p(D)T) * \phi = T * p(D)\phi$.

8. One can use the Fourier transform to Show each tempered distribution is the difference of a boundary value of a holomorphic function in the upper half plane and one in the lower half plane. This exercise establishes this for a special class of distributions. Let T be a tempered distribution on the real line such that $\hat{T}$ is a continuous function bounded by a polynomial $p(\omega)$. Define $F_\pm$ by

$$F_+(z) := \int_0^\infty e^{2\pi i z\omega}\hat{T}(\omega)\,d\omega\,, \qquad \mathrm{Im}(z) > 0$$

and

$$F_-(z) := \int_{-\infty}^0 e^{2\pi i z\omega}\hat{T}(\omega)\,d\omega\,, \qquad \mathrm{Im}(z) < 0\,.$$

Show the following:

(a) The function F_+ is holomorphic in $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and F_- is holomorphic in $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$.

(b) Define

$$\langle T_\pm, \varphi\rangle = \lim_{y\to 0\pm}\int_{-\infty}^\infty \varphi(x)F_\pm(x+iy)\,dx\,.$$

Then T_+ and T_- are distributions and

$$T = T_+ - T_-\,.$$

(c) Show if $T = \delta_0$ then $T_+ = \frac{-1}{2\pi i}\frac{1}{x+i0}$ and $T_- = \frac{-1}{2\pi i}\frac{1}{x-i0}$.

(d) Assume that $\hat{T}(\omega) = \omega^n$. Find $T_\pm$.

7. Convolution of Distributions

We have seen that it is possible to convolve functions f and g under special conditions on f and g and in this case

$$f * g(x) = g * f(x) = \int f(y)g(x-y)\,dy = \int g(y)f(x-y)\,dy.$$

The usefulness of convolution has already been established and since distributions are "generalized functions", it turns out to be quite useful to convolve distributions when a suitable definition can be made. In order to do this, let us start by a formal manipulation. Recall T_f is the distribution defined by

$$T_f(\phi) = \int \phi(y)f(y)\,dy.$$

Thus to make a suitable definition of $T_f * T_g$, one would like

$$T_f * T_g(\phi) = T_{f*g}(\phi)$$

when f, g, and $f*g$ give distributions. Hence formally if $T = T_f$ and $S = T_g$, one would have:

$$\begin{aligned}
(T * S)(\phi) &= T_f * T_g(\phi) \\
&= \int \phi(x) f * g(x)\,dx \\
&= \int \phi(x) \int f(y)g(x-y)\,dy\,dx \\
&= \int f(y) \int \phi(x)g(x-y)\,dx\,dy \\
&= \int \left(\int \phi(-x+y)g(-x)\,dx \right) f(y)\,dy \\
&= \int \left(\int \check{g}(x)\phi(y-x)\,dx \right) f(y)\,dy \\
&= \int (\check{g} * \phi(y)) f(y)\,dy \\
&= T_f(\check{S} * \phi) \\
&= T(\check{S} * \phi)
\end{aligned}$$

where

$$\begin{aligned}\check{S} * \phi(y) &= \int \check{g}(x)\phi(y-x)\,dx \\ &= \int g(-x)\phi(y-x)\,dx \\ &= \int \phi(y+x)g(x)\,dx \\ &= S(\lambda(-y)\phi).\end{aligned}$$

Since $\lambda(-y)\phi$ is a test function when ϕ is a test function, we clearly see that $\check{S} * \phi$ is defined. The problem is it may not be a test function.

Definition 4.49. *Suppose ϕ is in $\mathcal{D}(\mathbb{R}^n)$ and S is a distribution or ϕ is in $\mathcal{S}(\mathbb{R}^n)$ and S is a tempered distribution. Then $S * \phi$ is the function defined by*

$$S * \phi(y) = \int \phi(y-x)\,dS(x) = S((\lambda(y)\check{\phi}))$$

Definition 4.50. *Let T and S be distributions on $\mathbb{R}^n$. Then $T * S$ is defined if $\phi \mapsto \check{S} * \phi$ is continuous on $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}(\mathbb{R}^n)$ and then $T * S$ is the distribution defined by*

$$T * S(\phi) = T(\check{S} * \phi)$$

Similarly if T is a tempered distribution and $\phi \mapsto \check{S} * \phi$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$, then $T * S$ is the tempered distribution defined by

$$T * S(\phi) = T(\check{S} * \phi).$$

Since $\phi \mapsto \check{S} * \phi$ is continuous if and only if $\phi \mapsto S * \phi$ is continuous, a central aspect of convolution depends on the continuity of the transformation $\phi \mapsto S * \phi$.

We shall first show the mapping $\phi \mapsto S * \phi$ is continuous into $\mathcal{E}(\mathbb{R}^n)$. To do this we shall use the following lemmas.

Lemma 4.51. *Suppose Ω is an open subset of $\mathbb{R}^n$ and $F \in \mathcal{D}(\Omega)$. For $j \in \{1, 2, \ldots, n\}$ define F_h for $h > 0$ by*

$$F_h(x) = \frac{F(x + he_j) - F(x)}{h}.$$

Then F_h is in $\mathcal{D}(\Omega)$ for h near 0 and $F_h \to \partial_j F$ in $\mathcal{D}(\Omega)$ as $h \to 0$.

Proof. Let K be the compact support of F. Since $K \subseteq \Omega$, there is a $\delta > 0$ such that $x + he_j \in \Omega$ if $|h| \leqslant \delta$. Hence $F_h(x) = 0$ if $x \notin K + [-\delta, \delta]e_j$. So F_h has compact support inside K' where $K' = K + [-\delta, \delta]e_j$. To show

$F_h \to \partial_j F$ in $\mathcal{D}(\Omega)$, it suffices to show $F_h \to \partial_j F$ in $\mathcal{D}_{K'}(\Omega)$. Now using the Mean Value Theorem, we see

$$\begin{aligned} |F_h - \partial_j F|_{K',\alpha} &= \max_{x \in K'} |(D^\alpha F_h - D^{\alpha+e_j} F)(x)| \\ &= \max_{x \in K'} \left| \frac{D^\alpha F(x+he_j) - D^\alpha F(x)}{h} - \partial_j D^\alpha F(x) \right| \\ &= \max_{x \in K'} \left| \frac{D^\alpha F(x+he_j) - D^\alpha F(x)}{h} - \partial_j D^\alpha F(x) \right| \\ &= \max_{x \in K'} \left| \frac{h \partial_j D^\alpha F(x + h^*(x) e_j)}{h} - \partial_j D^\alpha F(x) \right| \\ &= \max_{x \in K'} |\partial_j D^\alpha F(x + h^*(x)e_j) - \partial_j D^\alpha F(x)| \end{aligned}$$

where $|h^*(x)| < h$ for each x in K'. It follows by the uniform continuity of $\partial_j D^\alpha F$ on K' that $|F_h - \partial_j F|_{K',\alpha} \to 0$ as $h \to 0$. □

Lemma 4.52. *Let $F \in \mathcal{S}(\mathbb{R}^n)$ and suppose $j \in \{1, 2, \ldots, n\}$. Define F_h for $h \neq 0$ by*

$$F_h(x) = \frac{F(x+he_j) - F(x)}{h}.$$

Then F_h is in $\mathcal{S}(\mathbb{R}^n)$ and $F_h \to \partial_j F$ in $\mathcal{S}(\mathbb{R}^n)$ as $h \to 0$.

Proof. Since $F_h = \frac{1}{h}(\lambda(-he_j)F - F)$, we see $F_h \in \mathcal{S}(\mathbb{R}^n)$ for all $h \neq 0$.

The Mean Value Theorem argument in the proof of the previous lemma shows for each $x \in \mathbb{R}^n$, there is an $h^*(x) \in (-h, h)$ such that

$$|D^\alpha(F_h - F)(x)| = |\partial_j D^\alpha F(x + h^*(x)e_j) - \partial_j D^\alpha F(x)|.$$

Hence using the Mean Value Theorem again, we see

$$(1+|x|^2)^N |D^\alpha(F_h - F)(x)| = (1+|x|^2)^N |h^*(x)| \partial_j^2 D^\alpha F(x + k^*(x)e_j)|$$

for some $k^*(x)$ with $|k^*(x)| < |h^*(x)| < h$. Now an adaptation of Exercise 2.3.4 shows $\frac{1+|x|^2}{1+|x+k^*(x)e_j|^2} \leqslant 2(1+|k^*(x)e_j|^2)$. Consequently

$$\begin{aligned} (1+|x|^2)^N |D^\alpha(F_h - F)(x)| &\leqslant |h| \frac{(1+|x|^2)^N}{(1+|x+k^*(x)e_j|^2)^N} |F|_{N,\alpha+2e_j} \\ &\leqslant 2^N (1+|k^*(x)e_j|^2)^N |h|\, |F|_{N,\alpha+2e_j} \\ &\leqslant 2^N |h| (1+h^2)^N |F|_{N,\alpha+2e_j}. \end{aligned}$$

Thus $|F_h - F|_{N,\alpha} \to 0$ as $h \to 0$ for each N and each α. □

Lemma 4.53. *Let Ω_1 and Ω_2 be nonempty open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Suppose T is a distribution on Ω_2 and let $\Phi \in \mathcal{D}(\Omega_1 \times \Omega_2)$. Then the function Φ_1 given by*

$$\Phi_1(x) = T(\Phi_x) = \int \Phi(x,y)\, dT(y)$$

is smooth, has compact support, and satisfies

$$D^\alpha \Phi_1(x) = T(D_x^\alpha \Phi) = \int D_x^\alpha \Phi(x, y)\, dT(y)$$

for all $\alpha \in \mathbb{N}_0^m$. *Furthermore, the mapping* $\Phi \mapsto \Phi_1$ *is continuous from* $\mathcal{D}(\Omega_1 \times \Omega_2)$ *into* $\mathcal{D}(\Omega_1)$.

Proof. Choose compact subsets K_1 of Ω_1 and K_2 of Ω_2 such that $\operatorname{supp}\Phi \subseteq K_1 \times K_2$. Thus Φ_x where $\Phi_x(y) = \Phi(x, y)$ is smooth and has support in K_2 for all x. Hence $T(\Phi_x)$ is defined for all x. Moreover, if $x \notin K_1$, then $\Phi_x = 0$ and consequently $T(\Phi_x) = 0$. Thus $x \mapsto T(\Phi_x)$ has compact support. Note Φ_1 is continuous. Indeed, by Exercise 2.4.16, we know if $x_k \to x$, then $\Phi_{x_k} \to \Phi_x$ in $\mathcal{D}(\Omega_2)$. Since T is continuous, we see $\Phi_1(x_k) \to \Phi_1(x)$. Using induction, to show $D^\alpha \Phi_1(x) = T(D_x^\alpha \Phi_1)$, one need only show

$$\frac{\partial}{\partial x_i}\Phi_1(x) = T\left[\left(\frac{\partial}{\partial x_i}\Phi\right)_x\right].$$

Now by Lemma 4.51, $\frac{\Phi(x+he_i,y)-\Phi(x,y)}{h} \to \frac{\partial}{\partial x_i}\Phi(x, y)$ in $\mathcal{D}(\Omega_1 \times \Omega_2)$ as $h \to 0$. This implies $\frac{\Phi_{x+he_i}-\Phi_x}{h} \to \left(\frac{\partial}{\partial x_i}\Phi\right)_x$ in $\mathcal{D}(\mathbb{R}^n)$ as $h \to 0$. Thus by the continuity of T, we see

$$\frac{\Phi_1(x + he_i) - \Phi_1(x)}{h} \to T(\left(\frac{\partial}{\partial x_i}\Phi\right)_x) \text{ as } h \to 0.$$

So $\frac{\partial}{\partial x_i}\Phi_1(x) = T\left[\left(\frac{\partial}{\partial x_i}\Phi\right)_x\right]$.

Finally, to see $\Phi \mapsto \Phi_1$ is continuous, it suffices to see it is continuous on $\mathcal{D}_{K_1\times K_2}(\Omega_1 \times \Omega_2)$. Let $\epsilon > 0$ and fix $\alpha \in \mathbb{N}_0^m$. Choose a finite set $F \subseteq \mathbb{N}_0^n$ and $\delta > 0$ such that $|T(\phi)| < \epsilon$ if $\phi \in \mathcal{D}_{K_2}(\Omega_2)$ and $|\phi|_{K_2,\beta} < \delta$ for $\beta \in F$. Then if $\Phi \in \mathcal{D}_{K_1\times K_2}(\Omega_1 \times \Omega_2)$ and $|\Phi|_{K_1\times K_{2,(\alpha,\beta)}} < \delta$ for $\beta \in F$, one has

$$\begin{aligned} |\Phi_1|_{K_1,\alpha} &= |D_x^\alpha \Phi_1|_{K_1} \\ &= \max_{x\in K_1} |T((D^\alpha\Phi)_x)| \\ &\leqslant \epsilon. \end{aligned}$$

□

Lemma 4.54. *Let* T *be a tempered distribution on* $\mathbb{R}^n$. *Let* Φ *be a Schwartz function on* $\mathbb{R}^m \times \mathbb{R}^n$. *Then* Φ_x *is a Schwartz function on* $\mathbb{R}^n$ *for each* $x \in \mathbb{R}^m$ *and the function* Φ_1 *defined by* $\Phi_1(x) = T(\Phi_x)$ *is a Schwartz function on* $\mathbb{R}^m$. *Moreover, the mapping* $\Phi \mapsto \Phi_1$ *is a continuous linear transformation of* $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ *into* $\mathcal{S}(\mathbb{R}^m)$.

Proof. Exercises 2.3.10 shows $\Phi \mapsto \Phi_x$ is a continuous linear transformation from $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. Moreover, Exercise 2.3.11 shows $x \mapsto \Phi_x$

is continuous from $\mathbb{R}^m$ into $\mathcal{S}(\mathbb{R}^n)$. Consequently, $\Phi_1(x)$ is a continuous function in x for each Φ in $\mathcal{S}(\mathbb{R}^m)$. Moreover, by Lemma 4.52,

$$\frac{\Phi_{he_j} - \Phi}{h} \to \frac{\partial}{\partial x_j}\Phi$$

in $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ for each $j = 1, 2, \ldots, m$. By continuity of $\Phi \mapsto \Phi_x$, one has

$$\frac{\Phi_{x+he_j - \Phi_x}}{h} = \left(\frac{\Phi_{he_j} - \Phi}{h}\right)_x \to \left(\frac{\partial}{\partial x_j}\Phi\right)_x$$

in $\mathcal{S}(\mathbb{R}^n)$. The linearity and continuity of T then shows

$$\frac{\partial}{\partial x_j}\Phi_1(x) = T((\frac{\partial}{\partial x_j}\Phi)_x).$$

Using induction it follows that $D^\alpha \Phi_1(x) = T((D^\alpha \Phi)_x)$ and we see Φ_1 is C^∞ on $\mathbb{R}^m$.

To obtain continuity, we note since T is continuous, that there is an natural number N' and a finite subset F of $\mathbb{N}_0^n$ and a $\delta > 0$ such that one has $|T(f)| \leqslant 1$ if $|f|_{N',\beta} \leqslant \delta$ for all $\beta \in F$. Thus if $N \in \mathbb{N}_0^m$ and $\alpha \in \mathbb{N}_0^m$, we see

$$\begin{aligned}(1+|y|^2)^{N'}|D^\beta(1+|x|^2)^N(D^\alpha\Phi)_x(y)| &= (1+|x|^2)^N(1+|y|^2)^{N'}|D^{(\alpha,\beta)}\Phi(x,y)| \\ &\leqslant (1+|x|^2+|y|^2)^{2N+2N'}|D^{(\alpha,\beta)}\Phi(x,y)| \\ &\leqslant |\Phi|_{2N+2N',(\alpha,\beta)}.\end{aligned}$$

Consequently, if $|\Phi|_{2N+2N',(\alpha,\beta)} \leqslant \delta$ for all x and all $\beta \in F$, then $|(1+|x|^2)^N(D^\alpha\Phi)_x|_{N',\beta} \leqslant \delta$ and hence

$$|\Phi_1|_{N,\alpha} = \sup_x (1+|x|^2)^N|D^\alpha\Phi_1(x)| = \sup_x |T\left((1+|x|^2)^N(D^\alpha\Phi)_x\right)| \leqslant 1.$$

Thus $\Phi \mapsto \Phi_1$ is continuous. □

Theorem 4.55. *Let T and S be distributions on nonempty subsets Ω_1 and Ω_2 of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Then there is a distribution $T \times S$ on $\Omega_1 \times \Omega_2$ such that*

$$(T \times S)(\phi \times \psi) = T(\phi)S(\psi)$$

if $\phi \in \mathcal{D}(\Omega_1)$ and $\psi \in \mathcal{D}(\Omega_2)$. It satisfies

$$(T \times S)(\Phi) = \int\int \Phi(x,y)\,dT(x)\,dS(y) = \iint \Phi(x,y)\,dS(y)\,dT(x)$$

for all $\Phi \in \mathcal{D}(\Omega_1 \times \Omega_2)$.

Proof. We show there is a distribution P that satisfies

$$P(\phi \times \psi) = T(\phi)S(\psi).$$

Indeed, by Lemma 4.53, the mapping $\Phi \mapsto \Phi_1$ where $\Phi_1(x) = \int \Phi(x,y)\,dS(y)$ is a continuous linear mapping from $\mathcal{D}(\Omega_1 \times \Omega_2)$ into $\mathcal{D}(\Omega_1)$. Thus P defined by $P(\Phi) = T(\Phi_1)$ is a distribution on $\Omega_1 \times \Omega_2$. Clearly it satisfies, $P(\phi \times \psi) = T(\phi)S(\psi)$ and is defined by

$$P(\Phi) = \int \int \Phi(x,y)\, dS(y)\, dT(x).$$

In a similar fashion, there is a distribution Q satisfying $Q(\phi \times \psi) = T(\phi)S(\psi)$ and

$$Q(\Phi) = \int \int \Phi(x,y)\, dS(y)\, dT(x).$$

Using Theorem 2.86, one sees $P = Q$. □

Theorem 4.56. *Let T and S be tempered distributions on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Then the distribution $T \times S$ on $\mathbb{R}^m \times \mathbb{R}^n$ is tempered; i.e., there is a tempered distribution $T \times S$ such that*

$$(T \times S)(\phi \times \psi) = T(\phi)S(\psi)$$

if $\phi \in \mathcal{S}(\mathbb{R}^m)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. It satisfies

$$(T \times S)(\Phi) = \int \int \Phi(x,y)\, dT(x)\, dS(y) = \iint \Phi(x,y)\, dS(y)\, dT(x)$$

for all $\Phi \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$.

Proof. We follow the previous argument. First there is a tempered distribution P that satisfies

$$P(\phi \times \psi) = T(\phi)S(\psi) \text{ for} \phi \in \mathcal{S}(\mathbb{R}^m) \text{ and} \psi \in \mathcal{S}(\mathbb{R}^n)\,.$$

Indeed, by Lemma 4.54, the mapping $\Phi \mapsto \Phi_1$ where $\Phi_1(x) = \int \Phi(x,y)\, dS(y)$ is a continuous linear mapping from $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^m)$. Thus P defined by $P(\Phi) = T(\Phi_1)$ is a tempered distribution $\mathbb{R}^{m+n}$. Clearly it satisfies, $P(\phi \times \psi) = T(\phi)S(\psi)$ and is defined by

$$P(\Phi) = \int \int \Phi(x,y)\, dS(y)\, dT(x).$$

In a similar fashion, there is a tempered distribution Q satisfying $Q(\phi \times \psi) = T(\phi)S(\psi)$ and

$$Q(\Phi) = \int \int \Phi(x,y)\, dS(y)\, dT(x).$$

Using Theorem 2.85, one sees $P = Q$. □

Proposition 4.57. *Let S be a distribution on $\mathbb{R}^n$. Then the mapping $\phi \mapsto S * \phi$ is a continuous mapping from $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{E}(\mathbb{R}^n)$. Moreover,*

$$D^\alpha(S * \phi) = (D^\alpha S) * \phi = S * (D^\alpha \phi).$$

Proof. We have $S * \phi(x) = S(\lambda(x)\check{\phi})$. Hence continuity will follow if $x \mapsto \lambda(x)\check{\phi}$ is continuous from $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}(\mathbb{R}^n)$. But this following from Exercise 2.4.15.

We calculate $\partial_j(S * \phi)$. By definition,

$$\begin{aligned}\partial_j(S * \phi)(x) &= \lim_{h \to 0} \frac{1}{h}(S(\lambda(x + he_j)\check{\phi} - \lambda(x)\check{\phi}) \\ &= \lim_{h \to 0} S(\frac{\lambda(x - he_j)\check{\phi} - \lambda(x)\check{\phi}}{-h}).\end{aligned}$$

Now by Lemma 4.51,

$$\begin{aligned}\frac{\lambda(x + he_j)\check{\phi}(y) - \lambda(x)\check{\phi}(y)}{h} &= \frac{\check{\phi}(y - x - he_j) - \check{\phi}(y - x)}{h} \\ &= \frac{(\lambda(x)\check{\phi})(y - he_j) - (\lambda(x)\check{\phi})(y)}{h} \\ &\to -\partial_j(\lambda(x)\check{\phi})(y)\end{aligned}$$

in $\mathcal{D}(\mathbb{R}^n)$ as $h \to 0$.

Thus

$$\partial_j(S * \phi)(x) = S(-\partial_j(\lambda(x)\check{\phi})) = (\partial_j S)(\lambda(x)\check{\phi}) = (\partial_j S) * \phi(x).$$

Next note since $\partial_j(\lambda(x)\check{\phi})(y) = -\lambda(x)(\partial_j\phi)\check{}(y)$, we also have

$$\partial_j(S * \phi)(x) = S(\lambda(x)(\partial_j\phi)\check{}) = S * (\partial_j\phi)(x).$$

Consequently, $D^\alpha(S * \phi) = (D^\alpha S) * \phi = S * (D^\alpha\phi)$ for all α and we have that $S * \phi$ is smooth.

To show the linear mapping $\phi \mapsto S * \phi$ is continuous from $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{E}(\mathbb{R}^n)$, it suffices to show for each compact subset $K \subseteq \mathbb{R}^n$, the mapping $\phi \mapsto S * \phi$ is continuous on $\mathcal{D}_K(\mathbb{R}^n)$. To see this let K' be a compact subset of $\mathbb{R}^n$ and let $\alpha \in \mathbb{N}_0^n$. Then $K' - K$ is a compact set and since $D^\alpha S$ is a distribution, there is a finite set $F \subseteq \mathbb{N}_0$ and a $\delta > 0$ such that

$$|D^\alpha S(\psi)| \leqslant 1$$

if $\psi \in \mathcal{D}_{K'-K}(\mathbb{R}^n)$ and $|\psi|_{K'-K,\beta} \leqslant \delta$ for $\beta \in F$.

Now let $\phi \in \mathcal{D}_K(\mathbb{R}^n)$ and suppose $|\phi|_{K,\beta} \leqslant \delta$ for $\beta \in F$. If $x \in K'$, then $\psi = \lambda(x)\check{\phi}$ has support in $K' - K$ and $D^\beta\psi(y) = (-1)^{|\beta|}(D^\beta\phi)(x - y)$. Thus $|\psi|_{K'-K,\beta} \leqslant \delta$ for $\beta \in F$ and hence

$$|D^\alpha(S * \phi)(x)| = |(D^\alpha S)(\lambda(x)\check{\phi})| = |D^\alpha S(\psi)| \leqslant 1.$$

Consequently,

$$|S * \phi|_{K',\alpha} = \max_{x \in K'} |(D^\alpha S)(\lambda(x)\check{\phi})| \leqslant 1$$

when $|\phi|_{K,\beta} \leqslant \delta$ for $\beta \in F$. This implies the continuity of $\phi \mapsto S * \phi$ on $\mathcal{D}_K(\mathbb{R}^n)$. □

Proposition 4.58. *Let S be a tempered distribution on $\mathbb{R}^n$. Then the mapping $\phi \mapsto S * \phi$ is a linear mapping from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'_\infty(\mathbb{R}^n)$. Moreover,*

$$D^\alpha(S * \phi) = (D^\alpha S) * \phi = S * (D^\alpha \phi)$$

for $\alpha \in \mathbb{N}_0^n$.

Proof. We have $S * \phi(x) = S(\lambda(x)\check{\phi})$. Hence continuity will follow if $x \mapsto \lambda(x)\check{\phi}$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. But this follows from Lemma 2.59.

We calculate $\partial_j(S * \phi)$. By definition,

$$\begin{aligned}\partial_j(S * \phi)(x) &= \lim_{h\to 0} \frac{1}{h}(S(\lambda(x + he_j)\check{\phi} - \lambda(x)\check{\phi}) \\ &= \lim_{h\to 0} S(\frac{\lambda(x - he_j)\check{\phi} - \lambda(x)\check{\phi}}{-h}).\end{aligned}$$

Now by Lemma 4.52,

$$\begin{aligned}\frac{\lambda(x + he_j)\check{\phi}(y) - \lambda(x)\check{\phi}(y)}{h} &= \frac{\check{\phi}(y - x - he_j) - \check{\phi}(y - x)}{h} \\ &= \frac{(\lambda(x)\check{\phi})(y - he_j) - (\lambda(x)\check{\phi})(y)}{h} \\ &\to -\partial_j(\lambda(x)\check{\phi})(y)\end{aligned}$$

in $\mathcal{S}(\mathbb{R}^n)$ as $h \to 0$. Thus

$$\partial_j(S * \phi)(x) = S(-\partial_j(\lambda(x)\check{\phi})) = (\partial_j S)(\lambda(x)\check{\phi}) = (\partial_j S) * \phi(x).$$

Again $\partial_j(\lambda(x)\check{\phi})(y) = -\lambda(x)(\partial_j\phi)\check{}(y)$, and so we have

$$\partial_j(S * \phi)(x) = S(\lambda(x)(\partial_j\phi)\check{}) = S * (\partial_j\phi)(x).$$

Consequently, $D^\alpha(S * \phi) = (D^\alpha S) * \phi = S * (D^\alpha\phi)$ for all α and we have that $S * \phi$ is smooth.

We next show $S * \phi \in \mathcal{S}'_\infty(\mathbb{R}^n)$. To do this since $D^\alpha(S * \phi) = S * D^\alpha\phi$, we need only show $S * \phi$ has polynomial growth for each $\phi \in \mathcal{S}(\mathbb{R}^n)$. Since S is continuous, there is an N and a $\delta > 0$ so that

$$|S(\phi)| \leqslant 1 \text{ if } |\phi|_{N,\alpha} \leqslant \delta \text{ for } |\alpha| \leqslant N.$$

Hence if $M = \frac{1}{\delta}$, we see by linearity of S that $|S(\phi)| \leqslant M$ if $|\phi|_{N,\alpha} \leqslant 1$ for $|\alpha| \leqslant N$. By using Exercise 2.3.4 one has

$$\begin{aligned}|(1 + |y|^2)^N D^\alpha\phi(x - y)| &= (1 + |x|^2)^N \frac{(1 + |y|^2)^N}{(1 + |x|^2)^N} \frac{(1 + |x - y|^2)^N}{(1 + |x - y|^2)^N} |D^\alpha\phi(x - y)| \\ &\leqslant 2^N(1 + |x|^2)^N |\phi|_{N,\alpha}\end{aligned}$$

for $|\alpha| \leqslant N$. Fix x and let

$$K = 2^N(1+|x|^2)^N \max_{|\alpha|\leqslant N} |\phi|_{N,\alpha}.$$

Then $|\frac{1}{K}\lambda(x)\check{\phi}|_{N,\alpha} \leqslant 1$ for $|\alpha| \leqslant N$. Hence $|\frac{1}{K}S(\lambda(x)\check{\phi})| \leqslant M$. Multiplying by K, we have

$$|S(\lambda(x)\check{\phi})| \leqslant 2^N M(1+|x|^2)^N \max_{|\alpha|\leqslant N} |\phi|_{N,\alpha},$$

and we see one has polynomial growth. □

Hence we see if S is a distribution and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then the function $S * \varphi(x) := \int \varphi(x-y)\, dS(y) = S(\lambda(x)\check{\varphi})$ is smooth and thus $S * \varphi$ is a distribution. Therefore $(S * \varphi) * \psi$ is defined for all $\psi \in \mathcal{D}(\mathbb{R}^n)$. Similarly if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and S is a tempered distribution, then $S * \varphi$ is a C^∞ functions all of whose derivatives have polynomial growth. Thus $S * \varphi$ is a tempered distribution, and therefore $(S*\varphi)*\psi$ is defined and smooth for all $\psi \in \mathcal{S}(\mathbb{R}^n)$.

Proposition 4.59. *Suppose S is a distribution on $\mathbb{R}^n$ and ϕ and ψ are in $\mathcal{D}(\mathbb{R}^n)$. Then*

$$(S * \phi) * \psi = S * (\phi * \psi).$$

The same statement holds if S is a tempered distribution and ϕ and ψ are in $\mathcal{S}(\mathbb{R}^n)$.

Proof. Fix x. Define $F(y,z) = \phi(y-z)\psi(x-y)$. Thus $F(y,z) = (\phi \otimes \lambda(x)\check{\psi})(y-z, y)$. By Proposition 2.69 and Theorem 2.86, $F \in \mathcal{D}(\mathbb{R}^{m+n})$. Moreover, by Theorem 4.55,

$$\begin{aligned}
(S * \phi) * \psi(x) &= \int S * \phi(y)\, \psi(x-y)\, dy \\
&= \int \left(\int \phi(y-z)\, dS(z) \right) \psi(x-y)\, dy \\
&= \int \int F(y,z)\, dS(z)\, dy \\
&= \int \int F(y,z)\, dy\, dS(z) \\
&= \int \left(\int \phi(y-z)\psi(x-y)\, dy \right)\, dS(z) \\
&= \int \left(\int \phi(y)\psi(x-z-y)\, dy \right)\, dS(z) \\
&= \int (\phi * \psi)(x-z)\, dS(z) \\
&= S(\phi * \psi).
\end{aligned}$$

The same argument works for tempered distributions if one uses Proposition 2.58, Theorem 2.85, and Theorem 4.56 instead of Proposition 2.69, Theorem 2.86 and Theorem 4.55. □

We are interested when $S * \phi \in \mathcal{D}(\mathbb{R}^n)$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$. In order to do this we shall use a theorem due to Lion. See Studia Math 81 (1985), "A Proof of the Theorem of Supports", p. 323-328.

For a subset E of $\mathbb{R}^n$, the *convex hull* $[E]$ of E is the intersection of all closed convex sets containing E. The following theorem we state without proof.

Theorem 4.60 (Leon)**.** *Let ϕ and ψ be in $\mathcal{D}(\mathbb{R}^n)$. Then*

$$[\mathrm{supp}\,(\phi * \psi)] = [\mathrm{supp}\phi] + [\mathrm{supp}\psi].$$

Proposition 4.61. *Let S be a distribution on $\mathbb{R}^n$. Then $S * \phi \in \mathcal{D}(\mathbb{R}^n)$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$ if and only if S has compact support. Moreover, if S has compact support, then linear mapping $\phi \mapsto S * \phi$ from $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}(\mathbb{R}^n)$ is a continuous.*

Proof. We already know $S * \phi$ is smooth. Suppose S has compact support K. Let ϕ be a test function with compact support F. Then if $\psi \in \mathcal{D}(\mathbb{R}^n)$ has support W where $W \cap (K + F) = \varnothing$, we see

$$\begin{aligned}
\langle \psi, S * \phi \rangle &= \int \psi(x)\, d(S * \phi)(x) \\
&= \int \psi(x) \int \phi(x - y)\, dS(y)\, dx \\
&= \int \int \psi(x)\phi(x - y)\, dS(y)\, dx \\
&= \int S(\lambda(x)\check{\phi})\, dx \\
&= \int 0\, dx
\end{aligned}$$

since if $\psi(x) \neq 0$, then $x \in W$ and so $\phi(x-y) \neq 0$ implies $x-y \in F$ and thus $y \in x-F \subseteq W-F$. So the support of $\lambda(x)\check{\phi}$ misses K for $(W-F) \cap K = \varnothing$. Thus $S * \phi$ has compact support.

Conversely, suppose $S * \phi$ has compact support for all ϕ. Then take an approximate identity h_t. Specifically, for $0 < t \leqslant 1$, set $h_t(x) = t^{-n}h(t^{-1}x)$ where $h \geqslant 0$ is C^∞, has integral one, and has support in the closed unit ball $|x| \leqslant 1$. We may also assume $\check{h} = h$.

Then since $(S * h_t * h_1) = (S * h_1) * h_t$, we see by Lion's Theorem that $[\mathrm{supp}\,(S*h_t)]+[\mathrm{supp}\,(h_1)] = [\mathrm{supp}(S*h_1)]+[\mathrm{supp}\,h_t] \subseteq [\mathrm{supp}(S*h_1)]+\{x \mid$

$|x| \leqslant 1\}$. In particular, $S * h_t$ has compact support in the compact set $K = [\operatorname{supp}(S * h_1)] + \{x \mid |x| \leqslant 1\}$ for $0 < t \leqslant 1$.

Now suppose ϕ has compact support disjoint from K. Then $(S*h_t)(\phi) = 0$ for all all $0 < t \leqslant 1$. But $S * h_t(\phi) \to S(\phi)$ as $t \to 0+$. Indeed,

$$\begin{aligned}
(S * h_t)(\phi) &= \int \phi(x)\, d(S * h_t)(x) \\
&= \int \phi(x) \int h_t(x-y)\, dS(y)\, dx \\
&= \int \left(\int h_t(x-y)\phi(x)\, dx \right) dS(y) \\
&= \int \left(\int h_t(y-x)\, \phi(x)\, dx \right) dS(y) \\
&= S(h_t * \phi).
\end{aligned}$$

So $S(h_t * \phi) = 0$ for $0 < t \leqslant 1$ and since by Lemma 2.83, $h_t * \phi \to \phi$ in $\mathcal{D}(\mathbb{R}^n)$ as $t \to 0+$, we obtain $S(\phi) = 0$ for all ϕ whose support misses K. Hence $\operatorname{supp} S \subseteq K$.

Assume now S has compact support. Consider the linear mapping $\phi \mapsto S * \phi$. By Proposition 4.57, this mapping is continuous from $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{E}(\mathbb{R}^n)$. Hence if K is a compact subset of $\mathbb{R}^n$, then the mapping $\phi \mapsto S*\phi$ is a continuous linear mapping of $\mathcal{D}_K(\mathbb{R}^n)$ into $\mathcal{D}_{K+\operatorname{supp} S}(\mathbb{R}^n)$. But by Lemma 2.66, the relative topology of $\mathcal{D}(\mathbb{R}^n)$ on $\mathcal{D}^\infty_{K+\operatorname{supp} S}(\mathbb{R}^n)$ is the topology of $\mathcal{D}_{K+\operatorname{supp} S}(\mathbb{R}^n)$. Thus the mapping $\phi \mapsto S * \phi$ is a continuous linear mapping of $\mathcal{D}_K(\mathbb{R}^n)$ into $\mathcal{D}(\mathbb{R}^n)$ for all compact subsets K of $\mathbb{R}^n$. Since $\mathcal{D}(\mathbb{R}^n)$ is the inductive limit topology, Proposition 2.15 shows $\phi \mapsto S * \phi$ is continuous on $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}(\mathbb{R}^n)$. □

Proposition 4.62. *Let S be a distribution on $\mathbb{R}^n$ with compact support. Then $S * \phi \in \mathcal{S}(\mathbb{R}^n)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ and the linear mapping $\phi \mapsto S * \phi$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Let K be the support of S. Fix α and N. Let $\epsilon > 0$. Since S has compact support K, there is a finite subset $F \subseteq \mathbb{N}_0^n$ and a $\delta > 0$ such that $|S(\phi)| < \frac{\epsilon}{2^N \max_{y\in K}(1+|y|^2)^N}$ if $|\phi|_{K,\beta} = \max_{x\in K} |D^\beta \phi(x)| < \delta$ for $\beta \in F$. Then by Exercise 2.3.4,

$$\begin{aligned}
(1+&|x|^2)^N |D^\alpha(S * \phi)(x)| \\
&= \left| \int \frac{(1+|x|^2)^N}{(1+|x-y|^2)^N (1+|y|^2)^N} (1+|y|^2)^N (D^\alpha \phi)(x-y)\, dS(y) \right| \\
&\leqslant 2^N \max_{y\in K} (1+|y|^2)^N \left| \int (D^\alpha \phi)(x-y)\, dS(y) \right| \\
&< \epsilon
\end{aligned}$$

if $|\phi|_{0,\alpha+\beta} = |D^\alpha \phi|_{0,\beta} < \delta$ for $\beta \in F$. From this one has

$$|S * \phi|_{N,\alpha} < \epsilon \text{ if } |\phi|_{0,\gamma} < \delta \text{ for } \gamma \in \alpha + F.$$

Consequently $S * \phi \in \mathcal{S}(\mathbb{R}^n)$ and $\phi \mapsto S * \phi$ is a continuous linear mapping of $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. □

Conditions that make $S * \phi$ rapidly decreasing for all rapidly decreasing ϕ depend on S being "rapidly decreasing". An example of such an S is $S = \sum_{k=1}^{\infty} e^{-k}\delta_k$.

Putting the prior two propositions together with Definition 4.50 gives the following theorem.

Theorem 4.63. *Let S be a distribution on $\mathbb{R}^n$ having compact support. If $T \in \mathcal{D}'(\mathbb{R}^n)$, then $T * S$ defined by $T * S(\phi) = T(\check{S} * \phi)$ is a distribution on $\mathbb{R}^n$. If T is a tempered distribution, then $T * S$ is a tempered distribution.*

Exercise Set 4.6

1. Let T be a tempered distribution and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Show $\widehat{T * \varphi} = \hat{\varphi}\hat{T}$.

2. Show $\delta_0 * \phi = \phi$ for all test functions ϕ.

3. Let H be the Heaviside function. Show $H * \varphi(x) = \int_{-\infty}^{x} \varphi(y)\,dy$ for $\varphi \in \mathcal{S}(\mathbb{R})$.

4. Suppose that S is a compactly supported distribution. Show $\varphi \mapsto S * \varphi$ is a continuous linear transformation of $\mathcal{E}(\mathbb{R}^n)$ into $\mathcal{E}(\mathbb{R}^n)$.

5. Suppose T and S are distributions on $\mathbb{R}^n$ with compact support. Show $\operatorname{supp}(S * T) \subseteq \operatorname{supp} S + \operatorname{supp} T$.

6. Let T be a tempered distribution on $\mathbb{R}^n$. Suppose $\lambda(x)T = T$ for all x. Show $T = c\,dx$; i.e. T is a multiple of Lebesgue measure.

8. The Sobolev Lemma

In solving PDE's one needs to be able to tell when a distribution is actually a function and when it is differentiable. In this section we shall establish a fundamental tool in handling such problems.

Lemma 4.64. *Suppose $\gamma \leqslant \alpha$ where γ and α are in $\mathbb{N}_0^n$. Then*

$$\sum_{\gamma \leqslant \beta \leqslant \alpha} (-1)^\alpha \binom{\alpha}{\beta}\binom{\beta}{\gamma} = \begin{cases} 0 & \text{if } \gamma \neq \alpha \\ (-1)^{|\alpha|} & \text{if } \gamma = \alpha. \end{cases}$$

Proof. It suffices to do the one dimensional case. Now note

$$\begin{aligned}\sum_{p\leqslant q\leqslant r}(-1)^q\binom{r}{q}\binom{q}{p} &= \sum_{0\leqslant q\leqslant r-p}(-1)^{q+p}\binom{r}{q+p}\binom{q+p}{p}\\ &= \sum_{0\leqslant q\leqslant r-p}(-1)^{q+p}\frac{r!}{(q+p)!(r-q-p)!}\frac{(q+p)!}{p!q!}\\ &= \frac{r!}{p!(r-p)!}\sum_{0\leqslant q\leqslant r-p}(-1)^{q+p}\frac{(r-p)!}{(r-p-q)!q!}\\ &= \frac{(-1)^p r!}{p!(r-p)!}\sum_{0\leqslant q\leqslant r-p}(-1)^q\binom{r-p}{q}\\ &= \frac{(-1)^p r!}{p!(r-p)!}(1-1)^{r-p}\end{aligned}$$

which is 0 if $r > p$ and is $(-1)^r$ if $r = p$. The formula follows from this and the distributive law. □

Lemma 4.65. *Let Ω be a nonempty open subset of $\mathbb{R}^n$. Suppose ψ is a C^∞ function on Ω and f is a locally L^1 function. Then*

$$D^\alpha(\psi T_f) = \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}(D^{\alpha-\beta}\psi)D^\beta T_f.$$

In particular we shall write

$$D^\alpha(\psi f) = \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}(D^{\alpha-\beta}\psi)D^\beta f$$

where this equality is understood in the distributional sense.

Proof. We start by calculating the right hand side on a test function ϕ in $\mathcal{D}(\Omega)$. Using the Leibnitz Formula, one sees

$$\begin{aligned}\text{RHS} &= \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}(D^{\alpha-\beta}\psi)(D^\beta T_f)(\phi)\\ &= \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}D^\beta T_f((D^{\alpha-\beta}\psi)\phi)\\ &= \sum_{\beta\leqslant\alpha}(-1)^{|\beta|}\binom{\alpha}{\beta}T_f(D^\beta((D^{\alpha-\beta}\psi)\phi))\\ &= \sum_{\beta\leqslant\alpha}(-1)^{|\beta|}\binom{\alpha}{\beta}T_f(\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma}(D^{\beta-\gamma}(D^{\alpha-\beta}\psi))D^\gamma\phi)\\ &= \sum_{\beta\leqslant\alpha}(-1)^{|\beta|}\binom{\alpha}{\beta}T_f(\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma}(D^{\alpha-\gamma}\psi)D^\gamma\phi)\end{aligned}$$

Continuing we see with the help of Lemma 4.64 that:

$$\begin{aligned}
\text{RHS} &= \sum_{\gamma \leqslant \alpha} \left(\sum_{\gamma \leqslant \beta \leqslant \alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} \binom{\beta}{\gamma} \right) T_f((D^{\alpha-\gamma}\psi) D^{\gamma}\phi) \\
&= (-1)^{|\alpha|} T_f(\psi D^{\alpha}\phi) \\
&= (-1)^{|\alpha|} (\psi T_f)(D^{\alpha}\phi) \\
&= D^{\alpha}(\psi T_f).
\end{aligned}$$

□

Lemma 4.66. *Let T be a distribution on $\mathbb{R}^n$ such that all $D^{\alpha}T$ for $|\alpha| \leqslant p$ are distributions given by continuous functions. Then $T = T_g$ where g is in $C^p(\mathbb{R}^n)$ and $D^{\alpha}T = T_{D^{\alpha}g}$ for $|\alpha| \leqslant p$.*

Proof. For simplicity we take $n = 1$ and $p = 1$. The general case follows easily by the same reasoning. We then have $T = T_f$ and $DT = T_g$ where f and g are continuous. We claim

$$f(x) = f(0) + \int_0^x g(t)\,dt.$$

Let ϕ be a test function. Set $c = \int \phi(x)\,dx$ and define F by

$$F(x) = f(0) + \int_0^x g(t)\,dt.$$

Let h be a C^{∞} function with compact support and integral 1. Set $\psi(x) = \int_{-\infty}^x (\phi(t) - ch(t))\,dt$. Note ψ is C^{∞} with compact support and $\psi'(x) = \phi(x) - ch(x)$. Thus

$$\begin{aligned}
T(\phi - ch) = T(\psi') = -DT(\psi) = -T_g(\psi) &= -\int g(x)\psi(x)\,dx \\
&= -\int F'(x)\psi(x)\,dx = \int F(x)\psi'(x)\,dx \\
&= \int F(x)(\phi(x) - ch(x))\,dx \\
&= \int F(x)\phi(x)\,dx - c\int F(x)h(x)\,dx.
\end{aligned}$$

Thus

$$T(\phi) = \int F(x)\phi(x)\,dx + cT(h) - c\int F(x)h(x)\,dx.$$

Since $T = T_f$, we have

$$T(\phi) = \int F(x)\phi(x)\,dx + c\int h(x)(f(x) - F(x))\,dx$$

for all h in $C_c^\infty(\mathbb{R})$ with $\int h(t)\,dt = 1$. This implies $\int h(x)(f(x) - F(x))\,dx$ is independent of h and consequently $f(x) - F(x)$ is constant in x. Since $f(0) - F(0) = 0$, we see $f = F$ and thus f is differentiable and has derivative g. $\square$

Theorem 4.67 (The Sobolev Lemma)**.** *Let T be a distribution on Ω where Ω is an open subset of $\mathbb{R}^n$. Suppose r and p are integers with $p \geqslant 0$ and $r > p + \frac{n}{2}$. If $D^\alpha T$ is locally an L^2 function for each α with $0 \leqslant |\alpha| \leqslant r$, then there is an $f \in C^p(\Omega)$ such that $T = T_f$.*

Proof. Let $D^\alpha T = g_\alpha \in L^2_{\text{loc}}(\Omega)$ for $0 \leqslant |\alpha| \leqslant r$. Extend all these g_α to all of $\mathbb{R}^n$ by defining them to be 0 off Ω. Note $T(D^\alpha \varphi) = (-1)^{|\alpha|} \int g_\alpha(x)\varphi(x)\,d_n x$ for any $\varphi \in \mathcal{D}(\Omega)$. Set $g = g_0$. Let V be an open subset of Ω having compact closure in Ω. Let ψ be in $\mathcal{D}(\Omega)$ with $\psi = 1$ on $\bar{V}$. Define $f(x) = \psi(x)g(x)$. Extend f to all of $\mathbb{R}^n$ by setting $f(x) = 0$ for $x \notin \operatorname{supp}\psi$. Then f is both an L^2 and an L^1 function on $\mathbb{R}^n$. By Lemma 4.65, $D^\alpha f = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta}\psi\, D^\beta g$. More precisely, this is an equality of distributions; i.e. $D^\alpha T_f = \sum \binom{\alpha}{\beta}(D^{\alpha-\beta}\psi) D^\beta T_g$ and this holds on $\mathbb{R}^n$. Thus f and $D^\alpha f$ as distributions are given by functions which are both L^2 and L^1. Thus $\mathcal{F}(D^\alpha f)(y) = (2\pi i y)^\alpha \hat{f}(y)$ is continuous and L^2. This gives $\int |y^{2\alpha} \hat{f}(y)|^2\,dy < \infty$ if $|\alpha| \leqslant r$ and since $\hat{f}$ is bounded, it follows by an easy argument that $E := \int (1 + |y|^2)^r |\hat{f}(y)|^2\,dy < \infty$. (Indeed, one need only show $\int_{|y| \geqslant 1} |y|^{2k} |\hat{f}(y)|^2\,dy < \infty$ whenever $k \leqslant r$. But $|y|^{2k}$ is a finite sum of terms of form $y^{2\alpha}$ where $|\alpha| \leqslant k$.) Now using the Cauchy-Schwarz inequality and integration in polar coordinates, one has

$$\begin{aligned}
\int (1 + |y|^2)^{p/2} |\hat{f}(y)|\,dy &= \int \left((1 + |y|^2)^{\frac{r}{2}} |\hat{f}(y)| \right) (1 + |y|^2)^{\frac{p-r}{2}}\,dy \\
&\leqslant E^{1/2} \left(\int (1 + |y|^2)^{p-r}\,dy \right)^{1/2} \\
&= E^{1/2} \left(\sigma(S^{n-1}) \int_0^\infty (1 + r^2)^{p-r} r^{n-1}\,dr \right)^{1/2} \\
&= E^{1/2} \left(\sigma(S^{n-1}) \int_0^\infty (1 + r^2)^{p-r} r^{n-1}\,dr \right)^{1/2}
\end{aligned}$$

which is finite if $\int_1^\infty r^{2p-2r} r^{n-1}\,dr < \infty$. This occurs if $2p - 2r + n < 0$; that is if $p < r - n/2$. We thus see $y^\alpha \hat{f}(y) \in L^1(\mathbb{R}^n)$ whenever $|\alpha| \leqslant p$. Since $\mathcal{F}^{-1}((2\pi y)^\alpha \hat{f}) = D^\alpha f$, we see $D^\alpha f$ as distribution is a continuous function when $|\alpha| \leqslant p$. Lemma 4.66 implies f is equal to a C^p function a.e. and the derivatives of this C^p function up to order p give the derivatives of the distribution f. Consequently, the same is true for g and the g_α on V. $\square$

EXERCISE SET 4.7

1. Let $f \geqslant 0$ be a continuous function such that $\int_{\mathbb{R}^n} f\,dx = 1$. Define $f_m(x) := m^n f(mx)$. Show

$$\lim_{m\to\infty} T_{f_m} = \delta_0.$$

2. Let $s \in (-\infty, \infty)$. Define $\mathcal{H}_s$ to be the Hilbert space of tempered distributions obtained by Fourier transforming the Hilbert space $L^2(\mathbb{R}^n, (1 + |x|^2)^s\,dx)$; i.e.,

$$\mathcal{H}_s = \{\mathcal{F}f \mid \int |f(x)|^2(1+|x|^2)^s\,dx < \infty\}$$

with

$$(\mathcal{F}f, \mathcal{F}g) = \int f(x)\bar{g}(x)(1+|x|^2)^s\,dx.$$

$\mathcal{H}_s$ is the Sobolev space of order s on $\mathbb{R}^n$; it is sometimes denoted by $W^{s,2}(\mathbb{R}^n)$.

(a) Show $\mathcal{H}_s \supseteq \mathcal{H}_{s'}$ if $s < s'$ and $\mathcal{H}_s \subseteq L^2(\mathbb{R}^n)$ if and only if $s \geqslant 0$;

(b) Given an integer $p \geqslant 0$, find the smallest index s such that $\mathcal{H}_{s'} \subseteq C^p(\mathbb{R}^n)$ if $s' > s$.

(c) Show the differential operator D^α is a bounded linear operator from $\mathcal{H}_s$ into $\mathcal{H}_{s-|\alpha|}$.

(c) Let $\alpha \in \mathbb{N}_0^n$. Determine which $\mathcal{H}_s$ contain the distribution $D^\alpha\epsilon_0$ where ϵ_0 is point mass at 0.

(d) Show $(1 - \frac{1}{4\pi^2}\Delta))$ is a unitary mapping of $\mathcal{H}_s$ onto $\mathcal{H}_{s-4}$.

(e) Show $\mathcal{H}_s(+) := \cup_{s'>s}\mathcal{H}_{s'}$ and $\mathcal{H}_s(-) := \cap_{s'<s}\mathcal{H}_s$ are proper subspaces of $\mathcal{H}_s$.

(f) Show $\mathcal{H}_\infty = \cap_s\mathcal{H}_s$ is the subspace of $C^\infty(\mathbb{R}^n)$ consisting of all f with $D^\alpha f \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$.

9. Schwartz Bases and Spaces of Rapidly Decreasing Sequences

We have been using $\mathbb{N}_0^n$ to denote the collection of all n tuples α of nonnegative integers. In this section, to minimize notation, we shall denote this set by A. Recall for $\alpha = (a_1, a_2, \ldots, a_n) \in A$, one has $|\alpha| = a_1 + a_2 + \cdots + a_n$, $x^\alpha = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$, and $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{a_n}$. Also as before we have $D_i = \partial_i = \frac{\partial}{\partial x_i}$. A n-multi index sequence is a complex valued function $\lambda : A \to \mathbb{C}$. This sequence converges to 0 as $\alpha \to \infty$ if $\lambda_\alpha \to 0$ as $|\alpha| \to \infty$. The space $\mathcal{R}$ of rapidly decreasing sequences is the space of all n-multi index sequences λ satisfying

$$p(\alpha)\lambda_\alpha \to 0 \text{ as } \alpha \to \infty$$

for each polynomial $p(x)$ in n variables. This is clearly a complex vector space.

For each polynomial $p(x)$, we define a seminorm $|\cdot|_p$ on $\mathcal{R}$ by

$$|\lambda|_p = \max_{\alpha \in A} |p(\alpha)\lambda_\alpha|.$$

These seminorms define a separated locally convex vector space topology on $\mathcal{R}$. We call it the Schwartz topology on $\mathcal{R}$. By Exercise 4.8.1 the space $\mathcal{R}$ with this topology is a Fréchet space.

The **Weyl algebra** $\mathcal{W}$ is the collection of all differential operators D on $C^\infty(\mathbb{R}^n)$ which can be written as a finite sum of multiples of the operators $x^\alpha D^\beta$ where α, $\beta \in A$. Thus if $D \in \mathcal{W}$, on has

$$D = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha D^\beta.$$

where $c_{\alpha,\beta} \in \mathbb{C}$.

The Weyl algebra is closely related to the Heisenberg group which we study in a later chapter. We introduce it here because it contains the Hermite operator whose spectral decomposition we shall use to characterize Schwartz functions. Recall from Equation 3.9 that we are taking $H = 4\pi|x|^2 - \Delta$. This operator is in $\mathcal{W}$.

Definition 4.68. *We say an orthonormal basis $\{e_\alpha \mid \alpha \in A\}$ of $L^2(\mathbb{R}^n)$ consisting of Schwartz functions is a Schwartz basis if a function $f \in L^2(\mathbb{R}^n)$ is Schwartz if and only if the sequence $\lambda_\alpha = (f, e_\alpha)_2$ is rapidly decreasing.*

Theorem 4.69. *Let $\{e_\alpha\}_{\alpha \in A}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$ satisfying:*

(a) *There is a k such that $(x_i e_\alpha, e_\beta)_2 = (D_i e_\alpha, e_\beta)_2 = 0$ if $|\alpha - \beta| > k$ and $1 \leqslant i \leqslant n$.*

(b) *There is a nonnegative polynomial Q so that*

$$\begin{matrix} |(x_i e_\alpha, e_\beta)_2| \leqslant Q(\alpha) \\ |(D_i e_\alpha, e_\beta)_2| \leqslant Q(\alpha) \end{matrix} \quad \text{for all } \alpha \text{ and } \beta \text{ in } A \text{ and } 1 \leqslant i \leqslant n.$$

(c) *There is an $H \in \mathcal{W}$ and an $\epsilon > 0$ such that $He_\alpha = c_\alpha e_\alpha$ where $|c_\alpha| \geqslant |\alpha|^\epsilon$ for all $\alpha \in A$.*

Then $\{e_\alpha\}_{\alpha \in A}$ is a Schwartz basis and the mapping Λ defined on $\mathcal{R}$ by $\Lambda(\lambda) = \sum \lambda_\alpha e_\alpha$ is a bicontinuous one-to-one linear transformation from $\mathcal{R}$ onto $\mathcal{S}(\mathbb{R}^n)$.

Proof. Suppose $\lambda = \{\lambda_\alpha\}_{\alpha \in A}$ is in $\mathcal{R}$. Set $p(x) = |x|^{2n}$. Since $p(\alpha)\lambda_\alpha \to 0$ as $|\alpha| \to \infty$, there is a constant M such that $|\alpha|_2^{2n}\,|\lambda_\alpha| \leqslant M$ for all α. Hence $|\lambda_\alpha|^2 \leqslant \frac{M}{|\alpha|^{4n}}$ when $|\alpha| > 0$. By Exercise 4.8.8, we see $\sum |\lambda_\alpha|^2 < \infty$. Thus $f_\lambda = \sum \lambda_\alpha e_\alpha$ exists in $L^2(\mathbb{R}^n)$ and hence is a tempered distribution. We calculate the distributional derivative $D_i f_\lambda$. Note if ϕ is a Schwartz function $\langle \phi, D_i f_\lambda \rangle = -\langle D_i \phi, f_\lambda \rangle = -\sum_\alpha \int \lambda_\alpha e_\alpha(x) D_i \phi(x)\, dx$. Hence

$$\begin{aligned}
\langle \phi, D_i f_\lambda \rangle &= \sum_\alpha \lambda_\alpha \langle D_i e_\alpha, \phi \rangle \\
&= \sum_\alpha \lambda_\alpha \langle \sum_{|\beta-\alpha| \leqslant k} (D_i e_\alpha, e_\beta)_2 e_\beta, \phi \rangle \\
&= \sum_\beta \sum_{|\beta-\alpha| \leqslant k} \lambda_\alpha (D_i e_\alpha, e_\beta)_2 \langle e_\beta, \phi \rangle.
\end{aligned}$$

The rearrangement in summation follows by absolute summability. Indeed, by the Cauchy-Schwarz inequality and since

$$|D_i e_\alpha|_2^2 = \sum_{|\beta-\alpha| \leqslant k} |(D_i e_\alpha, e_\beta)_2|^2 \leqslant \sum_{|\beta-\alpha| \leqslant k} Q(\alpha)^2 \leqslant k^n Q(\alpha)^2,$$

we see

$$\begin{aligned}
&\sum_\alpha \sum_{|\beta-\alpha| \leqslant k} |\lambda_\alpha (D_i e_\alpha, e_\beta)_2 \langle e_\beta, \phi \rangle| \leqslant \\
&\qquad \sum_\alpha |\lambda_\alpha| \left(\sum_{|\beta-\alpha| \leqslant k} |(D_i e_\alpha, e_\beta)_2|^2 \right)^{1/2} \left(\sum_{|\beta-\alpha| \leqslant k} |\langle e_\beta, \phi \rangle|^2 \right)^{1/2} \\
&\quad \leqslant |\phi|_2 \sum_\alpha |\lambda_\alpha|\, |D_i e_\alpha|_2 \\
&\quad \leqslant k^{\frac{n}{2}} |\phi|_2 \sum |\lambda_\alpha|\, Q(\alpha) < \infty.
\end{aligned}$$

Thus:

$$\langle \phi, D_i f_\lambda \rangle = \sum_\beta \left(\sum_{|\alpha-\beta| \leqslant k} \lambda_\alpha (D_i e_\alpha, e_\beta)_2 \right) \langle e_\beta, \phi \rangle.$$

Set $\lambda'_\beta = \sum_{|\alpha-\beta| \leqslant k} \lambda_\alpha (D_i e_\alpha, e_\beta)_2$. We show $\lambda' \in \mathcal{R}$. Indeed,

$$|\lambda'_\beta| \leqslant \sum_{|\alpha-\beta| \leqslant k} |\lambda_\alpha| Q(\alpha).$$

Now let $p(x)$ be any positive polynomial.

Define polynomial $P(x)$ by

$$P(x) = \sum_{\gamma \in A, |\gamma| \leqslant k} P(x - \gamma).$$

Then $p(\beta) \leqslant P(\alpha)$ if $|\alpha - \beta| \leqslant k$. Hence

$$p(\beta)|\lambda'_\beta| \leqslant p(\beta) \sum_{|\alpha-\beta|\leqslant k} |\lambda_\alpha| Q(\alpha) \leqslant \sum_{|\alpha-\beta|\leqslant k} |\lambda_\alpha| P(\alpha) Q(\alpha).$$

Consequently, $p(\beta)|\lambda'_\beta| \to 0$ as $\beta \to \infty$ and we see $\lambda' \in \mathcal{R}$.

Thus $D_i f_\lambda = f_{\lambda'}$ where $\lambda' \in \mathcal{R}$. Repeating, we see for any $\alpha \in A$, one has $D^\alpha f_\lambda = f_{\lambda'}$ for some $\lambda' \in \mathcal{R}$. Since $f_{\lambda'}$ is L^2 for each $\lambda' \in \mathcal{R}$, we see all distributions $D^\alpha f_\lambda$ are L^2 functions. By the Sobolev Lemma, $f_\lambda \in C^\infty(\mathbb{R}^n)$.

Using the same argument as above, one has $x_i f_\lambda = f_{\lambda'}$ for some $\lambda' \in \mathcal{R}$ for each $\lambda \in \mathcal{R}$. Thus by induction, pf_λ is a $f_{\lambda'}$ for some $\lambda' \in \mathcal{R}$ and hence is both C^∞ and an L^2 function for any polynomial p.

Using polar coordinates, (see Corollary 2.26), one has $x \mapsto (1 + |x|^2)^{-m}$ is in $L^2(\mathbb{R}^n)$ for $m > \frac{n}{4}$. Thus if m is an integer larger than $\frac{n}{4}$, and $p(x) = (1 + |x|^2)^m$, then $f_\lambda = (pf_\lambda)(\frac{1}{p})$ is in $L^1(\mathbb{R}^n)$. So f_λ is in $L^1(\mathbb{R}^n)$ for each $\lambda \in \mathcal{R}$. It's Fourier transform is thus a bounded continuous function. Thus the Fourier transforms of all $p(x)D^\alpha f_\lambda(x)$ are bounded L^2 functions. Hence $p(y)\hat{f}_\lambda(y)$ is L^2 for all polynomials $p(y)$. As above, we see this implies $\hat{f}_\lambda$ is L^1. Consequently, $f_\lambda = \mathcal{F}^{-1}(\hat{f}_\lambda)$ is bounded. Since $p(x)D^\alpha f_\lambda = f_{\lambda'}$ for some λ', $|f_\lambda|_{p,\alpha} < \infty$ for all polynomials $p(x)$ and $\alpha \in A$. Thus f_λ is a Schwartz function.

Let f be a Schwartz function. Define λ_α by $\lambda_\alpha = (f, e_\alpha)_2$. We claim $\lambda \in \mathcal{R}$. Indeed, $H^k f = \sum c_\alpha^k \lambda_\alpha e_\alpha$ for all k. Thus $c_\alpha^k \lambda_\alpha$ is square summable over $\alpha \in A$ for all k. Using (c), we see $|\alpha|^{k\epsilon}|\lambda_\alpha|$ is in $l_2(A)$ for all k. Let p be a polynomial of order m. Note if $m < k\epsilon$ and $|\beta| \leqslant m$, then $|\alpha^\beta| \leqslant (\max|\alpha_j|)^{|\beta|} \leqslant |\alpha|^m$. Hence $|\alpha^\beta \lambda_\alpha| \leqslant |\alpha|^{k\epsilon}|\lambda_\alpha| \to 0$ as $\alpha \to \infty$. Thus $p(\alpha)\lambda_\alpha \to 0$ and we see λ is rapidly decreasing. Clearly $f_\lambda = f$. Hence the mapping $\lambda \mapsto f_\lambda$ is onto.

We thus see the linear mapping Λ from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{R}$ defined by $\Lambda(f)(\alpha) = (f, e_\alpha)_2$ is one-to-one and onto. We show Λ is continuous. Define

$$\Lambda_m : \mathcal{S}(\mathbb{R}^n) \to \mathcal{R}$$

by

$$\Lambda_m(f)(\alpha) = \begin{cases} 0 & \text{for } |\alpha| > m \text{ and} \\ \Lambda(f)(\alpha) & \text{for } |\alpha| \leqslant m \end{cases}$$

Note each Λ_m is continuous. Moreover, $\Lambda_m(f) \to \Lambda(f)$ in $\mathcal{R}$ as $m \to \infty$ for each $f \in \mathcal{S}(\mathbb{R}^n)$. Indeed, if p is a polynomial, $|\Lambda_m(f) - \Lambda(f)|_p = \max_{|\alpha|>m} |p(\alpha)\Lambda(f)(\alpha)| \to 0$ as $m \to \infty$ for $\lim_{\alpha\to\infty} p(\alpha)\Lambda(f)(\alpha) = 0$. By the Principle of Uniform Boundedness, Theorem 2.18, Λ is continuous.

But since Λ is continuous and onto, we can apply the Open Mapping Theorem 2.19. Consequently Λ is an open mapping. Hence Λ is a topological isomorphism. $\square$

Corollary 4.70. *Let $\{e_\alpha\}_{\alpha\in A}$ be a Schwartz basis for $\mathcal{S}(\mathbb{R}^n)$. Suppose D is a differential operator in the Weyl algebra $\mathcal{W}$ and $De_\alpha = c_\alpha e_\alpha$ for all α. If $\inf_\alpha |c_\alpha| > 0$, then $D : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a homeomorphism.*

Proof. We already know D is continuous. Let $c = \inf_\alpha |c_\alpha| > 0$. Take $f \in \mathcal{S}(\mathbb{R}^n)$ and define λ by $\lambda_\alpha = (f, e_\alpha)$. Then $f = f_\lambda$ and $\{\lambda_\alpha\}_{\alpha\in A}$ is rapidly decreasing. For $\alpha \in A$, set $\Lambda_\alpha = \frac{\lambda_\alpha}{c_\alpha}$. We note Λ is rapidly decreasing. Indeed, if P is polynomial,

$$\begin{aligned}
|P(\lambda)\Lambda_\alpha| &= \frac{|P(\lambda)\lambda_\alpha|}{|c_\alpha|} \\
&\leqslant \frac{|P(\lambda)\lambda_\alpha|}{c} \\
&\leqslant c^{-1}|\lambda|_P \\
&< \infty.
\end{aligned}$$

Thus $f_\Lambda \in \mathcal{S}(\mathbb{R}^n)$, $D(f_\Lambda) = f$, and $|\Lambda|_P \leqslant c^{-1}|\lambda|_P$. Hence D is onto and one-to-one and the mapping $f \mapsto D^{-1}f$ is given by $\lambda \mapsto \Lambda$ in $\mathcal{R}$, we have that D^{-1} is continuous. $\square$

Recall from 3.8 that the Hermite functions $h_\alpha(x)$ are defined by

$$h_\alpha(x) = H_\alpha(x)e^{-\pi|x|^2}$$

where H_α is the Hermite polynomial

$$H_\alpha(x) = (-1)^{|\alpha|}e^{2\pi|x|^2}D^\alpha(e^{-2\pi|x|^2}).$$

By Theorem 3.36, the normalized Hermite functions

$$e_\alpha = \frac{2^{n/4}}{\sqrt{(4\pi)^{|\alpha|}\alpha!}}h_\alpha,\ \alpha \in A$$

form an orthonormal basis for $L^2(\mathbb{R}^n)$. It in fact is a Schwartz basis. Indeed, from equations (3.11) and (3.12) we know:

$$\begin{aligned}
D_je_\alpha(x) &= \sqrt{\pi\alpha_j}\, e_{\alpha-e_j}(x) + \sqrt{\pi(\alpha_j+1)}\, e_{\alpha+e_j}(x) \text{ and} \\
x_je_\alpha(x) &= \sqrt{\frac{\alpha_j}{4\pi}}\, e_{\alpha-e_j}(x) + \sqrt{\frac{\alpha_j+1}{4\pi}} e_{\alpha+e_j}(x).
\end{aligned}$$

Moreover, if H is the Hermite operator $4\pi^2|x|^2 - \Delta$, then Proposition 3.35 gives

$$He_\alpha = c_\alpha e_\alpha \text{ where } c_\alpha = 2\pi(2|\alpha| + n).$$

In particular, we see

$$(x_j e_\alpha, e_\beta) = (D_j e_\alpha, e_\beta) = 0 \text{ if } |\alpha - \beta| > 1$$

and

$$|(x_j e_\alpha, e_\beta)| \leqslant \pi(\alpha_j + 1)^2 \text{ and } |(D_j e_\alpha, e_\beta)| \leqslant \pi(\alpha_j + 1)^2.$$

Since, in addition, $c_\alpha > |\alpha|$, we may apply Theorem 4.69.

Theorem 4.71. *The orthonormal collection $e_\alpha = \frac{2^{n/4}}{\sqrt{(4\pi)^{|\alpha|}\alpha!}} h_\alpha$ where $\alpha \in \mathbb{N}_0^n$ is a Schwartz basis of $L^2(\mathbb{R}^n)$.*

From Lemma 3.23 we have

$$h_\alpha(x) = h_\alpha(x_1, x_2, \ldots, x_n) = h_{\alpha_1}(x_1) h_{\alpha_2}(x_2) \cdots h_{\alpha_n}(x_n)$$

and thus $h_\alpha = h_{\alpha_1} \otimes h_{\alpha_2} \otimes \cdots \otimes h_{\alpha_n}$. Consequently, $h_{\alpha,\beta} = h_\alpha \otimes h_\beta$ for $(\alpha, \beta) \in \mathbb{N}_0^m \times \mathbb{N}_0^n = \mathbb{N}_0^{m+n}$. Since $c_{\alpha,\beta} = \frac{2^{(m+n)/4}}{\sqrt{(4\pi)^{|\alpha|+|\beta|}\alpha!\beta!}} = c_\alpha c_\beta$, we see

$$e_{\alpha,\beta} = e_\alpha \otimes e_\beta \text{ for } (\alpha, \beta) \in \mathbb{N}_0^m \times \mathbb{N}_0^m = \mathbb{N}_0^{m+n}. \tag{4.19}$$

Proposition 4.72. *The Hermite operator $4\pi^2|x|^2 + \Delta$ is a linear homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$.*

Proof. This follows immediately from Corollary 4.70 since $\inf |c_\alpha| = 2\pi n$. □

The next proposition was proved earlier using an alternate method in Theorem 2.85.

Proposition 4.73. *The linear span of the simple tensors $f \otimes h$ where $f \in \mathcal{S}(\mathbb{R}^m)$ and $h \in \mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^{m+n})$ in the Schwartz topology.*

Proof. Since the $e_{\alpha,\beta} = c_{\alpha,\beta} h_{\alpha,\beta}$ for $\alpha \in \mathbb{N}_0^m$ and $\beta \in \mathbb{N}_0^n$ form a Schwartz basis for $L^2(\mathbb{R}^{m+n})$, we see the linear span of the functions $h_{\alpha,\beta}$ is a Schwartz dense subspace of $\mathcal{S}(\mathbb{R}^{m+n})$. But $h_{\alpha,\beta} = h_\alpha \otimes h_\beta$. Thus the linear span of the tensors $f \otimes h$ where $f \in \mathcal{S}(\mathbb{R}^m)$ and $h \in \mathcal{S}(\mathbb{R}^n)$ contains the linear span of the Hermite functions $h_{\alpha,\beta}$. □

We shall make use of the following theorem.

Theorem 4.74. *Let $K \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then the operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by*

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy$$

is a trace class operator and

$$\mathrm{Tr}(K) = \int_{\mathbb{R}^n} K(x, x)\, dx.$$

Proof. Define $\lambda_{\alpha,\beta} = (K, e_\alpha \otimes e_\beta)_2$. By Theorem 4.71, we know λ is a member of $\mathcal{R}_{2n}$, the space of rapidly decreasing multi $2n$-sequences. In $L^2(\mathbb{R}^{2n})$ one has $K = \sum_{\alpha,\beta\in\mathbb{N}_0^{m+n}} \lambda_{\alpha,\beta} e_\alpha \otimes \bar{e}_\beta$. We can conclude T is trace class by referring to Exercise 2.2.29.

As an alternative, one can proceed directly using Definition 2.37. Note $\sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{p(\alpha)} < \infty$ where p is the polynomial given by

$$p(x_1, x_2, \ldots, x_n) = (1 + x_1^2)(1 + x_2^2)\cdots(1 + x_n^2).$$

Define $w_\alpha = \sum_\beta \sqrt{p(\alpha)}\bar{\lambda}_{\alpha,\beta} e_\beta$. Let $Q(x, y)$ be the polynomial defined by

$$Q(x, y) = p(x)^2 p(y) \text{ for } x, y \in \mathbb{R}^n.$$

Then

$$\begin{aligned}
\sum_\alpha |w_\alpha|_2^2 &= \sum_{\alpha,\beta} p(\alpha)|\lambda_{\alpha,\beta}|^2 \\
&= \sum_{\alpha,\beta} \frac{1}{p(\alpha)p(\beta)} |p^2(\alpha)p(\beta)\lambda_{\alpha,\beta}|^2 \\
&\leqslant |\lambda|_Q^2 \sum_{\alpha,\beta} \frac{1}{p(\alpha)p(\beta)} \\
&= |\lambda|_Q^2 \left(\sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{p(\alpha)} \right)^2 \\
&< \infty.
\end{aligned}$$

Moreover, if $v_\alpha = \frac{1}{p(\alpha)} e_\alpha$, then $\sum_\alpha |v_\alpha|^2 = \sum \frac{1}{p(\alpha)^2} < \infty$. Thus $\sum v_\alpha \otimes \bar{w}_\alpha$ is trace class. It is T for

$$\begin{aligned}
T(f) &= \sum \lambda_{\alpha,\beta}(f, e_\beta)_2 e_\alpha \\
&= \sum_\alpha (f, \sum_\beta \bar{\lambda}_{\alpha,\beta} e_\beta)_2 e_\alpha \\
&= \sum_\alpha (f, p(\alpha) \sum \bar{\lambda}_{\alpha,\beta} e_\beta)_2 \frac{1}{p(\alpha)} e_\alpha \\
&= \sum_\alpha v_\alpha \otimes \bar{w}_\alpha(f).
\end{aligned}$$

Hence by definition, T is a trace class operator. Moreover, by Theorem 2.44, $\mathrm{Tr}(T) = \int K(x, x)\, dx$. □

Definition 4.75. *A multi-index complex valued sequence $\{c_\alpha\}_{\alpha\in A}$ is tempered if there is a polynomial $p(x)$ such that $|c_\alpha| \leqslant |p(\alpha)|$ for all α.*

Corollary 4.76. *Let $|\cdot|$ be any continuous seminorm on $\mathcal{S}(\mathbb{R}^n)$ and suppose $\{e_\alpha \mid \alpha \in A\}$ is a Schwartz basis. Then the sequence $|e_\alpha|$ is tempered.*

Proof. Under the mapping $f \mapsto \lambda$, $e_\alpha \to \delta_\alpha \in \mathcal{R}$ where

$$\delta_\alpha(\beta) = \begin{cases} 0 & \text{if } \beta \neq \alpha \\ 1 & \text{if } \beta = \alpha. \end{cases}$$

Thus we need only show $\{|\delta_\alpha|\}_{\alpha \in A}$ is tempered for any continuous seminorm on $\mathcal{R}$. But if $|\cdot|$ is a continuous seminorm on $\mathcal{R}$, there exists polynomials $p_1(x)$, $p_2(x)$, ..., $p_m(x)$ and a $\delta > 0$ such that $|\lambda| \leqslant 1$ if $|\lambda|_{p_k} \leqslant \delta$ for $k = 1, 2, \ldots, m$. Set $p(x) = 1 + \sum p_k(x)\bar{p}_k(x)$. Note $|\lambda|_{p_k} \leqslant |\lambda|_p$ for each k. Hence $|\lambda|_p \leqslant \delta$ implies $|\lambda| \leqslant 1$. But $|\frac{\delta}{p(\alpha)}\delta_\alpha|_p \leqslant \delta$ for all α. Thus $|\frac{\delta}{p(\alpha)}\delta_\alpha| \leqslant 1$ for all α. We thus have $|\delta_\alpha| \leqslant \frac{1}{\delta}p(\alpha)$ for all α. Consequently, the sequence $\alpha \mapsto |\delta_\alpha|$ is tempered. □

Exercise Set 4.8

1. Show the space $\mathcal{R}$ of rapidly decreasing n-multi indexed sequences with the Schwartz topology is a Fréchet space.

2. Let e_n, $n = 0, 1, 2, \ldots$ be a Schwartz basis of $L^2(\mathbb{R})$. Show there is a positive polynomial p with $\max_{x \in \mathbb{R}} |e_n(x)| \leqslant p(n)$ for all n.

3. Show the dual of $\mathcal{R}$ is the vector space of all tempered multi-indexed sequences $\{c_\alpha\}_{\alpha \in A}$.

4. Show a rearrangement of a rapidly decreasing sequence need not be rapidly decreasing.

5. Let e_n be the orthonormal sequence of normalized Hermite functions on $\mathbb{R}$. Show $n \mapsto \sup_x |p(x)D^k e_n(x)|_{p,k}$ is tempered for all polynomials p and nonnegative integers k.

6. Show the Weyl algebra $\mathcal{W}$ is an algebra; i.e., show it is closed under addition, multiplication by scalars, and composition.

7. Show $\mathcal{S}(\mathbb{R}^n)$ is the collection of all $f \in L^2(\mathbb{R}^n)$ such that as distributions, $Df \in L^2(\mathbb{R}^n)$ for all D in the Weyl algebra $\mathcal{W}$.

8. Let A be the collection of all n-tuples of nonnegative integers. Recall $|\alpha| = \sum a_i$ if $\alpha = (a_1, a_2, \ldots, a_n) \in A$. Show

$$\sum_{|\alpha|>0} \frac{1}{|\alpha|^k} < \infty \text{ if and only if } k > n.$$

9. Let A be the collection of all n-tuples consisting of nonnegative integers. Show $\{e_\alpha\}_{\alpha \in A}$ is a Schwartz basis if and only if the mapping $\Lambda : \mathcal{R} \to \mathcal{S}(\mathbb{R}^n)$ defined by $\Lambda(\lambda) = \sum_{\alpha \in A} \lambda_\alpha e_\alpha$ is a topological isomorphism.

10. Heisenberg Uncertainty Principle

One of the inherent problems with the Fourier transform is that it is not possible to localize both f and $\hat{f}$ at the same time. An extreme situation arises when $f = T = \delta_0$ which has support $\{0\}$ and is as well localized as possible. But $\hat{T} = 1$ is evenly distributed and thus has no localization. The same phenomenon can been seen by the dilation properties of the Fourier transform. If $f \geqslant 0$ is an integrable function with support in the closed ball $\overline{B_R(0)}$ and $a > 0$, then $f_a(x) = a^{-1}f(a^{-1}x)$ has support in $\overline{B_{aR}(0)}$ and $f_a \to \delta_0$ as $a \to 0+$ whereas $\widehat{f_a}(\omega) = \hat{f}(a\omega) \to \hat{f}(0) = 1$ for each ω. Moreover, since $\hat{f}$ is holomorphic and therefore cannot vanish on any open set, we see each $\hat{f}_a$ has support $\mathbb{R}$. This non-local property of the Fourier transform implies that a short term change in the function f has a global change in $\hat{f}$. As an example consider a variation of f by $a\chi_{(-\epsilon,\epsilon)}$, $\epsilon > 0$:

$$f_\epsilon = f + a\chi_{(-\epsilon,\epsilon)}\,.$$

Then since

$$\widehat{f_\epsilon}(\omega) = \hat{f}(\omega) + \frac{a\sin(2\pi\epsilon\omega)}{\pi\omega},$$

we see there is a global change which vanishes at infinity in the frequency representation of f. A partial explanation for this is the fact that the exponential function

$$e^{2\pi i\omega x} = \cos(2\pi\omega x) + i\sin(2\pi\omega x)$$

is a *uniform* wave on $\mathbb{R}$ and hence the integrals

$$\hat{f}(\omega) = \int f(x)e^{-2\pi i\omega x}\,dx$$

and

$$f(x) = \int \hat{f}(\omega)e^{2\pi i\omega x}\,d\omega$$

both involve global information on f and $\hat{f}$, respectively.

The Heisenberg Uncertainty Principle is a general statement about the joint distribution of f and $\hat{f}$. This principle can efficiently be expressed using ideas from probability theory. Let μ be a probability measure on $\mathbb{R}$, i.e., $\mu(\mathbb{R}) = 1$. Then one defines the *mean* of μ to be the number

$$m(\mu) := \int x\,d\mu(x)$$

if the function $\mathrm{id} : x \mapsto x$ is integrable. If the mean is finite, then the *variance* of μ is defined to be the number

$$\nu(\mu) := \int (x - m(\mu))^2\,d\mu(x)$$

in case $x \mapsto (x - m(\mu))^2$ is integrable. Otherwise $\nu(\mu) = \infty$. The number $\sigma(\mu) = \sqrt{\nu(\mu)}$ is the *standard deviation* of μ. If f is a nonzero member of $L^2(\mathbb{R})$, the functions

$$x \mapsto \frac{|f(x)|^2}{|f|_2^2}, \qquad \omega \mapsto \frac{\left|\hat{f}(\omega)\right|^2}{|f|_2^2}$$

define probability measures μ_f and $\mu_{\hat{f}}$ by

$$\mu_f(A) = \frac{1}{|f|_2^2}\int_A |f(x)|\ ^2 dx \text{ and } \mu_{\hat{f}}(A) = \frac{1}{\left|\hat{f}\right|_2^2}\int_A \left|\hat{f}(\omega)\right|\ ^2 dx\,.$$

To simplify the discussion, we shall work with functions f in $\mathcal{S}(\mathbb{R})$. Then the means and the variances of both μ_f and $\mu_{\hat{f}}$ are finite. We shall simply write

$$\begin{aligned} m(f) &= m(\mu_f) = \frac{1}{|f|_2^2}\int x\,|f(x)|^2\ dx \\ \nu(f) &= \nu(\mu_f) = \frac{1}{|f|_2^2}\int (x - m(f))^2\,|f(x)|^2\ dx \end{aligned} \tag{4.20}$$

and similarly for $\hat{f}$. Then $\nu(f)$ and $\nu(\hat{f})$ give information about the spread of f and $\hat{f}$.

Example 4.77. *For $\mu \in \mathbb{R}$ and $\sigma > 0$, let $\gamma_{\mu,\sigma}$ denote the Gaussian*

$$\gamma_{\mu,\sigma}(x) = \frac{1}{(2\pi\sigma^2)^{1/4}}\exp(-\frac{(x-\mu)^2}{4\sigma^2})\,.$$

Then

$$|\gamma_{\mu,\sigma}(x)|^2 = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(x-\mu)^2}{2\sigma^2})\,.$$

The substitution $u = \frac{x-\mu}{\sqrt{2\pi}\sigma}$ yields:

$$\begin{aligned} |\gamma_{\mu,\sigma}|_2^2 &= \frac{1}{\sqrt{2\pi}\sigma}\int \exp[-\frac{(x-\mu)^2}{2\sigma^2}]\,dx \\ &= \int e^{-\pi u^2}\,du \\ &= 1 \end{aligned}$$

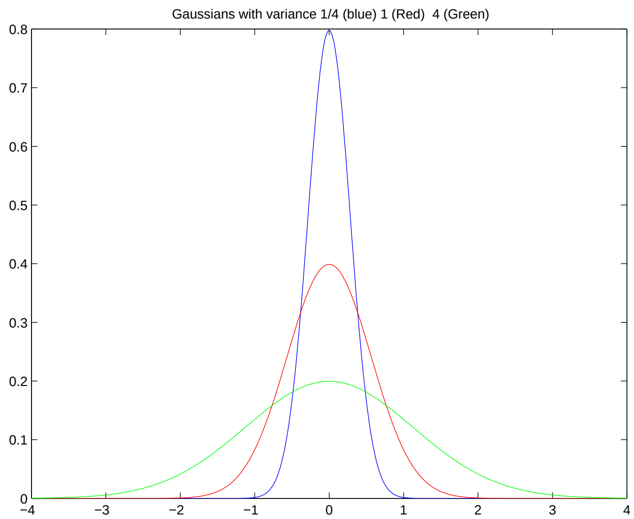

Figure 1. $|\gamma(0,\sigma)|^2$ for $\sigma = \frac{1}{2}$ (blue) , 1 (red) , 2 (green).

and

$$\begin{aligned} m(\gamma_{\mu,\sigma}) &= \frac{1}{\sqrt{2\pi}\sigma}\int x \exp[-\frac{(x-\mu)^2}{2\sigma^2}]\,dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int (x+\mu)\exp[-\left(\frac{x}{\sqrt{2}\sigma}\right)^2]\,dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int x \exp[-\left(\frac{x}{\sqrt{2}\sigma}\right)^2]\,dx + \frac{\mu}{\sqrt{2\pi}\sigma}\int \exp[-\left(\frac{x}{\sqrt{2}\sigma}\right)^2]\,dx \\ &= \mu\int e^{-\pi u^2}\,dx \\ &= \mu\,. \end{aligned}$$

Here we used $x \mapsto e^{-\frac{x^2}{2\sigma^2}}$ *is even, the substitution* $u = \frac{x}{\sqrt{2\pi}\sigma}$, *and* $\int e^{-\pi u^2}\,du = 1$. *Finally for the variance:*

$$\begin{aligned} \nu(\gamma_{\mu,\sigma}) &= \frac{1}{\sqrt{2\pi}\sigma}\int (x-\mu)^2\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \\ &= \sigma\sqrt{\frac{2}{\pi}}\int\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2\exp[-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2]\,dx = 2\sigma^2\int \pi u^2 e^{-\pi u^2}du \\ &= -\sigma^2\int u\frac{d}{du}e^{-\pi u^2}\,du = \sigma^2\int e^{-\pi u^2}\,du = \sigma^2. \end{aligned}$$

Using the fact that $\hat{h} = h$ for $h(x) = e^{-\pi x^2}$, we see

$$\begin{aligned}
\hat{\gamma}_{\mu,\sigma}(\omega) &= \frac{1}{(2\pi\sigma^2)^{1/4}} \int \exp(-\frac{(x-\mu)^2}{4\sigma^2}) e^{-2\pi i \omega x}\, dx \\
&= \frac{e^{-2\pi i \omega \mu}}{(2\pi\sigma^2)^{1/4}} \int \exp(-\frac{x^2}{4\sigma^2}) e^{-2\pi i \omega x}\, dx \\
&= \frac{2\sigma\sqrt{\pi} e^{-2\pi i \omega \mu}}{(2\pi\sigma^2)^{1/4}} \int \exp(-\frac{(2\sigma\sqrt{\pi}x)^2}{4\sigma^2}) e^{-2\pi i \omega 2\sigma\sqrt{\pi}x}\, dx \\
&= 2^{3/4}\pi^{1/4}\sqrt{\sigma} e^{-2\pi i \omega \mu} \int \exp(-\pi x^2) e^{-2\pi i \omega (2\sigma\sqrt{\pi}x)}\, dx \\
&= 2^{3/4}\pi^{1/4}\sqrt{\sigma} e^{-2\pi i \omega \mu} \hat{h}(2\sqrt{\pi}\sigma\omega) \\
&= 2^{3/4}\pi^{1/4}\sqrt{\sigma} e^{-2\pi i \omega \mu} \exp(-4\pi^2\sigma^2\omega^2) \\
&= e^{-2\pi i \omega \mu} \frac{1}{(2\pi(4\pi\sigma)^{-2})^{1/4}} \exp(-\frac{\omega^2}{4(4\pi\sigma)^{-2}}) \\
&= e^{-2\pi i \omega \mu} \gamma_{0,\frac{1}{4\pi\sigma}}(\omega).
\end{aligned}$$

Hence

$$m(\widehat{\gamma_{\mu,\sigma}}) = 0 \quad \textit{and} \quad \nu\,(\widehat{\gamma_{\mu,\sigma}}) = \frac{1}{16\pi^2\sigma^2}\,.$$

It follows that

$$\nu(\gamma_{\mu,\sigma})\nu(\widehat{\gamma_{\mu,\sigma}}) = \frac{1}{16\pi^2}\,.$$

Theorem 4.78 (Heisenberg Uncertainty Principle)**.** *Assume that $f \in \mathcal{S}(\mathbb{R})$. Then*

$$\nu(f)\nu(\hat{f}) \geqslant \frac{1}{16\pi^2}\,.$$

Furthermore $\nu(f)\nu(\hat{f}) = \frac{1}{16\pi^2}$ if and only if there exists $\mu \in \mathbb{R}$ and a pair A and a of nonzero complex numbers such that $\mathrm{Re}(a) > 0$ *and*

$$f(x) = Ae^{-a(x-\mu)^2}.$$

Proof. We can assume without loss of generality that $|f|_2 = 1$. Set $m = m(f)$ and $\hat{m} = m(\hat{f})$. Then we can assume that $m = \hat{m} = 0$. Indeed, if we replace f by g where $g(x) = e^{-2\pi i \hat{m} x} f(x+m)$, then $m(g) = 0$ and $\nu(g) = \nu(f)$. Furthermore, since

$$\hat{g}(\omega) = e^{2\pi i(\omega+\hat{m})m} \hat{f}(\omega + \hat{m}),$$

we see $m(\hat{g}) = 0$ and $\nu(\hat{g}) = v(\hat{f})$. As f is rapidly decreasing, we can use integration by parts and the fact that

$$\frac{d}{dx}\,|f(x)|^2 = \frac{d}{dx} f(x)\overline{f(x)} = 2\mathrm{Re}(f(x)\overline{f'(x)})$$

to see that

$$\int x\mathrm{Re}(f(x)\overline{f'(x)})dx = \frac{1}{2}\int x\frac{d}{dx}|f(x)|^2\,dx = -\frac{1}{2}\int |f(x)|^2\,dx = -\frac{1}{2}\,.$$

Now Hölder's inequality and $\mathcal{F}(Df)(\omega) = 2\pi i\omega f(\omega)$ imply

$$\begin{aligned}\frac{1}{2} &= -\int x\mathrm{Re}(f(x)\overline{f'(x)})\,dx \\ &= -\mathrm{Re}\left(\int xf(x)\overline{f'(x)}\,dx\right) \\ &\leqslant |xf|_2\,|f'|_2 \\ &= 2\pi\,|xf|_2\left|\omega\hat{f}\right|_2 \\ &= 2\pi\sigma(f)\sigma(\hat{f}).\end{aligned}$$

Thus

$$\nu(f)\nu(\hat{f}) \geqslant \frac{1}{16\pi^2}$$

and we have equality if and only if

$$|xf|_2\,|f'|_2 = -\mathrm{Re}\left(\int xf(x)\overline{f'(x)}\,dx\right) \leqslant |\int xf(x)\overline{f'(x)}\,dx| \leqslant |xf|_2\,|f'|_2\,.$$

Thus equality is equivalent to $|\int xf(x)\overline{f'(x)}\,dx| = |xf|_2\,|f'|_2$, and since equality occurs in Hölder's inequality precisely when the functions are linearly dependent, we see $\nu(f)\nu(\hat{f}) = \frac{1}{16\pi^2}$ if and only if there is a constant b with such that

$$f'(x) = 2bxf(x)$$

and thus $f(x) = Ae^{bx^2}$.

Finally the argument in Example 4.77 can be used to Show $f(x) = Ae^{-a(x-\mu)^2}$ with $\mathrm{Re}(a) > 0$ satisfies $\nu(f)\nu(\hat{f}) = 1/16\pi^2$. $\square$

Exercise Set 4.9

1. Let $f \in \mathcal{S}(\mathbb{R})$. Let $a, b \in \mathbb{R}$. Define g by $g(x) = e^{-2\pi iax}f(x+b)$. Show $m(g) = m(f) - b$, $m(\hat{g}) = m(\hat{f}) - a$, $\nu(g) = \nu(f)$ and $\nu(\hat{g}) = \nu(\hat{f})$.

2. Let $f \in L^2(\mathbb{R})$. Suppose $\omega \mapsto \omega\hat{f}(\omega)$ is an L^2 function. Show the distribution Df is an L^2 function f' and

$$\int |f'(x)|^2\,dx = 4\pi^2\int \omega^2|\hat{f}(\omega)|^2\,d\omega.$$

3. Suppose f and xf are in $L^2(\mathbb{R})$ and the distribution Df is an L^2 function f'. Show there is a sequence f_k in $\mathcal{D}(\mathbb{R})$ such that $f_k \to f$, $f_k' \to f'$, and $xf_k \to xf$ in $L^2(\mathbb{R})$. (**Hint:** Consider $h_q * (k_pf)$ where h_q is an approximate identity and $k_p(x) = k(\frac{x}{p})$ where k is a C^∞ function of compact support in $[-1,1]$, $0 \leqslant k(x) \leqslant 1$, and $k(x) = 1$ for $-1/2 \leqslant x \leqslant 1/2$.

4. Assume $f \in L^2(\mathbb{R})$ with $|f|_2 = 1$. Show using the previous exercise that

$$\left(\int x^2|f(x)|^2\,dx\right)\left(\int \omega^2|\hat{f}(\omega)|^2\,d\omega\right) \geqslant \frac{1}{16\pi^2}.$$

Also show one has equality if and only if $f = Ae^{bx}$ where $\operatorname{Re} b < 0$. (**Hint:** Show $|f(x)|^2$ is absolutely continuous and use integration by parts.)

5. Let $f \in L^2(\mathbb{R})$. Let $a, b \in \mathbb{R}$. Show

$$\left(\int (x-a)^2|f(x)|^2\,dx\right)\left(\int (\omega-b)^2|\hat{f}(\omega)|^2\,d\omega\right) \geqslant \frac{|f|^4}{16\pi^2}.$$

11. The Windowed Fourier Transform

One of the first proposals for dealing with the non-local behavior of the Fourier transform was made by D. Gabor in 1946. His idea was to multiply f by a well localized window function ψ, i.e., to consider the function

$$f_\psi(x,u) = f(x)\psi(x-u)\,.$$

If ψ is localized around 0, then $x \mapsto \psi(x-u)$ is localized around u and hence $x \mapsto f_\psi(\cdot,u)$ contains *local* information about f around u. A natural requirement is that ψ has mean zero and that ψ and $\hat{\psi}$ have finite momentum, i.e., that $x \mapsto x^\alpha\psi(x)$, $\omega \mapsto \omega^\beta\hat{\psi}(\omega)$ are in $L^2(\mathbb{R})$ for all α, $\beta > 0$. The second condition is clearly satisfied for all rapidly decreasing functions. But as all proofs are the same for $\psi \in L^2(\mathbb{R}^n)$ we will assume so if nothing else is stated. Let $\psi_{u,\omega}(x) = \psi(x-u)e^{2\pi i x\cdot\omega}$. Define the *windowed*, or *short time*, *Fourier transform* $\mathcal{S}(f)(u,\omega) = \mathcal{S}_\psi(f)(u,\omega)$ by

$$\mathcal{S}_\psi(f)(u,\omega) := \int f(x)\overline{\psi\,(x-u)}e^{-2\pi i\omega\cdot x}\,dx = (f \mid \psi_{u,\omega})_2. \tag{4.21}$$

Before we discuss the L^2-theory of this transform note that when f and ψ belong to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\int \overline{\psi(x)}\,dx = C \neq 0$, then $(x,u) \mapsto f(x)\overline{\psi(x-u)}$ is integrable on $\mathbb{R}^{2n}$ and

$$C^{-1}\int \mathcal{S}_\psi(f)(\omega,u)du = \int f(x)e^{-2\pi i\omega\cdot x}\,dx = \hat{f}(\omega)\,.$$

Thus the windowed Fourier transform decomposes the Fourier transform of f into an integral of a localized Fourier transform given by the window function ψ. Furthermore, as the Fourier transform is a unitary isomorphism, and

$$\widehat{\psi_{u,\omega}}(\eta) = e^{2\pi i u\cdot(\omega-\eta)}\hat{\psi}(\eta-\omega) = e^{2\pi i u\cdot\omega}(\hat{\psi})_{\omega,-u}(\eta)$$

one sees we obtain the following Lemma.

Lemma 4.79 (Fundamental identity)**.** *Let $f, \psi \in L^2(\mathbb{R}^n)$. Then*

$$\mathcal{S}_\psi(f)(u,\omega) = e^{-2\pi i u\cdot\omega}\mathcal{S}_{\hat{\psi}}(\hat{f})(\omega,-u)\,. \tag{4.22}$$

This identity shows that the short time Fourier transform giving local information about f is expressible in terms of $\hat{f}$ and $\hat{\psi}$. It is called *the fundamental identity of time-frequency analysis*. Other properties for the short time Fourier transform are presented in the exercise set following this section. We will focus on the inversion formula for the transform. We start with the Plancherel formula.

Theorem 4.80 (Plancherel Formula)**.** *Let $\psi, \varphi \in L^2(\mathbb{R}^n)$. Then*

$$(\mathcal{S}_\psi(f), \mathcal{S}_\varphi(g))_{\mathrm{L}^2(\mathbb{R}^{2n})} = (f,g)_2(\varphi,\psi)_2$$

for all $f, g \in L^2(\mathbb{R}^n)$. In particular the following hold:

(a) $\mathcal{S}_\psi(f) \in L^2(\mathbb{R}^{2n})$ *and* $\mathcal{S}_\psi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ *is continuous.*

(b) *If $\|\psi\|_2 = 1$, then $\mathcal{S}_\psi$ is an unitary isomorphism from $L^2(\mathbb{R}^n)$ onto its image.*

(c) *If $(\varphi,\psi)_2 \neq 0$, then $f = (\varphi,\psi)_2^{-1}\mathcal{S}_\varphi^*\mathcal{S}_\psi(f)$ for all $f \in L^2(\mathbb{R}^n)$.*

Remark 4.81. In Chapter 7 we study the Heisenberg group and its representation theory. The windowed (short term) Fourier transform turns out to be a matrix coefficient of an irreducible unitary representation and many of the formulas involving the windowed Fourier transform have reinterpretations in terms of formulas for the Heisenberg group. In particular, see Exercise 7.2.2.

Proof. We notice first that

$$\mathcal{S}_\psi(f)(u,\omega) = \mathcal{F}(f\overline{\lambda(u)\psi})(\omega)\,.$$

Thus by taking the inverse Fourier transform in the ω variable and applying Tonelli's Theorem one sees

$$\begin{aligned}
\iint\limits_{\mathbb{R}^{2n}} \mathcal{S}_\psi f(u,\omega)\overline{\mathcal{S}_\psi f(u,\omega)}\,d(\omega,u) &= \int (\mathcal{F}(f\,\overline{\lambda(u)\psi}),\, \mathcal{F}(f\,\overline{\lambda(u)\psi}))_{\mathrm{L}^2(\mathbb{R}^n)}\,du \\
&= \int (f\,\overline{\lambda(u)\psi},\, f\,\overline{\lambda(u)\psi})_2\,du \\
&= \int \left(\int f(x)\overline{\lambda(u)\psi(x)\,f(x)}\,\lambda(u)\psi(x)dx\right)du \\
&= \int |f(x)|^2\left(\int |\psi(x-u)|^2\,du\right)dx \\
&= \|f\|_2^2\,\|\psi\|_2^2\,.
\end{aligned}$$

Hence $\mathcal{S}_\psi f \in L^2(\mathbb{R}^{2n})$ and $f \mapsto |\psi|_2^{-1}\mathcal{S}_\psi(f)$ is an isometry. Now knowing that $\mathcal{S}_\psi(f)$ and $\mathcal{S}_\varphi(g)$ are in $L^2(\mathbb{R}^{2n})$, the same argument except with Fubini's instead of Tonelli's Theorem shows

$$
\begin{aligned}
(\mathcal{S}_\psi(f), \mathcal{S}_\varphi(g))_{\mathrm{L}^2(\mathbb{R}^{2n})} &= \iint\limits_{\mathbb{R}^{2n}} \mathcal{S}_\psi f(u,\omega)\overline{\mathcal{S}_\varphi g(u,\omega)}\, d(\omega,u) \\
&= \int (\mathcal{F}(f\,\overline{\lambda(u)\psi}),\, \mathcal{F}(g\,\overline{\lambda(u)\varphi}))_2\, du \qquad (4.23)\\
&= (f,g)_2\,(\varphi,\psi)_2\,.
\end{aligned}
$$

The rest is now formal. In particular (a) and (b) follows directly form (4.23). For (c) notice that (4.23) implies that

$$
\begin{aligned}
(\varphi,\,\psi)_2^{-1}(\mathcal{S}_\varphi^*\mathcal{S}_\psi(f),\, g)_2 &= (\varphi,\,\psi)_2^{-1}(\mathcal{S}_\psi(f),\, \mathcal{S}_\varphi(g))_2 \\
&= (\varphi,\,\psi)_2^{-1}(f,\, g)_2(\varphi,\,\psi)_2 \\
&= (f,\, g)_2
\end{aligned}
$$

for all $g \in L^2(\mathbb{R}^n)$. Thus $(\varphi,\,\psi)_2^{-1}\mathcal{S}_\varphi^*\mathcal{S}_\psi(f) = f$. □

The last statement shows that a formula for the adjoint of the operator $\mathcal{S}_\phi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ would allow one to recover f from $\mathcal{S}_\psi(f)$.

Theorem 4.82. *Let $\varphi \in L^2(\mathbb{R}^n)$ and $\mathcal{F}_2$ denote the Fourier transform on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ in the second variable. Then*

$$\mathcal{S}_\varphi^*(F)(x) = \int_{\mathbb{R}^n} \mathcal{F}_2 F(u,-x)\varphi(x-u)\, du \text{ for a.e. } x.$$

In particular, if $f, \psi, \varphi \in L^2(\mathbb{R}^n)$ are such that $(\varphi,\,\psi)_2 = 1$, then for almost all x in $\mathbb{R}^n$

$$f(x) = \int_{\mathbb{R}^n} \mathcal{F}_2\mathcal{S}_\psi(f)(u,-x)\varphi(x-u)\, du \qquad (4.24)$$

Proof. For $F \in L^2(\mathbb{R}^{2n})$ denote by F_u the function $v \mapsto F(u,v)$. Note $\mathcal{F}_2$ is a unitary isomorphism of L^2 and $\mathcal{F}_2^*(F)(u,x) = \mathcal{F}_2 F(u,-x)$ a.e. (u,x). From (4.21), one sees if $G(u,x) = f(x)\overline{\psi(x-u)}$, then $S_\psi f(u,\omega) = \mathcal{F}_2 G(u,\omega)$. Now $|G|_{L^2(\mathbb{R}^{2n})} = |f|_2|\psi|_2 < \infty$. Consequently

$$
\begin{aligned}
(S_\psi(f),\, F)_{L^2(\mathbb{R}^{2n})} &= (\mathcal{F}_2 G, F) \\
&= (G, \mathcal{F}_2^* F) \\
&= \int\int G(u,x)\overline{\mathcal{F}_2 F(u,-x)}\, du\, dx \qquad (4.25)\\
&= \int f(x)\overline{\int \psi(x-u)\mathcal{F}_2 F(u,-x)\, du}\, dx\,.
\end{aligned}
$$

This implies for a.e. x, the function $S_\psi^* F$ in $L^2(\mathbb{R}^n)$ is given by $S_\psi^* F(x) = \int \psi(x-u)\mathcal{F}_2 F(u,-x)\,du$.

Assume $\phi \in L^2(\mathbb{R}^n)$ and $(\phi,\psi)_2 \neq 0$. By (c) of the Plancherel Theorem, we have

$$f = \frac{1}{(\phi,\psi)_2} S_\phi^* S_\psi f.$$

Thus for a.e. x, we have

$$f(x) = \frac{1}{(\phi,\psi)_2}\int \phi(x-u)\mathcal{F}_2 S_\psi f(u,-x)\,du.$$

$\square$

Remark 4.83. Note for $\psi \in L^2(\mathbb{R}^n)$, since

$$S_\psi^*(F)(x) = \int \psi(x-u)\mathcal{F}_2 F(u,-x)\,du$$

for a.e. x, Cauchy-Schwarz implies

$$\begin{aligned}
|S_\psi^* F|_2^2 &= \int \left|\int \psi(x-u)\mathcal{F}_2 F(u,-x)\,du\right|^2 dx \\
&\leqslant \int \left(\int |\psi(x-u)|^2\,du \int |\mathcal{F}_2 F(u,-x)|^2\,du\right) dx \\
&= \int |\psi(u)|^2\,du \int\int |\mathcal{F}_2 F(u,-x)|^2\,du\,dx \\
&= |\psi|_2^2 |F|_2^2.
\end{aligned}$$

Thus as a linear operator from $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^n)$ one has

$$||S_\psi^*|| \leqslant |\psi|_2.$$

Note from Equation (4.24) we can write

$$f(x) = \iint\limits_{\mathbb{R}^{2n}} \mathcal{S}_\psi(f)(u,\omega)e^{2\pi i\omega\cdot x}\phi(x-u)\,d\omega du\,, \tag{4.26}$$

but of course the inner integral does not necessarily exist. However, it can can be interpreted as a weak integral in a Hilbert space. To define $\iint S_\psi(f)(u,\omega)e^{2\pi i\omega\cdot x}\phi(x-u)\,d\omega\,du$ weakly, set $H(u,\omega)$ to be the function in $L^2(\mathbb{R}^n)$ given by

$$H(u,\omega)(x) = S_\psi f(u,\omega)e^{2\pi i x\cdot\omega}\phi(x-u).$$

Note $H(u,\omega)$ is in $L^2(\mathbb{R}^n)$ and if $h \in L^2(\mathbb{R}^n)$, then

$$\begin{aligned}
(h, H(u,\omega))_2 &= \int \overline{S_\psi f(u,\omega)}h(x)e^{-2\pi i x\cdot\omega}\overline{\phi(x-u)}\,dx \\
&= S_\phi h(u,\omega)\overline{S_\psi f(u,\omega)}
\end{aligned}$$

which by the Plancherel Formula is integrable on $\mathbb{R}^{2n}$. Moreover, by Theorem 4.80,

$$\int (h, H(u,\omega))_2 \, d(u,\omega) = (\psi, \phi)_2 (h, f)_2.$$

Hence, by the Riesz representation theorem, there is a unique vector

$$\iint H(u,\omega)\, d(u,\omega) = \iint S_\psi f(u,\omega) e^{2\pi i x\cdot\omega} \phi(x-u)\, d(u,\omega)$$

in $L^2(\mathbb{R}^n)$ such that

$$(h, \iint H(u,\omega)\, d(u,\omega))_2 = \iint (h, H(u,\omega))_2 \, d(u,\omega) = (h, (\phi,\psi)_2 f)_2$$

and this vector is said to be the weak L^2 integral of the $L^2(\mathbb{R}^n)$ valued function $(u,\omega) \mapsto H(u,\omega)$.

Summarizing we obtain

$$(\phi,\psi)_2 f = \iint_{\mathbb{R}^{2n}} S_\psi f(u,\omega)[\tau(\omega)\lambda(u)\phi]\, d(u,\omega)$$

as a weak L^2 integral. The formal inversion formula (4.26) can now be stated in an elegant way as:

Theorem 4.84. *Let ψ and φ be window functions such that $(\varphi,\, \psi)_2 \neq 0$. Then*

$$f = \frac{1}{(\varphi,\, \psi)_2} \iint_{\mathbb{R}^{2n}} \mathcal{S}_\psi(f)(u,\omega)\, [\tau(\omega)\lambda(u)\phi]\; d(u,\omega) \tag{4.27}$$

for all $f \in L^2(\mathbb{R}^n)$.

Two remarks are central here. First (4.27) is an identification of elements in $L^2(\mathbb{R}^n)$. As it stands it **does not** say anything about the pointwise identity $f(x) = (\varphi,\, \psi)^{-1} \iint \mathcal{S}_\psi(f)\tau(\omega)\lambda(u)\varphi(x)\, d(u,\omega)$. This might or might not be correct! Secondly we can not write the Fourier transform in a similar way because in this case we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\omega) e_\omega(x)\, d\omega$$

where as usual $e_\omega(x) = e^{2\pi i x\cdot\omega}$. But the "basis functions" e_ω are **not** in $L^2(\mathbb{R}^n)$!

Exercise Set 4.10

1. Let f and g be in $L^2(\mathbb{R}^n)$. Show $\int |f(x)\lambda(u)g(x)|\, dx \to 0$ as $u \to \infty$.

2. Show if $f, \psi \in L^2(\mathbb{R}^n)$, then the windowed Fourier transform $S_\psi f$ is a bounded uniformly continuous function on $\mathbb{R}^n \times \mathbb{R}^n$.

3. Let $\psi \in L^2(\mathbb{R}^n)$. Show the range $S_\psi(L^2(\mathbb{R}^n))$ is a closed subspace of $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

4. Let f and ψ be L^2 functions on $\mathbb{R}^n$. Show

$$\mathcal{S}_\psi(f)(u,\omega) = e^{-\pi i u\cdot\omega} \int_{\mathbb{R}^n} f(x+\frac{u}{2})\,\overline{\psi(x-\frac{u}{2})}\,e^{-2\pi i x\cdot\omega}\,dx\,.$$

5. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be nonzero. Show the mapping S_ψ is a continuous linear linear homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto its range which is a closed subspace of $\mathcal{S}(\mathbb{R}^{2n})$. **Hint:** Use Propositions 2.62 and 4.73.

6. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Show S_ϕ^* is a continuous linear mapping from $\mathcal{S}(\mathbb{R}^{2n})$ into $\mathcal{S}(\mathbb{R}^n)$. Conclude using the previous exercise that if ϕ and ψ are in $\mathcal{S}(\mathbb{R}^n)$ with $(\phi,\psi)_2 = 1$, then S_ψ is a homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto the closed subspace $\mathcal{F} = S_\psi(\mathcal{S}(\mathbb{R}^n))$ of $\mathcal{S}(\mathbb{R}^{2n})$ which has inverse $S_\phi^*|_{\mathcal{F}}$. Moreover,

$$f(x) = \iint S_\psi f(u,\omega) e^{2\pi i\omega\cdot x}\phi(x-u)\,d(u,\omega) \text{ for all } x.$$

7. Let $\psi \in L^2(\mathbb{R}^n)$. Show $\mathrm{Im}(\mathcal{S}_\psi)$ is a reproducing Hilbert space with reproducing kernel

$$K((u_1,\omega_1),(u,\omega)) = \int \psi(x-u)\overline{\psi(x-u_1)}e^{2\pi i x\cdot(\omega-\omega_1)}dx\,.$$

8. Assume $\mathcal{H}$ is a separable Hilbert space, Ω is open in $\mathbb{R}^d$, and $F:\Omega\to\mathcal{H}$ is weakly integrable; i.e. $x\mapsto(v,F(x))$ is integrable for each $v\in\mathcal{H}$. Show if $\{e_j\}$ is an orthonormal basis for $\mathcal{H}$ then $F(x) = \sum_j (F(x),\, e_j)e_j$ for a.e. x and the weak Hilbert space integral satisfies

$$\int F(x)dx = \sum_j \left(\int_\Omega (F(x),\, e_j)\,dx\right) e_j\,.$$

12. The Continuous Wavelet Transform

In this section we briefly discuss a simple form of the *continuous wavelet transform*. In Section 18 of Chapter 6 we will give a more sophisticated discussion and explanation based on *the representation theory of topological groups*. That discussion will clarify the importance of abstract harmonic analysis in general for transforms of many types; one which is not widely known in applied science. This approach is inclusive to both the windowed Fourier transform and the continuous wavelet transform.

The windowed Fourier transform uses a window function ψ and two parameters u and ω, where u indicates where we are localizing our functions and ω is the dual frequency parameter coming form the Fourier transform. But after introducing both a new parameter and a “localizing” window function ψ one can just as well take the final step and completely forget about the exponential function in the decomposition. Instead we both dilate and translate the function ψ to get information about f. First we introduce some notation that we will use for the n-dimensional case. For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, let $M(a)$ be the multiplication operator

$$M(a)(x) = (a_1x_1, \ldots, a_nx_n).$$

Then

$$\det M(a_1, \ldots, a_n) = a_1a_2 \cdots a_n\,.$$

We will therefore simply write

$$\det(a) = a_1a_2 \cdots a_n$$

for $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. We obviously have $\det(a) \neq 0$ if and only if $a \in (\mathbb{R}^*)^n$. Define for $a \in (\mathbb{R}^*)^n$ and $b \in \mathbb{R}^n$ the function $\psi_{a,b}$ by $\psi_{a,b}(x) = |\det a|^{-1/2}\, \psi(M(a)^{-1}(x-b))$ and then define $W_\psi f : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$\begin{aligned} W_\psi f(a,b) =& (f,\, \psi_{a,b})_2 \\ &= |\det a|^{-1/2} \int f(x)\overline{\psi\left(M(a)^{-1}(x-b)\right)}\, dx \\ &= |\det a|^{1/2} \int \hat{f}(\omega)\overline{\hat{\psi}(M(a)\omega)} e^{2\pi i b\cdot\omega}\, d\omega \end{aligned} \tag{4.28}$$

where we have used that the function $\psi_{a,b}(x) = |\det a|^{-1/2}\psi(M(a)^{-1}(x-b))$ has Fourier transform

$$\widehat{\psi_{a,b}}(\omega) = |\det a|^{1/2} e^{-2\pi i b\cdot\omega}\hat{\psi}(M(a)\omega)\,. \tag{4.29}$$

If the function ψ is fixed, then we simply write Wf for $W_\psi f$.

Definition 4.85. *A nonzero function $\psi \in L^2(\mathbb{R}^n)$ is called a **wavelet function** if and only if*

$$\hat{\psi} \in L^2((\mathbb{R}^*)^n, \frac{d\omega}{|\det \omega|}). \tag{4.30}$$

The transform

$$L^2(\mathbb{R}^n) \ni f \mapsto W_\psi f \in \mathcal{C}((\mathbb{R}^*)^n \times \mathbb{R}^n)$$

*is called the **wavelet transform**.*

Example 4.86. *Let $\psi = \chi_{(-1/2,0)} - \chi_{[0,1/2)}$. Then*

$$\begin{aligned} \hat{\psi}(\omega) &= \int_{-1/2}^{0} e^{-2\pi i\omega x} dx - \int_0^{1/2} e^{-2\pi i \omega x}\, dx \\ &= \frac{i}{\pi\omega}\left[\cos(\pi\omega) - 1\right]\,. \end{aligned}$$

As $\left|\hat{\psi}\right|$ *is even it follows that* $\int_0^\infty \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|}\,d\omega = \int_{-\infty}^0 \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|}\,d\omega$*. Note*

$$\int_{|\omega|\geqslant 1} \left|\frac{\cos(\pi\omega)-1}{\pi\omega}\right|^2 / |\omega| \leqslant \frac{2}{\pi^2}\int_{|\omega|\geqslant 1} \frac{1}{|\omega|^3}\,d\omega < \infty\,.$$

On the other hand close to zero we have

$$\frac{\cos(\pi\omega)-1}{\pi\omega} = \omega F(\omega)$$

where F *is the analytic function*

$$F(\omega) = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{\pi^{2k+1}\omega^{2k+1}}{(2(k+1))!}\,.$$

Hence

$$\left|\frac{\cos(\pi\omega)-1}{\pi\omega}\right|^2 \cdot \frac{1}{|\omega|} = |\omega|\,|F(\omega)|^2$$

is clearly integrable on $[-1,1]$*. Together these imply*

$$\frac{1}{\pi^2}\int_{\mathbb{R}} \frac{|\cos(\pi\omega)-1|^2}{|\omega|^3}\,d\omega < \infty\,.$$

It follows that (4.30) holds. Notice that for positive a *we have*

$$\chi_{(\alpha,\beta)}\left(\frac{x-b}{a}\right) = \chi_{(b+a\alpha,b+a\beta)}(x)\,.$$

Thus for $a > 0$

$$\begin{aligned} Wf(a,b) &= a^{-1/2}\int_{b-a/2}^{b} f(x)\,dx - a^{-1/2}\int_b^{b+a/2} f(x)\,dx \\ &= \frac{\sqrt{a}}{2}\left(\frac{2}{a}\int_{b-a/2}^{b} f(x)\,dx - \frac{2}{a}\int_b^{b+a/2} f(x)\,dx\right)\,. \end{aligned}$$

Thus the wavelet transform associated to this choice of wavelet function gives us with a factor of $\frac{\sqrt{a}}{2}$ *the* ***change*** *of average value by going from the interval* $(b-a/2,b)$ *to the next interval* $(b,b+a/2)$*.*

We will see in a moment why this condition is reasonable. But let us first give the following simple replacement for the wavelet condition in the one dimensional case. Suppose that $\psi \in \mathcal{S}(\mathbb{R})$. Then $\hat{\psi}$ is continuous. Assume that $\hat{\psi}(0) = \int \psi(x)\,dx \neq 0$. Then there exists $\epsilon, \delta > 0$ such that $\left|\hat{\psi}(\omega)\right| > \delta$ for $|\omega| < \epsilon$. But then

$$\int_0^\infty \frac{\left|\hat{\psi}(\omega)\right|^2}{\omega}d\omega \geqslant \int_0^\epsilon \frac{\delta}{\omega}\,d\omega = \infty\,.$$

On the other hand, if $\hat{\psi}(0) = 0$ then $\Phi(x) = \int_{-\infty}^{x} \psi(u)\, du$ is rapidly decreasing (see Exercise 2.3.9). As $\Phi' = \psi$ it follows that

$$\hat{\psi}(\omega) = 2\pi i\omega \hat{\Phi}(\omega)$$

and hence

$$\frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} = 2\pi\, |\omega| \left|\hat{\Phi}(\omega)\right| \in L^2(\mathbb{R})\,.$$

We have thus proved the following Lemma.

Lemma 4.87. *Suppose that $\psi \in \mathcal{S}(\mathbb{R})$ is real valued. Then the following are equivalent:*

(a) *ψ is a wavelet function;*

(b) *$\hat{\psi}(0) = \int \psi(u)\, du = 0$;*

(c) *$\psi \in \frac{d}{dx}\mathcal{S}(\mathbb{R})$.*

Before we go on to prove the Plancherel formula and inversion formula, we will need to give simple reformulations of the Wavelet transform. Recall first that

$$\widehat{\psi_{a,b}}(\omega) = e^{-2\pi i b\cdot\omega} |\det a|^{1/2} \hat{\psi}(M(a)\omega) = e^{-2\pi i b\cdot\omega}\Psi_a(\omega)$$

where we are taking

$$\Psi_a(\omega) = \det(a)^{1/2}\hat{\psi}(M(a)\omega). \tag{4.31}$$

The following Lemma follows then form the fact that the Fourier transform is an unitary isomorphism.

Lemma 4.88. *Let ψ be a wavelet function. Then*

$$\begin{aligned} W_\psi f(a,b) &= (\hat{f}, \widehat{\psi_{a,b}}) \\ &= |\det a|^{1/2} \int \hat{f}(\omega)\, \overline{\hat{\psi}(M(a)\omega)}\, e^{2\pi i b\cdot\omega}\, d\omega \\ &= \mathcal{F}^{-1}(\hat{f}\overline{\Psi}_a)(b)\,. \end{aligned} \tag{4.32}$$

We will now consider the wavelet condition (4.30). For that we notice the following fact:

Lemma 4.89. *Let $\psi \in L^2(\mathbb{R})$. Let f be a nonzero L^2 function on $\mathbb{R}^n$. Then $W_\psi f \in L^2((\mathbb{R}^*)^n \times \mathbb{R}^n, \frac{da\, db}{|\det a|^2})$ if and only if $\hat{\psi} \in L^2((\mathbb{R}^*)^n\,, \frac{d\omega}{|\det \omega|})$, i.e., if and only if ψ is a wavelet.*

Proof. First note by (4.32) one has

$$\begin{aligned}\int \left|W_\psi f(a,b)\right|^2\, db &= (\mathcal{F}^{-1}(\hat{f}\widetilde{\overline{\Psi}}_a),\, \mathcal{F}^{-1}(\hat{f}\widetilde{\overline{\Psi}}_a)) \\ &= (\hat{f}\cdot\overline{\Psi}_a,\, \hat{f}\cdot\overline{\Psi}_a) \\ &= |\det a| \int |\hat{f}(\omega)|^2 \left|\hat{\psi}(M(a)\omega)\right|^2\, d\omega.\end{aligned}$$

Fubini's Theorem and the fact that $\mathbb{R}^n\backslash(\mathbb{R}^*)^n$ has measure zero now gives

$$\begin{aligned}\iint |W_\psi f(a,b)|^2\, \frac{dadb}{|\det a|^2} &= \iint \left|\hat{f}(b)\right|^2 \frac{\left|\hat{\psi}(M(a)b)\right|^2}{|\det a|}\, dadb \\ &= \iint\limits_{(\mathbb{R}^*)^{2n}} \left|\hat{f}(b)\right|^2 \frac{\left|\hat{\psi}(M(a)b)\right|^2}{|\det a|}\, dadb \\ &= \int_{(\mathbb{R}^*)^n} |\hat{f}(b)|^2 \int_{(\mathbb{R}^*)^n} |\hat{\psi}(a_1b_1,\ldots,a_nb_n)|^2 \frac{da}{|\det a|}\, db \\ &= \int_{(\mathbb{R}^*)^n} |\hat{f}(b)|^2 \int_{(\mathbb{R}^*)^n} |\hat{\psi}(u_1,\ldots,u_n)|^2 \frac{du}{|\det u|}\, db \\ &= |\hat{\psi}|^2_{L^2((\mathbb{R}^*)^n,\frac{da}{|\det a|})} \int_{(\mathbb{R}^*)^n} |\hat{f}(b)|^2\, db \\ &= |\hat{\psi}|^2_{L^2((\mathbb{R}^*)^n,\frac{da}{|\det a|})} |f|^2_{L^2(\mathbb{R}^n)}\end{aligned}$$

where we have used the change of coordinates $u_j = a_jb_j$ and

$$\frac{du_j}{|u_j|} = \frac{da_j}{|a_j|}\,.$$

Hence $W_\psi f \in L^2((\mathbb{R}^*)^n \times \mathbb{R}^n, \frac{dadb}{|\det a|^2})$ if and only if

$$C_\psi := |\hat{\psi}|^2_{L^2((\mathbb{R}^*)^n,\frac{da}{|\det a|})} < \infty$$

□

For the remainder of this section we consider $X = (\mathbb{R}^*)^n \times \mathbb{R}^n$ with the measure $d\mu(a,x) = |\det a|^{-2} dadx$.

Theorem 4.90 (Plancherel Theorem)**.** *Let ψ and φ be wavelet functions. Then*

$$(W_\psi f,\, W_\varphi g)_{L^2(X,d\mu)} = (f,\, g)_{L^2(\mathbb{R}^n)} (\widehat{\varphi},\, \widehat{\psi})_{L^2((\mathbb{R}^*)^n,\frac{da}{|\det a|})}$$

for all $f, g \in L^2(\mathbb{R}^n)$. In particular the following holds:

(a) *$W_\psi : L^2(\mathbb{R}^n) \to L^2(X, d\mu)$ is continuous.*

(b) *If* $C_{\psi,\varphi} := (\widehat{\varphi},\, \widehat{\psi})_{L^2((\mathbb{R}^*)^n, \frac{da}{|\det a|})} \neq 0$, *then*

$$f = \frac{1}{C_{\psi,\varphi}} W_\varphi^* W_\psi f$$

for all $f \in L^2(\mathbb{R}^n)$.

(c) *If* $\int_{(\mathbb{R}^*)^n} \left|\hat{\psi}(\omega)\right|^2 \frac{d\omega}{|\det \omega|} = 1$ *then* $W_\psi : L^2(\mathbb{R}^n) \to \operatorname{Im}(W_\psi)$ *is an unitary isomorphism.*

Proof. By Lemma 4.89 we know that $W_\varphi g$ and $W_\psi f$ are in $L^2(X, d\mu)$. Hence

$$(a,b) \mapsto W_\psi f(a,b)\overline{W_\varphi g(a,b)}$$

is integrable with respect to μ. We can therefore use Fubini's Theorem without hesitation and repeat the arguments in the proof of Lemma 4.89. Thus with Φ_a defined in the same way as Ψ_a in (4.31)

$$\begin{aligned}
(W_\psi f,\, W_\varphi g)_{L^2(X,d\mu)} &= \int (\mathcal{F}^{-1}(\hat{f}\overline{\Psi}_a),\, \mathcal{F}^{-1}(\hat{g}\overline{\Phi}_a))\, \frac{da}{|\det a|^2} \\
&= \int (\hat{f}\overline{\Psi}_a,\, \hat{g}\overline{\Phi}_a)\, \frac{da}{|\det a|^2} \\
&= \iint \hat{f}(\omega)\overline{\hat{g}(\omega)}\, |\det a|\, \overline{\Psi(M(a)\omega)}\Phi(M(a)\omega)\, d\omega \frac{da}{|\det a|^2} \\
&= \int \hat{f}(\omega)\overline{\hat{g}(\omega)} \left(\int \hat{\varphi}(M(a)\omega)\, \overline{\hat{\psi}(M(a)\omega)}\, \frac{da}{|\det a|} \right) d\omega \\
&= \int_{(\mathbb{R}^*)^n} \hat{f}(\omega)\overline{\hat{g}(\omega)} \left(\int_{(\mathbb{R}^*)^n} \hat{\varphi}(a)\, \overline{\hat{\psi}(a)}\, \frac{da}{|\det a|} \right) d\omega \\
&= (f,\, g)_{L^2(\mathbb{R}^n)} (\hat{\varphi},\, \hat{\psi})_{L^2((\mathbb{R}^*)^n, \frac{da}{|\det a|})}\,.
\end{aligned}$$

The proof of the remaining statements is exactly the same as in the proof of Theorem 4.80 and is therefore left to the reader. □

The Plancherel formula states that the inversion formula is given by W_ψ^*, but instead of finding this operator we state the inversion formula using the weak integral.

Theorem 4.91. *Let* ψ *and* φ *be wavelet functions. Assume*

$$C_{\psi,\varphi} = (\widehat{\varphi},\, \widehat{\psi})_{L^2((\mathbb{R}^*)^n, \frac{da}{|\det a|})} \neq 0.$$

Then

$$f = \frac{1}{C_{\psi,\varphi}} \iint_{\mathbb{R}^n \times (\mathbb{R}^*)^n} W_\psi f(a,b) \varphi_{a,b}\, \frac{da db}{|\det a|^2}\,.$$

Proof. Let $F(a,b) = \frac{1}{C_{\psi,\varphi}} W_\psi f(a,b)\varphi_{a,b}$. Then $F : X \to L^2(\mathbb{R}^n)$. Let $g \in L^2(\mathbb{R}^n)$. Then

$$\begin{aligned}\int_X (F(a,b),g)_{L^2(\mathbb{R}^n)}\,d\mu(a,b) &= \frac{1}{C_{\psi,\varphi}}\int_X W_\psi f(a,b)\left(\overline{\int_{\mathbb{R}^n} g(x)\overline{\varphi_{a,b}(x)}\,dx}\right)d\mu(a,b)\\ &= \frac{1}{C_{\psi,\varphi}}\int_X W_\psi f(a,b)\overline{W_\varphi g(a,b)}\,d\mu(a,b)\\ &= \frac{1}{C_{\psi,\varphi}}(W_\psi f,\, W_\varphi g)\\ &= (f,\, g).\end{aligned}$$

Thus $g \mapsto \int (F(a,b),\, g)\,d\mu$ is a continuous antilinear form and hence the weak integral $\frac{1}{C_{\psi,\varphi}} \iint_{\mathbb{R}^n \times (\mathbb{R}^*)^n} W_\psi f(a,b)\varphi_{a,b}\,\frac{dadb}{|\det a|^2}$ exists. As

$$\int_X (F(a,b),g)_{L^2(\mathbb{R}^n)}\,d\mu(a,b) = (f,\, g)_2$$

for all $g \in L^2(\mathbb{R}^n)$, it follows that $\int F(a,b)\,d\mu(a,b) = f$ weakly. □

Exercise Set 4.11

1. In many applications one would prefer to use positive numbers for the dilation. In this exercise we show what changes are necessary. For $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1,1\}^n$ let

$$\mathbb{R}^n_\epsilon = \{x \in (\mathbb{R}^*)^n \mid \forall j \in \{1,\ldots,n\} \,:\, \mathrm{sign}(x_j) = \epsilon_j\}\,.$$

Define a wavelet function to be a function $\psi \in L^2(\mathbb{R}^n)$ such that there exists a constant $0 < C_\psi < \infty$ such that for all $\epsilon \in \{-1,1\}^n$ we have

$$\int_{\mathbb{R}^n_\epsilon} \frac{|\hat{\psi}(\omega)|^2}{|\det \omega|}\,d\omega = C_\psi\,.$$

Show the Plancherel and Inversion formula still hold.

2. Let ψ be a wavelet function such that $C_{\psi,\psi} = 1$. Prove the following:

(a) $H_\psi := \mathrm{Im}(W_\psi)$ is a closed subspace of $L^2(X, d\mu)$ and hence is a Hilbert space.

(b) $H_\psi \subset \mathcal{C}(X)$ and the point evaluation maps $\mathrm{ev}_x : H_\psi \to \mathbb{C}$, $F \mapsto F(x)$ are continuous for each $x \in X$.

(c) The reproducing kernel for H_ψ is given by

$$K(x,y) = W_\psi \psi_y(x)\,.$$

13. Shannon Sampling Theorem

The transforms that we have considered up to now, except the Fourier transform on the torus, express a function as a continuous superposition of functions with coefficients given by the corresponding transform. In many applications it is necessary, or at least preferable, to deal with countable sums instead of integrals. Thus we would need to convert a function defined on the line or an interval into a sequence of numbers. We can do that for periodic functions on the line by using the Fourier series. For functions defined on the line the Fourier transform is again a function on the line, so that does not work directly. But in most applications we only need a finite band width of frequency information, so one can combine the Fourier transform on the line and the Fourier transform of a periodic function to recover a function by a discrete sample of values of $\hat{f}$.

Definition 4.92. *We say that an L^2 function $f : \mathbb{R}^n \to \mathbb{C}$ is band-limited if $\hat{f}$ has compact essential support.*

If f is band-limited, say $\operatorname{supp}(f) \subset B_r(0)$ for some $r > 0$, then $\hat{f} = \hat{f}\,\chi_{B_r(0)} \in L^1$ and hence for almost every x

$$f(x) = \int_{B_r(0)} \hat{f}(\omega) e^{2\pi i x\cdot\omega}\, d\omega.$$

By the same methods as in the section on the Paley-Wiener theorem, the right hand side is a smooth function that extends to a holomorphic function. Now holomorphic functions are determined by their values at **any** sequence having at least one accumulation point. Here we give a discussion of this which does not use the Paley Wiener theorem. For $T = (T_1, \ldots, T_n) \in \mathbb{R}^n$, let us write $T > 0$ if $T_j > 0$ for $j = 1, \ldots, n$. For $T > 0$ we introduce the notation $Q_T = \{x \in \mathbb{R}^n \mid -T_j \leqslant x_j \leqslant T_j\}$. We say that a function (or better class) $f \in L^2(\mathbb{R}^n)$ is continuous, smooth, etc if there exists a function g in the same class that is continuous, smooth, etc. Moreover, we shall use χ_T to denote the characteristic function of Q_T and Γ to denote the lattice of points

$$\Gamma = \frac{\mathbb{Z}^n}{2T} = \{(\frac{k_1}{2T_1}, \frac{k_2}{2T_2}, \ldots, \frac{k_n}{2T_n}) \mid k_i \in \mathbb{Z}\}$$

Lemma 4.93. *Let $f \in L^2(\mathbb{R}^n)$ be such that ess-supp$(\widehat{f}) \subset Q_T$ for some $T > 0$. Then $f \in C^\infty(\mathbb{R}^n)$. In particular,*

$$D^\alpha f(x) = (2\pi i)^{|\alpha|} \int_{Q_T} \omega^\alpha \hat{f}(\omega) e^{2\pi i\omega\cdot x}\, dx$$

for each $\alpha \in \mathbb{N}_0^n$.

Proof. By our assumption on $\hat{f}$ it follows that $\omega \mapsto \omega^\alpha \hat{f}(\omega)$ is integrable for all $\alpha \in \mathbb{N}_0^n$. In particular we have

$$f(x) = \int \hat{f}(\omega) e^{2\pi i \omega \cdot x}\, d\omega = \int_{Q_T} \hat{f}(\omega) e^{2\pi i x \cdot \omega}\, dy$$

is continuous and

$$\frac{f(x + he_j) - f(x)}{h} = \int_{Q_T} \hat{f}(\omega) e^{2\pi i x \cdot \omega} \frac{e^{2\pi i h \omega_j} - 1}{h}\, d\omega\,.$$

But

$$\left| \frac{e^{2\pi i h \omega_j} - 1}{h} \right| \leqslant 2\pi\, |\omega_j|$$

and hence

$$\left| \hat{f}(\omega) e^{2\pi i x \cdot \omega} \frac{e^{2\pi i h \omega_j} - 1}{h} \right| \leqslant 2\pi \left| \omega_j \hat{f}(\omega) \right| \leqslant 2\pi T_j |\hat{f}(\omega)|$$

which is integrable. It follows that f is once continuously differentiable with

$$\frac{\partial}{\partial x_j} f(x) = 2\pi i \int_{Q_T} \omega_j \hat{f}(\omega) e^{2\pi i \omega \cdot x}\, d\omega\,.$$

By induction we see that $D^\alpha f$ exists for all $\alpha \in \mathbb{N}_0^n$ and

$$D^\alpha f(x) = (2\pi i)^{|\alpha|} \int_{Q_T} \omega^\alpha \hat{f}(\omega) e^{2\pi i \omega \cdot x}\, dx$$

is continuous. It follows that f is smooth. □

This lemma allows us to define $L^2_T(\mathbb{R}^n)$ by

$$L^2_T(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) \mid \int |f(x)|^2\, dx < \infty,\ \mathrm{supp}\hat{f} \subseteq Q_T \} \tag{4.33}$$

Proposition 4.94. *For $\gamma = (\frac{k_1}{2T_1}, \frac{k_2}{2T_2}, \ldots, \frac{k_n}{2T_n}) \in \Gamma = \frac{\mathbb{Z}^n}{2T}$, set $e_\gamma(x) = e^{2\pi i \gamma \cdot x}$ for $x \in Q_T$. The functions e_γ for $\gamma \in \frac{\mathbb{Z}^n}{2T}$ are an orthonormal basis of $L^2(Q_T, \frac{dx}{\mathrm{Vol}(Q_T)})$.*

Proof. By Theorem 1.20, the functions $e_k(z) = z^k$ form an orthonormal basis of $L^2(\mathbb{T})$. Now the mapping $W : L^2(\mathbb{T}) \to L^2[-\frac{1}{2}, \frac{1}{2}]$ defined by $Wf(\theta) = f(e^{2\pi i \theta})$ is a unitary isomorphism. Hence the functions We_k for $k \in \mathbb{Z}$ form an orthonormal basis of $L^2[-\frac{1}{2}, \frac{1}{2}]$. For each $j \in \{1, 2, \ldots, n\}$, the operator $V_j : L^2[-\frac{1}{2}, \frac{1}{2}] \to L^2([-T_j, T_j], \frac{dx}{2T_j})$ defined by

$$V_j f(x) = f(\frac{x}{2T_j})$$

is a unitary isomorphism of $L^2[-\frac{1}{2}, \frac{1}{2}]$ onto $L^2([-T_j, T_j], \frac{dx}{2T_j})$. Thus the functions $V_j(We_k)$ for $k \in \mathbb{Z}$ form an orthonormal basis of $L^2[-T_j, T_j]$.

Repeatedly using Exercise 2.2.18 we see the functions $e_\gamma = V_1(We_{k_1}) \times V_2(We_{k_2}) \times \cdots \times V_n(We_{k_n})$ where $\gamma = (\frac{k_1}{2T_1}, \frac{k_2}{2T_2}, \ldots, \frac{k_n}{2T_n}) \in \frac{\mathbb{Z}^n}{2T}$ form an orthonormal basis of $L^2(Q_T, \frac{dx}{\mathrm{Vol}(Q_T)})$. $\square$

In Exercise 3.1.2 the function sinc was defined. Namely one has

$$\mathrm{sinc}\,(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{\sin x}{x} & \text{if } x \neq 0. \end{cases}$$

Note sinc(x) is a analytic function on $\mathbb{R}$.

For $T > 0$ we define the sinc-function sinc_T by

$$\mathrm{sinc}_T(x) := \prod_{j=1}^{n} \frac{\sin(2\pi T_j x_j)}{2\pi T_j x_j}.$$

Lemma 4.95. sinc_T *is the Fourier transform of* $\frac{1}{\mathrm{Vol}(Q_T)}\chi_{Q_T}$.

Proof. By Example 3.1, the Fourier transform of χ_T is given by

$$\begin{aligned} \widehat{\chi_T}(\omega) = \pi^{-n} \prod \frac{\sin(2\pi T_j\omega_j)}{\omega_j} &= \mathrm{Vol}(Q_T) \prod_{j=1}^{n} \frac{\sin(2\pi T_j\omega_j)}{2\pi T_j} \\ &= \mathrm{Vol}(Q_T)\mathrm{sinc}_T(\omega). \end{aligned}$$

$\square$

As can be seen using Exercise 4.1.21, we see since $L^2_T(\mathbb{R}^n)$ is a Hilbert space of continuous functions for which the point evaluation maps are continuous, there exists a reproducing kernel $K(x, y)$ having the property $f(y) = (f, K_y)$ where $K_y(x) = K(x, y)$. Indeed, since ev_y is bounded, ev_y^* is a bounded linear transformation from Hilbert space $\mathbb{C}$ into Hilbert space $L^2_T(\mathbb{R}^n)$. Consequently,

$$K(x, y) = \mathrm{ev}_x(\mathrm{ev}_y^*(1)) = \mathrm{ev}_y^*(1)(x)$$

is well defined and

$$f(y) = \mathrm{ev}_y(f) = (\mathrm{ev}_y(f), 1)_{\mathbb{C}} = (f, \mathrm{ev}_y^*(1))_2 = (f, K_y)_2.$$

To find K we start with a continuous f in $L^2_T(\mathbb{R}^n)$ and see:

$$\begin{aligned}
(f, \mathrm{ev}_y^*(1)) = f(y) &= \int_{Q_T} \hat{f}(x) e^{2\pi i x \cdot y}\, dy \\
&= \int \chi_T(x) \hat{f}(x) e^{2\pi i x \cdot y}\, dy \\
&= \int \mathcal{F}(e_y \chi_T)(x) f(x)\, dx \\
&= \int \mathcal{F}(\chi_T)(x-y) f(x)\, dx \\
&= \mathrm{Vol}(Q_T) \int \mathrm{sinc}_T(x-y) f(x)\, dx \\
&= \mathrm{Vol}(Q_T) \int f(x) \overline{\mathrm{sinc}_T(x-y)}\, dx \\
&= (f, \mathrm{Vol}(Q_T)\tau(y)\mathrm{sinc}_T)_2.
\end{aligned}$$

Hence

$$K(x,y) = \mathrm{Vol}(Q_T)\tau(y)\mathrm{sinc}_T(x) = \mathrm{Vol}(Q_T)\mathrm{sinc}_T(x-y) \tag{4.34}$$

After this preparation we can now state and prove the Shannon sampling Theorem:

Theorem 4.96 (Shannon Sampling Theorem)**.** *Let $f \in L^2_T(\mathbb{R}^n)$. Then f is C^∞, each $D^\alpha f$ is in $L^2_T(\mathbb{R}^n)$, and the series*

$$\sum_{\gamma \in \Gamma} f(\gamma) D^\alpha \left(\mathrm{sinc}_T(x-\gamma)\right)$$

converges in L^2 and pointwise uniformly to $D^\alpha f$. Moreover,

$$\sum_{\gamma \in \Gamma} |f(\gamma)|^2 = \mathrm{Vol}(Q_T)|f|_2^2. \tag{Parseval}$$

Proof. By Lemma 4.93, we know f is C^∞, $D^\alpha f$ is $L^2_T(\mathbb{R}^n)$, and $\mathcal{F}(D^\alpha f)(\omega) = (2\pi i \omega)^\alpha \hat{f}(\omega)$. Using Proposition 4.94, it follows that $\mathcal{F}(f)$ has an L^2 expansion on Q_T in terms of the orthonormal basis $\frac{1}{\sqrt{\mathrm{Vol}(Q_T)}} e_\gamma$ for $\gamma \in \Gamma$. Specifically,

$$\mathcal{F}(f)|_{Q_T} = \frac{1}{\mathrm{Vol}(Q_T)} \sum_\gamma (\mathcal{F}(f), e_\gamma)_2 e_\gamma$$

where this convergence is in L^2 and

$$\frac{1}{\mathrm{Vol}(Q_T)} \sum_\gamma |(\mathcal{F}(f), e_\gamma)_2|^2 = |\mathcal{F}(f)|_{Q_T}|_2^2 = |\mathcal{F}(f)|_2^2 = |f|_2^2.$$

Using $(\mathcal{F}f, e_\gamma)_2 = \int_{Q_T} \hat{f}(\omega)\overline{e_\gamma(\omega)}\, d\omega = \int \hat{f}(\omega)e^{-2\pi i\gamma\cdot\omega}\, d\omega = f(-\gamma)$, we have

$$\mathcal{F}(f)|_{Q_T} = \frac{1}{\mathrm{Vol}(Q_T)} \sum_\gamma f(-\gamma)e_\gamma$$

in $L^2(Q_T)$ and Parseval's equality $\sum_{\gamma\in\Gamma} |f(-\gamma)|^2 = \mathrm{Vol}(Q_T)|f|_2^2$.

Define g by $g(\omega) = (2\pi i\omega)^\alpha$. Using the boundedness of g on Q_T, we see the convergence in $L^2(Q_T)$ of $\frac{1}{\mathrm{Vol}(Q_T)} \sum_\gamma f(-\gamma)e_\gamma$ to $\mathcal{F}(f)|_{Q_T}$ implies the convergence of $\frac{1}{\mathrm{Vol}(Q_T)} \sum_\gamma f(-\gamma)ge_\gamma\chi_T$ to $g\mathcal{F}(f)$ in $L^2(\mathbb{R}^n)$. From this we can conclude since $\mathcal{F} : L^2_T(\mathbb{R}^n) \to \mathcal{F}(L^2(Q_T))$ is unitary with inverse $\mathcal{F}^{-1}$, that

$$\frac{1}{\mathrm{Vol}(Q_T)} \sum_\gamma f(-\gamma)\mathcal{F}^{-1}(ge_\gamma\chi_T) \to \mathcal{F}^{-1}(g\mathcal{F}f) = D^\alpha f$$

in $L^2_T(\mathbb{R}^n)$. Since $\mathcal{F}^{-1}(\frac{1}{\mathrm{Vol}(Q_T)}\chi_T) = \mathrm{sinc}_T$, we see $\sum_\gamma f(-\gamma)D^\alpha \mathrm{sinc}_T(x+\gamma)$ converges in L^2 to $D^\alpha f$.

Furthermore, the evaluation maps ev_x on $L^2_T(\mathbb{R}^n)$ for $x \in \mathbb{R}^n$ have a uniform bound M. Since

$$\sum_\gamma f(-\gamma)(ge_\gamma) \to g\mathcal{F}(f)|_{Q_T}$$

in $L^2(Q_T)$ we have

$$\sum_{\gamma\in F} f(-\gamma)\mathcal{F}^{-1}(ge_\gamma\chi_T) \to \mathcal{F}^{-1}(g\mathcal{F}f) = D^\alpha f \text{ in } L^2_T(\mathbb{R}^n)$$

where F is a finite subset of Γ and the limit is as F increases.

Consequently

$$|\mathrm{ev}_x(D^\alpha f) - \mathrm{ev}_x(\sum_{\gamma\in F} f(-\gamma)D^\alpha\mathcal{F}(e_\gamma\chi_T)| \leqslant M|D^\alpha f - \sum_{\gamma\in F} D^\alpha\mathcal{F}(e_\gamma\chi_T)|_2 \to 0$$

independently of x as F increases. □

Remark 4.97. For $x \in \mathbb{R}^n$, the series $\sum_{\gamma\in\Gamma} f(-\gamma)D^\alpha\mathrm{sinc}_T(x+\gamma)$ is absolutely summable for the sum is the same after any rearrangement.

One can actually establish more. Namely we shall show the series $\sum_{\gamma\in\Gamma} f(-\gamma)D^\alpha\mathrm{sinc}_T(x+\gamma)$ converges absolutely uniformly on all compact subsets of $\mathbb{R}^n$.

Lemma 4.98. *All derivatives of* $\mathrm{sinc}(x)$ *are bounded. Moreover, for each* k*, there is an* $M > 0$ *such that* $\left|\frac{d^k}{dx^k}\mathrm{sinc}(x)\right| \leqslant \frac{M}{|x|}$ *for* $|x| > 1$*.*

Proof. Let $f(x) = \frac{\sin x}{x}$. Clearly f is bounded. Now $f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x} - \frac{1}{x}\frac{\sin x}{x}$. Thus $f'(x)$ is bounded. We now inductively show $f^{(k)}(x) =$

$P(\frac{1}{x})\sin x + Q(\frac{1}{x})\cos x$ for some polynomials P and Q both having 0 constant term. This is true for $k = 0$ and $k = 1$. Now note

$$f^{(k+1)}(x) = P'(\frac{1}{x})\left(\frac{-1}{x^2}\right)\sin x + P(\frac{1}{x})\cos x + Q'(\frac{1}{x})\left(\frac{-1}{x^2}\right)\cos x - Q(\frac{1}{x})\sin x.$$

Since $P(\frac{1}{x})$ and $Q(\frac{1}{x})$ are bounded for $|x|$ large and $f^{(k)}$ is continuous on $\mathbb{R}$, one has $f^{(k)}$ is a bounded function for all k.

Moreover, since $P(\frac{1}{x}) = \frac{1}{x}R(\frac{1}{x})$ and $Q(\frac{1}{x}) = \frac{1}{x}S(\frac{1}{x})$ where R and S are polynomials, we have $|R(\frac{1}{x})|$ and $|S(\frac{1}{x})|$ are bounded for $|x| \geqslant 1$. Consequently, there is an M such that $|R(\frac{1}{x})\sin x + S(\frac{1}{x})\cos x| \leqslant M$ for $|x| > 1$. This gives $|f^{(k)}(x)| \leqslant \frac{M}{|x|}$ for $|x| > 1$. $\square$

Lemma 4.99. *Let $M > 0$. Suppose $a > 0$, $b > 0$, and f is a bounded function on $\mathbb{R}$ satisfying $|f(x)| < \frac{M}{|x|}$ for all $|x| > b$. Then*

$$\sum_{k\in\mathbb{Z}} \sup_{x\in[-m,m]} |f(x-ka)|^2 < \infty \text{ for all } m \in \mathbb{N}.$$

Proof. Let $B > 0$ be an upperbound for $|f|$. Then if $-m \leqslant x \leqslant m$, we have

$$\begin{aligned}
\sum_k \sup_{-m\leqslant x\leqslant m} |f(x-ka)|^2 &= \sum_{ka>m+b} \sup|f(x-ka)|^2 + \sum_{ka<-m-b} \sup|f(x-ka)|^2 \\
&\quad + \quad | \sum_{-m-b\leqslant ka\leqslant m+b} \sup|f(x-ka)|^2 \\
&\leqslant \sum_{ka>m+b} \sup|f(x-ka)|^2 + \sum_{ka<-m-b} \sup|f(x-ka)|^2 + (\frac{2m+2b}{a}+1)B \\
&\leqslant \sum_{ka>m+b} \sup\frac{M}{|x-ka|^2} + \sum_{ka<-m-b} \sup\frac{M}{|x-ka|^2} + (\frac{2m+2b}{a}+1)B \\
&\leqslant \sum_{ka>m+b} \frac{M}{|m-ka|^2} + \sum_{ka<-m-b} \frac{M}{|-m-ka|^2} + (\frac{2m+2b}{a}+1)B \\
&\leqslant 2\sum_{ka>m+b}^{\infty} \frac{M}{(ka-m)^2} + (\frac{2m+2b}{a}+1)B \\
&= \frac{2}{a^2}\sum_{k>\frac{m+b}{a}} \frac{M}{(k-\frac{m}{a})^2} + (\frac{2m+2b}{a}+1)B \\
&\leqslant \frac{2}{a^2}\sum_{j=1}^{\infty} \frac{M}{(j-1+\frac{b}{a})^2} + (\frac{2m+2b}{a}+1)B.
\end{aligned}$$

$\square$

Theorem 4.100. *Suppose $f \in L^2_T(\mathbb{R}^n)$. If K is a compact subset of $\mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$, the series*

$$\sum_{\gamma\in\Gamma} f(\gamma)D^\alpha \mathrm{sinc}_T(x-\gamma)$$

converges absolutely uniformly on K.

Proof. Note we may assume $K = \prod_{j=1}^{n}[-m, m]$ where $m \in \mathbb{N}$. We show there is a square summable function $b : \Gamma \to [0, \infty)$ such that for each $\gamma \in \Gamma$,

$$\sup_{x \in K} |D^{\alpha}\mathrm{sinc}_T(x - \gamma)| \leqslant b(\gamma).$$

Since

$$\mathrm{sinc}_T(x - \gamma) = \prod_{j=1}^{n} \mathrm{sinc}(2\pi T_j(x_j - \gamma_j)),$$

it suffices to find a square summable function

$$b_j : \mathbb{Z} \to [0, \infty)$$

satisfying

$$|\left(\frac{d}{dt}\right)^{\alpha_j} \mathrm{sinc}(2\pi T_j(t - \frac{k}{2T_j})| \leqslant b_j(k) \text{ for all } t \in [-m, m].$$

Consider the function f defined by

$$f(t) = \left(\frac{d}{dt}\right)^{\alpha_j} \mathrm{sinc}(2\pi T_j t) = (2\pi T_j)^{\alpha_j}\mathrm{sinc}^{(\alpha_j)}(2\pi T_j t).$$

By Lemma 4.98, we know $|\mathrm{sinc}^{(\alpha_j)}(t)|$ is bounded and there is a constant $B_j > 0$ such that $|\mathrm{sinc}^{(\alpha_j)}(x)| < \frac{B_j}{|x|}$ for $|x| > 1$. Hence f is a bounded function and $|f(t)| < (2\pi T_j)^{\alpha_j} \frac{B_j}{|2\pi T_j t|}$ if $|2\pi T_j t| > 1$. Taking $M = (2\pi T_j)^{\alpha_j - 1}B_j$, we see

$$|f(t)| < \frac{M}{|t|} \text{ for } |t| > \frac{1}{2\pi T_j}.$$

By Lemma 4.99,

$$\sum_{k \in \mathbb{Z}} \left(\sup_{t \in [-m,m]} |f(t - \frac{k}{2T_j})|\right)^2 < \infty.$$

So we take

$$b_j(k) = \sup_{t \in [-m,m]} |f(t - \frac{k}{2T_j})|.$$

Setting

$$b(\frac{m_1}{2T_1}, \ldots, \frac{m_n}{2T_n}) = b_1(m_1)b_2(m_2)\cdots b_n(m_n),$$

we obtain a square summable function b on Γ with

$$|D^{\alpha}\mathrm{sinc}_T(x - \gamma)| \leqslant b(\gamma)$$

for all $x \in K$ and all γ. Since $\sum_{\gamma\in\Gamma} |f(\gamma)|^2 < \infty$ by Parseval's equality, we see $\gamma \mapsto f(\gamma)b(\gamma)$ is absolutely summable. This implies the uniform absolute convergence for $x \in K$ of

$$\sum_{\gamma} f(\gamma) D^{\alpha} \mathrm{sinc}_T(x-\gamma).$$

$\square$

EXERCISE SET 4.12

1. Use the Fourier transforms given in Exercise 3.2.3 to show if $f \in L^2(\mathbb{R}^n)$, then $\mathcal{F}f = \mathcal{F}_{2\pi}f$ has essential support in Q_T where $T > 0$ if and only if $\mathcal{F}_a f$ has essential support in $Q_{\frac{a}{2\pi}T}$.

2. Suppose f is a C^∞ square integrable function on $\mathbb{R}$ with ess-supp$\hat{f} \subseteq [a, a+2b]$. Show

$$f(x) = \sum_k e^{-\pi i k(\frac{a}{b}+1)} f(\frac{k}{b}) e^{2\pi i(a+b)x} \mathrm{sinc}(2\pi b x - k\pi)$$

pointwise uniformly and in $L^2(\mathbb{R})$.

3. Show

$$\frac{\sin \pi x}{2x} + \frac{\cos \pi x - 1}{\pi x^2} = \frac{4x \sin \pi x}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \left(\frac{1}{(2n+1)^2 - x^2} \right)$$

pointwise uniformly in x. **Hint:** Apply Shannon's sampling to the function f where $\hat{f}(\omega) = |2\pi\omega| \chi_{[-\frac{1}{2},\frac{1}{2}]}(\omega)$.

14. The Poisson Summation Formula

In this section we discuss the Poisson summation formula, which shows in particular that for certain functions f we have the relation $\sum_{k\in\mathbb{Z}^n} f(x-k) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{2\pi i k\cdot x}$ between the function f and its Fourier transform $\hat{f}$. In particular

$$\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\,.$$

We will not try to prove this in most generality, but refer to the exercises. Let $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ and $Q = [0,1]^n$. We can then identify $L^p(\mathbb{T})$ with $L^p(Q)$ in the usual way.

Lemma 4.101. *Let $a > 0$. Then $\sum_{k\in\mathbb{Z}^n} \frac{1}{(a+|k|^2)^p}$ converges if and only if $p > \frac{n}{2}$.*

Proof. It suffices to establish when $\sum_{k\in\mathbb{N}_0^n} \frac{1}{(a+|k|^2)^p}$ converges. If $Q = [0,1]^n$, we note if $f(x) = \frac{1}{(a+|x|^2)^p}$, then $f(x) \leqslant \frac{1}{(a+|k|^2)^p}$ on $k+Q$ for $k \in \mathbb{N}_0^n$. Thus $\sum_{k\in\mathbb{N}^n} \frac{1}{(a+|k|^2)^p} \geqslant \sum_k \int_{k+Q} f(x)\, dk = \int_{[0,\infty)^n} \frac{1}{(a+|x|^2)^p}\, dx$. Using polar coordinates (Corollary 2.26), we see $\sigma(S_+^{n-1}) \int_0^\infty \frac{r^{n-1}}{(a+r^2)^p}\, dr = \infty$ if $-2p+n-1 \geqslant -1$. Hence the series diverges for $p \leqslant \frac{n}{2}$.

Now assume $p > \frac{n}{2}$. The series $\sum_{k\in\mathbb{N}_0^n} \frac{1}{(a+|k|^2)^p}$ converges if $\sum_{|k|\geqslant 2\sqrt{n}} \frac{1}{(a+|k|^2)^p}$ converges. Let $g(x) = \frac{1}{(\alpha+\alpha|x|^2)^p}$ where $0 < \alpha < \min\{a, \frac{4}{9}\}$. Note if $x \in Q+k$, $|x| \leqslant |k| + |(1,1,\ldots,1)| = |k| + \sqrt{n}$. Thus for $x \in Q+k$,

$$\begin{aligned}
\alpha + \alpha|x|^2 &\leqslant \alpha + \alpha(|k|^2 + 2\sqrt{n}|k| + n) \\
&< a + \alpha(|k|^2 + |k|^2 + \frac{|k|^2}{4}) \\
&= a + \frac{9}{4}\alpha|k|^2 \\
&< a + |k|^2.
\end{aligned}$$

Thus $g(x) \geqslant \frac{1}{(a+|k|^2)^p}$ for $x \in Q+k$ for all $k \in \mathbb{N}_0^n$ with $|k| \geqslant 2\sqrt{n}$. Consequently $\int_{[0,\infty)^n} g(x)\, dx \geqslant \sum_{|k|\geqslant 2\sqrt{n}} \frac{1}{(a+|k|^2)^p}$. Now again using polar coordinates, $\int_{[0,\infty)^n} g(x)\, dx < \infty$ if and only if $p > \frac{n}{2}$. $\square$

Proposition 4.102. *Let f be in $\mathcal{S}(\mathbb{R}^n)$. Define f_a by*

$$f_a(x) = \sum_{k\in\mathbb{Z}^n} f(x+k).$$

Then f_a is C^∞ and is $\mathbb{Z}^n$ periodic on $\mathbb{R}^n$. Moreover, the series $\sum_{k\in\mathbb{Z}^n} f(x+k)$ and each series of its derivatives are absolutely uniformly convergent and converge uniformly to f_a and f_a's derivatives.

Proof. First we show if $f \in \mathcal{S}(\mathbb{R}^n)$, the series $\sum_{k\in\mathbb{Z}^n} f(x+k)$ is uniformly absolutely convergent on $Q = [0,1]^n$. Since f is Schwartz we can take positive A and $p > \frac{n}{2}$ with $|f(x)| \leqslant \frac{A}{(1+|x|^2)^p)}$ for all x. We note if $x \in Q$, $|x+k| \geqslant ||x|-|k||$. Hence $M(k) := \sup_{x\in Q} |f(x+k)| \leqslant \sup_{x\in Q} \frac{A}{(1+(|k|-|x|)^2)^p}$. Now $\sum_{k\in\mathbb{Z}^n} |f(x+k)|$ converges uniformly on Q if $\sum_k M(k)$ is finite.

But if $x \in Q$ and $|k| \geqslant 2\sqrt{n}$, we have $|k| - |x| \geqslant |k| - \sqrt{n} \geqslant \frac{|k|}{2}$ and thus

$$M(k) \leqslant \frac{A}{(1+\left(\frac{|k|}{2}\right)^2)^p} = \frac{4^p A}{(4+|k|^2)^p}.$$

Thus $\sum M(k)$ is finite by Lemma 4.101 for $p > \frac{n}{2}$.

Consequently, $\sum |f(x+k)|$ is uniformly summable in x on Q and by $\mathbb{Z}^n$ periodicity is uniformly convergent on $\mathbb{R}^n$. Since $D^\alpha f$ is again Schwartz for all α, we can apply the above and obtain the series $\sum_k D^\alpha f(x+k)$ is uniformly absolutely summable.

To finish we need only show $\frac{\partial}{\partial x_i} f_a = \left(\frac{\partial}{\partial x_i} f\right)_a$ and use induction to show $D^\alpha f_a = (D^\alpha f)_a$ for higher order α. Now $f_a(x+te_i) = \sum_k f(x+k+te_i) = \sum_k (f(x+k) + \int_0^t \frac{\partial}{\partial x_i} f(x+k+se_i)\,ds)$. Since $\sum_k f(x+k)$ converges and $\sum_k \frac{\partial}{\partial x_i} f(x+k+se_i)$ converges uniformly in s, we can interchange summation and integration and obtain

$$\begin{aligned} f_a(x+te_i) &= \sum_k f(x+k) + \int_0^t \sum_k \frac{\partial}{\partial x_i} f(x+k+se_i)\,dx \\ &= f_a(x) + \int_0^t \left(\frac{\partial}{\partial x_i} f\right)_a (x+se_i)\,ds. \end{aligned}$$

This gives $\frac{\partial}{\partial x_i} f_a = \left(\frac{\partial}{\partial x_i} f\right)_a$. □

Theorem 4.103 (Poisson Summation Formula)**.** *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\sum_{k \in \mathbb{Z}^n} D^\alpha f(x+k) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) D^\alpha e^{2\pi i k \cdot x}$$

where both sides converge absolutely uniformly and both sides give $D^\alpha f_a$ where $f_a(x) = \sum_{k \in \mathbb{Z}^n} f(x+k)$. In particular,

$$\sum_{k \in \mathbb{Z}^n} f(k) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k)$$

Proof. By Proposition 4.102, each series $\sum_k D^\alpha f(x+k)$ converges uniformly absolutely and gives $D^\alpha(f_a)$. Since f_a is C^∞ and $\mathbb{Z}^n$ periodic, Theorem 1.22, Corollary 1.23, and their generalizations to $\mathbb{R}^n$ in Exercises 1.3.14 and 1.3.15, show one has

$$\sum_k \mathcal{F}(f_a)(k) e^{2\pi i k \cdot x}$$

and $\sum_k \mathcal{F}(f_a)(k) D^\alpha e^{2\pi i k \cdot x}$ converge uniformly absolutely and their limits are f_a and $D^\alpha(f_a)$.

To finish, we note:

$$\begin{aligned}\mathcal{F}(f_a)(k) &= \int_Q f_a(y)e^{2\pi i y\cdot k}\,dy \\ &= \int_Q \sum_{j\in\mathbb{Z}^n} f(y+j)\,e^{-2\pi i y\cdot k}\,dy \\ &= \sum_j \int_Q f(y+j)e^{-2\pi i (y+j)\cdot k}\,dy \\ &= \sum_j \int_{Q+j} f(y)e^{-2\pi i y\cdot k}\,dy \\ &= \int_{\mathbb{R}^n} f(y)e^{-2\pi i y\cdot k}\,dy \\ &= \hat{f}(k).\end{aligned}$$

□

Corollary 4.104. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathbb{R}^n_+$. Then*

$$\sum_{\gamma\in T\mathbb{Z}^n} f(\gamma) = \frac{1}{T_1T_2\dots T_n}\sum_{\sigma\in\frac{\mathbb{Z}^n}{T}} \hat{f}(\sigma)\,.$$

Proof. Define $\delta(T)$ on $\mathcal{S}(\mathbb{R}^n)$ for $T > 0$ by

$$\delta(T)f(x) = \sqrt{T_1T_2\cdots T_n}\,f(T_1x_1, T_2x_2, \dots, T_nx_n).$$

Then δ is a homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ with inverse $\delta(\frac{1}{T}) = \delta(\frac{1}{T_1}, \frac{1}{T_2}, \dots, \frac{1}{T_n})$. Moreover, $\mathcal{F}(\delta(T)f) = \delta(\frac{1}{T})\mathcal{F}(f)$. Thus

$$\sum_{\gamma\in\mathbb{Z}^n} \delta(T)f(\gamma) = \sum_{\sigma\in\mathbb{Z}^n} \delta(\frac{1}{T})\hat{f}(\sigma).$$

This gives the result. □

Example 4.105. *Let $n = 1$. As seen by Example 3.2, we know $\hat{g} = g$ for the Gaussian $g(x) = e^{-\pi x^2}$. Thus the Poisson summation formula for g is trivial. Even so, Corollary 4.104 gives*

$$T\sum_{n\in\mathbb{Z}} e^{-\pi T^2k^2} = \sum_{n\in\mathbb{Z}} e^{-\frac{\pi k^2}{T^2}}.$$

The function $\theta(t) := \sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$, $t > 0$, is called a (Jacobi) theta-function and is important in number theory. By choosing $T = \sqrt{t}$ the Poisson summation formula shows that the theta functions satisfies the functional equation

$$\sqrt{t}\theta(t) = \theta(1/t)\,.$$

Example 4.106. *Now take* $f(x) = e^{-2\pi a|x|}$*,* $a > 0$*. Then*

$$\hat{f}(\omega) = \frac{a}{\pi}\frac{1}{a^2+\omega^2}\,.$$

The more general version of the Poisson summation formula in Exercise 4.13.1 shows that

$$1 + 2\sum_{k=1}^{\infty} e^{-2\pi ak} = \frac{1}{\pi a} + \frac{a}{\pi}\sum_{k=1}^{\infty}\frac{2}{a^2+k^2}\,.$$

Exercise Set 4.13

1. Show the Poisson summation formula holds if we assume that there exists positive constants $A, B > 0$, and $p > n/2$, such that

$$|f(x)| \leqslant A(1+|x|^2)^{-p}\,, \quad \left|\hat{f}(\omega)\right| \leqslant B(1+|\omega|^2)^{-p}$$

More specifically show $\sum_{k\in\mathbb{Z}^n} f(x+k)$ and $\sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{2\pi i x\cdot k}$ converge uniformly absolutely and

$$\sum_{k\in\mathbb{Z}^n} f(x+k) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{2\pi i x\cdot k}.$$

In particular,

$$\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k).$$

2. Let $f : \mathbb{R}^n \to \mathbb{C}$ be $\mathbb{Z}^n$ periodic and satisfy

$$\sum_{k\in\mathbb{Z}^n} |\hat{f}(k)|\,|k|^m < \infty.$$

Show $f \in C^m(\mathbb{R}^n)$ and

$$\sum_{k} \hat{f}(k)(2\pi i k)^{\alpha} e^{2\pi i k\cdot x}$$

converges uniformly to $D^{\alpha} f$ if $|\alpha| \leqslant m$.

3. Suppose f is an L^2 function on $\mathbb{R}^n$, $m \in \mathbb{N}$, and there are positive constants A, $B > 0$, and $p > \frac{m+n}{2}$ with

$$|f(x)| \leqslant A(1+|x|^2)^p, \quad |\hat{f}(\omega)| \leqslant B(1+|\omega|^2)^p.$$

Show f is in $C^m(\mathbb{R}^n)$ and for $|\alpha| \leqslant m$ one has

$$D^{\alpha}\left(\sum_{k\in\mathbb{Z}^n} f(x+k)\right) = \sum_{k\in\mathbb{Z}^n} D^{\alpha} f(x+k) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)(2\pi i k)^{\alpha} e^{2\pi i k\cdot x}.$$

4. Introduce the *Wiener space* $W(\mathbb{R}^n)$ as the space of functions $f \in L^{\infty}(\mathbb{R}^n)$ such that

$$||f||_W := \sum_{k\in\mathbb{Z}^d} ||\lambda(k) f|_Q||_{L^{\infty}(Q)} < \infty\,.$$

Show the following:

(a) $W(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.

(b) $\mathcal{S}(\mathbb{R}^n) \subset W(\mathbb{R}^n)$.

(c) If f and $\hat{f}$ are in $W(\mathbb{R}^n)$ then $\sum_{k \in \mathbb{Z}^n} f(x-k)$ and $\sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i k \cdot x}$ converge uniformly absolutely.

(d) The Poisson summation formula holds if $f, \hat{f} \in W(\mathbb{R}^n)$; i.e.,

$$\sum_{k \in \mathbb{Z}^n} f(x-k) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i k \cdot x}$$

for $x \in \mathbb{R}^n$.

Chapter 5

Topological Groups

In this chapter we develop the basic theory of topological groups, but having in mind the material for which it will be used in this text. A topological group has two compatible structures; the *algebraic* structure of a group, and the *analytic/geometric* structure of a topological space. Here compatibility requires the algebraic operations are continuous.

We shall make use of standard notions in both topology and algebra. In many instances, we will recall appropriate definitions and well known results. Terminology will be standard and thus we hope students with basic courses in algebra and topology will have no difficulty in following the presentation.

1. Topological Groups

Definition 5.1. *Let G be a group and a topological space. Then G is a topological group if*

(a) $G \times G \ni (x, y) \mapsto xy \in G$ *and*
(b) $G \ni x \to x^{-1} \in G$

are continuous functions, (here $G \times G$ has the product topology).

We note that the definition can be regiven by replacing (a) and (b) by

$$(A) \qquad G \times G \ni (x, y) \mapsto xy^{-1} \in G \text{ is continuous.}$$

Indeed, it is easy to show the equivalence. See Exercise 5.1.1.

If G is a topological group, then the inversion $\iota : G \to G$, $\iota(x) = x^{-1}$ is a homeomorphism with inverse ι. Since groups are not necessarily commutative, one must distinguish between *left translation* $\lambda(a)(x) = ax$ and *right* *translation* $\rho(a)(x) = xa$. When convenient, we will also use the notation $\lambda_a(x)$ for $\lambda(a)x$ and $\rho_a(x)$ for $\rho(a)x$.

Lemma 5.2. *Let G be a topological group and let $a, b \in G$. Then $\lambda(a)$ and $\rho(a)$ are homeomorphisms with inverse $\lambda(a^{-1})$, respectively $\rho(a^{-1})$. Furthermore $\lambda(e) = \rho(e) = \mathrm{id}$, $\lambda(ab) = \lambda(a) \circ \lambda(b)$ and $\rho(ab) = \rho(b) \circ \rho(a)$.*

Proof. We can write $\lambda(a)$ in the following way

$$\begin{array}{ccccc} G & \to & G \times G & \to & G \\ x & \mapsto & (a, x) & \mapsto & ax. \end{array}$$

The first map is continuous by the definition of the product topology and the second map is continuous by part (a) in the definition of a topological group. Hence $\lambda(a) : G \to G$ is continuous. We have

$$\lambda(ab)(x) = (ab)x = a(bx) = \lambda(a)(\lambda(b)(x)) = (\lambda(a) \circ \lambda(b))(x).$$

Obviously $\lambda(e)(x) = ex = x$, so $\lambda(e)$ is the identity map. Finally $\lambda(a^{-1})\lambda(a) = \lambda(a^{-1}a) = \mathrm{id} = \lambda(a) \circ \lambda(a^{-1})$. It follows that $\lambda(a)$ is a homeomorphism with inverse $\lambda(a^{-1})$. The case $\rho(a)$ is treated similarly. □

Let A and B be non-empty subsets of G, and let $x \in G$. Define

$$\begin{aligned} AB &= \{ab \mid a \in A,\, b \in B\} \\ A^{-1} &= \{a^{-1} \mid a \in A\} \\ xA &= \{x\}A = \{xa \mid a \in A\} = \lambda(x)(A) \\ Ax &= A\{x\} = \{ax \mid a \in A\} = \rho(x)(A) \\ A^1 &= A \quad \text{and} \quad A^{n+1} = AA^n\,,\ n \geqslant 1\,. \end{aligned}$$

Lemma 5.2 implies that if $a \in G$ and U is an open neighborhood of the identity, then aU and Ua are open neighborhoods of a.

Lemma 5.3. *Assume that $A, B \subset G$ and that B is open. Then AB, BA, and B^{-1} are open.*

Proof. We have $AB = \bigcup_{a \in A} aB$ and each aB is open according to Lemma 5.2. That BA is open follows in the same way. The last statement follows since inversion is a homeomorphism. □

For x in a topological space, $\mathcal{N}(x)$ will denote the neighborhood system at x; i.e., $\mathcal{N}(x)$ is the collection of all neighborhoods of x. A subset A of a group said to be *symmetric* if $A = A^{-1}$.

Corollary 5.4. *Let $U \in \mathcal{N}(e)$, then there exists a symmetric open set $V \in \mathcal{N}(e)$ such that $V \subset U$.*

Proof. Take an open subset W of U containing e. Set $V = W \cap W^{-1}$ Then V is open, $e \in V$, and V is symmetric. $\square$

Corollary 5.5. *Let $a \in G$. Then*

$$\mathcal{N}(a) = \{aU \mid U \in \mathcal{N}(e)\} = \{Ua \mid A \in \mathcal{N}(e)\}.$$

Proof. This is immediate for λ_a and ρ_a are homeomorphisms. $\square$

Proposition 5.6. *Let G and H be topological groups. Then $G \times H$ with the product topology is a topological group.*

Proof. Let W be a neighborhood of (ab^{-1}, cd^{-1}) where $a, b \in G$ and $c, d \in H$. Then W contains a set of form $U \times V$ where U is an open neighborhood of ab^{-1} in G and V is an open neighborhood of cd^{-1} in H. Since G and H are topological groups, there are open neighborhoods U_a and U_b of a and b in G and open neighborhoods V_c and V_d of c and d in H satisfying $U_a U_b^{-1} \subseteq U$ and $V_c V_d^{-1} \subseteq V$. Thus $U_a \times V_c$ and $U_b \times V_d$ are open neighborhoods of (a, c) and (b, d) in $G \times H$ satisfying $(U_a \times V_c)(U_b \times V_d)^{-1} \subseteq U \times V \subseteq W$. Thus the mapping $((a, c), (b, d)) \mapsto (a, c)(b, d)^{-1}$ is continuous. By (A), $G \times H$ is a topological group. $\square$

As just seen a product of topological groups with the product topology is a topological group. This can be used to give some simple examples of topological groups. A more extensive list of examples are given in Section 5.

Example 5.7 (Normed vector spaces)**.** *Let X be any topological vector space. Then X is a topological group with respect to addition. In particular this holds for any normed vector space X. As X is abelian it follows that the maps $\lambda(x)$ and $\rho(x)$ agree and are given by translation by x, $\lambda_x(y) = y + x$. Furthermore,*

$$\begin{aligned} AB &= \{x + y \mid x \in A, y \in B\} = A + B \\ A^{-1} &= -A \\ A^n &= A + A + \cdots + A\,. \end{aligned}$$

In particular, the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are topological groups.

Example 5.8 ($\mathbb{C}^{*n}$ and the torus $\mathbb{T}^n$)**.** *Let $\mathbb{C}^*$ be the nonzero complex numbers under multiplication, and let $\mathbb{T}$ be the unit circle in $\mathbb{C}^*$. Then $\mathbb{C}^*$ is a topological group and $\mathbb{T}$ is a compact subgroup of $\mathbb{C}^*$. Indeed, since*

$$|zw - z_0 w_0| \leqslant |z|\,|w - w_0| + |z - z_0|\,|w_0|,$$

one sees multiplication is continuous. Moreover,

$$|\frac{1}{z} - \frac{1}{z_0}| = \frac{1}{|zz_0|}|z - z_0|$$

implies the inverse mapping $\mathfrak{i}(z) = \frac{1}{z}$ *is continuous. Hence* $\mathbb{C}^*$ *is a topological group. Since* $\mathbb{T}$ *has the relative topology of* $\mathbb{C}^*$*, it follows that* $\mathbb{T}$ *is a topological group. Hence* $\mathbb{C}^{*n}$ *and* $\mathbb{T}^n$ *with the product topologies are topological groups and their topologies are the relative topologies from the space* $\mathbb{C}^n$*.*

Example 5.9 (The Heisenberg Groups)**.** *Let* B *be an alternating bilinear form on* $\mathbb{R}^{2n}$*; thus*

$$\begin{aligned} B(x,y) &= -B(y,x) \\ B(ax+by,z) &= aB(x,z)+bB(y,z) \end{aligned}$$

for all $x, y \in \mathbb{R}^{2n}$ *and* $a, b \in \mathbb{R}$*. Note if* $e_k = (\delta_{k,i})_{i=1}^n$ *is the usual standard basis of* $\mathbb{R}^n$*, then*

$$B(x,y) = \sum_{i,j} B_{ij} x_i y_j$$

where $B_{i,j} = B(e_i, e_j)$*. This implies* B *is continuous on* $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$*. Moreover,* $B_{i,j} = -B_{j,i}$*; i.e., the square matrix* $[B_{i,j}]$ *is skew symmetric. If this square matrix is invertible, the alternating form* B *is said to be nondegenerate; and then if one takes*

$$H = \mathbb{R}^{2n} \times \mathbb{R}$$

with multiplication defined by

$$(x,t)(y,s) = (x+y, t+s+\frac{1}{2}B(x,y)),$$

one obtains a noncommutative group called the $2n+1$ *dimensional Heisenberg group. We note* $(0,0)$ *is the multiplicative identity for this group and* $(x,t)^{-1} = (-x,-t)$ *for* $B(x,x) = 0$*. Since* B *is continuous, we see* H *with the topology of* $\mathbb{R}^{2n+1}$ *is a topological group; for both multiplication and inversion are continuous. Note the commutant of the group elements* (x,t) *and* (y,s) *is* $(0, B(x,y))$*. Indeed,*

$$\begin{aligned} (x,t)(y,s)(x,t)^{-1}(y,s)^{-1} &= ((x,t)(y,s))\,((-x,-t)(-y-s)) \\ &= (x+y, t+s+\frac{1}{2}B(x,y))\,(-x-y, -t-s+\frac{1}{2}B(x,y)) \\ &= (0, B(x,y)+\frac{1}{2}B(x+y,-x-y)) \\ &= (0, B(x,y)). \end{aligned}$$

The Heisenberg group has multiplication the commutative addition of $\mathbb{R}^{2n+1}$ *but skewed in the last component by the alternating bilinear form* B*. Studying this minor alteration and understanding its rather dramatic consequences is the aim of Chapter 7.*

Let G and H be groups. Recall a subgroup $N \subset G$ is a *normal subgroup* if $aNa^{-1} \subset N$ for all $a \in G$ and a map $\varphi : G \to H$ is a *homomorphism* if

$\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$. Note that $\varphi(G)$ is a subgroup of H and $\mathrm{Ker}(\varphi) = \varphi^{-1}(e) = \{a \in G \mid \varphi(a) = e\}$ is a normal subgroup of G. If G and H are topological groups, then G and H are *isomorphic* if there exists a homomorphism $\varphi : G \to H$ which is also a homeomorphism. In that case $\varphi^{-1} : H \to G$ is also an isomorphism. Finally, recall a topological space is T_1 if all sets with one element are closed. We will discuss the separation axioms in more detail in Section 4.

Lemma 5.10. *Let G and H be topological groups and $\varphi : G \to H$ be a homomorphism. Then the following hold:*

(a) *φ is continuous if and only if φ is continuous at $e \in G$.*

(b) *If φ is continuous and H is T_1, then $\mathrm{Ker}(\varphi)$ is a closed normal subgroup of G.*

Proof. Assume ϕ is continuous at e. Then $\phi(x) = \lambda_{\phi(a)}\phi(\lambda(a^{-1})x)$ is continuous at a for $\lambda(a^{-1})$ and $\lambda_{\phi(a)}$ are homeomorphisms. Hence ϕ is continuous everywhere if ϕ is continuous at e, and we see (a) holds.

Note (b) follows from the preimage of a closed set under a continuous function is a closed set and $\{e\}$ is a closed set when H is T_1. $\square$

Theorem 5.11. *Let G be a group and let $\mathcal{U}$ be a non-empty family of subsets of G such that the following hold:*

(a) *$e \in U$ for all $U \in \mathcal{U}$;*

(b) *If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;*

(c) *If $U \in \mathcal{U}$, then there is a $V \in \mathcal{U}$ such that $VV \subset U$;*

(d) *If $U \in \mathcal{U}$, then $V^{-1} \in \mathcal{U}$;*

(e) *If $U \in \mathcal{U}$ and $a \in G$, then $aUa^{-1} \in \mathcal{U}$.*

Call a set A in G open if for each $a \in A$ there exits a $U \in \mathcal{U}$ such that $aU \subset A$. Let τ be the collection of all open sets. Then τ is a topology on G that makes G into a topological group.

Proof. It is clear that $\varnothing$ and G are both open. Let U and U' be open sets. We have to show $U \cap U'$ is open. If $U \cap U'$ is empty, then there is nothing to prove. Otherwise let $a \in U \cap U'$. Choose $V, V' \in \mathcal{U}$ such that $aV \subset U$ and $aV' \subset U'$. Then by (b) it follows that $V'' := V \cap V' \in \mathcal{U}$. Finally $aV'' \subset aV \subset U$ and $aV'' \subset aV' \subset U'$. Hence $aV'' \subset U \cap U'$, and it follows that $U \cap U'$ is open. Let $\{U_i\}_{i \in I}$ be a collection of open sets. Let $a \in \cup_{i \in I} U_i$. Then there exists a i_0 such that $a \in U_{i_0}$. Let $V \in \mathcal{U}$ be such that $aV \subset U_{i_0}$, then $aV \subset \cup_i U_i$ and it follows that $\cup_i U_i$ is open.

Next we note $\mathcal{U}$ is a neighborhood base at e for this topology. Clearly, if N is a neighborhood of e, then there is a $U \in \mathcal{U}$ with $U = eU \subseteq N$.

Conversely, let $U \in \mathcal{U}$. Set $N = \{a \in U \mid aV \subseteq U$ for some $V \in \mathcal{U}\}$. Since $eU \subseteq U$, $e \in N$. Moreover, if $a \in N$ then $aV \subseteq U$ for some $V \in \mathcal{U}$. By (c), we can choose $W \in \mathcal{U}$ with $W^2 \subseteq V$. Thus $(aW)W \subseteq U$. Hence $aW \subseteq N$. Since a is arbitrary, N is open.

We have just shown that τ is a topology on G, but we still have to show that G is a topological group. According to (A), it suffices to show that the map $G \times G \ni (a,b) \overset{\psi}{\mapsto} ab^{-1} \in G$ is continuous. Let $a, b \in G$ and let U be an open set containing ab^{-1}. Let $W \in \mathcal{U}$ be such that $ab^{-1}W \subset U$. Let V be such that $VV^{-1} \subset b^{-1}Wb$. Then if $x \in aV$ and $y \in bV$,

$$xy^{-1} \in aVV^{-1}b^{-1} \subset ab^{-1}W \subset U\,.$$

It follows that ψ is continuous. $\square$

Exercise Set 5.1

1. Let G be a group with topology τ. Show that G is a topological group if and only if the map

$$G \times G \ni (x,y) \mapsto xy^{-1} \in G$$

is continuous.

2. Let G be a topological group. Show G is T_1 if and only if $\{e\}$ is a closed set in G.

3. Let G be a topological group. Let $U \subset G$ be open and let $a \in U$. Show there exists an open set $V \ni e$ such that $aV \subset U$.

4. Let G and H be topological groups. Show a function $f : G \to H$ is continuous if and only if for each x in G and each neighborhood V of the identity in H, there is a neighborhood U of the identity in G with

$$f(xU) \subseteq f(x)V.$$

5. Let G be a topological group. Show that $\rho(a) : G \to G$ is a homeomorphism.

6. Let G be a topological group and $H \subset G$ an open subgroup. Show H is also closed.

7. Let H be a subgroup of a topological group G. Show the closure $\bar{H}$ is a group and the interior H° of H is either empty or is a subgroup of G.

2. Group Actions

The importance of many groups comes from their connection to geometry as a symmetry group of a geometric structure and their *action on sets or manifolds.* We recall the main definitions. Let G be a group and X a set.

An *action* or more precisely a *left-action* of G on X is a map $\mu : G \times X \to X$, $\mu(g, x) = g \cdot x$, such that the following hold:

(1) $e \cdot x = x$ for all $x \in X$;

(2) $(ab) \cdot x = a \cdot (b \cdot x)$ for all $a, b \in G$ and all $x \in X$.

Similarly a *right-action* is a map $G \times X \to X$, $(g, x) \mapsto x \cdot g$ such that $x \cdot e = x$ for all $x \in X$ and $x \cdot (ab) = (x \cdot a) \cdot b$. If μ is a left action then we can define a right action μ_R by $\mu_R(a, x) = \mu(a^{-1}, x)$. If $x \in X$ and $a \in G$, let

$$\mu_x : G \to X, \quad g \mapsto g \cdot x$$

and

$$\lambda_a : X \to X, \quad x \mapsto a \cdot x\,.$$

We also use the notation $\lambda(a) = \lambda_a$. If G is a topological group and X is a topological space, then the action is *continuous* if μ is continuous where $G \times X$ has the product topology. In this case we say that X is a continuous G-space. The action is *separately continuous* if for each $x \in X$ and $a \in G$ the maps μ_x and λ_a are continuous. Let $x \in X$, $Y \subset X$, and $B \subset G$ be non-empty. Then

$$\begin{aligned} Y^B &:= \{y \in Y \mid \forall g \in B : g \cdot y = y\}\,, \\ B_Y &:= \{g \in B \mid \forall y \in Y : g \cdot y = y\}\,, \\ B_x &:= B_{\{x\}} = \{g \in B \mid g \cdot x = x\}\,. \end{aligned}$$

The space X is a *transitive* G space if there exists a $x \in X$ such that $X = G \cdot x$, and in this case we say G *acts transitively on* X. If G acts transitively, then $X = G \cdot y$ for every $y \in X$. A function ϕ between G spaces X and Y is G*-equivariant* if $\varphi(a \cdot x) = a \cdot \varphi(x)$ for all $a \in G$ and all $x \in X$. If G acts separately continuously on X and Y, then a G*-map* is a continuous G-equivariant function from X into Y; we use $M_G(X, Y)$ to denote the set of all G-maps $X \to Y$. If $M_G(X, Y)$ contains a homeomorphism, the G spaces X and Y are said to be G *isomorphic*, and we write $X \simeq_G Y$.

Lemma 5.12. *Assume that G acts separately continuously on X. Then the following hold:*

(a) *Each mapping $\lambda(a) : X \to X$, $x \mapsto a \cdot x$ is a homeomorphism with inverse $\lambda(a^{-1})$.*

(b) *Assume that X is Hausdorff. If $Y \subset X$ is closed and $B \subset G$, then Y^B is a closed subset of X; and if $B \subset G$ is closed in G and $Y \subset X$ is non-empty, then $B_Y \subset G$ is a closed subset of G.*

(c) *For $Y \subseteq X$, G_Y is a subgroup of G.*

Proof. To see (a), note both $\lambda(a) : X \to X$ and $\lambda(a^{-1}) : X \to X$ are continuous. Moreover, $\lambda(a)\lambda(a^{-1}) = \lambda(aa^{-1}) = \lambda(e) = \mathrm{id}$ and $\lambda(a)^{-1}\lambda(a) = \lambda(a^{-1}a) = \lambda(e) = \mathrm{id}$.

We show (b). Let $g \in B$ and suppose $g \cdot x \neq x$. Then there exist open sets $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(g \cdot x)$ with $U \cap V = \varnothing$. As $\lambda(g) : X \to X$ is continuous, it follows that there exists an open set $U_1 \in \mathcal{N}(x)$ such that $\lambda(g)(U_1) \subset V$. If $y \in U_1$ is such that $g \cdot y = y$, then this would imply that $y \in U_1 \cap V \subset U \cap V = \varnothing$, which is a contradiction. Hence $x \in U_1 \subset \{y \in X \mid g \cdot y \neq y\}$; and it follows that $\{x \in X \mid g \cdot x \neq x\}$ is open. Hence $\{x \in X \mid g \cdot x = x\}$ is closed. As Y is closed in X and

$$Y^B = \{x \in Y \mid \forall g \in B : g \cdot x = x\} = \left(\bigcap_{g \in B} \{x \in X \mid g \cdot x = x\} \right) \cap Y,$$

it follows that Y^B is closed in X.

Now fix x. Assume that $b \cdot x \neq x$. Let $U \in \mathcal{N}(x)$ and $W \in \mathcal{N}(b \cdot x)$ be open and satisfy $U \cap W = \varnothing$. As

$$G \ni a \mapsto a \cdot x \in X$$

is continuous, it follows that we can find a neighborhood V of b such that $V \cdot x \subset W$. Assume that $a \in V$ and $a \cdot x = x$. Then $a \cdot x \in W \cap U = \varnothing$, a contradiction. Hence $\{g \in G \mid g \cdot x \neq x\}$ is open in G. Thus $\{g \in G \mid g \cdot x = x\}$ is closed in G. As B is closed in G, it follows that

$$B_Y = \bigcap_{x \in Y} \{b \in G \mid b \cdot x = x\} \cap B$$

is closed in G.

Finally, let Y be a nonempty subset of X. Note $e \in G_Y$ and if a and b are in G_Y, then $(ab) \cdot y = a \cdot (b \cdot y) = a \cdot y = y$ and $a^{-1} \cdot y = a^{-1} \cdot (a \cdot y) = (a^{-1}a) \cdot y = y$ for $y \in Y$. So G_Y is a subgroup of G.

□

3. Homogeneous Spaces

Assume G acts on a space X. Define a equivalence relation $\sim$ on X by

$$x \sim y \text{ if and only if there exists a } g \in G \text{ with } g \cdot x = y\,.$$

If $x \in X$, let $[x]$ denote the equivalence class containing x. This set is called the *orbit* of x and equals the set $G \cdot x$. Then $G\backslash X = \{[x] \mid x \in X\}$ is called the *orbit space* of the action. We define X/G in a similar manner if G acts from the right. Let $H \subset G$ be a subgroup. Then H acts on G on the right by $g \cdot h = gh$. The orbits are the "left cosets" gH and the space G/H is the space of left cosets. The group G acts on G/H by $a \cdot (bH) = (ab)H$.

Let $\kappa : G \to G/H$ be the canonical map $a \mapsto aH$. Assume that G is a topological group. The quotient topology on G/H is defined by requiring a set $U \subset G/H$ to be open if and only if $\kappa^{-1}(U)$ is open in G. Note a subset U of G/H is open if and only if the union of the cosets in G/H is open in G. That this is a topology is an easy exercise. See Exercise 5.2.3.

Lemma 5.13. *Let G be a topological group and H be a subgroup. Then the following hold:*

(a) *The canonical map $\kappa : G \to G/H$ is continuous and open.*

(b) *The action of G on G/H is continuous.*

(c) *G/H is Hausdorff if and only if H is closed in G.*

Proof. The quotient map is continuous by definition. Let $U \subset G$ be open. Then

$$\kappa^{-1}(\kappa(U)) = UH = \bigcup_{h \in H} Uh$$

which is open as Uh is open in G for all $h \in H$. Hence $\kappa(U) \subset G/H$ is open by the definition of the topology on G/H. Thus (a) holds.

To do (b) we need to show $(x, aH) \mapsto xaH$ is continuous. Let U be a nonempty open subset of G/H, and let (x_0, a_0H) be such that x_0a_0H is in U. Then x_0a_0 is contained in the open set $\kappa^{-1}(U)$. Since $(x, a) \mapsto xa$ is continuous from $G \times G$ into G, there are open sets V and W in G with $x_0 \in V$, $a_0 \in W$, and $VW \subseteq \kappa^{-1}(U)$. Then (a) implies $\kappa(W)$ is open in G/H, and clearly $a_0H \in \kappa(W)$. Moreover, the image of $V \times \kappa(W)$ in G/H under the mapping $(x, aH) \mapsto xaH$ is contained in U.

Finally to see (c), assume first that G/H is Hausdorff. Then $\{\kappa(e)\} = \{H\} \in G/H$ is closed in G/H. Therefore

$$\kappa^{-1}(G/H - \{H\}) = G - H$$

is open in G. It follows that H is closed. Suppose that H is closed in G. Let $\mu : G \times G \to G$ be the map $(a, b) \mapsto a^{-1}b$. Then $\mu^{-1}(H)$ is closed in $G \times G$. Assume that $aH \neq bH$. Then $\mu(a, b) = a^{-1}b \notin H$. As $\mu^{-1}(H)$ is closed in $G \times G$, there exist open subsets U and V of G with $a \in U$, $b \in V$, and $(U \times V) \cap \mu^{-1}(H) = \varnothing$. Furthermore $\kappa(U)$ is an open neighborhood of aH and $\kappa(V)$ is an open neighborhood of bH. Assume that $x \in \kappa(U) \cap \kappa(V)$. Then we can write

$$x = cH = dH$$

with $c \in U$ and $d \in V$. It follows that $c^{-1}d = \mu(c, d) \in H$ or $(c, d) \in \mu^{-1}(H) \cap (U \times V) = \varnothing$, a contradiction. Hence G/H is Hausdorff. $\square$

A natural problem to consider is which G-spaces are of the form G/H. We will not attempt to answer this question in its most generality. Clearly the action of G must be transitive and at minimum the action must be separately continuous.

Assume X is a topological space and G acts transitively and separately continuously on X and $x \in X$. Let $H = G_x = \{a \in G \mid a \cdot x = x\}$ be the stabilizer of x. Then the mapping

$$G \to X, \quad a \overset{\kappa_x}{\mapsto} a \cdot x$$

factors to a bijection

$$G/H \ni aH \overset{\pi_x}{\longleftrightarrow} a \cdot x \in X,$$

i.e., $\kappa_x = \pi_x \circ \kappa$. As the action of G is continuous and because of the definition of the topology on G/H it follows that π_x is continuous. Thus $G/H \simeq X$ if and only if π_x is open. This *in general* is not the case. Notice that π_x is open if and only if κ_x is open. We show that π_x is a homeomorphism under the assumptions that G is a *locally compact Lindelöff space* and X is *locally compact.*

Recall a topological space X is **locally compact** if every point x has a compact neighborhood. It is a **Lindelöff space** if every open covering $\{U_i\}_{i\in I}$ of X has a countable subcovering, i.e., there exists a countable subset $J \subset I$ with $X = \cup_{j\in J} U_j$.

We use the following standard results from topology. Let X and Y be topological spaces and suppose $K \subset X$ is compact. Then the following hold:

- If $L \subset K$ is closed, then L is compact.
- If $f : X \to Y$ is continuous, then $f(K)$ is compact in Y.
- If X is Hausdorff, then K is closed.

We will use the the following lemma. An easier proof can be given when the space is a topological group.

Lemma 5.14. *Let X be a locally compact Hausdorff space. Suppose U is a neighborhood of a point x in X. Then there is a compact neighborhood W of x with $W \subseteq U$.*

Proof. We may assume U is open. Let K be a compact neighborhood of x. Since $K - U$ is closed, it is compact. For each y in $K - U$, choose open neighborhoods $N(y)$ and $N_y(x)$ of y and x with $N(y) \cap N_y(x) = \varnothing$ and $N_y(x) \subseteq U \cap K$. Then $N(y)$ for $y \in K - U$ cover $K - U$. By compactness, there are $y_1, y_2, \ldots, y_m$ with $K - U \subseteq \cup_{j=1}^m N(y_j)$. Set $W = \cap \overline{N_{y_j}(x)}$. Then W is a closed neighborhood of x. Note if $z \in W$ and $z \notin U$, then $z \notin N(y_j)$ for each j. So $z \notin K - U$. Since $z \notin U$, $z \notin K$. So $X - K$ is an open neighborhood of z missing K. Since each $N_{y_j}(x) \subseteq K$, $z \notin \overline{N_{y_j}(x)}$ for each j. Thus $z \notin W$. But we are assuming $z \in W$. Hence $W \subseteq U$. Since each $N_{y_j}(x) \subseteq K$, $W \subseteq K$. Hence W is compact. $\square$

For the proof of our next theorem we need the Baire category theorem. A set is **nowhere dense** if its closure has no interior.

Baire Category Theorem: *A complete metric space or a locally compact Hausdorff space is* **second category***; i.e., it cannot be written as a countable union of nowhere dense subsets.*

Sets which are countable unions of nowhere dense sets are said to be of **first category**. They are also called **meagre**. An obvious consequence of the definition is that a closed set A with no interior is of first category. Furthermore, a countable union of sets of first category is of first category. Note there are Cantor like sets in $\mathbb{R}$ that are closed, have no interior, but can have positive Lebesgue measure.

Theorem 5.15. *Let G be a locally compact Hausdorff topological group and let X be a locally compact Hausdorff space or a complete metric space. Assume that G acts separately continuously and transitively on X and that G is a Lindelöff space. Then $\pi_x : G/G_x \to X$ is a homeomorphism for each $x \in X$ and G acts continuously on X.*

Proof. Let $x \in X$. We only have to show that κ_x is an open mapping. Let $H = G_x$ and U be an open subset of G. Let $g \in U$ and $y = \kappa_x(g) = g \cdot x \in \kappa_x(U)$. Let V be a compact symmetric neighborhood of e with $gV^2 \subset U$. We have $G = \cup_{a \in G} aV^\circ$. Since G is a Lindelöff space, there is a countable set J such that $G = \cup_j a_j V^\circ$. As the map κ_x is surjective, it follows that

$$X = \bigcup_{j \in J} \kappa_x(a_j V) = \bigcup_{j \in J} a_j \cdot \kappa_x(V).$$

The continuity of κ_x implies each of the sets $\kappa_x(a_j V)$ is compact. Since X is Hausdorff, they are closed. By the Baire Category Theorem, there is a j such that $\kappa_x(a_j V)^\circ \neq \varnothing$. The maps $\lambda(a^{-1}) : X \to X$ being homeomorphisms imply that $\kappa_x(V)^\circ \neq \varnothing$. Let $z \in \kappa_x(V)^\circ$. Then $z = v \cdot x$ for some $v \in V$. Hence $x = v^{-1} \cdot z$ is an interior point in $\lambda(v^{-1})\kappa_x(V)$. Consequently, $y = g \cdot x$ is interior to $\lambda(gv^{-1})\kappa_x(V)$. But $\lambda(gv^{-1})\kappa_x(V) \subseteq \kappa_x(gV^2) \subseteq \kappa_x(U)$. Thus y is an interior point in $\kappa_x(U)$. As y was an arbitrary point in $\kappa_x(U)$, it follows that $\kappa_x(U)$ is open and hence κ_x is an open map. □

4. Separation in Topological Groups

Recall that a space is T_0 or *Kolomorgoroff* if given two distinct points, there is an open set U containing exactly one of these points. We have already used T_1-separability. This occurs if every set with one element is closed. A space which is T_1 is said to be *Fréchet*. A space is T_2 if it is Hausdorff. A space is *regular* if given any closed set A and a point x not in A, there are open sets U and V with $A \subseteq U$, $x \in V$, and $U \cap V = \varnothing$. A space which is both regular and T_1 is said to be a T_3-space. A space is *completely regular* if for a given closed set A and a point x not in A, there is a continuous real valued function f with $f(x) = 1$ and $f = 0$ on A. If a space is completely regular and T_1, then it is said to be a T_{3a}-*space*. Such spaces are also called *Tichonov*. Finally a space is said to be *normal* if given two disjoint closed sets A and B, there are open sets U and V with $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$. It is a T_4-*space* if it is normal and T_1.

Our aim in this section is to show that any T_0 topological group is completely regular. We start with two simple consequences of Lemma 5.13.

Lemma 5.16. *Let N be a closed normal subgroup of a topological group G. Then G/N is a Hausdorff topological group.*

Proof. By Lemma 5.13, G/N is Hausdorff and a continuous G-space. In particular the mapping $(g, xN) \mapsto gxN$ is continuous. This implies the mapping $(gN, xN) \mapsto gxN$ is continuous on $G/N \times G/N$. Finally, $xN \mapsto x^{-1}N$ is continuous for the preimage of an open set U in G/N under this mapping is U^{-1} and thus $\kappa^{-1}(U^{-1}) = \kappa^{-1}(U)^{-1}$ is open in G. □

Lemma 5.17. *Let G be a topological group. Then the following are equivalent:*

(a) *G is T_0.*

(b) *The set $\{e\}$ is closed in G.*

(c) *G is T_1.*

(d) *G is Hausdorff.*

Proof. Assume G is T_0. We show $\{e\}$ is a closed set. By Lemma 5.13, since G and $G/\{e\}$ have the same topology, it then would follow that G is Hausdorff and we would be done.

Take $x \neq e$. Pick an open set U containing precisely one of the two points x and e. If $e \in U$, replace U by the open set xU^{-1}. Then U would contain x but not e. Consequently, x could not be a limit point of $\{e\}$, and the set $\{e\}$ would be closed. □

The positive dyadic rationals are all rationals of form $\frac{n}{2^m}$ where n is a natural number and m is an integer.

Lemma 5.18. *Let G be a group. Suppose for each $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $U(\frac{1}{2^n})$ is a nonempty subset containing the identity and*

$$U(\frac{1}{2^n})U(\frac{1}{2^n}) \subseteq U(\frac{1}{2^{n-1}}) \text{ for } n \geqslant 1.$$

Then there is an extension of U to the positive dyadic rationals with the following properties:

(a) $U(\frac{1}{2^n})U(\frac{k}{2^n}) \subseteq U(\frac{k+1}{2^n})$ *for all integers n and natural numbers k.*

(b) $U(r) \subseteq U(s)$ *if* $r \leqslant s$.

(c) $U(r) = G$ *if* $r > 1$.

Proof. Define $U(r) = G$ for all dyadic rationals $r > 1$. So (c) holds. We now give an inductive definition for $U(\frac{m}{2^n})$ when $n \geqslant 1$. Suppose $U(\frac{k}{2^n})$ has been defined for all k for all $n < m$.

Then $U(\frac{k}{2^m})$ is already defined if k is even, $k = 1$, or $k \geqslant 2^m$. If $k = 1 + 2l < 2^m$ where $l > 0$, define $U(\frac{k}{2^m})$ by

$$U(\frac{k}{2^m}) = U(\frac{1}{2^m})U(\frac{l}{2^{m-1}}).$$

Thus $U(r)$ is defined for all positive dyadic rationals r.

We show (a) by induction on n. Note if $n \leqslant 1$, (a) is clear. Hence assume (a) is true for all $n < m$ where $m > 1$. First note from the definition that $U(\frac{k+1}{2^m}) \supseteq U(\frac{1}{2^m})U(\frac{k}{2^m})$ if k is even or $k + 1 > 2^m$. Hence we may assume k is odd and $k + 1 \leqslant 2^m$. Then $k + 1 = 2l$ where $l \geqslant 1$. If $k = 1$, we have by hypothesis that $U(\frac{1}{2^m})U(\frac{1}{2^m}) \subseteq U(\frac{1+1}{2^m})$. Thus we may assume $l \geqslant 2$. Hence

$$\begin{aligned}
U(\frac{1}{2^m})U(\frac{2l-1}{2^m}) &= U(\frac{1}{2^m})U(\frac{2(l-1)+1}{2^m}) \\
&= U(\frac{1}{2^m})U(\frac{1}{2^m})U(\frac{l-1}{2^{m-1}}) \\
&\subseteq U(\frac{1}{2^{m-1}})U(\frac{l-1}{2^{m-1}}) \\
&\subseteq U(\frac{1}{2^{m-1}} + \frac{l-1}{2^{m-1}}) \text{ by induction} \\
&= U(\frac{l}{2^{m-1}}) \\
&= U(\frac{2l}{2^m}) \\
&= U(\frac{1}{2^m} + \frac{k}{2^m}).
\end{aligned}$$

Thus (a) holds.

Using (a) and $e \in U(s)$ for all s, one has $U(\frac{m}{2^n}) \subseteq U(\frac{1}{2^n})U(\frac{m}{2^n}) \subseteq U(\frac{m+1}{2^n})$ for any natural number m. Repeating, we see $U(\frac{m}{2^n}) \subseteq U(\frac{l}{2^n})$ whenever $m < l$. This implies (b). □

Definition 5.19. *A function $f : G \to \mathbb{C}$ is* ***uniformly continuous from the right*** *if for each $\epsilon > 0$ there is a $U \in \mathcal{N}(e)$ such that*

$$|f(x) - f(y)| < \epsilon$$

for all $x, y \in G$ with $x^{-1}y \in U$. f is ***uniformly continuous from the left*** *if for each $\epsilon > 0$ there is a $U \in \mathcal{N}(e)$ such that*

$$|f(x) - f(y)| < \epsilon$$

for all $x, y \in G$ with $xy^{-1} \in U$. f is ***uniformly continuous*** *if it is uniformly continuous from the left and the right.*

Notice that a function that is uniformly continuous from the left or the right is obviously continuous.

Lemma 5.20. *Let U_n, $n \in \mathbb{N}_0$, be nonempty open symmetric sets in $\mathcal{N}(e)$ with $U_{n+1}^2 \subset U_n$. Let H be a subgroup of G. Then there exists a $f \in C(G)$ such that:*

(a) $f(G) \subset [0, 1]$.

(b) $f(e) = 0$;

(c) *If $a^{-1}b \in H$, then $f(a) = f(b)$.*

(d) *If $n \in \mathbb{N}_0$ and $f(x) < 2^{-n}$, then $x \in U_n H$.*

(e) *f is uniformly continuous from the left.*

Proof. Let D be the set of positive dyadic rationals. We can find sets $U(r)$ with $U(\frac{1}{2^n}) = U_n$ which satisfy Lemma 5.18. The construction in the Lemma shows we may assume the sets $U(r)$ are open neighborhoods of e.

For $x \in G$ let $F_x = \{r \in D \mid x \in U(r)H\}$. Define $f(x)$ by $f(x) = \inf F_x$. Clearly $f(x) \geqslant 0$ and since $U(r) = G$ for $r > 1$, we have $f(x) \leqslant 1$. As $e \in U(r)$ for all r it follows that $f(e) = 0$. If $aH = bH$ then $F_a = F_b$ and thus $f(a) = f(b)$. Suppose now that $f(x) < 2^{-n}$. Then there exists a $s < 2^{-n}$ such that $x \in U(s)H$. By (b) of Lemma 5.18, we see $x \in U(2^{-n})H = U_n H$. Thus (a), (b), (c), and (d) hold.

We show (e). Let $\epsilon > 0$. Choose n such that $2^{-n} < \epsilon$. Assume that $x, y \in G$ are such that $xy^{-1} \in U_n$. Hence $x \in U(\frac{1}{2^n})y$. Now suppose $r \in F_y$. Then $y \in U(r)H$. Choose k with $\frac{k-1}{2^n} < r \leqslant \frac{k}{2^n}$. By using (a) of Lemma 5.18, $x \in U(\frac{1}{2^n})U(\frac{k}{2^n})H \subseteq U(\frac{1+k}{2^n})H$. Hence $f(x) \leqslant r + \frac{1}{2^n}$ for all $r \in F_y$. Consequently, $f(x) \leqslant \inf F_y + \frac{1}{2^n} = f(y) + \frac{1}{2^n}$. But if $xy^{-1} \in U_n$, then $yx^{-1} \in U_n^{-1} = U_n$, and we see $f(y) \leqslant f(x) + \frac{1}{2^n}$. Hence

$$|f(x) - f(y)| < \epsilon \text{ if } xy^{-1} \in U_n$$

and f is left uniformly continuous. □

Theorem 5.21. *Let H be a subgroup of G. Then G/H is completely regular. If H is closed in G, then G/H is a T_{3a}-space.*

Proof. Let $A \subset G/H$ be closed and let $x = aH \in G/H$ be a point not in A. We have to show that there is a continuous function $g : G/H \to [0,1]$ such that $g|A = 0$ and $g(x) = 1$. Let $U := a^{-1}(G - \kappa^{-1}(A))$. Note U is an open neighborhood of e. By continuity of multiplication and inversion, there is a sequence U_n, $n = 0, 1, 2, \ldots$, of open symmetric neighborhoods of e satisfying $U_0 \subseteq U$ and $U_{n+1}^2 \subseteq U_n$. Let $f : G \to [0,1]$ be the function obtained in Lemma 5.20. Note $f(e) = 0$ and if $b \notin U$, then $b \notin U(1) = U_0$. This implies $f(b) = 1$ for $b \notin U$. Define $g(bH) = 1 - f(a^{-1}b)$. This function is well defined by (c) of Lemma 5.20. Moreover, since $b \mapsto 1 - f(a^{-1}b)$ is continuous on G and is constant on cosets bH, g is continuous. Note $g(aH) = 1 - f(e) = 1$; and if $y \in A$, then $y = bH$ where $a^{-1}b \notin U$. Consequently, $g(y) = 1 - f(a^{-1}b) = 0$. We thus see G/H is a completely regular space.

If H is closed, then (c) of Lemma 5.13 shows G/H is T_2. In particular G/H is T_1 and completely regular. So it is $\mathrm{T}_{3\mathrm{a}}$. □

Corollary 5.22. *Let G be a topological group. Then G is completely regular. Furthermore G is a T_{3a}-space if and only if $\{e\}$ is closed in G.*

If f is a continuous real or complex valued function on a topological space X, then the **support** of the function f is the closure of $\{x \mid f(x) \neq 0\}$. It is denoted by $\mathbf{supp(f)}$. The space of continuous functions on X with compact support is defined by

$$C_c(X) = \{f \mid f \text{ is continuous and } \operatorname{supp}(f) \text{ is compact}\}.$$

Proposition 5.23. *Let X be locally compact Hausdorff space. Suppose K is a compact subset of an open subset V of X. Then there is a function $f \in C_c(X)$ such that $0 \leqslant f \leqslant 1$, $f(x) = 1$ if $x \in K$, and $\operatorname{supp} f \subseteq V$. If K is a G_δ, f may be chosen with $f^{-1}(1) = K$.*

Proof. We repeatedly use the result in Exercise 5.2.14. We start by choosing an open subset V_1 containing K whose closure is a compact subset of V. Then choose an open subset $V_{\frac{1}{2}}$ containing K whose closure is a compact subset of V_1. Then choose open subsets $V_{\frac{1}{4}}$ and $V_{\frac{3}{4}}$ having compact closures satisfying

$$K \subseteq V_{\frac{1}{4}} \subseteq \bar{V}_{\frac{1}{4}} \subseteq V_{\frac{1}{2}} \subseteq \bar{V}_{\frac{1}{2}} \subseteq V_{\frac{3}{4}} \subseteq \bar{V}_{\frac{3}{4}} \subseteq V.$$

Repeating in this manner, we can for each positive dyadic rational $d = \frac{m}{2^n}$ where $1 \leqslant d \leqslant 2^n$, obtain a family of precompact open subsets V_d satisfying

$$K \subseteq V_d \subseteq \bar{V}_d \subseteq V_{d'} \subseteq \bar{V}_{d'} \subseteq V_1 \subseteq V$$

if $d < d'$. When K is a G_δ, we can write $K = \cap_n W_n$ where W_n is a decreasing sequence of open subsets of V. In this case, the choices for $V_{\frac{1}{2^n}}$ can be made with the requirement that $V_{\frac{1}{2^n}} \subseteq W_n$.

Now using Exercise 5.2.13, the function $f(x) = 1 - \inf\{d \mid x \in V_d\}$ has the desired properties. □

Lemma 5.24. *Let G be a locally compact Hausdorff group. Suppose $f \in C_c(G)$. Then f is uniformly continuous.*

Proof. It suffices to show it is left uniformly continuous. Let K be the compact support of f. For each $x \in G$, choose an open neighborhood N_x of e such that $|f(y) - f(x)| < \frac{\epsilon}{2}$ if $y \in N_x x$. Then choose an open neighborhood W_x of e with $W_x^{-1} W_x \cup W_x^2 \subseteq N_x$. The $W_x x$ cover G and hence K. So there are $x_1, \ldots, x_n$ with $K \subseteq \cup_{k=1}^n W_{x_k} x_k$. Take $U = \cap_{k=1}^n W_{x_k}$. Now let $x \in G$ and $y \in U$. If $x \notin K$ and $yx \notin K$, then $|f(yx) - f(x)| = 0$. Suppose $x \notin K$ and $yx \in K$. Then $yx \in W_{x_i} x_i$ for some i. Thus $x \in W_{x_i}^{-1} W_{x_i} x_i$. So both yx and x are in $N_{x_i} x_i$. Thus

$$|f(yx) - f(x)| \leqslant |f(yx) - f(x_i)| + |f(x_i) - f(x)| < \epsilon.$$

Similarly, if $x \in K$ and $y \in U$, then $x \in W_{x_i} x_i \subseteq N_{x_i} x_i$ for some i and $yx \in W_{x_i}^2 x_i \subseteq N_{x_i} x_i$. Hence

$$|f(yx) - f(x)| \leqslant |f(yx) - f(x_i)| + |f(x_i) - f(x)| < \epsilon.$$

□

Exercise Set 5.2

1. Let X be a topological space. Show the following:

(a) $T_4 \Rightarrow T_3$;

(b) $T_3 \Rightarrow T_2$;

(c) Completely regular $\Rightarrow$ regular.

2. Show any subspace of a Lindelöf space is Lindelöf.

3. Suppose that G is a topological group and H a subgroup. Let τ be the collection of all sets $U \subset G/H$ such that $\kappa^{-1}(U) \subset G$ is open in G. Show that τ is a topology on G/H.

4. Let H be a subgroup of a topological group G, and let X be a topological space.

(a) Show that a function $f : G/H \to X$ is continuous if and only if the map $f \circ \kappa : G \to X$ is continuous.

(b) Suppose a function f maps G/H onto X. Show f is open if and only if the map $f \circ \kappa : G \to X$ is open.

(c) Assume that G acts separately continuously and transitively on X. Show that the following are equivalent:
- There exists a $x \in X$ such that κ_x is open;
- There exists a $x \in X$ such that $\pi_x : G/G_x \to X$ is a homeomorphism;
- κ_x is open for all $x \in X$;
- $\pi_x : G/G_x \to X$ is a homeomorphism for all $x \in X$.

5. Let $G = \mathbb{R}$, and let τ be an irrational number.

(a) Show $G = \mathbb{Z} + \mathbb{Z}\tau$ is dense in $\mathbb{R}$. (**Hint:** Show G has no smallest positive member.)

(b) Show that G acts continuously on $\mathbb{T}^2$ by

$$x \cdot (e^{iu}, e^{iv}) := (e^{i(u+x)}, e^{i(v+\tau x)}) \text{ for } x, u, v \in \mathbb{R}\,.$$

(c) Let $z \in \mathbb{T}^2$. Show $G_z = \{0\}$ and $G \cdot z$ is dense in $\mathbb{T}^2$.

(d) Show $\kappa_z : x \mapsto x \cdot z$ is not an open mapping.

6. Prove the basic topological statements highlighted on page 256.

7. Suppose G is a topological group and X and Y are G-spaces. Assume that $\varphi : X \to Y$ is an G-isomorphism. Show that $\varphi^{-1} : Y \to X$ is also a G-isomorphism.

8. Let G be a topological group and let X be a topological G-space. Set $G\backslash X = \{Gx \mid x \in X\}$ to be the space of orbits. Let $\pi : X \to G\backslash X$ by $\pi(x) = Gx$. Define a set U in $G\backslash X$ to be open if $\pi^{-1}(U)$ is open in X.

(a) Show the set of open sets is a topology on X. It is called the *quotient topology.*

(b) Show $\pi : X \to G\backslash X$ is an open mapping.

(c) Give an example where G and X are Hausdorff, but $G\backslash X$ is not Hausdorff.

(d) Suppose G and X are locally compact Hausdorff spaces. Show $G\backslash X$ is compact if and only if there is a compact set $C \subseteq X$ with $GC = X$.

9. Let G be a topological group and let H be a subgroup with the relative topology.

(a) Show if H is locally compact and Hausdorff, then H is closed in G. In particular if H is discrete in the relative topology, then H is closed.

(b) Give an example of a discrete subspace M of some topological space X which is not closed.

10. Suppose G is a locally compact Hausdorff group and K is a compact subgroup. Show if A is compact in G/K, then $\kappa^{-1}(A)$ is compact in G.

11. Let G be a compact Hausdorff group. Show a subgroup D has discrete relative topology if and only if D if finite.

12. Let D be a subgroup of G where G is locally compact and Hausdorff. Show D is discrete if and only if $D \cap K$ is finite for every compact subset of G.

13. A dyadic rational is a number of form $\frac{k}{2^n}$. Consider a topological space X, and let F be a closed set and G be an open set with $F \subseteq G$. Suppose V_d is an open set for each $d \in D$, $V_0 = \varnothing$, $V_1 = X$, $F \subseteq V_d \subseteq G$ when $0 < d < 1$, and

$$\bar{V}_d \subseteq V_{d'}$$

for $d < d'$ in D. Show the function f defined by $f(x) = \inf\{d \mid x \in V_d\}$ is a continuous function that satisfies $0 \leqslant f \leqslant 1$, $f = 0$ on F, and $f = 0$ off G. Moreover, if $\cap_{d>0} G_d = F$, show $f(x) = 0$ if and only if $x \in F$.

14. Let X be a locally compact Hausdorff space and let V be an open set containing compact subset K. Show there is a an open subset G of V with compact closure such that $K \subseteq \bar{G} \subseteq V$.

15. Let X be a σ-compact locally compact Hausdorff space. Show there is a sequence U_n of precompact open subsets which cover X and has the property $\bar{U}_n \subseteq U_{n+1}$.

16. Let X be a locally compact Hausdorff space. Show a subset K is a compact G_δ set if and only if there is a continuous function $0 \leqslant \phi \leqslant 1$ on X having compact support with $\phi^{-1}\{1\} = K$.

5. Examples

In this section we give some examples of topological groups and homogeneous spaces. We have met some of them before. Others play important roles in harmonic analysis.

5.1. The general linear group GL(n). Let $\mathbb{F}$ be the field of real or complex numbers. Elements in $\mathbb{F}^n$ will be written as column vectors. Let $M(m \times n, \mathbb{F}) = \left\{(x_{ij})_{i,j=1}^{m,n} \mid x_{ij} \in \mathbb{F}\right\}$ be the space of m by n matrices. If $n = m$, we simply write $M(n, \mathbb{F})$. $M(m \times n, \mathbb{F})$ is linearly isomorphic to $\mathbb{F}^{mn}$ under the mapping

$$X = [x_{i,j}] \mapsto (x_{1,1}, x_{1,2}, \ldots, x_{1,n}, x_{2,1}, x_{2,2}, \ldots, x_{2,n}, \ldots, \ldots, x_{m,n})^t.$$

$\mathbb{F}^{mn}$ is a product space with the product topology. Using the above isomorphism we see there is a natural ‘product’ topology on the vector space

$M(m \times n, \mathbb{F})$. By Exercise 5.3.1, this topology is obtained from the norm defined by:

$$||X||_2 = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |x_{ij}|^2 \right)^{1/2}$$

The determinant function $\det : M(n, \mathbb{F}) \to \mathbb{F}$,

$$X \mapsto \det(X) = \sum_{\sigma \in \Gamma_n} \operatorname{sign}(\sigma) x_{1\sigma(1)} x_{2\sigma(2)} \cdots x_{n\sigma(n)}$$

is a polynomial in the coordinates and hence continuous. It follows that

$$\mathrm{GL}(n, \mathbb{F}) := \{X \in M(n, \mathbb{F}) \mid \det(X) \neq 0\}$$

is open and dense in $M(n, \mathbb{F})$. As $X \in M(n, \mathbb{F})$ is invertible if and only if $\det(X) \neq 0$, $\det(XY) = \det(X)\det(Y)$, and $\det(X^{-1}) = \det(X)^{-1}$, it follows that $\mathrm{GL}(n, \mathbb{F})$ is a group. It is called *the general linear group*. Let $X = (x_{ij})$ and $Y = (y_{ij})$. Note

$$(XY)_{ij} = \sum_{\nu=1}^{n} x_{i\nu} y_{\nu j}.$$

Hence multiplication is continuous. Since $X_{ij}^{-1} = (-1)^{i+j} \frac{\det(M_{ji})}{\det(X)}$, where M_{ij} is the i, j^{th} minor, we see $X \mapsto X^{-1}$ is continuous. Hence $\mathrm{GL}(n, \mathbb{F})$ is a topological group. Since $\det : \mathrm{GL}(n, \mathbb{F}) \to \mathbb{F}$ is a continuous homomorphism, the kernel

$$\mathrm{SL}(n, \mathbb{F}) := \{A \in \mathrm{GL}(n, \mathbb{F}) \mid \det(A) = 1\}$$

is a closed normal subgroup of $\mathrm{GL}(n, \mathbb{F})$ and hence is a locally compact Hausdorff topological group. It is called the *special linear group*.

The group $\mathrm{SL}(2, \mathbb{F})$ acts continuously on $\mathbb{F}^2$ by $(A, x) \mapsto Ax$. The set $\{0\}$ is obviously an orbit. Let $x = (x_1, x_2)^t \in \mathbb{F}^2 \setminus \{0\}$. If $A = (a_1, a_2)$ with column vectors $a_1, a_2 \in \mathbb{F}^2$, then $Ae_1 = a_1$. If $x_1 \neq 0$, let

$$A = \begin{pmatrix} x_1 & 0 \\ x_2 & 1/x_1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{F}).$$

If $x_1 = 0$, then $x_2 \neq 0$. Let

$$A = \begin{pmatrix} 0 & -1/x_2 \\ x_2 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{F})$$

Then $Ae_1 = x$. It follows that $\mathbb{F}^2 \setminus \{0\} = \mathrm{SL}(2, \mathbb{F})e_1$ is one orbit. Hence $\mathbb{F}^2$ decomposes into exactly two orbits. Assume that $Ae_1 = e_1$. Then A must have the form

$$A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ where } x \in \mathbb{F}.$$

It follows that

$$G_{e_1} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F} \right\} =: N$$

and $\mathbb{F}^2 \backslash \{0\} \simeq \mathrm{SL}(2,\mathbb{F})/N$. Note Theorem 5.15 shows this is a topological isomorphism for $\mathrm{SL}(2,\mathbb{F})$ is a locally compact Lindelöf space (see Exercise 5.3.4) and $\mathbb{F}^2 \backslash \{0\}$ is locally compact.

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{C})$ and $z \in \mathbb{C}$ let

$$A \cdot z = \frac{az+b}{cz+d}$$

if $az + d \neq 0$. Let

$$H_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}\,.$$

Lemma 5.25. *Let $A \in \mathrm{SL}(2,\mathbb{R})$ and $z \in H_+$. Then $A \cdot z$ is well defined, and $A \cdot z \in H_+$. Furthermore $\mathrm{SL}(2,\mathbb{R}) \times H_+ \ni (A,z) \mapsto A \cdot z \in H_+$ defines a continuous action of $\mathrm{SL}(2,\mathbb{R})$ on H_+. The action is transitive and* $\mathrm{SL}(2,\mathbb{R})_i = \mathrm{SO}(2,\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$.

Proof. Write $z = x + iy \in H_+$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$. Then $cz + d = (cx + d) + icy$. If $cy = 0$ then $c = 0$ as $y > 0$. But then $d \neq 0$ and $cz + d = d \neq 0$. Furthermore

$$\mathrm{Im}\, A \cdot z = \frac{y}{(cx+d)^2 + c^2y^2} > 0\,.$$

Hence $A \cdot z \in H_+$. As $(A,z) \mapsto A \cdot z$ is a rational function in a, b, c, d, and z, it follows that $(A,z) \mapsto A \cdot z$ is continuous. We refer to the Exercise 5.3.21 to see that $(AB) \cdot z = A \cdot (B \cdot z)$.

Let $x + iy \in H_+$. Then $y > 0$. Let $A = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$. Then

$$A \cdot i = \frac{i\sqrt{y} + x/\sqrt{y}}{1/\sqrt{y}} = x + iy\,.$$

Hence the action is transitive. Assume that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot i = \frac{ai+b}{ci+d} = i\,.$$

Then

$$ai + b = di - c\,.$$

Thus $a = d$ and $b = -c$. As $\det A = ad - bc = a^2 + b^2 = 1$ it follows that we can write $a = \cos(\theta)$ and $b = \sin(\theta)$ for some $\theta \in \mathbb{R}$. $\square$

Let A be the subgroup of $SL(2,\mathbb{R})$ consisting of all matrices of form $\begin{pmatrix} x & 0 \\ 0 & \frac{1}{x} \end{pmatrix}$ where $x > 0$ and N be the subgroup of $\mathrm{SL}(2,\mathbb{R})$ of all matrices of form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $x \in \mathbb{R}$. The argument just given shows for each $z \in H_+$, there are unique $a \in A$ and $n \in N$ with $(an) \cdot i = z$. Exercise 5.3.22 shows the subgroup AN is topologically isomorphic to H_+.

Corollary 5.26. *The mapping* $\mathrm{SO}(2,\mathbb{R}) \times A \times N \ni (k,a,n) \mapsto kan$ *is a homeomorphism onto* $\mathrm{SL}(2,\mathbb{R})$.

Proof. Clearly $(k,a,n) \mapsto kan$ is continuous. Let $z = g^{-1} \cdot i$. Then z is a continuous function of g and since H_+ is homeomorphic to AN by the correspondence $(an) \cdot i = z$, there are unique functions a and n depending continuously on g such that $(a(g)n(g)) \cdot i = g^{-1} \cdot i$. Hence $g(a(g)n(g))^{-1} = k(g) \in \mathrm{SO}(2,\mathbb{R})$ where $g \mapsto k(g)$ is continuous. Thus the inverse mapping $g \mapsto (k(g), a(g), n(g))$ is continuous. □

5.2. The classical linear groups. There are many closed subgroups of the general linear group. Those most important to us are the *classical linear groups*. We list some of these here. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let V be a vector space over $\mathbb{F}$.

Definition 5.27. *A map* $\beta : V \times V \to \mathbb{F}$ *is*

(a) ***bilinear*** *if for each fixed* $u \in V$ *the maps*

$$V \ni x \mapsto \beta(x,u) \in \mathbb{F} \quad \text{and} \quad V \ni x \mapsto \beta(u,x) \in \mathbb{F}$$

are linear;

(b) ***sesquilinear*** *if for each fixed* $u \in V$ *the maps*

$$V \ni x \mapsto \beta(x,u) \in \mathbb{F} \quad \text{and} \quad V \ni x \mapsto \overline{\beta(u,x)} \in \mathbb{F}$$

are linear;

(c) ***symmetric*** *if it is bilinear and* $\beta(u,v) = \beta(v,u)$ *for all* $u, v \in V$*;*

(d) ***Hermitian*** *if it is sesquilinear and* $\beta(u,v) = \overline{\beta(v,u)}$ *for all* $u, v \in V$*;*

(e) ***skew-symmetric*** *if it is bilinear and* $\beta(u,v) = -\beta(v,u)$ *for all* $u, v \in V$*;*

(f) ***skew-Hermitian*** *if it is sesquilinear and* $\beta(u,v) = -\overline{\beta(v,u)}$ *for all* $u, v \in V$*;*

(g) ***non-degenerate*** *if for each* $u \neq 0$ *there exists a* $v \in V$ *such that* $\beta(u,v) \neq 0$ *and for each* $v \neq 0$ *there is a vector* u *with* $\beta(u,v) \neq 0$.

(h) ***positive*** *if* $\beta(u,u) > 0$ *for all* $u \in V$, $u \neq 0$.

(i) *an* ***inner product*** *on* V *if* β *is Hermitian and positive.*

If $\mathbb{F} = \mathbb{R}$, we prefer to use the terms sesquilinear, Hermitian, and skew-Hermitian even though these are same as the more descriptive terms bilinear, symmetric, and skew-symmetric. This is done so that we do not have to make the distinction between $\mathbb{R}$ and $\mathbb{C}$. Notice also that if $\mathbb{F} = \mathbb{C}$ and β is sesquilinear, then

$$\begin{aligned}\beta(u, \lambda x + y) &= \overline{\overline{\beta(u, \lambda x + y)}}\\ &= \overline{\lambda\overline{\beta(u, x)} + \overline{\beta(u, y)}}\\ &= \bar{\lambda}\beta(u, x) + \beta(u, y)\,.\end{aligned}$$

Hence β is *conjugate linear* in the second variable.

If V is a Hilbert space and β is either sesquilinear or bilinear, define

$$||\beta|| := \sup_{u,v \in V,\, ||u|| = ||v|| = 1} |\beta(u, v)|\,.$$

We say that β is bounded if $||\beta|| < \infty$. In that case $||\beta||$ is called norm of β. Note β satisfies:

$$|\beta(u, v)| \leqslant ||\beta||\,||u||\,||v|| \text{ for all } u, v \in V \tag{5.1}$$

Proposition 5.28. *Assume that V is a Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$) with inner product $(\cdot\,,\cdot)$ and corresponding norm $||\cdot||$. Suppose that β is a bounded sesquilinear form on V. Then there exists a bounded linear transformation $T : V \to V$ such that*

$$\beta(u, v) = (Tu, v)$$

for all $u, v \in V$. Furthermore $||\beta|| = ||T||$. T is injective with dense range if and only if β is non-degenerate. We have $\beta(u, v) = \overline{\beta(v, u)}$, respectively $\beta(u, v) = -\overline{\beta(v, u)}$, for all $u, v \in V$, if and only if T is self adjoint, i.e., $T^* = T$, respectively skew-adjoint, $T^* = -T$.

Proof. For each $u \in V$, define a linear mapping f_u by

$$f_u(v) = \overline{\beta(u, v)}.$$

Note $|f_u(v)| \leqslant ||\beta||\,||u||\,||v||$. By the Riesz Representation Theorem, there is a unique vector $Tu \in V$ with $\overline{\beta(u, v)} = f_u(v) = (v, Tu)$ for all u. Taking conjugates gives

$$\beta(u, v) = (Tu, v)$$

for all u and v. T is linear in u for $\beta(u, v)$ is linear in u. Exercise 5.3.5 shows $||\beta|| = ||T||$.

Suppose T is injective with dense range. If $u \neq 0$, then $Tu \neq 0$ and thus $\beta(u, Tu) = (Tu, Tu) \neq 0$. Also if $v \neq 0$, then since T has dense range, there is a u with $\beta(u, v) = (Tu, v) \neq 0$. Thus β is nondegenerate. Conversely, let β be nondegenerate. If $Tu = 0$, then $\beta(u, v) = 0$ for all v. So $u = 0$ and T

is injective. If $v \in T(V)^\perp$, then $\beta(u,v) = 0$ for all u. But then $v = 0$. So $T(V)^\perp = \{0\}$ and thus $T(V)$ is dense.

Finally β is Hermitian if and only if $(Tu,v) = \overline{(Tv,u)} = (u,Tv)$ for all u and v if and only if $T^* = T$. Similarly β is skew-Hermitian if and only if $T^* = -T$. □

Definition 5.29. *Let V be a complex Hilbert space. A continuous conjugate linear transformation j of V satisfying $j^2 = I$ is called a* ***conjugation*** *on V.*

Let j be a conjugation on V. Then $V_\mathbb{R} := \{v \in V \mid j(v) = v\}$ is a real Hilbert space such that $V_\mathbb{R} \oplus iV_\mathbb{R} = V$ and $iV_\mathbb{R} = \{v \in V \mid j(v) = -v\}$. Indeed, $v = v_1 + v_2$ where $v_1 = \frac{1}{2}(v + jv) \in V_\mathbb{R}$ and $v_2 = \frac{1}{2}(v - jv) = -\frac{i}{2}(iv + jiv) \in iV_\mathbb{R}$. On the other hand any decomposition $V = V_\mathbb{R} \oplus iV_\mathbb{R}$ with $V_\mathbb{R}$ a real sub-Hilbert space gives rise to a conjugation by defining j to be the identity on $V_\mathbb{R}$ and $-\mathrm{id}$ on $iV_\mathbb{R}$. Note this decomposition need not be orthogonal; see Exercise 5.3.10.

Lemma 5.30. *Let V be a Hilbert space. Then there exists a conjugation $j : V \to V$.*

Proof. Let $\{e_m\}_{m\in I}$ be a orthonormal basis for V. Define $j : V \to V$ by

$$j\left(\sum \lambda_m e_m\right) := \sum \bar{\lambda}_m e_m\,.$$

Then j is conjugate linear and $j^2 = I$. It is continuous since it is an isometry. □

As an example, the usual conjugation on $\mathbb{C}^n$ is the mapping $j : \mathbb{C}^n \to \mathbb{C}^n$ given by

$$j((z_1, \ldots, z_n)^t) = (\bar{z}_1, \ldots, \bar{z}_n)^t.$$

In the above construction, it comes from the standard basis $e_1 = (1, 0, \ldots, 0)^t$, $\ldots$, $e_n = (0, 0, \ldots, 0, 1)^t$ of $\mathbb{C}^n$. Not all conjugations on a Hilbert space are obtained from an orthonormal basis as in the proof of this Lemma. See Exercise 5.3.7.

Lemma 5.31. *Let j be a conjugation on a Hilbert space V. Define β_j by*

$$\beta_j(u,v) = (u, jv).$$

Then

(a) *β_j is a nondegenerate bilinear form on V.*

(b) *If A is a bounded complex linear transformation on V, there is a unique complex linear operator A^t of V such that*

$$\beta_j(Au,v) = \beta_j(A^t v,u) \text{ for } u,v \in V.$$

Proof. We first note $\beta_j(u, ju) = (u, u) \neq 0$ and $\beta_j(ju, u) \neq 0$ if $u \neq 0$. Hence β_j is nondegenerate.

Let A be a bounded complex linear transformation of V. Define $\beta(v, u) = (Aju, jv)$. Then β is a bounded sesquilinear form. By Proposition 5.28, there is a unique complex linear transformation A^t satisfying $(Aju, jv) = (A^t v, u)$. Replacing u by ju, we see

$$(Au, jv) = (A^t v, ju)$$

for all u and v. Thus $\beta_j(Au, v) = \beta_j(A^t v, u)$. □

If j is a conjugation on V, the form β_j is called the **canonical bilinear form** on V determined by j. The conjugation j is said to be **symmetric** if β_j is symmetric.

Lemma 5.32. *If j is a symmetric conjugation on a complex Hilbert space V, then $A^t = jA^*j$.*

Proof. Since $\beta_j(Au, v) = \beta_j(A^t v, u)$ and β_j is symmetric, we have $(Au, jv) = \beta_j(u, A^t v) = (u, jA^t v)$. Hence $(u, A^* jv) = (u, jA^t v)$ for all u and v. So $jA^t = A^* j$. Thus $A^t = jA^* j$. □

Lemma 5.33. *Let j be a conjugation on a complex Hilbert space V. Then the following are equivalent.*

(a) *j is symmetric.*

(b) *$(ju, jv) = (v, u)$ for all u and v.*

(c) *j is an isometry.*

(d) *There is an orthonormal basis $\{e_m\}_{m \in I}$ of V with $je_m = e_m$ for all m.*

Proof. Suppose β_j is symmetric. Then $(u, jv) = (v, ju)$ for all u and v. Thus $(ju, jv) = (v, j^2 u) = (v, u)$ for all u and v. If $(ju, jv) = (v, u)$ for all u and v. Then $||ju||^2 = (ju, ju) = (u, u) = ||u||^2$ and we see j is an isometry.

Assume j is an isometry. Define $(u, v)_{\mathbb{R}} = \mathrm{Re}(u, v)$. Then V with this inner product is a real Hilbert space. We know $(ju, ju)_{\mathbb{R}} = (u, u)_{\mathbb{R}}$ for all u. Hence

$$(j(u + v), j(u + v))_{\mathbb{R}} - (j(u - v), j(u - v))_{\mathbb{R}} = (u + v, u + v)_{\mathbb{R}} - (u - v, u - v)_{\mathbb{R}}.$$

This implies $(ju, jv)_{\mathbb{R}} + (jv, ju)_{\mathbb{R}} = (u, v)_{\mathbb{R}} + (v, u)_{\mathbb{R}}$. Since $(\cdot\,,\cdot)_{\mathbb{R}}$ is symmetric, we have $(ju, jv)_{\mathbb{R}} = (u, v)_{\mathbb{R}}$ for all u and v. This implies $V_{\mathbb{R}} \perp_{\mathbb{R}} (iV_{\mathbb{R}})$. Indeed, if $ju = u$ and $jv = -v$, then $2(u, v)_{\mathbb{R}} = (u, v)_{\mathbb{R}} + (ju, jv)_{\mathbb{R}} = (u, 0)_{\mathbb{R}} = 0$. Now let $\{e_m\}_{m \in I}$ be an orthonormal basis of the real Hilbert space $V_{\mathbb{R}}$ under the inner product $(\cdot\,,\cdot)_{\mathbb{R}}$. Then since $ie_m \in iV_{\mathbb{R}}$, we see $(e_m, ie_n)_{\mathbb{R}} = 0$. Hence $\mathrm{Im}(e_n, e_m) = 0$ for all $m, n \in I$ and the vectors e_m are orthonormal. They form a complete basis for $V_{\mathbb{R}} \oplus (iV_{\mathbb{R}}) = V$. Since

$je_m = e_m$ for all m, we have $j(\sum \lambda_m e_m) = \sum \bar{\lambda}_m e_m$ whenever $\sum |\lambda_m|^2 < \infty$; and one sees $\beta_j(u,v) = \beta_j(v,u)$ for all u and v and thus β_j is symmetric. $\square$

Proposition 5.34. *Assume that V is a complex Hilbert space with inner product $(\cdot,\cdot)$ and conjugation j. Suppose that β is a bounded bilinear form on V. Then there exists a unique bounded complex linear map $A : V \to V$ with $\beta(u,v) = \beta_j(Au,v)$ for all $u,v \in V$. β is non-degenerate if and only if A is injective with dense range. β is symmetric if and only if $A^t = A$ and skew-symmetric if and only if $A^t = -A$. If in addition j is symmetric, then $||\beta|| = ||A||$.*

Proof. Define $\beta'(u,v) = \beta(u,jv)$. Note β' is sesquilinear. It is also bounded for

$$\begin{aligned} |\beta'(u,v)| &= |\beta(u,jv)| \\ &\leqslant ||\beta||\,||u||\,||jv|| \\ &\leqslant ||\beta||\,||u||\,||j||\,||v||. \end{aligned}$$

By Proposition 5.28, there exists a unique complex linear mapping A satisfying $||A|| = ||\beta'||$ and

$$\beta'(u,v) = (Au,v) \text{ for } u,v \in V.$$

Thus $\beta(u,v) = \beta'(u,jv) = (Au,jv) = \beta_j(u,v)$. Now β' is nondegenerate if and only if β is nondegenerate. Hence again by Proposition 5.28, β is nondegenerate if and only if A is injective with dense range.

Now β is symmetric if and only if $\beta_j(Au,v) = \beta_j(Av,u)$ if and only if $A^t = A$. Similarly, β is skew-symmetric if and only if $A^t = -A$.

Finally suppose j is symmetric. Then by (c) of Lemma 5.33, j is an isometry. Hence

$$\begin{aligned} ||\beta'|| &= \sup_{||u||=1,\,||v||=1} |\beta'(u,v)| \\ &= \sup_{||u||=1,\,||v||=1} |\beta(u,jv)| \\ &= \sup_{||u||=1,\,||v||=1} |\beta(u,v)| \\ &= ||\beta||. \end{aligned}$$

$\square$

Example (The Orthogonal Groups). In the examples we present here, the space V will be either $\mathbb{R}^n$ or $\mathbb{C}^n$ with their usual inner products. We let j be the standard symmetric conjugation on $\mathbb{C}^n$ given by $j\left((z_1, z_2, \ldots, z_n)^t\right) = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)^t$.

Set $\mathrm{Symb}_n(\mathbb{F})$ to be the space of symmetric bilinear forms on $\mathbb{F}^n$. This is a finite dimensional vector space and thus has a unique vector space topology. If $\beta \in \mathrm{Symb}_n(\mathbb{F})$ and $a \in \mathrm{GL}(n, \mathbb{F})$, define $a \cdot \beta$ by

$$a \cdot \beta(u, v) = \beta(a^t u, b^t v).$$

Then

$$\begin{aligned}(ab) \cdot \beta(u, v) &= \beta((ab)^t u, (ab)^t v) \\ &= \beta(b^t a^t u, b^t a^t v) \\ &= b \cdot \beta(a^t u, a^t v) \\ &= a \cdot (b \cdot \beta)(u, v).\end{aligned}$$

Thus $\mathrm{GL}(n, \mathbb{F})$ acts on $\mathrm{Symb}_n(\mathbb{F})$ and this action is continuous; see Exercise 5.3.12. Thus the stabilizer $\mathrm{O}(\beta, \mathbb{F})$ of β is a closed subgroup of $\mathrm{GL}(n, \mathbb{F})$. The subgroup $\mathrm{SO}(\beta, \mathbb{F})$ is the subgroup of $\mathrm{O}(\beta, \mathbb{F})$ consisting of those a's in $\mathrm{O}(\beta, \mathbb{F})$ with $\det(a) = 1$.

Lemma 5.35. *Let V be a vector space over $\mathbb{F}$.*

(a) *If β is a symmetric bilinear form on a vector space V, then*

$$\beta(u, v) = \frac{1}{2}\left(\beta(u + v, u + v) - \beta(u, u) - \beta(v, v)\right).$$

(b) *If $\mathbb{F} = \mathbb{C}$ and β is an Hermitian sesquilinear form, then*

$$\beta(u, v) = \frac{1}{4}\sum_{j=0}^{3} i^j \beta(u + i^j v, u + i^j v).$$

Proof. Note (a) is immediate. For (b) note

$$\frac{1}{4}\sum_{j=0}^{3} i^j \beta(u + i^j v, u + i^j v) = \frac{1}{4}\sum_{j=0}^{3} i^j \beta(u, u) + \frac{1}{4}\sum_{j=0}^{3} i^j \beta(u, i^j v) +$$

$$\frac{1}{4}\sum_{j=0}^{3} i^j \beta(i^j v, u) + \frac{1}{4}\sum_{j=0}^{3} i^j \beta(i^j v, i^j v) =$$

$$\frac{1}{4}\left(0 + \sum_{j=0}^{3} \beta(u, v) + \sum_{j=0}^{3} (-1)^j \beta(v, u) + 0\right) = \beta(u, v).$$

$\square$

The formulas in (a) and (b) are called *polarizations* of the forms.

The case $\mathbb{F} = \mathbb{C}$ is simpler and we shall deal with it first. The main reason it is simpler is the existence of square roots in $\mathbb{C}$.

Lemma 5.36. *Let β be a symmetric bilinear form on $\mathbb{C}^n$. Then there is a basis $e_1, e_2, \ldots, e_r, e_{r+1}, e_{r+2}, \ldots, e_n$ of $\mathbb{C}^n$ such that*

$$\beta(e_i, v) = 0 \text{ for all } v \text{ if } i \leqslant r$$
$$\beta(e_i, e_j) = \delta_{i,j} \text{ if } i, j > r.$$

Proof. The set $R := \{v \mid \beta(v, w) = 0 \text{ for all } w \in \mathbb{C}^n\}$ is a linear subspace of $\mathbb{C}^n$. Take a basis $e_1, e_2, \ldots, e_r$ of R. If $r = n$, we are done. Hence suppose $e_1, e_2, \ldots, e_s$ where $r \leqslant s < n$ are linearly independent and satisfy $\beta(e_i, x) = 0$ for all x if $i \leqslant r$ and $\beta(e_i, e_j) = \delta_{i,j}$ if $i, j > r$. Let V be the linear span of the vectors e_1, e_2, $\ldots$, e_s and define $W := \{w \mid \beta(w, v) = 0 \text{ for all } v \in V\}$. Note $V + W = \mathbb{C}^n$, for if $x \in \mathbb{C}^n$, then $w = x - \sum_{i=1}^{s} \beta(x, e_i)e_i \in W$. Indeed $\beta(w, e_j) = \beta(x, e_j) - \sum_{i=1}^{s} \beta(x, e_i)\beta(e_i, e_j) = 0$ for $j = 1, 2, \ldots, s$. By Lemma 5.35, $\beta|_{W\times W} = 0$ or there is a $w \in W$ with $\beta(w, w) \neq 0$. If $\beta|_{W\times W} = 0$, then $\beta(w, v + w') = 0$ for all $v \in V$, $w, w' \in W$ and thus $W \subseteq R$. Consequently $V = R$ and $s = n$, a contradiction. Hence there is a $w \in W$ with $\beta(w, w) \neq 0$. Clearly $w \notin V$. Define $e_{s+1} = \frac{1}{\sqrt{\beta(w,w)}}w$. Then e_1, e_2, $\ldots$, e_{s+1} has the property $\beta(e_i, e_j) = \delta_{i,j}$ for $r < i, j \leqslant s + 1$. $\square$

The form $\beta_I(u, v) = u^t v = (u, jv)$ is the standard symmetric nondegenerate bilinear form on $\mathbb{F}^n$. Note if $\mathbb{F} = \mathbb{C}$, then $\beta_I(u, v) = (u, jv)$. In the complex case we write $\mathrm{O}(n, \mathbb{C})$ for $\mathrm{O}(\beta_I, \mathbb{F})$ while in the real case we write $\mathrm{O}(n)$.

Proposition 5.37. *If β is a nondegenerate symmetric bilinear form on $\mathbb{C}^n$, then there exists an $a \in \mathrm{GL}(n, \mathbb{F})$ with $\beta = a \cdot \beta_I$. Moreover, the mapping*

$$b \mapsto a^{-1}ba$$

is a group isomorphism from $\mathrm{O}(\beta, \mathbb{C})$ onto $\mathrm{O}(n, \mathbb{C})$.

Proof. Since β is nondegenerate, Lemma 5.36 shows there is a basis $\{v_k\}_{k=1}^n$ of $\mathbb{C}^n$ satisfying $\beta(v_k, v_l) = \delta_{k,l}$. Let e_1, e_2, $\ldots$, e_n be the usual basis of $\mathbb{C}^n$. Define $a \in \mathrm{GL}(n, \mathbb{F})$ by $a^t v_k = e_k$. Then

$$\begin{aligned} a \cdot \beta_I(v_k, v_l) &= \beta_I(a^t v_k, a^t v_l) \\ &= \beta_I(e_k, e_l) \\ &= \delta_{k,l}. \end{aligned}$$

Since $a \cdot \beta_I$ and β are bilinear, we then have $a \cdot \beta_I = \beta$.

Finally $b \in \mathrm{GL}(n, \mathbb{F})_\beta$ if and only if $a \cdot \beta_I(b^t v, b^t w) = a \cdot \beta_I(v, w)$ if and only if $\beta_I(a^t b^t v, a^t b^t w) = \beta_I(a^t v, a^t w)$ if and only if $\beta_I(a^t b^t (a^t)^{-1} v, a^t b^t (a^t)^{-1} w) = \beta_I(v, w)$ if and only if $\beta_I((a^{-1}ba)^t v, (a^{-1}ba)^t w) = \beta_I(v, w)$ if and only if $a^{-1}ba \in \mathrm{GL}(n, \mathbb{F})_{\beta_I}$. Thus $\mathrm{O}(n, \mathbb{C}) = a^{-1}\mathrm{O}(\beta, \mathbb{C})a$. $\square$

Let $\mathrm{Symb}_n^*(\mathbb{C})$ denote the space of nondegenerate symmetric bilinear forms on $\mathbb{C}^n$. We then have

$$\mathrm{Symb}_n^*(\mathbb{C}) = \mathrm{GL}(n,\mathbb{C}) \cdot \beta_I,$$

and since the stabilizer of β_I is $\mathrm{O}(n,\mathbb{C})$, we see

$$\mathrm{Symb}_n^*(\mathbb{C}) \cong \mathrm{GL}(n,\mathbb{C})/\mathrm{O}(n,\mathbb{C}).$$

The situation for symmetric bilinear forms on $\mathbb{R}^n$ is slightly more complicated.

Lemma 5.38. *Let β be a symmetric bilinear form on $\mathbb{R}^n$. Then there are nonnegative integers r, p, and q with $r+p+q=n$ and a basis $e_1, e_2, \ldots, e_r$, $e_{r+1}, \ldots, e_{r+p}$, e_{r+p+1}, $e_{r+m+2}, \ldots, e_{r+p+q}$ of $\mathbb{R}^n$ such that*

$$\begin{aligned}
&\beta(e_i, e_j) = 0 \text{ if } i \neq j\\
&\beta(e_i, e_i) = 0 \text{ if } i \leqslant r\\
&\beta(e_i, e_i) = 1 \text{ if } r < i \leqslant r+p\\
&\beta(e_i, e_i) = -1 \text{ if } r+p+1 < i \leqslant n.
\end{aligned}$$

Proof. Let R be the linear subspace of all vectors v such that $\beta(v,x) = 0$ for all $x \in \mathbb{R}^n$. Choose a basis $e_1, e_2, \ldots, e_r$ of R. Let M be a maximal collection of vectors satisfying

$$\begin{aligned}
&\beta(v,v)^2 = 1 \text{ for } v \in M\\
&\beta(v,v') = 0 \text{ if } v, v' \in M \text{ and } v \neq v'.
\end{aligned}$$

We claim $B := \{e_1, e_2, \ldots, e_r\} \cup M$ is a basis of $\mathbb{R}^n$. Note this set is linearly independent. Indeed, if $x = \sum_{i=1}^r a_i e_i + \sum_{v \in M} a_v v = 0$ and $v' \in M$, then

$$\beta(v', x) = a_{v'} \beta(v', v') = 0.$$

Since $\beta(v', v') = \pm 1$, $a_{v'} = 0$. Thus $\sum_{i=1}^r a_i e_i = 0$. Since $\{e_i\}_{i=1}^r$ is a basis of R, $a_1 = a_2 = \cdots = a_r = 0$.

We now show B spans $\mathbb{R}^n$. Let V be the linear span of B. Suppose $V \neq \mathbb{R}^n$. Let $W = \{w \mid \beta(w,v) = 0 \text{ for all } v \in V\}$. Then $V + W = \mathbb{R}^n$. In fact, for each $v \in M$, set $\epsilon_v = \beta(v,v)$. Then ϵ_v is either 1 or -1. If $x \in \mathbb{R}^n$, set $w = x - \sum_{v \in M} \epsilon_v \beta(x,v) v$. Clearly, $\beta(w, e_i) = 0$ for $i = 1, 2, \ldots, r$ and if $v' \in M$, then $\beta(w,v') = \beta(x,v') - \epsilon_{v'} \beta(x,v')\beta(v',v') = \beta(x,v')(1 - \beta(v',v')^2) = 0$. Hence $w \in W$ and thus $x = \sum_{v \in M} \epsilon_v \beta(x,v) v + w \in V + W$. So $V + W = \mathbb{R}^n$. Note $\beta|_{W \times W} \neq 0$, for otherwise $\beta|_{\mathbb{R}^n \times W} = 0$ and consequently $W \subseteq R$. By Lemma 5.35, there is a $w \in W$ with $\beta(w,w) \neq 0$. Set $v = \frac{1}{\sqrt{|\beta(w,w)|}} w$. Note $\beta(v,v)^2 = 1$ and $\beta(v,v') = 0$ for all $v' \in V$. $M' = M \cup \{v'\}$ has the properties of M and thus M was not maximal. This is a contradiction and we see B is a basis of $\mathbb{R}^n$. We can now enumerate

M as e_{r+1}, e_{r+1}, ..., e_{r+p}, e_{r+p+1}, ..., e_{r+p+q} where $\beta(e_i, e_i) = 1$ for $i = r+1, \ldots, r+p$ and $\beta(e_i, e_i) = -1$ for $i = r+p+1, \ldots, n$. □

The number r is called the **nullity** of the bilinear form, the number $p+q$ is called the **rank**, and the number $p-q$ is called the **signature**. Note these numbers are unique. See Exercise 5.3.14.

Corollary 5.39. *Symmetric bilinear forms on* $\mathbb{R}^n$ *are in the same* $\mathrm{GL}(n, \mathbb{R})$ *orbit if and only if they have the same nullity, rank, and signature.*

Proof. Suppose β and β' have the same nullity, rank, and signature. Then there exist bases e_1, e_2, ..., e_r, e_{r+1}, ..., e_{r+p}, e_{r+p+1}, ..., e_{r+p+q} and e'_1, e'_2, ..., e'_r, e'_{r+1}, ..., e'_{r+p}, e'_{r+p+1}, ..., e'_{r+p+q} with the properties in Lemma 5.38 for β and β', respectively. There is a unique matrix $a \in \mathrm{GL}(n, \mathbb{R})$ with $a^t e'_i = e_i$ for $i = 1, \ldots, n$. Hence $a \cdot \beta(e'_i, e'_j) = \beta(a^t e'_i, a^t e'_j) = \beta(e_i, e_j) = \beta'(e'_i, e'_j)$ for all i and j. Since $a \cdot \beta$ and β' are bilinear and the $\{e'_i\}_{i=1}^n$ is a basis, we see $a \cdot \beta = \beta'$. The converse follows by the inverse of this argument. □

The orbits of β of full rank are described by pairs (p, q) where $0 \leqslant p \leqslant n$, $0 \leqslant q \leqslant n$, and $p + q = n$. Fix p, q with $p + q = n$. Define $\beta_{p,q} \in \mathrm{Symb}_n(\mathbb{F})$ by

$$\beta_{p,q}(v, w) = \sum_{i=1}^{p} v_i w_i - \sum_{i=p+1}^{p+q} v_i w_i.$$

The stabilizer of $\beta_{p,q}$ in $\mathrm{GL}(n, \mathbb{R})$ is denoted by $\mathrm{O}(p, q)$. The subgroup of $\mathrm{O}(p, q)$ consisting of those a with $\det(a) = 1$ is called $\mathrm{SO}(p, q)$.

If $\mathrm{Symb}^*_{p,q}(\mathbb{R})$ denotes the space of symmetric bilinear mappings on $\mathbb{R}^n$ with rank n and signature $p - q$, then

$$\mathrm{Symb}^*_{p,q}(\mathbb{R}) = \mathrm{GL}(n, \mathbb{R}) \cdot \beta_{p,q}.$$

Since the stabilizer $\mathrm{GL}(n, \mathbb{R})_{\beta_{p,q}}$ of $\beta_{p,q}$ is $\mathrm{O}(p, q)$, we see

$$\mathrm{Symb}^*_{p,q}(\mathbb{R}) \cong \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(p, q).$$

Let $\mathrm{Sym}_n(\mathbb{F})$ be the vector space of $n \times n$ symmetric matrices with entries from the field $\mathbb{F}$.

Proposition 5.40. *Let* e_1, e_2, ..., e_n *be the standard basis of* $\mathbb{F}^n$. *The mapping* $\beta \mapsto B$ *where* $B = [\beta(e_i, e_j)]$ *is a one-to-one linear isomorphism of* $\mathrm{Symb}_n(\mathbb{F})$ *onto* $\mathrm{Sym}_n(\mathbb{F})$. *Moreover, if* $a \in \mathrm{GL}_n(\mathbb{F})$ *and* $\beta \mapsto B$, *then* $a \cdot \beta \mapsto aBa^t$.

Proof. First note if β is symmetric, then since $\beta(e_i, e_j = \beta(e_j, e_i)$, the matrix B is symmetric. Clearly the map is linear. Moreover, if $B = 0$, then $\beta(e_i, e_j) = 0$ for all i and j. Since β is bilinear, one then would have

$\beta = 0$. Hence this mapping is one-to-one. Moreover, if $B \in \mathrm{Sym}_n(\mathbb{F})$, define $\beta \in \mathrm{Symb}_n(\mathbb{F})$ by

$$\beta(\sum a_i e_i, \sum b_j e_j) = \sum_{i,j} a_i b_j B_{i,j}.$$

Then $\beta \mapsto B$ and the mapping is onto.

Suppose $a \in \mathrm{GL}(n, \mathbb{F})$ and $\beta \mapsto B$. Then $a \cdot \beta \mapsto a \cdot B$ where $a \cdot B$ is the matrix $\big(\beta(a^t e_i, a^t e_j)\big)$. But $a^t e_i = \sum_k a_{i,k} e_k$ and $a^t e_j = \sum_l a_{j,l} e_l$. Thus $(a \cdot B)_{i,j} = \sum_k \sum_l a_{i,k} \beta(e_k, e_l) a_{j,l} = \sum_k \sum_l a_{i,k} B_{k,l} a_{j,l} = (aBa^t)_{i,j}$. □

We note if $\beta \mapsto B$, then $\beta(v, w) = v^t B w$. Indeed, $v = \sum_i v_i e_i$ and $w = \sum_j w_j e_j$. Thus $\beta(v, w) = \sum_{i,j} v_i w_j \beta(e_i, e_j) = \sum_{i,j} v_i B_{i,j} w_j = v^t B w$. In particular, β_I corresponds to the identity matrix I for $\beta_I(u, v) = u^t I v$ and consequently, one has

$$\begin{aligned} \mathrm{O}(n, \mathbb{F}) &= \{A \in \mathrm{GL}(n, \mathbb{F}) \mid A \cdot \beta_I = \beta_I\} \\ &= \{A \in \mathrm{GL}(n, \mathbb{F}) \mid AA^t = I\}. \end{aligned}$$

Note $\mathrm{O}(n) = \mathrm{O}(n, 0)$ and if $p + q = n$, then $\beta_{p,q}$ corresponds to the matrix

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

for $\beta_{p,q}(e_i, e_j) = 0$ if $i \neq j$ and $\beta_{p,q}(e_i, e_i) = 1$ for $i \leqslant p$ and $\beta_{p,q}(e_i, e_i) = -1$ for $i > p$. Thus

$$\begin{aligned} \mathrm{O}(p, q) &= \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A \cdot \beta_{p,q} = \beta_{p,q}\} \\ &= \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A I_{p,q} A^t = I_{p,q}\}. \end{aligned}$$

Moreover,

$$\mathrm{Symb}^*_{p,q}(\mathbb{R}) \cong \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(p, q).$$

Example (The Symplectic Groups). Again we let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Set $\mathrm{Alt}_n(\mathbb{F})$ be the finite dimensional vector space of skew symmetric bilinear forms β on V. As before the mapping $\mathrm{GL}(n, \mathbb{F}) \times \mathrm{Alt}_n(\mathbb{F}) \ni (a, \beta) \mapsto a \cdot \beta$ where

$$a \cdot \beta(u, v) = \beta(a^t u, a^t v)$$

satisfies $a \cdot (b \cdot \beta) = (ab) \cdot \beta$. This action is continuous and thus one has

$$\mathrm{GL}(n, \mathbb{F}) \cdot \beta \cong \mathrm{GL}(n, \mathbb{F})/\mathrm{GL}(n, \mathbb{F})_\beta.$$

We denote the stabilizer $\mathrm{GL}(n, \mathbb{F})_\beta$ by $\mathrm{Sp}(\beta, \mathbb{F})$. Of greatest interest are the stabilizers of nondegenerate skew symmetric β. To describe these we need the following result.

Lemma 5.41. *Let β be a skew symmetric bilinear form on $\mathbb{F}^n$. Then there are nonnegative integers r and m with $r+2m=n$ and a basis $z_1, z_2, \ldots, z_r, e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_m$ of $\mathbb{F}^n$ such that*

$$\begin{aligned}
\beta(z_i, v) &= 0 \text{ for all } v \in \mathbb{F}^n \\
\beta(e_i, e_j) &= \beta(f_i, f_j) = 0 \text{ for all } i, j \\
\beta(e_i, f_j) &= \delta_{i,j}.
\end{aligned}$$

Proof. Let $R = \{x \mid \beta(x, v) = 0 \text{ for all } v \in \mathbb{F}^n\}$. R is a linear subspace and if r is its dimension, we can choose a basis $z_1, z_2, \ldots, z_r$ of R. If $r = n$ we are done. Suppose we have chosen $e_1, e_2, \ldots, e_l$ and $f_1, f_2, \ldots, f_l$ having the properties listed. If $r + 2l = n$, we are done. Otherwise, if W is the linear span of the set $\{z_1, \ldots, z_r, e_1, \ldots, e_l, f_1, \ldots, f_l\}$, then W is a proper linear subspace of $\mathbb{F}^n$. Choose $v \notin W$. Set $e_{l+1} = v - \sum_{i=1}^{l} \beta(v, f_i)e_i + \sum_{i=1}^{l} \beta(v, e_i)f_i$. Note

$$\begin{aligned}
\beta(e_{l+1}, e_k) &= \beta(v, e_k) - \sum_{i=1}^{l} \beta(v, f_i)\beta(e_i, e_k) + \sum_{i=1}^{l} \beta(v, e_i)\beta(f_i, e_k) \\
&= \beta(v, e_k) + \beta(v, e_k)\beta(f_k, e_k) \\
&= \beta(v, e_k) - \beta(v, e_k) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\beta(e_{l+1}, f_k) &= \beta(v, f_k) - \sum_{i=1}^{l} \beta(v, f_i)\beta(e_i, f_k) + \sum_{i=1}^{l} \beta(v, e_i)\beta(f_i, f_k) \\
&= \beta(v, f_k) - \beta(v, f_k)\beta(e_k, f_k) \\
&= \beta(v, e_k) - \beta(v, e_k) \\
&= 0.
\end{aligned}$$

Since $v \notin W$, $e_{l+1} \notin R$. Thus there is a vector v' with $\beta(e_{l+1}, v') = 1$. Since $\beta(e_{l+1}, w) = 0$ for $w \in W$, $v' \notin W$. Define f_{l+1} by

$$f_{l+1} = v' - \sum_{i=1}^{l} \beta(v', f_i)e_i + \sum_{i=1}^{l} \beta(v', e_i)f_i.$$

Note $\beta(e_{l+1}, f_{l+1}) = \beta(e_{l+1}, v') = 1$ and as before we have $\beta(f_{l+1}, w) = 0$ for $w \in W$. Repeating one finally obtains the desired basis. $\square$

The number r is called the **nullity** of the skew symmetric bilinear form β. The number $n - r$ is β's **rank**. Analogous to Corollary 5.39, skew symmetric bilinear forms on V are in the same $\mathrm{GL}(n, \mathbb{F})$ orbit if and only if they have the same rank. Moreover, note there are nondegenerate skew symmetric bilinear forms on V if and only if the dimension of V is even.

Fix n and let $V = \mathbb{F}^{2n}$. Set $\mathrm{Alt}_n^*(\mathbb{F})$ to be the space of all nondegenerate skew symmetric bilinear forms on V. Define J on

$$\mathbb{F}^{2n} = \mathbb{F}^n \times \mathbb{F}^n = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{F}^n \right\}$$

by

$$J \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

Note $J \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Set $\beta_J(u, v) = u^t J v$. Note

$$\beta_J(u, v) = (u^t J v)^t = v^t J^t u = v^t(-J)u = -\beta_J(v, u)$$

and thus β_J is a skew symmetric nondegenerate bilinear form on $\mathbb{F}^{2n}$. We call this particular skew symmetric form ω. In terms of components, we have

$$\omega(u, v) = \sum_{i=1}^{n} (u_i v_{n+i} - u_{i+n} v_i).$$

We let

$$\begin{aligned} \mathrm{Sp}(n, \mathbb{F}) &:= \{A \in \mathrm{GL}(2n, \mathbb{F}) \mid A \cdot \omega = \omega\} \\ &= \{A \in \mathrm{GL}(2n, \mathbb{F}) \mid AJA^t = J\}. \end{aligned}$$

The group $\mathrm{Sp}(n, \mathbb{F})$ is the *symplectic group*. We let $\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{SO}(2n)$. Then $\mathrm{Sp}(n)$ is a compact subgroup of $\mathrm{Sp}(n, \mathbb{R})$.

5.3. The sphere S^n. The classical linear groups act continuously on $\mathbb{F}^n$ by $A \cdot x = Ax$. Since these groups are closed subgroups of the general linear group, they are locally compact and Lindelöf. Hence by Theorem 5.15, any closed orbit is topologically isomorphic to the quotient space G/H where H is a stabilizer of a point in the orbit. Perhaps the most well known example is the sphere $S^n = \{x \mid x \in \mathbb{R}^{n+1}, |x| = 1\}$. It is clearly closed and is the orbit of the point $e_1 = (1, 0, 0, \cdots, 0)$ under both $\mathrm{O}(n+1)$ and $\mathrm{SO}(n+1)$.

Indeed, if $x_1 \in S^n$, then x_1 is a unit vector. One can then extend to an orthonormal basis $x_1, x_2, \ldots, x_{n+1}$ of $\mathbb{R}^{n+1}$. Let A be the matrix with column vectors $x_1, x_2, \ldots, x_{n+1}$. Thus $A = \big(x_1, x_2, \ldots, x_{n+1}\big)$. Note $Ae_1 = x_1$. Since $\{x_i\}_{i=1}^n$ is an orthonormal basis, one has $AA^t = I$, and we see $A \in \mathrm{O}(n+1)$. Moreover, $\det(A)^2 = \det(AA^t) = 1$. Hence $\det(A) = \pm 1$. If the determinant is -1, then by changing x_{n+1} to $-x_{n+1}$, we have $\det(A) = 1$. Thus A is in $\mathrm{SO}(n+1)$. The stabilizer in $\mathrm{O}(n+1)$ of e_1 consists of those matrices $A \in \mathrm{O}(n+1)$ whose first column is e_1. Since $A^t A = I$, we have $(A^t A)_{i,1} = \delta_{i,1}$. Thus $\sum a_{i,j}^t a_{j,1} = a_{i,1}^t = \delta_{i,1}$. Hence

$a_{1,2} = a_{1,3} = \ldots = a_{1,n+1} = 0$. This implies $A = (e_1, x_2, \ldots, x_{n+1})$ where the first elements in the column vectors $x_2, \ldots, x_{n+1}$ are zero. Hence

$$A = \begin{pmatrix} 1 & 0_n^t \\ 0_n & B \end{pmatrix}$$

where 0_n is the zero column vector in $\mathbb{R}^n$. Since $AA^t = I_{n+1}$, $BB^t = I_n$, and we see $B \in \mathrm{O}(n)$. In the case where $A \in \mathrm{SO}(n+1)$, then $\det(B) = 1$ and so $B \in \mathrm{SO}(n)$.

Lemma 5.42. $S^n \cong \mathrm{O}(n+1)/\mathrm{O}(n) \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$.

5.4. The flag manifolds. Let V be the vector space $\mathbb{F}^n$ over $\mathbb{F}$. Suppose $1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant n$ with $n_j \in \mathbb{N}$. Let $\mathbf{n} = (n_1, \ldots, n_k)$. Let $\mathrm{Flag}_{\mathbb{F}}(\mathbf{n}) = \mathrm{Flag}(\mathbf{n}) = \mathrm{Flag}(n_1, \ldots, n_k)$ be the set of nested chains $V_1 \subset \ldots \subset V_k$ of vector subspaces with $\dim V_j = n_j$. Then $\mathrm{GL}(n, \mathbb{F})$ acts on $\mathrm{Flag}(\mathbf{n})$ by

$$A \cdot (V_1, \ldots, V_k) = (AV_1, \ldots, AV_k)\,.$$

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{F}^n$. Let $E_j = \sum_{i=1}^{n_j} \mathbb{F}e_i$. Let $\mathbf{E} := (E_1, \ldots, E_k) \in \mathrm{Flag}(\mathbf{n})$. Let $\mathbf{V} = (V_1, \ldots, V_k) \in \mathrm{Flag}(\mathbf{n})$. Choose $f_1, \ldots, f_n$ a basis of V such that $f_1, \ldots, f_{n_j}$ is a basis of V_j. Then there exists a $A \in \mathrm{GL}(n, \mathbb{F})$ such that $Ae_j = f_j$. It follows that $A \cdot \mathbf{E} = \mathbf{V}$. Hence $\mathrm{GL}(n, \mathbb{R})$ acts transitively on $\mathrm{Flag}(\mathbf{n})$. We can even say more: if we choose $f_1, \ldots, f_n$ orthonormal with the same orientation as the standard basis, we see that we can even choose $A \in \mathrm{SO}(n)$ if $\mathbb{F} = \mathbb{R}$ and $A \in \mathrm{SU}(n)$ if $\mathbb{F} = \mathbb{C}$. Hence $\mathrm{SO}(n)$ (respectively $\mathrm{SU}(n)$) acts transitively on $\mathrm{Flag}(\mathbf{n})$. Assume that $A \cdot \mathbf{E} = \mathbf{E}$. Then $AV_1 = V_1$. It follows that A has the form

$$A = \begin{pmatrix} A_1 & B \\ 0 & C \end{pmatrix}$$

with $A_1 \in \mathrm{GL}(n_1, \mathbb{F})$, $C \in \mathrm{GL}(n - n_1, \mathbb{F})$, and B an arbitrary $n_1 \times (n - n_1)$-matrix. Using $AV_j \subset V_j$ and repeating the argument, one sees A has the form

$$A = \begin{pmatrix} A_1 & * & * & * & * \\ 0 & A_2 & * & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & A_{k-1} & * \\ 0 & 0 & 0 & 0 & A_k \end{pmatrix}$$

with $A_j \in \mathrm{GL}(n_j - n_{j-1}, \mathbb{F})$, $1 \leqslant j \leqslant k$. Here $*$ stands for an arbitrary matrix of the correct size. Let $P(\mathbf{n})$ be the group $\mathrm{GL}(n, \mathbb{F})_{\mathbf{E}}$. Then $P(\mathbf{n})$ is a *parabolic subgroup* of $\mathrm{GL}(n, \mathbb{F})$. Hence $\mathrm{Flag}(\mathbf{n})$ is isomorphic to $\mathrm{GL}(n, \mathbb{F})/\mathrm{P}(\mathbf{n})$. But $\mathrm{SO}(n)$ ($\mathrm{SU}(n)$ in the case $\mathbb{F} = \mathbb{C}$) also acts transitively on $\mathrm{Flag}(\mathbf{n})$. In this case the stabilizer of $\mathbf{E}$ is $\mathrm{SO}(n) \cap P(\mathbf{n})$ ($\mathrm{SU}(n) \cap P(\mathbf{n})$ in the complex case). These stabilizers can be described in the following manner.

Let $G_j \subset \mathrm{GL}(n_j - n_{j-1}, \mathbb{F})$, $j = 1, 2, \ldots, k$, be subgroups. Denote by $\mathrm{S}(G_1 \times \cdots \times G_k)$ the subgroup of $\mathrm{GL}(n, \mathbb{F})$ consisting of the matrices

$$\begin{pmatrix} A_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & A_{k-1} & 0 \\ 0 & 0 & 0 & 0 & A_k \end{pmatrix}$$

where $A_j \in G_j$ and $\det(A_1) \cdots \det(A_k) = 1$; so

$$\mathrm{S}(G_1 \times \cdots \times G_k) = (G_1 \times \cdots \times G_k) \cap \mathrm{SL}(n, \mathbb{F}).$$

There are several ways of giving a topology to $\mathrm{Flag}(\mathbf{n})$. The most obvious is to use the quotient topologies from either $\mathrm{GL}(n, \mathbb{F})/\mathrm{GL}(n, \mathbb{F})_E$ or from $\mathrm{SO}(n)/(\mathrm{SO}(n) \cap \mathrm{P}(\mathbf{n}))$ or $\mathrm{SU}(n)/(\mathrm{SU}(n) \cap \mathrm{P}(\mathbf{n}))$. The later make the flag manifolds into compact homogeneous spaces. Exercises 3.23 and 3.24 show this topology is the same as the quotient topology from $\mathrm{GL}(n, \mathbb{F})/\mathrm{GL}(n, \mathbb{F})_E$ and is a natural metric space topology.

Theorem 5.43. *The space* $\mathrm{Flag}(\mathbf{n})$ *is compact. In fact we have:*

(a) $\mathrm{Flag}_{\mathbb{R}}(\mathbf{n}) \simeq \mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(n_1) \times \mathrm{O}(n_2 - n_1) \times \cdots \times \mathrm{O}(n_k - n_{k-1}))$;

(b) $\mathrm{Flag}_{\mathbb{C}}(\mathbf{n}) \simeq \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(n_1) \times \mathrm{U}(n_2 - n_1) \times \cdots \times \mathrm{U}(n_k - n_{k-1}))$.

5.5. Motion groups. Let H be a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$. There is a natural multiplication on the set $G = \mathbb{R}^n \times H$ given by

$$(x, A) \cdot (y, B) = (x + Ay, AB).$$

Using the product topology and this multiplication $\mathbb{R}^n \times H$ is a locally compact Lindelöf group. It is denoted by $\mathbb{R}^n \rtimes H$. This construction is an example of a semi-direct product group. (See Exercise 5.3.25). Moreover, there is a natural action of $\mathbb{R}^n \rtimes H$ on $\mathbb{R}^n$. It is defined by

$$(x, A) \cdot y = x + Ay.$$

Note

$$\begin{aligned} [(x, A) \cdot (x', A')] \cdot y &= (x, A) \cdot (x' + A'y) \\ &= x + Ax' + AA'y \\ &= (x + Ax', AA') \cdot y. \end{aligned}$$

Since $(x, A) \cdot 0 = x$, this action is transitive. Moreover, the stabilizer G_0 is H. Also the natural mapping

$$(x, A)H \mapsto x$$

is a topological equivariant isomorphism of G/H onto $\mathbb{R}^n$.

We point out two important examples. In the case when $H = \mathrm{O}(n)$, the group $E(n) := \mathbb{R}^n \rtimes \mathrm{O}(n)$ is the group of rigid motions of Euclidean space and is known as the Euclidean group. In the case when $H = \mathrm{SO}(n-1,1)$, one obtains the Poincaré group $\mathbb{R}^n \rtimes \mathrm{SO}(n-1,1)$, a group important in the study of special relativity. Other examples of semi-direct products occur in the exercises.

Exercise Set 5.3

1. Show the topology defined by the norm $||\cdot||_2$ on $M(m \times n, \mathbb{F})$ is the product topology.

2. Show that the standard norm on $M(n \times m, \mathbb{F})$ is given by

$$||X|| = \sqrt{\mathrm{Tr}(XX^*)}, \quad X \in M(n \times m, \mathbb{F})$$

where $X^* = (\bar{x}_{ji}) = \bar{X}^t$.

3. Prove Lemma 5.34.

4. Let $\mathbb{F}$ be the reals or the complexes.

(a) Show $\mathrm{GL}(n, \mathbb{F})$ is locally compact and Lindelöf.

(b) Show all closed subgroups of $\mathrm{GL}(n, \mathbb{F})$ are locally compact and Lindelöf.

5. Suppose β is a sesquilinear form on a Hilbert space V given by

$$\beta(u, v) = (Tu, v)$$

where T is a linear transformation of V. Show $||\beta|| = ||T||$.

6. Let $\mathbb{F} = \mathbb{C}$ and let V be a complex vector space. Let β be a Hermitian bilinear form on V. Show that $\beta(u, u) \in \mathbb{R}$ for all $u \in V$.

7. Show there is a conjugation on a Hilbert space which is not symmetric; i.e., (u, jv) need not always be (v, ju). In view of Lemma 5.33, there is no orthonormal basis $\{e_i\}_{i \in I}$ with $je_i = e_i$.

8. Suppose j is a conjugation on a Hilbert space V defined by $j(\sum \lambda_i e_i) = \sum \bar{\lambda}_i e_i$ where $\{e_i\}$ is an orthonormal basis. Let A be a complex bounded linear operator on V. Show the matrix of A^t is the transpose of the matrix for A; i.e., show $(Ae_j, e_i) = (A^t e_i, e_j)$

9. Let V be an infinite dimensional complex Hilbert space. Show there is a conjugate linear transformation j of V satisfying $j^2 = I$ that is not a conjugation.

10. Let j be a conjugation on a complex Hilbert space V. Show $V_{\mathbb{R}}$ and $iV_{\mathbb{R}}$ are orthogonal relative to the real inner product $(v, v')_{\mathbb{R}} = \mathrm{Re}(v, v')$ if and only if j is symmetric.

11. Show that $\mathrm{O}(\beta, \mathbb{F})$ is a group.

12. Show the action of $\mathrm{GL}(n, \mathbb{F})$ on $\mathrm{Symb}_n(\mathbb{F})$ given by

$$a \cdot \beta(u, v) = \beta(a^t u, a^t v)$$

is continuous.

13. Let β be a bilinear symmetric form on $\mathbb{C}^n$. Show there is a unique r such that β is in the $\mathrm{GL}(n, \mathbb{C}^n)$ orbit of the bilinear form β_r defined on $\mathbb{C}^n$ by

$$\beta_r(u, v) = \sum_{j=r+1}^{n} u_j v_j.$$

14. Let β be a symmetric bilinear form on $\mathbb{R}^n$.

(a) Show the rank and signature of β are unique. (**Hint:** Show if V and W are subspaces of $\mathbb{R}^n$ and β is positive definite on V and negative definite on W, then the sum $V + W$ is direct.)

(b) Show if β is a symmetric bilinear form of rank $n - r$ and signature $p - q$, then there is an $a \in \mathrm{GL}(n, \mathbb{R})$ such that

$$a \cdot \beta = \beta_{r,p,q}$$

where $\beta_{r,p,q}$ is the bilinear form defined by

$$\beta_{r,p,q}(u, v) = \sum_{k=r+1}^{r+p} u_k v_k - \sum_{k=r+p+1}^{r+p+q} u_k v_k.$$

15. Let β be a bilinear form on a finite dimensional vector space V. Show β is nondegenerate if and only if $\beta(v, w) = 0$ for all w implies $v = 0$.

16. Let $\beta_A(u, v) = \beta(Au, v)$ where $\beta(u, v) = \sum_{j=1}^{n} u_i u_j$. Show the following:

(a) Assume that A and B are symmetric and $B = aAa^t$ for some $a \in \mathrm{GL}(n, \mathbb{F})$. Then

$$\mathrm{O}(\beta_B, \mathbb{F}) = a\mathrm{O}(\beta_A, \mathbb{F})a^t$$

(b) Assume that A is in $\mathrm{GL}(n, \mathbb{F})$. Then $\mathrm{O}(\beta_A, \mathbb{F}) := \{a \in \mathrm{GL}(n, \mathbb{F}) \mid a^t A a = a\}$, where a^t stands for the transposed matrix $a^t_{ij} = a_{ji}$.

17. Let $A \in \mathrm{O}(p, q)$. Show $\det(A) = \pm 1$, and if $A \in \mathrm{Sp}(n, \mathbb{R})$ then $\det(A) = 1$.

18. Let $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^j$ be such that $n_1 < n_2 < \ldots < n_k$. Let G_j be a subgroup of $\mathrm{GL}(n_j - n_{j-1}, \mathbb{R})$. Define

$$S(G_1 \times \ldots \times G_k) := \left\{ \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_k \end{pmatrix} \mid A_j \in G_j,\ \det(A_1) \cdot \ldots \cdot \det(A_k) = 1 \right\}.$$

Show that $\mathrm{SO}(n)$ acts transitively on $\mathrm{Flag}(\mathbf{n})$ and that

$$\mathrm{Flag}(\mathbf{n}) \simeq \mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(n_1) \times \ldots \times \mathrm{O}(n_k - n_{k-1}))).$$

19. Show that $E(n)$ is the group of distance preserving affine transformation of $\mathbb{R}^n$.

20. Let $\mathrm{Sym}^+(n, \mathbb{R})$ be the set of positive definite symmetric matrices. Show that $\mathrm{Sym}^+(n, \mathbb{R}) \simeq \mathrm{GL}(n, \mathbb{F})/\mathrm{O}(n)$.

21. Show that the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$ satisfies $AB \cdot z = A \cdot (B \cdot z)$ whereever defined.

22. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$ be the action of $\mathrm{SL}(2, \mathbb{R})$ on the upper half plane H_+. Let $A = \{a = \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) \mid x > 0\}$ and $N = \{n = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{R}\}$.

(a) Show AN is a closed subgroup of $\mathrm{SL}(2, \mathbb{R})$.

(b) Show the mapping $g \mapsto g \cdot i$ is a homeomorphism from AN onto H_+.

23. Let $\mathcal{V}(l)$ be the collection of all subspaces of the Hilbert space $\mathbb{F}^n$ having dimension l. Consider the collection of all ordered orthonormal sets $\mathcal{F} = \{f_1, f_2, \ldots, f_l\}$ in $\mathbb{F}^n$. Let $V, V' \in \mathcal{V}(l)$. Define

$$\rho_l(V, V') = \inf_{\mathcal{F} \subseteq V, \mathcal{F}' \subseteq V'} \sum_{i=1}^{l} ||f_i - f_i'||.$$

(a) Show ρ_l is a metric on $\mathcal{V}(l)$.

(b) Suppose the requirement that the ordered sets $f_1, f_2, \ldots, f_l$ be orthonormal is weakened so that they need only be linearly independent and have length one. Show that changing the metric in this way does not change the resulting topology.

24. Define d on $\mathrm{Flag}(\mathbf{n})$ by

$$d((V_1, V_2, \ldots, V_k), (V_1', V_2', \ldots, V_k')) = \sum_{i=1}^{k} \rho_{n_i}(V_i, V_i').$$

where ρ_{n_i} is the metric defined in Exercise 5.3.23.

(a) Show d is a metric.

(b) Let $\mathbb{F} = \mathbb{R}$. Show the topology on $\mathrm{Flag}(\mathbf{n})$ given by the metric d is the topology on $\mathrm{Flag}(\mathbf{n})$ given by the quotient topology on $\mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(n_1) \times \mathrm{O}(n_2 - n_1) \times \cdots \times \mathrm{O}(n_k - n_{k-1}))$ under the isomorphism given in Theorem 5.43.

(c) Show the topology on $\mathrm{Flag}(\mathbf{n})$ is also the quotient topology of $\mathrm{GL}(n, \mathbb{R})/\mathrm{P}(\mathbf{n})$. (Similar statements hold in the case $\mathbb{F} = \mathbb{C}$.)

25. Let H and K be topological Hausdorff groups. Suppose $\alpha : H \to \mathrm{Aut}(K)$ is a homomorphism of H into the automorphism group of K and the mapping $(h, k) \mapsto \alpha(h)(k)$ is continuous. Define $G = K \times_\alpha H$ to be the set $K \times H$ with product topology and multiplication defined by

$$(k, h) \cdot (k', h') = (k\alpha_h(k'), hh').$$

(a) Show G is a topological Hausdorff group.

(b) Show the mapping $(k, h) \cdot k' = k\alpha_h(k')$ defines a continuous G action on K.

(c) Show

$$G/H \cong K$$

as G spaces; i.e.; the map $(k, h)H \mapsto k$ is a bicontinuous G map. The group $K \times_\alpha H$ is a **semi-direct product group** of K and H.

26. Let H be the direct product group $\mathbb{R} \times \mathrm{O}(n)$ and let K be the group $\mathbb{R}^n \times \mathbb{R}^n$ under addition. Define $\alpha_{(t,A)}(x, v) = (Ax + tAv, Av)$ for $(t, A) \in H$.

(a) Show $\alpha : H \to \mathrm{Aut}(K)$ is a homomorphism and determine multiplication in the group $K \times_\alpha H$.

(b) Show $((x, v), (t, A)) \cdot (y, s) = (Ay - (s - t)v - x, s - t)$ is a continuous action of the Gallilean group $K \times_\alpha H$ on $\mathbb{R}^n \times \mathbb{R}$ or Gallilean space-time.

27. Let $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathbb{C}^2, +)$ by

$$\alpha_t(z_1, z_2) = (e^{it} z_1, e^{\pi it} z_2).$$

Show this defines a semi-direct product group $\mathbb{C}^2 \times_\alpha \mathbb{R}$. Write out explicitly the multiplication for this semi-direct product and find $(z_1, z_2, t)^{-1}$.

28. Let ϕ be a transformation of $\mathbb{R}^n$ onto $\mathbb{R}^n$ that preserves the inner product $(\cdot, \cdot)$. Show there is an matrix $T \in \mathrm{O}(n, \mathbb{R})$ so that $\phi(x) = Tx$ for $x \in \mathbb{R}^n$.

Chapter 6

Basic Representation Theory

1. Invariant Integrals and Measures

In this section we shall state the existence of a Haar integral on a locally compact Hausdorff group and use some general results from measure theory. For a detailed proof, see for example the texts of [**31**, Loomis] or [**22**, Hewitt and Ross].

Let X be a locally compact Hausdorff space and let $C_c(X)$ be the space of continuous complex valued functions on X having compact support. We start by recalling some standard results regarding integrals and measures on X. First, the σ-algebra of Baire sets is the smallest σ-algebra on X such that each $f \in C_c(X)$ is measurable. It is a subalgebra of the σ-algebra of Borel sets on X. This σ algebra is the smallest σ-algebra on X which contains every compact G_δ set. A **Radon** measure is a measure μ on the Baire sets satisfying $\mu(K) < \infty$ for each compact G_δ set K and for each Baire set E, $\mu(E)$ is the supremum of all $\mu(K)$ where K is a compact G_δ subset of E. If I is a positive integral on $C_c(X)$, i.e., a complex valued linear functional on $C_c(X)$ satisfying $I(f) \geqslant 0$ for any real valued nonnegative function $f \in C_c(X)$, then there is a unique Radon measure μ on X such that

$$I(f) = \int_X f(x)\, d\mu(x) \text{ for each } f \in C_c(X).$$

Moreover, every Radon measure μ extends to a unique Borel measure which we also call μ that has the following properties:

(a) $\mu(K) < \infty$ for all compact sets K

(b) If E is Borel and $\mu(E) < \infty$ or E is open, then

$$\mu(E) = \sup\{\mu(K) \mid K \text{ is compact}, K \subseteq E\}.$$

(c) $\mu(E) = \inf\{\mu(U) \mid U \text{ is open}, E \subseteq U\}$ for each Borel set E.

A Borel measure will be called **regular** if μ satisfies (a), (b), and (c). The following theorem due to F. Riesz characterizes the positive linear functionals I on the space $C_c(X)$.

Theorem 6.1 (Riesz)**.** *Let $I : C_c(X) \to \mathbb{C}$ be a linear functional satisfying $I(f) \geqslant 0$ if $f \geqslant 0$. Then there is a unique Radon measure μ such that*

$$I(f) = \int_X f(x)\, d\mu(x)$$

for all $f \in C_c(X)$.

The Haar integral is a major tool in extending harmonic analysis on locally Euclidean spaces to general locally compact Hausdorff groups. Recall $\lambda(a)f(x) = f(a^{-1}x)$ if f is a function on a group G.

Theorem 6.2 (Haar Integral)**.** *Let G be a locally compact Hausdorff group. Then there is a nonzero positive integral I on $C_c(G)$ such that $I(\lambda(a)f) = I(f)$ for each $f \in C_c(X)$ and $a \in G$. Moreover, if J is another such integral, there is a constant $c > 0$ such that $J = cI$.*

The resulting Radon measure m is called a left Haar measure for the left invariant integral I. It has the property

$$m(aE) = m(E)$$

for all a and all Baire sets E. Moreover, it has a unique regular extension to the Borel sets having the same invariance property. This measure is also called a left Haar measure.

We now establish some facts for a left invariant Haar integral I and its corresponding Haar measure m.

Proposition 6.3. *Let I be a left Haar integral for a locally compact Hausdorff group G and let m be the corresponding left invariant Haar measure on the σ-algebra of Borel subsets of G.*

(a) *$I(f) > 0$ if $f \geqslant 0$ and $f \neq 0$.*

(b) *$m(U) > 0$ for every nonempty open set U.*

(c) *$\int f(gx)\, dm(x) = \int f(x)\, dm(x)$ for each nonnegative Borel function f.*

(d) *For each $g \in G$, there is a $\Delta(g) > 0$ such that*

$$m(Eg) = \Delta(g)m(E)$$

for all Borel sets E.

(e) $\int f(xg)\, dm(x) = \Delta(g^{-1}) \int f(x)\, dm(x)$ *for* $g \in G$ *and nonnegative Borel functions* f.

(f) $\Delta : G \to (0, \infty)$ *is a continuous homomorphism of* G *into the multiplicative group of positive real numbers.*

(g) $\int f(x^{-1})\Delta(x^{-1})\, dm(x) = \int f(x)\, dm(x)$ *for nonnegative Borel functions* f.

Proof. Assume f is a nonzero nonnegative continuous function with compact support. Since $I \neq 0$, we can choose $h \in C_c(G)$ such that $I(h) \neq 0$. We note $I(h_1) \leqslant I(h_2)$ if $h_1 \leqslant h_2$. This implies $I(|h|) > 0$. Thus we may assume $h \geqslant 0$. By replacing f by $\frac{f}{||f||_\infty}$, we may assume f has maximum 1. Choose x_0 such that $f(x_0) = 1$. Set $K = \operatorname{supp} h$. For $x \in K$, consider $U_x = \{y \mid (||h||_\infty + 1)f(x_0 x^{-1} y) > h(y)\}$. Note $x \in U_x$ and U_x is open. Since K is compact, we can find a finite set $x_1, \ldots, x_k$ such that $K \subseteq \cup U_{x_i}$. In particular

$$h \leqslant \sum_{i=1}^{k} (||h||_\infty + 1)\lambda(x_i x_0^{-1})f.$$

Thus

$$0 < I(h) \leqslant \sum_{i=1}^{k} (||h||_\infty + 1)I(f).$$

So $I(f) > 0$.

For (b), note if U is a nonempty open set, there is a nonempty compact subset K of U. Proposition 5.23 implies there is a nonzero $f \in C_c(G)$ such that $0 \leqslant f \leqslant 1$ and $\operatorname{supp} f \subseteq U$. Thus $m(U) > I(f) > 0$.

For (c), note one can show using $m(gE) = m(E)$ for any $g \in G$ and any Borel set E that $\int s(gx)\, dm(x) = \int s(x)\, dm(x)$ for any simple nonnegative Borel function s. Now if $f \geqslant 0$ is Borel, $f(x) = \lim s_n(x)$ for all x where s_n is a pointwise increasing sequence of simple Borel functions. Thus the Monotone Convergence Theorem gives $\int f(gx)\, dm(x) = \int f(x)\, dm(x)$.

To see (d) and (e), let m be a left Haar measure. Then m' defined by $m'(E) = m(Eg)$ is left invariant and regular, positive on nonempty open sets, and finite on compact sets. This implies m' is also a left Haar measure and corresponds uniquely to a positive left invariant integral on $C_c(G)$. By Haar's Theorem, there is a $\Delta(g) > 0$ such that $m(Eg) = \Delta(g)m(E)$. An easy calculation shows $\int s(xg^{-1})\, dm(x) = \Delta(g) \int s(x)\, dm(x)$ for simple nonnegative Borel functions s. Taking limits as in the argument for (c) shows $\int f(xg^{-1})\, dm(x) = \Delta(g) \int f(x)\, dm(x)$ for Borel functions $f \geqslant 0$.

For (f), we already know $\Delta(g) > 0$ for all g, and clearly $\Delta(e) = 1$. Since $\Delta(g_1 g_2)m(E) = m(E g_1 g_2) = \Delta(g_2)m(E g_1) = \Delta(g_2)\Delta(g_1)m(E)$ for Borel sets E, we have

$$\Delta(g_1 g_2) = \Delta(g_1)\Delta(g_2).$$

To see continuity, by Lemma 5.10, we only need to show Δ is continuous at e. Using Proposition 5.23, we can find compact neighborhoods U and V of e and a function $f \in C_c(G)$ such that $f = 1$ on U, $0 \leqslant f \leqslant 1$, and $\mathrm{supp} f \subseteq UV$. Recall $\rho(y)f(x) = f(xy)$ for $x, y \in G$. Let $\epsilon > 0$. By right uniform continuity, see Lemma 5.24, there is an open neighborhood W of e contained in U^{-1} such that $|f(xy) - f(x)| < \frac{\epsilon I(f)}{m(UVU)}$ for all x and for $y \in W$. Note the support of f and $\rho(y)f$ are both contained in UVU. Consequently, if $y \in W$, then

$$\begin{aligned} |I(\rho(y)f) - I(f)| &= |I(\rho(y)f - f)| \\ &= |\int (f(xy) - f(x))\, dm(x)| \\ &\leqslant \int_{UVU} |f(xy) - f(x)|\, dm(x) \\ &\leqslant \epsilon I(f). \end{aligned}$$

But $I(\rho(y)f) = \Delta(y^{-1})I(f)$. Consequently, $|\Delta(y^{-1}) - 1| \leqslant \epsilon$ for $y \in W$. So Δ is continuous at e.

Finally we show (g). Define $J(f) = \int f(x^{-1})\Delta(x^{-1})\, dm(x)$ for $f \in C_c(G)$. Clearly J is positive. We show J is left invariant. Indeed, by (f),

$$\begin{aligned} J(\lambda(g)f) &= \int f(g^{-1}x^{-1})\Delta(x^{-1})\, dm(x) \\ &= \Delta(g)^{-1} \int f(g^{-1}(xg^{-1})^{-1})\Delta((xg^{-1})^{-1})\, dm(x) \\ &= \Delta(g)^{-1} \int f(x^{-1})\Delta(x^{-1})\Delta(g)\, dm(x) \\ &= J(f). \end{aligned}$$

Thus by uniqueness of left Haar integrals, there is a $c > 0$ with $J = cI$. Hence

$$\int f(x^{-1})\Delta(x^{-1})\, dm(x) = c \int f(x)\, dm(x)$$

for $f \in C_c(G)$. To see $c = 1$, note

$$\begin{aligned}\int f(x)\,dm(x) &= \frac{1}{c}\int f(x^{-1})\Delta(x^{-1})\,dm(x)\\ &= \frac{1}{c^2}\int f(x)\Delta(x)\Delta(x^{-1})\,dm(x)\\ &= \frac{1}{c^2}\int f(x)\,dm(x).\end{aligned}$$

Thus $c^2 \int f(x)\,dm(x) = \int f(x)\,dm(x)$. So $c = 1$.

□

The function Δ in Proposition 6.3 is called the modular function for the group G. If Δ is identically one, the group G is said to be **unimodular**. Thus a left Haar measure on G is right invariant if and only if G is unimodular.

In many of the most important instances, our locally compact Hausdorff spaces X will be second countable or σ-compact. The following results show that in these cases, the measure theory becomes a bit less complex. We start by showing if X is second countable, then the Borel sets and the Baire sets are the same.

Lemma 6.4. *Let X be a second countable locally compact Hausdorff space. Then the σ-algebra of Baire sets of X is the σ-algebra of Borel sets.*

Proof. Let K be a compact subset of X. Since X is second countable and locally compact, there exists a countable base U_i, $i = 1, 2, \ldots$ for the topology of X such that each $\bar{U}_i$ is compact. Now consider the collection of those U_i such that $\bar{U}_i$ misses K. This is countable and if $y \notin K$, then there is an open set U having compact closure such that $y \in \bar{U} \subseteq X - K$. In particular, there must be a U_i whose closure is compact and disjoint from K that contains y. Thus $X - K$ is the union of those U_i (call the corresponding index set of i's, I_0) with $\bar{U}_i \subseteq X - K$. Thus $K = \cap_{i \in I_0}(G - \bar{U}_i)$ is a G_δ. So every compact set is Baire. Since every open set is a countable union of compact sets, every open set is Baire. Thus every Borel set is Baire. □

Next we present some general results on Baire sets and use these to investigate σ-compactness. A subset E of X is said to be **σ-bounded** if it is a subset of a countable union of compact subsets of X. Note that Exercise 6.1.3 shows that every second countable locally compact Hausdorff space is σ-compact and hence every subset is σ-bounded.

Lemma 6.5. *The union of two compact G_δ subsets of a locally compact Hausdorff space is a compact G_δ.*

Proof. Let C and K be compact G_δ sets. Then $C \cup K$ is compact. Now since C and K are G_δ sets, there are decreasing sequences $\{U_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ of open subsets such that $C = \cap U_n$ and $K = \cap V_n$. Wet $W_n = U_n \cup V_n$. Each W_n is open and $C \cup K \subseteq W_n$ for all n. So $C \cup K \subseteq \cap W_n$. Moreover, if $x \in \cap W_n$, then x belongs to either infinitely many of the U_n or infinitely many of the V_n. Since these are decreasing sequences, x either belongs to all the U_n or x belongs to all of the V_n. Thus $x \in C$ or $x \in K$ and we see $\cap W_n \subseteq C \cup K$. □

Lemma 6.6. *Let X be a locally compact Hausdorff space and suppose F is a closed G_δ subset of X with the relative topology. Then the Baire sets for the topological space F consist of all $F \cap B$ where B is a Baire subset of X.*

Proof. The collection $\{B \in \mathrm{Baire}(X) \mid F \cap B \in \mathrm{Baire}(F)\}$ is a σ-algebra. If K is a compact G_δ in X, then $K \cap F = \cap(U_n \cap F)$ where U_n are open subsets of X and $\cap U_n = K$. Thus this collection contains the compact G_δ subsets of X. Hence it contains all Baire subsets of X. So $F \cap B$ is Baire in F for each Baire subset B of X.

Next consider the collection of all subsets $\{E \subseteq F \mid E = F \cap B, B \in \mathrm{Baire}(X)\}$. This is a σ-algebra on F. Now $F = \cap_{k=1}^\infty G_k$ where each G_k is an open subset of X. Hence, if K is a compact G_δ subset of the space F, then $K = \cap(U_n \cap F)$ where the U_n for $n \geqslant 1$ are open subsets of X. Thus $K = \cap_{k,n}(G_k \cap U_n)$ is a compact G_δ subset of X and hence is Baire for X. Thus K is in this collection. Hence every Baire set for F is in this collection. □

We thus see that for closed G_δ Baire subsets F of X, the Baire sets for F are the Baire sets of X contained in F.

Part of the definition for a Radon measure μ on the Baire subsets of a locally compact Hausdorff space is *inner regularity* for every Baire set E; i.e.,

$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ is a compact } G_\delta\}.$$

The following shows σ-bounded Baire sets are always inner regular.

Proposition 6.7. *Let μ be a measure on the Baire subsets of a locally compact Hausdorff space X. Assume $\mu(K) < \infty$ for all compact G_δ subsets K. Then μ is inner regular on every σ-bounded Baire set.*

Proof. We first assume X is compact. Set $\mathcal{A}$ be the collection of all Baire subsets E of X where both E and $X - E$ are inner regular. This collection is closed under complements. We also note it contains each compact G_δ subset K. Indeed, if K is a compact G_δ, then clearly K is inner regular. Moreover, by Proposition 5.23, there is an $f \in C(X)$ with $0 \leqslant f \leqslant 1$ and $f^{-1}(1) = K$. Consider the subsets $K_n = \{x \mid f(x) \leqslant 1 - \frac{1}{n}\}$. These are an increasing sequence of compact G_δ subsets of the complement of K whose union is $X - K$. Since $\mu(X - K) < \infty$, we have $\mu(X - K) = \lim_n \mu(K_n)$. Thus $X - K$ is inner regular.

Next we note $\mathcal{A}$ is closed under countable unions. Indeed, suppose E_n for $n \geqslant 1$ are in $\mathcal{A}$. Note $\mu(E_n) \leqslant \mu(X) < \infty$ for all n. Choose $K_n \subseteq E_n$ with $\mu(E_n - K_n) < \frac{\epsilon}{2^n}$ where K_n is a compact G_δ. Set $E = \cup_{n=1}^{\infty} E_n$. By Lemma 6.5, $\cup_{i=1}^{n} K_i$ is a finite union of compact G_δ sets and thus is a compact G_δ. Also

$$\begin{aligned}
\mu(\cup_{i=1}^{n} K_i) \leqslant \mu(E) &= \lim_{n\to\infty} \mu\left((\cup_{i=1}^{n} K_i) \cup (\cup_{i=1}^{n}(E_i - K_i)\right) \\
&\leqslant \lim_{n\to\infty} \mu(\cup_{i=1}^{n} K_i) + \limsup_{n\to\infty} \sum_{i=1}^{n} \mu(E_i - K_i) \\
&< \lim_{n\to\infty} \mu(\cup_{i=1}^{n} K_i) + \epsilon.
\end{aligned}$$

This true for all $\epsilon > 0$ implies $\lim_{n\mapsto\infty} \mu(\cup_{i=1}^{n} K_i) = \mu(E)$. Thus E is inner regular.

Now $X - E = \cap(X - E_n)$. Since each $X - E_n$ is inner regular, there is a compact G_δ subset $K_n \subseteq X - E_n$ with $\mu((X - E_n) - K_n) < \frac{\epsilon}{2^n}$. Set $K = \cap_n K_n$. Then K is a compact G_δ subset of $\cap_n(X - E_n)$. Also $(X - E) - K = (X - \cup E_n) - K \subseteq \cup((X - E_n) - K_n)$ and consequently,

$$\mu((X - E) - K) \leqslant \sum_{n=1}^{\infty} \mu((X - E_n) - K_n) < \sum_n \frac{\epsilon}{2^n} = \epsilon.$$

Hence $X - E$ is inner regular. We thus see $\mathcal{A}$ is closed under countable unions and thus is a σ-algebra containing the compact G_δ subsets of X. Thus $\mathcal{A}$ contains all the Baire subsets of X.

Now let X be a locally compact Hausdorff space and E be a σ-bounded Baire subset of X. Then E is contained in a countable union of a sequence $\{K_n\}_{n=1}^{\infty}$ of compact sets.

Now for each n, choose a function $f_n \in C_c(X)$ with $0 \leqslant f_n \leqslant 1$ and $f_n(x) = 1$ for $x \in K_n$. Set $F_n = \{x \mid f_n(x) \geqslant \frac{1}{2}\} = \cap_k x \mid f_n(x) > \frac{1}{2} - \frac{1}{k}\}$. Since each F_n is a compact G_δ set containing K_n, we see by replacing F_n by $F_1 \cup F_2 \cup \cdots \cup F_n$ that we have an increasing sequence of compact G_δ subsets of X which cover E. Let $\epsilon > 0$. By Lemma 6.6, $E \cap F_n$ is a Baire subset for F_n and by the first part of the proof,

$$\mu(E \cap F_n) = \sup\{\mu(K) \mid K \text{ is a compact } G_\delta,\ K \subseteq E \cap F_n\}.$$

Since $\mu(E) = \lim_{n\to\infty} \mu(E \cap F_n)$, we see E is inner regular. □

Corollary 6.8. *Let X be a locally compact Hausdorff space and suppose μ is a Radon measure on X. If ν is a measure on the Baire subsets of X and*

$\nu(K) = \mu(K)$ for each compact G_δ, then μ and ν are equal on the σ-bounded Baire sets. In particular, if X is σ-compact, then $\nu = \mu$.

As we have seen in Lemma 6.4, if X is a second countable locally compact Hausdorff space, the Baire sets and the Borel sets are the same. Moreover, every subset is σ bounded. Thus a Borel measure on X is Radon if and only if the measure of each compact subset is finite.

In particular, in the second countable locally compact case, any left invariant Borel measure which is finite on compact sets and nonzero is a left Haar measure. For example, Lebesgue measures on the line or on $\mathbb{R}^n$ are Haar measures for these groups.

Example 6.9 ($GL(n,\mathbb{R})$)**.** *Recall that* $\mathrm{GL}(n,\mathbb{R})$ *can be viewed as an open dense subset of $M(n,\mathbb{R})$ and $M(n,\mathbb{R})$ can be naturally identified with $\mathbb{R}^{n^2}$ by stacking the n column vectors of $n \times n$ matrices into a column vector of length n^2. Define a Radon measure μ on* $\mathrm{GL}(n,\mathbb{R})$ *by*

$$\begin{aligned}\int_{\mathrm{GL}(n,\mathbb{R})} f(X)\, d\mu(X) :=& \\ &\int_{\mathrm{GL}(n,\mathbb{R})} f([x_{i,j}])|\det([x_{i,j}])|^{-n} dx_{1,1}\cdots dx_{1,n}\cdots dx_{2,1}\cdots dx_{n,n} \\ &\qquad\qquad = \int_{\mathrm{GL}(n,\mathbb{R})} f(X)\,|\det(X)|^{-n} d\lambda(X)\end{aligned}$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^{n^2}$. Let $C, X \in$ $\mathrm{GL}(n,\mathbb{R})$ *and denote by $\mathbf{x}_1, \ldots, \mathbf{x}_n$ the column vectors of X. Then the matrix CX is given by*

$$CX = (C\mathbf{x}_1, \ldots, C\mathbf{x}_n)$$

Hence left multiplication by C on $\mathrm{GL}(n,\mathbb{R})$ *corresponds after stacking column vectors to the linear transformation on $\mathbb{R}^{n^2}$ having $n^2 \times n^2$ matrix*

$$L_C = \begin{pmatrix} C & & 0 \\ & \ddots & \\ 0 & & C \end{pmatrix}.$$

This transformation has determinant $\det(C)^n$*. It follows using Theorem 2.22 that*

$$\begin{aligned}\int f(CX)\, d\mu(X) &= \int f(CX)\,|\det(X)|^{-n}\, d\lambda(X) \\ &= |\det C|^n \int f(CX)\,|\det(CX)|^{-n}\, d\lambda(X) \\ &= \int f(X)\,|\det(X)|^{-n}\, d\lambda(X)\end{aligned}$$

Hence μ is a left Haar measure.

It will be important later to integrate continuous functions using Radon measures. To facilitate this, we will need to know continuous functions are Baire measurable.

Definition 6.10. *Let X and Y be locally compact Hausdorff spaces. A function $f : X \to Y$ is said to be Baire measurable if $f^{-1}(E)$ is a Baire subset of X whenever E is a Baire subset of Y.*

Since by Lemma 6.4, the Baire and Borel sets for $\mathbb{R}$ are the same and a real valued function f on a measurable space $(X, \mathcal{A})$ is measurable if and only if the $f^{-1}(E)$ is measurable for each Borel subset E of $\mathbb{R}$, we see if $\mathcal{A}$ is the σ-algebra consisting of the Baire subsets of a locally compact Hausdorff space X, then f is measurable if and only if f is Baire measurable.

Lemma 6.11. *Let X be a σ-compact locally compact Hausdorff space. If f is a continuous function on X, then f is Baire measurable.*

Proof. Note this is true if f has compact support. Suppose f has non compact support. Now there is an increasing sequence of compact G_δ subsets F_k whose union is X. By Proposition 5.23, for each k, we can choose $\phi_k \in C_c(X)$ such that $\phi_k = 1$ on F_k. So $f\phi_k \in C_c(X)$ and thus is Baire measurable. Since f is the pointwise limit of the sequence $f\phi_k$, f is Baire measurable. □

We end this section with a discussion of product spaces. Note by Exercise 6.1.25, if X and Y are locally compact Hausdorff spaces and $\mathcal{A}$ is the σ-algebra of Baire sets for X and $\mathcal{B}$ is the σ-algebra of Baire sets for Y, then $\mathcal{A} \times \mathcal{B}$ need not be the σ-algebra of Baire subsets for the product topology on $X \times Y$. Exercise 6.1.26 establishes a similar result for the Borel sets. However, one does have the following:

Proposition 6.12. *Let $\mathcal{A}$ be the σ-algebra of Baire sets for a σ-compact locally compact Hausdorff space X and let $\mathcal{B}$ be the σ-algebra of Baire sets for a locally compact Hausdorff space Y. Then $\mathcal{A} \times \mathcal{B}$ is the σ-algebra of Baire sets for the locally compact Hausdorff space $X \times Y$.*

Proof. Let $\mathcal{C}$ be the σ-algebra of Baire sets for $X \times Y$. First note if f and g are continuous complex valued functions on X and Y with compact support, then $f \times g$ defined by $f \times g(x, y) = f(x)g(y)$ is continuous and has compact support. Also it is easy to check $f \times g$ is $\mathcal{A} \times \mathcal{B}$ measurable. Now the linear span S of all functions $f \times g$ where $f \in C_c(X)$ and $g \in C_c(Y)$ is an algebra of continuous functions that is closed under conjugation, separates points, and vanishes at no point. By the Stone-Weierstrass Theorem, this algebra is uniformly dense in $C_c(X \times Y)$. Thus if $F \in C_c(X \times Y)$, there exists a sequence $F_n \in S$ with $F_n \to F$ uniformly and hence pointwise. Since each F_n is $\mathcal{A} \times \mathcal{B}$ measurable, the limit F is $\mathcal{A} \times \mathcal{B}$ measurable. Since the Baire sets for $X \times Y$ are the elements in the smallest σ-algebra making all $F \in C_c(X \times Y)$ measurable, we see $\mathcal{C} \subseteq \mathcal{A} \times \mathcal{B}$.

Conversely, we know $X = \cup F_n$ where each F_n is a compact G_δ subset of X. Now by Lemma 6.6, the Baire sets for the subspace F_n are the Baire sets for X contained in F_n. Thus to show $\mathcal{A} \times \mathcal{B} \subseteq \mathcal{C}$, it suffices to show all sets $U \times V$ where U is a Baire subset of F_n and V is a Baire set for Y are in $\mathcal{C}$. Let F be a compact G_δ subset of X and set $\mathcal{D}$ to be the collection of Baire subsets V of Y such that $F \times V$ is a Baire subset of $X \times Y$. $\mathcal{D}$ is a σ-algebra and if V is a G_δ compact subset of Y, then $F \times V$ is a compact G_δ subset of $X \times Y$. Thus $\mathcal{D}$ contains the compact G_δ subsets of Y. Consequently, $\mathcal{D} = \mathcal{B}$. Thus $F \times V \in \mathcal{C}$ for all $V \in \mathcal{B}$. Now for each n and any given $V \in \mathcal{B}$, let $\mathcal{E}$ be the collection of Baire subsets U of the space F_n such that $U \times V \in \mathcal{C}$. Again $\mathcal{E}$ is a σ-algebra. Moreover, if U is a compact G_δ subset of F_n, then as already been established we have $U \in \mathcal{E}$. Thus $\mathcal{E}$ contains the Baire subsets of F_n. So $U \times V$ is Baire in $X \times Y$ for any Baire subset U of F_n and any Baire subset V of Y. □

We remark that each locally compact group is a disjoint union of open σ-compact sets. This turns out to be sufficient to make the σ-algebra of Baire sets for $G \times H$ be the product of the σ-algebras of Baire sets for G and H whenever G and H are locally compact Hausdorff groups. Indeed, see Exercise 6.1.35. One cannot show in general that multiplication is Baire measurable. See Exercise 6.1.36. We do, however, have the following result.

Lemma 6.13. *Let G be a σ-compact locally compact Hausdorff group. Then the function $(x, y) \mapsto xy$ is Baire measurable from $G \times G$ into G.*

Proof. Let f be the function $f(x, y) = xy$. We need to show the preimage $f^{-1}(E)$ of any Baire subset E of G is a Baire subset of $G \times G$. But the collection of subsets E of G where $f^{-1}(E)$ is a Baire subset of $G \times G$ is a σ-algebra. Since the Baire subsets of G are generated by the compact G_δ subsets F of G, it suffices to show $f^{-1}(F)$ is Baire for each compact G_δ subset F. Now using the σ-compactness of G and Exercise 6.1.1, we see there are compact G_δ subsets F_k of G such that $G = \cup_{k=1}^{\infty} F_k$. Consequently the sets $F_j \times F_k$ are compact G_δ subsets of $G \times G$ and $f^{-1}(F) = \cup_{j,k} f^{-1}(F) \cap (F_j \times F_k)$. Thus it is sufficient to show $f^{-1}(F) \cap (F_j \times F_k)$ is a compact G_δ subset of $G \times G$ for each j and k. Since $F = \cap_{k=1}^{\infty} U_k$ where U_k are open and f is continuous, $f^{-1}(F) = \cap_k f^{-1}(U_k)$ is a closed G_δ subset of $G \times G$. Consequently, $f^{-1}(F) \cap (F_j \times F_k)$ is a compact G_δ subset of $G \times G$ for each pair of j and k. □

Relatively Invariant Measures. Let H be a closed subgroup of a locally compact Hausdorff group G. Then by Lemma 5.13, the homogeneous space G/H with quotient topology is Hausdorff and the mapping

$$(g, xH) \mapsto gxH$$

is a continuous action of G on G/H. Moreover, since the mapping $\kappa : G \to G/H$ is an open mapping, the space G/H is locally compact.

Lemma 6.14. *Use dh to denote a left Haar measure on the σ-algebra of Baire sets of H. The mapping $f \mapsto f_H$ defined by $f_H(xH) = \int f(xh)\,dh$ maps $C_c(G)$ onto $C_c(G/H)$. More specifically, for each compact subset W of G/H, there is a linear mapping T_W from the subspace $C_W(G/H) = \{f \in C_c(G/H) \mid \operatorname{supp} f \subseteq W\}$ into $C_c(G)$ satisfying $T_W(f_1) \geqslant T_W(f_2)$ if $f_1 \geqslant f_2$, $T_W(|f|) = |T(f)|$, and $(T_W f)_H = f$.*

Proof. Let m be the left Haar measure on H corresponding to dh. Suppose $f \in C_c(G)$. To see f_H is continuous, let $\epsilon > 0$. Choose a compact neighborhood N of e. By left uniform continuity of f, we choose a neighborhood N' of e contained in N such that

$$|f(ny) - f(y)| \leqslant \frac{\epsilon}{m(H \cap x^{-1}N^{-1}\operatorname{supp} f)} \text{ for all } y \in G \text{ for } n \in N'.$$

Let $n \in N'$. Then $f(nxh) = 0$ and $f(xh) = 0$ for $h \notin H \cap x^{-1}N^{-1}\operatorname{supp} f$. Hence

$$\begin{aligned} |f_H(nxH) - f_H(xH)| &\leqslant \int_H |f(nxh) - f(xh)|\,dh \\ &\leqslant \int_{H \cap x^{-1}N^{-1}\operatorname{supp} f} \frac{\epsilon}{m(H \cap x^{-1}N^{-1}\operatorname{supp} f)}\,dh \\ &= \epsilon. \end{aligned}$$

So f_H is continuous.

Moreover, recalling $\kappa : G \to G/H$ is the mapping $g \mapsto gH$, we have $\operatorname{supp}(f_H) \subseteq \kappa(\operatorname{supp} f)$. Hence $f_H \in C_c(G/H)$ for $f \in C_c(G)$.

Now choose an open set V in G with $\bar{V}$ compact and $\kappa(V) \supseteq W$. By Proposition 5.23, we can find a function t with $t = 1$ on V; $t \geqslant 0$, and $t \in C_c(G)$. Then $t_H > 0$ on $\kappa(V)$. Let $F \in C_W(G/H)$. Define $T_W f(x) = \frac{t(x)}{t_H(xH)} f(xH)$ where we are using $\frac{0}{0} = 1$. Note $T_W f$ is continuous where $t_H(xH) > 0$. If $f(xH) \neq 0$, then $xH \in VH$ and thus $t_H(xH) > 0$. Suppose $t_H(xH) = 0$. Then $xH \notin VH$ and so $xH \notin W$. Since W is closed and G/H has the quotient topology, it follows there is an open neighborhood U of x in G with $\kappa(U) \cap W = \varnothing$. So $f = 0$ on UH. Hence $T_W f$ is continuous on U which contains x. The mapping T_W is a linear mapping satisfying

$|T_W(f)| = T_W(|f|)$ and $T_W(f_1) \geqslant T_W(f_2)$ if $f_1 \geqslant f_2$. Finally

$$\begin{aligned}(T_W f)_H(xH) &= \int_H (T_W f)(xh)\, dh \\ &= \int \frac{t(x)}{t_H(xH)} f(xH)\, dh \\ \\ &= f(xH)\end{aligned}$$

for $x \in G$. □

Theorem 6.15. *Suppose H is a closed subgroup of a locally compact Hausdorff group G. Let ϕ be a positive continuous function on G satisfying $\phi(xh) = \phi(x)\frac{\Delta_H(h)}{\Delta_G(h)}$. Then I defined on $C_c(G/H)$ by $I(f_H) = \int \phi(x) f(x)\, dx$ where $f \in C_c(G)$ is well defined on $C_c(G/H)$ and defines a Radon measure μ on G/H satisfying*

$$d\mu(xyH) = \frac{\phi(xy)}{\phi(y)} d\mu(yH).$$

Thus

$$\int_{G/H} F(x^{-1}yH)\, d\mu(yH) = \int_{G/H} \frac{\phi(xy)}{\phi(y)} F(yH)\, d\mu(yH)$$

for $F \in C_c(G/H)$.

Proof. We first show I is well defined. Indeed, suppose $f_H = 0$. Choose $g \in C_c(G)$ with $g_H = 1$ on the compact set $\kappa(\operatorname{supp} f)$. Using the compactness of the supports of f and g one can use Fubini and obtain

$$\begin{aligned}0 &= \int\int \phi(x)g(x)f(xh)\, dh\, dx \\ &= \int\int \phi(x)g(x)f(xh)\, dx\, dh \\ &= \int\int \phi(xh^{-1})g(xh^c)f(x)\Delta_G(h^{-1})\, dx\, dh \\ &= \int\int \Delta_G(h)\Delta_H(h^{-1})\phi(xh)g(xh)f(x)\, dh\, dx \\ &= \int\int \phi(x)g(xh)f(x)\, dh\, dx \\ &= \int \phi(x)f(x)g_H(xH)\, dx \\ &= \int \phi(x)f(x)\, dx.\end{aligned}$$

So I is well defined. Now

$$\begin{aligned}\int f_H(yH)\,d\mu(xyH) &= \int f_H(x^{-1}yH)\,d\mu(yH)\\ &= \int f(x^{-1}y)\phi(y)\,dy\\ &= \int f(y)\phi(xy)\,dy\\ &= \int f(y)\frac{\phi(xy)}{\phi(y)}\,\phi(y)\,dy\\ &= \int \frac{\phi(xy)}{\phi(y)} f_H(yH)\,d\mu(yH).\end{aligned}$$

□

Positive functions ϕ satisfying $\phi(xh) = \phi(x)\frac{\Delta_H(h)}{\Delta_G(h)}$ are called **rho** functions. As seen above, if they are continuous or more generally measurable, one can obtain measures having translation given by Radon-Nikodym derivative $x \mapsto \frac{\phi(xy)}{\phi(y)}$; i.e.,

$$\mu(xE) = \int_E \frac{\phi(xy)}{\phi(y)}\,d\mu(y).$$

In general continuous rho functions may or may not exist. However, for paracompact groups they always exist.

Recall a Hausdorff space X is **paracompact** if every open covering has an open locally finite refinement. Examples include both metrizable spaces and compact Hausdorff spaces. Since second countable locally compact Hausdorff spaces are metrizable, homogeneous spaces G/H are paracompact if G is a second countable locally compact Hausdorff group and H is a closed subgroup. However, for groups more is true. Indeed, every locally compact Hausdorff group G is paracompact and so are their quotients G/H for closed subgroups H; see Exercise 6.1.31.

Lemma 6.16. *Let G be a locally compact Hausdorff group with closed subgroup H. Then there is a positive continuous function $\phi(x)$ with $\phi(xh) = \phi(x)\frac{\Delta_H(h)}{\Delta_G(h)}$ for all $x \in G$ and $h \in H$.*

Proof. We use G/H is paracompact. Since G/H is locally compact, we can find a locally finite cover $\mathcal{U}$ of G/H consisting of open sets U with each $\bar{U}$ compact. Now consider the collection of all open sets V with $\bar{V} \subseteq U$ for some $U \in \mathcal{U}$. This is an open cover of G/H. Hence it has a locally finite refinement $\mathcal{V}$ of open sets covering G/H.

For each open set U in $\mathcal{U}$, set $W_U = \bigcup\{V \in \mathcal{V} \mid \bar{V} \subseteq U\}$. The sets W_U for $U \in \mathcal{V}$ form an open cover for G/H. Since $W_U \subseteq U$, $\bar{W}_U \subseteq \bar{U}$. Thus each

$\bar{W}_U$ is compact. We finally note $\bar{W}_U \subseteq U$. Indeed, let $x \in \bar{W}_U$. Choose a neighborhood N_x of x that meets only finitely many V in $\mathcal{V}$. In particular, $\{V \in \mathcal{V} \mid \bar{V} \subseteq U,\ N_x \cap V \neq \varnothing\}$ consists of finitely many sets $V_1, V_2, \ldots, V_n$. This implies $x \in \overline{V_1 \cup V_2 \cup \cdots \cup V_n} = \cup_{k=1}^n \bar{V}_k \subseteq U$. So $\bar{W}_U \subseteq U$.

Now by Proposition 5.23, one can find for each $U \in \mathcal{U}$ a continuous function F_U of compact support inside U and satisfying $0 \leqslant F_U \leqslant 1$ and $F_U = 1$ on W_U. By Lemma 6.14 and its proof, there are nonnegative $f_U \in C_c(G)$ such that

$$F_U(xH) = \int f_U(xh)\,dh$$

for all xH. Define $f = \sum_{U \in \mathcal{U}} f_U$. Note if $x \in G$, there is an open set N in G/H with $xH \in N$ and N meets only finitely many U. Since F_U has compact support in U, this implies f_U is zero on $\kappa^{-1}(N)$ for all but finitely many U. Thus f is defined, nonnegative, and continuous. Moreover, for each x, $f_U(xh) > 0$ for some U and h; and the set of h with $f_U(xh) > 0$ is precompact.

Now set $\delta(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$. Define $\phi(x) = \int_H f(xh)\,\delta(h^{-1})\,dh$. Note ϕ is continuous for

$$\int_H f(yh)\delta(h^{-1})\,dh = \sum_{U \cap N \neq \varnothing} \int f_U(yh)\delta(h^{-1})\,dh \text{ when } y \in N.$$

Moreover, $\phi(xh') = \int f(xh'h)\delta(h^{-1})\,dh = \int \psi(xh)\delta(h^{-1}h')\,dh = \delta(h')\phi(x)$. $\square$

Theorem 6.17 (Relatively invariant measures)**.** *Assume G is locally compact and Hausdorff and H is a closed subgroup. Let δ be a continuous homomorphism from G into the positive multiplicative reals. Then there is a Radon measure μ on G/H satisfying $\mu(xE) = \delta(x)\mu(E)$ if and only if $\delta(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$ for $h \in H$.*

Proof. We have done one direction already. Suppose μ is a measure on G/H satisfying $\mu(xE) = \delta(x)\mu(E)$. For $f \in C_c(G)$, define $I(f) = \int (\check{\delta} f)_H\,d\mu$ where $\check{\delta}$ is defined by $\check{\delta}(x) = \delta(x^{-1})$. Define ${}_yf$ by ${}_yf(x) = f(yx)$. Note $(\check{\delta}_y f)_H(xH) = \int \check{\delta}(xh) f(yxh)\,dh = \delta(y) \int \check{\delta}(yxh) f(yxh)\,dh = \delta(y)_y(\check{\delta} f)_H(xH)$. Thus

$$\begin{aligned} I({}_yf) &= \int \delta(y)(\check{\delta} f)_H(yxH)\,d\mu(xH) \\ &= \int \delta(y)(\check{\delta} f)_H(xH)\,d\mu(y^{-1}xH) \\ &= \delta(y)\delta(y^{-1}) \int (\check{\delta} f)_H(xH)\,d\mu(xH) \\ &= I(f). \end{aligned}$$

Hence I is a left Haar integral for G and we have

$$\int\int \check{\delta}(xh)f(xh)\,dh\,d\mu(xH) = \int f(y)\,dy.$$

Next note

$$\begin{aligned}\Delta_G(h^{-1})\int f(y)\,dy &= \int f(yh)\,dy \\ &= \int\int \check{\delta}(xh')f(xh'h)\,dh'\,d\mu(xH) \\ &= \int\int \check{\delta}(xh'h^{-1})f(xh')\Delta_H(h^{-1})\,dh'\,d\mu(xH) \\ &= \delta(h)\Delta_H(h^{-1})\int\int \check{\delta}(xh')f(xh')\,dh'\,d\mu(xH) \\ &= \delta(h)\Delta_H(h^{-1})\int f(y)\,dy.\end{aligned}$$

Thus $\delta(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$ for $h \in H$. ☐

Corollary 6.18. *Any two nonzero δ covariant Radon measures on G/H are positive multiples of each other.*

Proof. Indeed,

$$\int (\check{\delta}f)_H(xH)\,d\mu(xH) = \int f(y)\,dy.$$

Thus $\int f_H\,d\mu = \int \delta(y)\,f(y)\,dy$ where dy is a left Haar measure. Since Haar measures are determined up to a multiple by a positive scalar, the result follows. ☐

Quasi-invariant measures on G/H. Let μ be a Radon measure on G/H. The measure $x\mu$ on the Baire sets of G/H given by

$$x\mu(E) = \mu(x^{-1}E) \tag{6.1}$$

is called the *left translate* of μ by x. Note

$$\begin{aligned}(xy)\mu(E) &= \mu(y^{-1}x^{-1}E) \\ &= (y\mu)(x^{-1}E) \\ &= x(y\mu)(E).\end{aligned}$$

Moreover, one can show (see Exercise 6.1.27)

$$\int f(xyH)\,d\mu(yH) = \int f(yH)\,d(x\mu)(yH) \tag{6.2}$$

for functions $f \in C_c(G/H)$.

Definition 6.19. *Let H be a closed subgroup of a locally compact Hausdorff group G. Then a left quasi-invariant measure μ on G/H is a Radon measure with the property that a Baire subset $E \subseteq G/H$ has μ measure 0 if and only if $\mu(xE) = 0$ for all $x \in G$.*

In particular, a Radon measure μ on G/H is quasi-invariant if and only if all the measures $x\mu$ have the same null sets. Furthermore, one has analogous notions on homogeneous spaces $H\backslash G$. Namely one takes $\mu x(E) = \mu(Ex^{-1})$ and one again has the property $(\mu x)y = \mu(xy)$. Thus a Radon measure μ on G is *right quasi-invariant* if $\mu(E) = 0$ if and only if $\mu(Ex) = 0$ for Baire subsets E and all $x \in G$.

Lemma 6.20. *Let X be a locally compact Hausdorff space and suppose K is a compact subset of an open subset V of X. Then there is a compact G_δ subset of V whose interior contains K. Moreover, if K is a compact G_δ, then there exists a pointwise decreasing sequence $F_n \in C_c(X)$ such that $F_n \to \chi_K$ pointwise.*

Proof. By Proposition 5.23, there is an $f \in C_c(X)$ with $0 \leqslant f \leqslant 1$ and $f(x) = 1$ for $x \in K$ and $\operatorname{supp} f \subseteq V$. Set $F = \{x \mid f(x) \geqslant \frac{1}{2}\}$. Clearly F is compact. Since $F = \cap_{k=2}^{\infty}\{x \mid f(x) > \frac{1}{2} - \frac{1}{k}\}$, we see F is a G_δ set. Also the interior of F contains the open subset $\{x \mid f(x) > \frac{1}{2}\}$ that contains K. Now if K is also a G_δ, then we can take f to have the additional property that $f(x) = 1$ implies $x \in K$. Since $0 \leqslant f(x) \leqslant 1$, we then would have $F_n = f^n$ decreases pointwise to x_K. □

If ϕ is a continuous rho function on G, the Radon measure μ given by

$$\int f_H(yH)\,d\mu(yH) = \int f(x)\phi(x)\,dx$$

in Theorem 6.15 is left quasi-invariant. Indeed, Theorem 6.15 and Equation 6.2 give

$$\int f(xyH)d\mu(yH) = \int f(yH)\frac{\phi(x^{-1}y)}{\phi(y)}\,d\mu(yH) = \int f(yH)\,d(x\mu)(yH)$$

for $f \in C_c(G/H)$. Together Lemma 6.20 and the Lebesgue Dominated Convergence Theorem imply $\mu(x^{-1}K) = \int_K \frac{\phi(x^{-1}y)}{\phi(y)}\,d\mu(yH)$ for compact G_δ sets K. Now ν defined by $\nu(E) = \int_E \frac{\phi(x^{-1}y)}{\phi(y)}\,d\mu(yH)$ is a Radon measure; inner regularity, i.e. $\mu(E) = \sup \mu(K)$ over the compact G_δ subsets K of E, can be shown using the Monotone Convergence Theorem. Since Radon measures which agree on compact G_δ sets are equal, one has $x\mu = \nu$. Thus

$$\mu(x^{-1}E) = \int_E \frac{\phi(x^{-1}y)}{\phi(y)}\,d\mu(yH)$$

for Baire sets E.

Proposition 6.21. *Let μ be a nonzero left quasi-invariant Radon measure and let ν be a nonzero right quasi-invariant Radon measure on a σ-compact locally compact group G. Then μ and ν are equivalent.*

Proof. Since μ and ν are finite on compact G_δ's and G is σ-compact and by Exercise 6.1.1 each compact subset is a subset of a compact G_δ, we see μ and ν are σ-finite and hence are equivalent to finite measures.

We may therefore assume μ and ν are finite measures. By Proposition 6.12, if $\mathcal{A}$ is the σ-algebra of Baire subsets of G, then $\mathcal{A}\times\mathcal{A}$ is the σ-algebra of Baire subsets of $G\times G$. Moreover, if E is a Baire subset of G, then by Lemma 6.13, $\{(x,y) \mid xy \in E\}$ is a Baire subset of $G\times G$. Consequently if $\mu\times\nu$ is the product measure on $\mathcal{A}\times\mathcal{A}$, we have by Fubini's Theorem, $\mu(E) = 0$ if and only if $\mu(x^{-1}E) = 0$ for all $x \in G$ if and only $\mu\{y \mid xy \in E\} = 0$ for all $x \in G$ if and only if $(\mu\times\nu)\{(x,y) \mid xy \in E\} = 0$ if and only if $\nu\{x \mid xy \in E\} = 0$ a.e. y if and only if $\nu(Ey^{-1}) = 0$ for all y if and only if $\nu(E) = 0$. $\square$

Corollary 6.22. *Let G be a locally compact group which is σ-compact. Then any two nonzero left quasi-invariant Radon measures on G are equivalent.*

Proof. Let μ and ν be such measures. Then ν^{-1} is a nonzero Radon measure which is right quasi-invariant. Thus $\mu \sim \nu^{-1}$ and $\nu \sim \nu^{-1}$. Hence $\mu \sim \nu$. $\square$

Lemma 6.23. *Let H be a closed subgroup of a σ-compact locally compact Hausdorff group G. Then $\kappa : G \to G/H$ is Baire; i.e., $\kappa^{-1}(E)$ is a Baire subset of G if E is Baire subset of G/H. Moreover, $F\circ\kappa$ is Baire measurable on G for each Baire measurable function F on G/H.*

Proof. The argument for the first statement here can be made with the same line of reasoning used in the proof of Lemma 6.13. Just replace f by κ and use G and G/H instead of $G\times G$ and G.

Now for the statement regarding the Baire measurable function F on G/H, one has $(F\circ\kappa)^{-1}(U) = \kappa^{-1}(F^{-1}(U))$ is a Baire subset of G for each open subset U of $\mathbb{R}$. $\square$

Theorem 6.24. *Let H be a closed subgroup of a locally compact σ-compact Hausdorff group G. Any two left quasi-invariant measures on G/H are equivalent.*

Proof. Let μ be a quasi-invariant Radon measure on G/H. Note G/H is σ-compact and hence μ is σ-finite. This implies there is an equivalent Radon probability measure. Hence we may assume $\mu(G/H) = 1$.

Define a Radon measure μ^* on G by

$$\int_G f(x)\,d\mu^*(x) := \int_{G/H} f_H(yH)\,d\mu(yH) \tag{6.3}$$

for $f \in C_c(G)$. We claim μ^* is left quasi-invariant. Indeed, if $A = \frac{d(x\mu)}{d\mu}$, then A is Baire measurable and nonnegative and

$$\begin{aligned}
\int f(xy)\,d\mu^*(y) &= \int\int_H f(xyh)dh\,d\mu(yH) \\
&= \int A(yH)\int_H f(yh)\,dh\,d\mu(yH) \\
&= \int A(yH) f_H(yH)\,d\mu(yH) \\
&= \int ((A\circ\kappa)f)_H(yH)\,d\mu(yH) \\
&= \int (A\circ\kappa)(y) f(y)\,d\mu^*(y)
\end{aligned}$$

for $f \in C_c(G)$.

It follows that if K is a compact G_δ subset of G, then since by Lemma 6.20 there is a decreasing sequence $f_n \in C_c(G)$ of functions $0 \leqslant f_n \leqslant 1$ with $f_n(x) \to \chi_K(x)$ for all x, that

$$\int \chi_K(xy)\,d\mu^*(y) = \int_K (A\circ\kappa)(y)\,d\mu^*(y).$$

So we see

$$\mu^*(x^{-1}K) = \int_K (A\circ\kappa)(y)\,d\mu^*(y).$$

Now define measure ν^* on the Baire subsets of G by

$$\nu^*(E) = \int_E (A\circ\kappa)(y)\,d\mu^*(y) = \int (A\circ\kappa)(y)\chi_E(y)\,d\mu^*(y).$$

Note the Radon measure $x\mu^*$ agrees with the Baire measure ν^* on the compact G_δ's. By Corollary 6.8, $\nu^* = x\mu^*$. Thus the Radon measure $x\mu^*$ is given by

$$(x\mu^*)(E) = \mu^*(x^{-1}E) = \int_E (A\circ\kappa)(y)\,d\mu^*(y)$$

for all Baire subsets E of G. This implies for each $x \in G$, $(x\mu^*)(E) = 0$ whenever $\mu^*(E) = 0$. Hence the Radon measure μ^* is left quasi-invariant on G. By Corollary 6.22, we then know μ^* is equivalent to Haar measure on G.

Now let K be a compact G_δ subset of G/H. By Lemma 6.20, there is a sequence $F_n \in C_c(G/H)$ so that pointwise F_n decreases with limit χ_K. Since $((F_n \circ \kappa)f)_H(yH) = F_n(yHf_H(yH)$ for all y, (6.3) gives

$$\int_G (F_n \circ \kappa)(x) f(x)\, d\mu^*(x) = \int_{G/H} F_n(yH) f_H(yH)\, d\mu(yH)$$

for all n. Letting $n \to \infty$, we thus see if $f \in C_c(G)$ and K is a compact G_δ subset of G/H, then

$$\int (\chi_K \circ \kappa)(x) f(x)\, d\mu^*(x) = \int \chi_K(yH) f_H(yH)\, d\mu(yH).$$

Now suppose $f \in C_c(G)$ and $f \geqslant 0$. Define measures ν_1 and ν_2 by

$$\nu_1(E) = \int \chi_E \circ \kappa(x) f(x)\, d\mu^*(x)$$
$$\nu_2(E) = \int \chi_E(yH) f_H(yH)\, d\mu(yH)$$

for Baire subsets E of G/H. We note $\nu_1(K) = \nu_2(K) < \infty$ for compact G_δ sets K. Since every subset is σ-bounded, by Proposition 6.7, we know $\nu_1 = \nu_2$. Thus

$$\int (\chi_E \circ \kappa)(x) f(x)\, d\mu^*(x) = \int \chi_E(yH) f_H(yH)\, d\mu(yH) \tag{6.4}$$

for every Baire subset E of G/H and all $f \in C_c(G)$.

By Lemma 6.23, $\kappa^{-1}(E)$ is a Baire subset of G for each Baire subset E of G/H. We claim that a Baire subset E of G/H has μ measure 0 if and only if $\kappa^{-1}(E)$ has Haar measure 0. This would then give the result.

First let E have μ measure 0 in G/H. Suppose $f \in C_c(G)$. Then by (6.4), $\int \chi_E(yH) f_H(yH)\, d\mu(yH) = \int (\chi_E \circ \kappa)(x) f(x)\, d\mu^*(x)$. So if E has μ measure 0, $\int (\chi_E \circ \kappa)(x) f(x)\, d\mu^*(x) = 0$ for all $f \in C_c(G)$. Using Lemma 6.20 to obtain a decreasing sequence $\{f_n\}$ in $C_c(G)$ converging pointwise to χ_K, we see $\int (\chi_E \circ \kappa)(x) \chi_K(x)\, d\mu^*(x) = 0$ for all compact G_δ subsets K of G. Thus for each compact G_δ subset K of $\kappa^{-1}(E)$, $\mu^*(K) = 0$. By inner regularity, $\mu^*(\kappa^{-1}(E)) = 0$.

Next suppose $\mu^*(\kappa^{-1}(E)) = 0$ where E is a Baire subset of G/H. Using (6.4), we see $\int (\chi_E \circ \kappa)(x) f(x)\, d\mu^*(x) = \int \chi_E(yH) f_H(yH)\, d\mu(yH) = 0$ for all $f \in C_c(G)$. Then again by Lemma 6.20, if K is a compact G_δ subset of E, there is a decreasing sequence F_n in $C_c(G/H)$ such that $F_n \to \chi_K$. Now by Lemma 6.14, we can find a decreasing sequence f_n in $C_c(G)$ such that $(f_n)_H = F_n$ and the f_n converge pointwise. Thus $\int \chi_E(yH) F_n(yH)\, d\mu(yH) = 0$ for all n. Taking a limit, we have $\mu(K) = \int \chi_E(yH) \chi_K(yH)\, d\mu(yH) = 0$. By inner regularity, $\mu(E) = 0$. $\square$

This theorem holds even when G is not σ-compact. See Exercises 1.33 and 1.34.

Corollary 6.25. *Let μ be a quasi-invariant Radon measure on G/H. Then a Baire subset E of G/H has μ measure 0 if and only if $\kappa^{-1}(E)$ has Haar measure 0.*

Exercise Set 6.1

In the following exercises, unless otherwise stated, X and Y denote locally compact Hausdorff spaces and G will denote a locally compact Hausdorff topological group.

1. Use Proposition 5.23 to show every every compact subset K of X is contained in a compact G_δ.

2. Show if E is a Baire subset of X, then E or $X - E$ is σ-bounded.

3. Let X be a second countable locally compact Hausdorff space.

(a) Show X has a countable dense subset.

(b) Show there is a countable base for the topology of X consisting of open sets whose closures are compact.

(c) Show X is σ-compact.

4. Let X be a locally compact Hausdorff space with Radon measure μ. Show if ν is the unique regular extension of μ to the Borel subsets of X, then for each Borel subset E of ν finite measure, there is a Baire subset E' such that $\nu(E\backslash E') + \nu(E'\backslash E) = 0$; i.e., E and E' are the same sets in the measure algebra of ν.

5. Let μ be a Radon measure on a locally compact Hausdorff space X and let ν be the regular Borel measure measure which extends μ. Show $L^p(\mu) = L^p(\nu)$ for $1 \leqslant p < \infty$ while $L^\infty(\mu)$ might be strictly smaller than $L^\infty(\nu)$.

6. Show m is a left invariant regular Borel measure on G if and only if m^* defined by

$$m^*(E) = m(E^{-1})$$

is a right invariant regular Borel measure on G. Then show the modular function for m^* and m satisfy $\Delta^* = \check{\Delta}$; i.e., show if $m(Eg) = \Delta(g)m(E)$ for all Borel sets E, then $m^*(gE) = \Delta(g^{-1})m^*(E)$ for all Borel sets E.

7. Show Haar measures on the multiplicative group of positive reals is given by $\frac{dx}{x}$. That is show $\int_{\mathbb{R}^+} f(x)\,dm(x) = \int_0^\infty \frac{f(x)}{x}\,dx$.

8. Let G be the $ax+b$ group; that is the subset $\{(a,b) \mid a>0,\, b \in \mathbb{R}\}$ of $\mathbb{R}^2$ with the relative topology and with multiplication defined by

$$(a,b)(a',b') = (aa', b+ab').$$

Show that I defined by

$$I(f) = \int_0^\infty \int_{-\infty}^\infty f(a,b)\,\frac{1}{a^2}\,db\,da$$

is a positive left invariant integral on $C_c(G)$. Then show the modular function for G is given by $\Delta(a,b) = \frac{1}{a}$. In particular, the $ax+b$ group is nonunimodular.

9. Let G be a discrete topological group. Determine the Baire sets for G. Then show counting measure on the Baire sets is both left and right invariant.

10. Let $C_0(X)$ be the space of all continuous functions f on X that vanish at ∞; i.e., for each $\epsilon > 0$, $\{x \mid |f(x)| \geqslant \epsilon\}$ is compact. Show $C_0(X)$ is a Banach space when equipped with norm $|\cdot|_\infty$ where $|f|_\infty = \max_{x\in X}|f(x)|$. Then show $C_c(X)$ is dense in $C_0(X)$.

11. Let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X and let $C(X_\infty)$ be the space of continuous functions on X_∞. Show that

$$C_0(X) = \{f|_X \mid f \in C(X_\infty) \text{ and } f(\infty) = 0\}$$

and that $C(X_\infty)$ corresponds to the space of bounded continuous functions on X such that $\lim_{x\to\infty} f(x)$ exists. Finally show that every finite Borel measure on X gives by integration a continuous linear functional on $C(X_\infty)$.

12. Let λ be the Lebesgue measure on $\mathbb{R}$.

(a) Show μ defined by $\mu(E) = \int_E \frac{1}{x}\,d\lambda(x)$ is a Radon measure on $\mathbb{R}^+$.

(b) Show the measure μ can not be extended to a Radon measure on $\mathbb{R}$.

13. Let G be a locally compact Hausdorff topological group and let m be a left Haar measure on G. Show the following:

(a) G is compact if and only if $m(G) < \infty$. (**Hint:** If G is not compact, show if V is a compact symmetric neighborhood of e, there is a sequence g_k where the sets g_kV are disjoint.)

(b) G is discrete if and only if $m(\{e\}) > 0$. (**Hint:** Show if $m(\{e\}) > 0$ and G is not discrete, then there is a compact subset of infinite measure.)

14. Let μ be a Radon measure on X. Show $C_c(X)$ is dense in $L^p(X)$ for $1 \leqslant p < \infty$.

15. Let K be a compact subgroup of G. Set $X = G/K$, and let $\kappa : G \to X$ be the canonical map $a \mapsto aH$. Fix a left Haar measure m for G. Define a linear functional I_K on $C_c(X)$ by

$$I_K(f) = \int f \circ \kappa(a)\, dm(a)\,.$$

Show that I_K is a positive, G-invariant integral on $C_c(X)$. Then determine an expression for the Radon measure μ on X for this integral in terms of m.

16. By Lemma 5.25, $\mathrm{SL}(2,\mathbb{R})$ acts on $H_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$, has stabilizer $\mathrm{SO}(2)$, and $H_+ = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. Define a linear functional I on $C_c(H_+)$ by

$$I(f) = \int_{-\infty}^{\infty}\int_0^{\infty} f(x+iy)\, \frac{dydx}{y^2}\,.$$

Show that I is positive, non-zero, and $\mathrm{SL}(2,\mathbb{R})$-invariant.

17. Let H be a locally compact Hausdorff group and suppose the modular function for H is Δ. Define a multiplication on $G = H \times \mathbb{R}$ by

$$(x,s)(y,t) = (xy, \Delta(y^{-1})s + t).$$

Then show if dy is a left invariant Haar measure on H, then

$$I(f) = \int_G \int_0^{\infty} f(y,t)\, dt\, dy$$

defines a left and right invariant integral on G. In particular, every locally compact Hausdorff group is a closed subgroup of a unimodular group.

18. A $2n+1$ dimensional Heisenberg group H is the Euclidean space $\mathbb{R}^{2n} \times \mathbb{R}$ with multiplication defined by

$$(x,t)(y,s) = (x+y, t+s+\frac{1}{2}B(x,y))$$

where B is a nondegenerate alternating bilinear form on $\mathbb{R}^{2n}$. See Example 5.9. Show Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}$ is both a left and right Haar measure for H. In particular, Heisenberg groups are unimodular.

19. Show extension groups of closed unimodular groups by compact groups are unimodular. More specifically, let H be a closed normal subgroup of a locally compact Hausdorff group G. Suppose H is unimodular and the group G/H is compact. Show G is unimodular.

20. Use Example 6.6.9 to show the group $\mathrm{GL}(n,\mathbb{R})$ is unimodular.

21. Let G be the group of invertible upper triangular matrices. Thus G is collection of all real $n\times n$ matrices with 0's below the diagonal having nonzero diagonal elements. This is naturally identified with an open subset of $\mathbb{R}^k$ where $k = \frac{1}{2}n(n+1)$. With the relative topology and matrix multiplication, this is a locally compact Hausdorff group. Define a homomorphism H on G into $\mathbb{R}^+$ by

$$H(A) = |A_{11}^n A_{22}^{n-1} \cdots A_{nn}^1|.$$

Show a left Haar integral on G is given by

$$I(f) = \int \frac{f(X)}{H(X)}\, dX$$

where dX is Lebesgue measure on G. Moreover, show the modular function Δ is given by

$$\Delta(X) = \prod_{i<j} \left|\frac{X_{j,j}}{X_{i,i}}\right|.$$

22. Show an infinite dimensional Hilbert space with the weak topology is an example of a σ-compact Hausdorff space X which is not locally compact.

23. Show every σ-compact open subset of a locally compact Hausdorff space X is a countable union of compact G_δ subsets and hence is Baire.

24. Let W be a σ-compact open subset of a locally compact Hausdorff space X. Show every Baire subset of the topological space W is a Baire subset of X.

25. Let X be an uncountable set with the discrete topology. Give $X \times X$ the product topology. Show the σ-algebra of Baire sets for $X \times X$ is strictly smaller than $\mathcal{A} \times \mathcal{A}$ where $\mathcal{A}$ is the σ-algebra of Baire sets on X.

26. Let X and Y be locally compact Hausdorff spaces and consider $X \times Y$ with the product topology. Let $\mathcal{A}$ be the algebra of Borel sets on X and $\mathcal{B}$ be the algebra of Borel sets for Y. Show $\mathcal{A} \times \mathcal{B}$ need not be the σ-algebra of Borel subsets for $X \times Y$. **Hint:** Let $X = Y$ be a set with cardinality greater than $\mathfrak{c}$ with the discrete topology. Show $D = \{(x,x) \mid x \in X\}$ is not in the product algebra by first establishing if not there are countably many measurable rectangles $A_i \times B_i$ such that D is in the smallest σ-algebra containing these rectangles.

27. Show if μ is a Radon measure on G/H where G is a locally compact Hausdorff group and H is a closed subgroup, then the measure $x\mu$ defined by (6.1) is a Radon measure on G/H and satisfies (6.2).

28. Let X be a locally compact Hausdorff space and suppose U is an open subset of X with the relative topology.

(a) Show every Baire subset of U has form $B \cap U$ where B is a Baire subset of X.

(b) Let $Y = [0, 1]$ with the discrete topology with one point compactification $\bar{Y}$. Consider $X = \{0, 1\} \times \bar{Y}$ with the product topology.

Show $\{0,1\} \times Y$ is an open subset of X, $B = \{1\} \times \bar{Y}$ is a Baire subset of X, and $B \cap (\{0,1\} \times Y)$ is not a Baire subset of $\{0,1\} \times Y$.

(c) If U is σ-compact, then every Baire set for U is a Baire subset of X.

29. Let H be a closed subgroup of a locally compact Hausdorff group G. Show if f is a Baire measurable function on G and F is a Baire measurable function on G/H, the function $f(F \circ \kappa)$ is a Baire function on G.

30. Show every σ-compact locally compact Hausdorff space X is paracompact. **Hint:** Use Exercise 5.2.15.

31. Let G be a locally compact Hausdorff group.

(a) Show the subgroup generated by a compact neighborhood of e is σ-compact and is open and closed.

(b) Show if H is a closed subgroup of G, then G/H is paracompact. **Hint:** Show G/H is a disjoint union of open and closed σ-compact sets.

32. Suppose X is a locally compact Hausdorff space which is a union of disjoint open Baire subsets X_α of X.

(a) Show the Baire sets for X consists of all those sets E such that $E \cap X_\alpha$ is Baire in X_α for all α and either $E \cap X_\alpha \neq \varnothing$ for countably many α or $(X - E) \cap X_\alpha \neq \varnothing$ for countably many α.

(b) Show a measure μ on X is a Radon measure if and only if there is for each α a Radon measure μ_α on X_α such that

$$\mu = \sum_\alpha \mu_\alpha$$

(c) A real valued function h on X is said to be Baire decomposable if $h|_{X_\alpha}$ is Baire measurable for each α. If $h \geqslant 0$ and μ is a Baire measure on X, then $\int h\,d\mu$ is defined to be $\sum_\alpha \int_{X_\alpha} h\,d\mu$. Show if ν and μ are Radon measures on X and $\nu \ll \mu$, then there is a Baire decomposable $f \geqslant 0$ on X such that

$$\nu(E) = \int_E f\,d\mu = \int \chi_E f\,d\mu$$

for all Baire subsets E of X.

33. Show if μ and ν are nonzero left invariant Radon measures on a locally compact Hausdorff group G, then μ and ν are equivalent. **Hint:** Use an open closed σ-compact subgroup of G.

34. Show any two nonzero left quasi-invariant Radon measures on G/H where G is a locally compact Hausdorff group and H is a closed subgroup are equivalent. More precisely, show the Baire sets of measure 0 in G/H are the sets whose preimages in G have Haar measure 0.

35. Let G and H be a locally compact Hausdorff groups and suppose $\mathcal{A}$ is the σ-algebra of Baire subsets of G and $\mathcal{B}$ is the σ-algebra of Baire subsets of H. Use Exercise 6.1.32 to show the σ-algebra of Baire subsets of $G \times H$ is $\mathcal{A} \times \mathcal{B}$.

36. F be the free abelian group with $\mathfrak{c}$ generators and with the discrete topology. Show multiplication $(x, y) \mapsto xy$ is not Baire measurable on F.

2. Representations for Groups and Algebras

We start in this section by presenting some basic notions for representations of groups and algebras on topological vector spaces.

Definition 6.26. *Let V be a locally convex Hausdorff topological vector space and let* $\mathrm{GL}(V)$ *be the group of continuous and continuously invertible linear transformations of V. If G is a topological group, a representation of G on V is a homomorphism $\pi : G \to \mathrm{GL}(V)$ satisfying*

$$g \mapsto \pi(g)v$$

is continuous from G in V for each $v \in V$. If $\mathcal{A}$ is an algebra over the reals or complexes, a representation π of $\mathcal{A}$ is an algebra homomorphism of $\mathcal{A}$ into the space of continuous linear transformations of V.

Definition 6.27. *Two representation π and π' on locally convex spaces V and V' are equivalent if there is a topological linear isomorphism $J : V \to V'$ such that*

$$\pi'(x)J = J\pi(x)$$

for all x in G (or in $\mathcal{A}$).

Example 6.28. *Let $G = \mathbb{R}$ and let $V = \mathcal{S}(\mathbb{R})$. Then $\lambda(x)f(y) = f(x - y)$ and $m(x)f(y) = e^{-2\pi ixy}f(y)$ are representations of $\mathbb{R}$. They are equivalent. Take J to be the Fourier transform $\mathcal{F}$.*

Let π be a representation of a topological group or an algebra on a locally convex topological vector space V. A linear subspace V_0 of V is said to be **invariant** if $\pi(x)v \in V_0$ for all $v \in V_0$ and all x. Define π_0 by $\pi_0(x)w = \pi(x)w$ for $w \in V_0$. Give V_0 the relative topology of V. Then π_0 is a representation on the locally convex topological vector V_0. The representation π_0 is said to be a **subrepresentation** of π. We usually shall be interested in subrepresentations obtained from closed invariant subspaces of V. The following lemma shows this is not much of a limitation.

Lemma 6.29. *Let π be a representation on a locally convex topological vector space V. Let V_0 be an invariant linear subspace. Then the closure $\bar{V}_0$ is an invariant subspace.*

Proof. Let w be in the closure of V_0 and let U be an open neighborhood of $\pi(x)w$ in V. Since $\pi(x)$ is continuous, $\pi(x)^{-1}U$ is an open neighborhood of w. Hence there is a v in $\pi(x)^{-1}U \cap V_0$. Since V_0 is invariant, $v' = \pi(x)v$ is in $U \cap V_0$. Hence $\pi(x)w$ is in the closure of V_0 and we see the closure is invariant. □

Definition 6.30. *A representation of a topological group or an algebra on a locally convex topological vector space V is irreducible if the only closed invariant subspaces of V are $\{0\}$ and V.*

3. Representations on Hilbert Spaces–Unitary Representations

We have been discussing representations of groups or algebras on topological vector spaces. In harmonic analysis, the most important cases occur when the vector spaces are Hilbert spaces over the complexes. In this case one can in many instances require stronger conditions on the representations; namely for groups we shall deal with unitary representations and for Banach $*$-algebras, we shall deal with $*$-representations. When dealing with Hilbert spaces or $*$ algebras, we assume the scalar field is the complexes.

Definition 6.31. *Let $\mathcal{A}$ be a Banach $*$ algebra. Suppose $\mathcal{H}$ is a Hilbert space. A (star) representation π is a homomorphism of $\mathcal{A}$ into the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear transformations of $\mathcal{H}$ which has the additional property that*

$$\pi(x^*) = \pi(x)^*$$

for each x in $\mathcal{A}$.

Whenever we have a $*$ algebra and a representation on a Hilbert space we shall assume unless otherwise stated that the representation is a star representation.

In Exercise 2.2.23, we introduced the strong operator topology on the space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. It is the locally convex topology on $\mathcal{B}(\mathcal{H})$ defined by the pseudonorms $|\cdot|_v$ where

$$|A|_v = |Av| \text{ for } A \in \mathcal{B}(\mathcal{H}).$$

This is the topology on $\mathcal{B}(\mathcal{H})$ for which representations of topological groups on $\mathcal{H}$ are continuous.

A unitary operator U on Hilbert space $\mathcal{H}$ is a bounded linear operator satisfying

$$U^*U = UU^* = I.$$

This is equivalent to U being linear, onto, and

$$(Uv, Uw) = (v, w) \text{ for all } v, w \in \mathcal{H}.$$

The **unitary group** on a Hilbert space $\mathcal{H}$ is the set $\mathcal{U}(\mathcal{H})$ consisting of all unitary operators on $\mathcal{H}$. It is easy to check it is closed under multiplication and the taking of inverses. Thus it is a group.

Proposition 6.32. *The unitary group $\mathcal{U}(\mathcal{H})$ with the strong operator topology is a Hausdorff topological group.*

Proof. It is Hausdorff for if $U_1 \neq U_2$, then we can choose $v \in \mathcal{H}$ with $U_1 v \neq U_2 v$. Thus $\{U \mid |U - U_1|_v < ||U_1 v - U_2 v||/2\}$ and $\{U \mid |U - U_2|_v < ||U_1 v - U_2 v||/2\}$ are disjoint open sets containing U_1 and U_2. To see multiplication is continuous, note if $U, V, U_0, V_0 \in \mathcal{U}(H)$ and $v \in \mathcal{H}$, then

$$\begin{aligned} |UV - U_0 V_0|_v &= |U(V - V_0)v + (U - U_0)V_0 v| \\ &\leqslant |U(V - V_0)v| + |U - U_0|_{V_0 v} \\ &= |V - V_0|_v + |U - U_0|_{V_0 v}. \end{aligned}$$

For inverses, we have:

$$\begin{aligned} |U^{-1} - U_0^{-1}|_v &= |U^{-1}v - U_0^{-1}v| \\ &= |U^{-1}U_0 U_0^{-1}v - U_0^{-1}v| \\ &= |U_0 U_0^{-1}v - U U_0^{-1}v| \\ &= |U_0 - U|_{U_0^{-1}v}. \end{aligned}$$

Thus $U \mapsto U^{-1}$ is continuous at U_0. □

Definition 6.33. *Let G be a topological group. A continuous homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ of G into the unitary group of a Hilbert space $\mathcal{H}$ with the strong operator topology is a* **unitary representation** *of G.*

Note a mapping F from a topological space X into $\mathcal{U}(\mathcal{H})$ is strongly continuous if and only if $x \mapsto F(x)v$ is continuous for each v. Now a homomorphism π is continuous if and only if $g \mapsto \pi(g)$ is continuous at e. Thus π is continuous if and only if $g \mapsto \pi(g)v$ is continuous for each v if and only if $g \mapsto \pi(g)v$ is continuous at e for each v.

Definition 6.34. *Unitary representations π_1 and π_2 of topological group G or (star) representations π_1 and π_2 of a Banach $*$ algebra on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are* **unitarily equivalent** *if there is a unitary isomorphism U from $\mathcal{H}_1$ onto $\mathcal{H}_2$ satisfying*

$$U\pi_1(x) = \pi_2(x)U \text{ for all } x.$$

We now have two notions of equivalence. The first is the existence of a continuous linear transformation T from $\mathcal{H}_1$ onto $\mathcal{H}_2$ having continuous inverse with the property

$$T\pi_1(g) = \pi_2(g)T \text{ for all } g.$$

This prima-facie is a less restrictive notion than unitary equivalence. However, as we shall see these two are equivalent.

Exercise Set 6.2

1. Let π be a representation of G on a locally convex topological vector space V. Let V^* be the dual space of V with the weak $*$ topology, see Exercise 2.1.17. Define $\check{\pi}$ on V^* by

$$\langle v, \check{\pi}(g)f\rangle = \langle \pi(g^{-1})v, f\rangle;$$

i.e., $\check{\pi}(g)f(v) = f(\pi(g^{-1}v))$. Show $\check{\pi}$ is a representation of G. It is the representation *contragredient* to the representation π.

2. Let π be a unitary representation of a topological group G on a Hilbert space $\mathcal{H}$. Recall the dual space $\mathcal{H}^*$ of $\mathcal{H}$ is the Hilbert space $\bar{\mathcal{H}}$ under the identification $\bar{v} \to f_v$ where $f_v(w) = (v, w)$. The representation $\bar{\pi}$ of G is defined by $\bar{\pi}(g)\bar{v} = \overline{\pi(g)v}$. Define a representation $\check{\pi}$ of G on $\mathcal{H}^*$ by $\check{\pi}(g)(f)(v) = f(\pi(g^{-1})v)$. Show $\check{\pi}$ and $\bar{\pi}$ are unitarily equivalent representations of G.

3. Let V be the space of analytic functions f on $\mathbb{R}$ with seminorms $|\cdot|_K$ where $|f|_K = \max_{x\in K}|f(x)|$ and K is a compact subset. Show the representation m given by $m(x)f(y) = e^{2\pi i xy}f(y)$ has no one dimensional invariant subspaces while the representation λ where $\lambda(x)f(y) = f(y-x)$ has an uncountable number of distinct one dimensional invariant subspaces.

4. Show if $\mathcal{H}$ is a Hilbert space and $\{v_n\}_{n=1}^\infty$ is a sequence in $\mathcal{H}$ satisfying $(v_n, w) \to (v, w)$ for all $w \in \mathcal{H}$ and $||v_n|| \to ||v||$, then $v_n \to v$ in $\mathcal{H}$.

5. Let $\mathcal{H}$ be a Hilbert space. The weak operator topology on $\mathcal{B}(\mathcal{H})$ is defined by the seminorms $|\cdot|_{v,w}$ where $v, w \in \mathcal{H}$ and

$$|T|_{v,w} = |\langle Tv, w\rangle|.$$

Show a sequence (more generally a net) T_n converges to T in the strong operator topology if and only if $T_n \to T$ weakly and $||T_n v|| \to ||Tv||$ for all $v \in \mathcal{H}$.

6. Show a homomorphism π of G into the unitary group of a Hilbert space is strongly continuous if and only if π is weakly continuous.

7. Let V be a locally convex Hausdorff topological vector space. Let $\mathcal{F} = \{|\cdot|_i \mid i \in I\}$ be the collection of all continuous seminorms on V. Let $\mathcal{B}(V)$

be the algebra of continuous linear transformations of V. Define seminorms $|\cdot|_{i,v}$ on $\mathcal{B}(V)$ by

$$|T|_{i,v} = |Tv|_i.$$

The topology defined by these seminorms where $v \in V$ and $i \in I$ is called the *strong operator* topology on $\mathcal{B}(V)$.

(a) Show $\mathcal{B}(V)$ is a Hausdorff.

(b) Show a homomorphism $\pi : G \to \mathrm{GL}(V)$ is continuous in the strong operator topology if and only if $g \mapsto \pi(g)v$ is continuous for each $v \in V$.

(c) Show $(A, B) \mapsto AB$ from $\mathcal{B}(V) \times \mathcal{B}(V)$ into $\mathcal{B}(V)$ is continuous in the strong operator topology.

(d) Define a subset $U \subseteq \mathrm{GL}(V)$ to be open if U and U^{-1} are open in the relative strong operator topology on $\mathrm{GL}(V)$. Show $\mathrm{GL}(V)$ with this topology is a Hausdorff topological group.

4. Orthogonal Sums of Representations

Definition 6.35. *Let π be a representation on a Hilbert space $\mathcal{H}$. If there exist pairwise orthogonal closed invariant subspaces $\mathcal{H}_i$ whose orthogonal direct sum $\bigoplus\mathcal{H}_i$ is $\mathcal{H}$, then π is said to the internal orthogonal direct sum of the subrepresentations π_i defined on $\mathcal{H}_i$ by $\pi_i(g)w = \pi(g)w$ for $w \in \mathcal{H}_i$. One writes:*

$$\pi = \bigoplus \pi_i.$$

For unitary representations or representations of $*$ algebras, one obtains internal orthogonal direct sum decompositions whenever one has a proper closed invariant subspace.

Lemma 6.36. *Let π be a unitary representation of a group or a representation of a $*$ algebra on a Hilbert space $\mathcal{H}$ and suppose $\mathcal{H}_0$ is a closed invariant subspace. Then $\mathcal{H}_0^\perp$ is a closed invariant subspace and π is an internal orthogonal direct sum of the subrepresentations $\pi_0 = \pi|_{\mathcal{H}_0}$ and $\pi_0^\perp = \pi|_{\mathcal{H}_0^\perp}$.*

Proof. We first note $\mathcal{H}_0$ is invariant under $\pi(x)^*$ for all x. Indeed, in the group case, since π is unitary, $\pi(x)^* = \pi(x^{-1})$; and in the algebra case, $\pi(x)^* = \pi(x^*)$ for π is a $*$ representation. In either case we then have

$$(\pi(x)w, v) = (w, \pi(x)^* v) = 0$$

for $v \in \mathcal{H}_0$ and $w \in \mathcal{H}_0^\perp$. Thus $\pi(x)\mathcal{H}_0^\perp \subseteq \mathcal{H}_0^\perp$ and so $\mathcal{H}_0^\perp$ is a closed invariant subspace. The remaining statements follows easily from $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$. $\square$

Analogously, one sometimes can form external orthogonal direct sums of representations. Namely if π_i is a representation on $\mathcal{H}_i$, set $\mathcal{H} = \bigoplus \mathcal{H}_i = \{(v_i)_{i \in I} \mid v_i \in \mathcal{H}_i, \sum ||v_i||^2_{\mathcal{H}_i} < \infty\}$. Then this is a Hilbert space under inner product

$$((v_i), (w_i)) = \sum (v_i, w_i)_{\mathcal{H}_i}.$$

Define π on $\mathcal{H}$ by $\pi(x)(v_i) = (\pi_i(g)v_i)$. If this π is a representation on $\mathcal{H}$, we say π is the "external" direct sum of the representations π_i and write $\pi = \bigoplus \pi_i$. For algebras, it is easy to check that $\bigoplus \pi_i$ is a representation if and only if the family π_i is pointwise bounded; i.e.,

$$\sup_i ||\pi_i(x)|| < \infty \text{ for each } x.$$

Moreover, if the π_i are pointwise bounded and each π_i is a $*$ representation, then $\bigoplus \pi_i$ is a $*$ representation. See Exercise 6.3.2. For groups, one has to be a little more careful.

Assume G is a topological group. The collection π_i is said to be *locally uniformly bounded* if there is a finite constant M and a neighborhood U of e with

$$||\pi_i(g)|| \leqslant M \text{ for all } i \in I \text{ and } g \in U.$$

Lemma 6.37. *Let $\{\pi_i \mid i \in I\}$ be a pointwise bounded collection of representations of a topological group G on Hilbert spaces $\mathcal{H}_i$. Define $\pi(g)(v_i) = (\pi_i(g)v_i)$ for $v = (v_i) \in \bigoplus \mathcal{H}_i$. If the π_i are locally uniformly bounded, then π is a representation of G.*

Proof. Set $\mathcal{H} = \bigoplus \mathcal{H}_i$. We first note $\pi(g)(v_i)$ belongs to $\mathcal{H}$ when $(v_i) \in \mathcal{H}$. Indeed, $\sum ||\pi_i(g)v_i||^2 \leqslant \sup ||\pi_i(g)||^2 \sum ||v_i||^2$. Thus $\pi(g)$ is bounded with norm at most $\sup_i ||\pi_i(g)||$. One easily checks $\pi(g_1)\pi(g_2) = \pi(g_1g_2)$ and $\pi(g)^{-1} = \pi(g^{-1})$. Thus we need only show π is strongly continuous.

Now choose a neighborhood U of e such that $M = \sup\{||\pi_i(g)|| \mid i \in I, g \in U\} = M < \infty$. Suppose $v = (v_i) \in \mathcal{H}$ and suppose $\epsilon > 0$. Fix g_0. Pick F a finite subset of i's with $\sum_{i \notin F} ||v_i||^2 < \frac{\epsilon^2}{8M^2||\pi(g_0)||^2}$. Choose a neighborhood g_0V of g_0 where $V \subseteq U$ such that $||\pi_i(g)v_i - \pi_i(g_0)v_i||_i^2 < \frac{\epsilon^2}{2|F|}$ when $g \in g_0V$ and $i \in F$. Then if $g = g_0g'$ where $g' \in V$, we have

$$\begin{aligned}
||\pi(g)v - \pi(g_0)v||^2 &= \sum_{i \in F} ||\pi_i(g_0g')v_i - \pi_i(g_0)v_i||_i^2 + \sum_{i \notin F} (||\pi_i(g_0g')v_i - \pi_0(g_0)v_i||_i)^2 \\
&< \frac{\epsilon^2}{2} + \sum_{i \notin F} ||\pi_i(g_0)||^2 ||\pi_i(g')v_i - v_i||^2 \\
&\leqslant \frac{\epsilon^2}{2} + ||\pi_i(g_0)||^2 \sum_{i \notin F} (2M)^2 ||v_i||^2 \\
&< \epsilon^2.
\end{aligned}$$

Thus π is strongly continuous. □

Corollary 6.38. *Let π_i be unitary representations on Hilbert spaces $\mathcal{H}_i$. Then $\pi = \oplus\pi_i$ is a unitary representation of G on $\oplus\mathcal{H}_i$.*

When one uses internal orthogonal direct sums, one is decomposing representations into smaller subrepresentations. When one forms external direct sums, one is building larger representations from smaller ones.

Exercise Set 6.3

1. Let G be a compact Hausdorff group. Suppose π is a representation of G on a Banach space X. Using the finiteness of Haar measure and the Uniform Boundedness Principle, show there is an equivalent norm on X such that each $\pi(g)$ is an isometry.

2. Let $\mathcal{H} = \oplus\mathcal{H}_i$ be an orthogonal direct sum of Hilbert spaces $\mathcal{H}_i$. For each i, let $A_i \in \mathcal{B}(\mathcal{H}_i)$. Define linear transformation $A = \oplus A_i$ on $\mathcal{H}$ by

$$A(v_i) = (A_i v_i).$$

(a) Show A is bounded if and only if $\sup_i ||A_i|| < \infty$ and then $||A|| = \sup_i ||A_i||$.

(b) Show if $\oplus A_i$ is bounded, then $(\oplus A_i)^* = \oplus(A_i^*)$.

(c) Show A is unitary if and only each A_i is unitary.

5. The Spectral Theorem, Intertwining Operators, and Schur's Lemma

One item we use from Functional Analysis is a von Neumann duality form of the spectral theorem. We state it next without proof.

Theorem 6.39 (Spectral)**.** *Let A be a bounded self adjoint operator on a Hilbert space $\mathcal{H}$. Then A is in the norm closure of the linear span of all orthogonal projections commuting with all the bounded linear operators commuting with A.*

We call the family of all orthogonal projections commuting with all bounded operators commuting with A the family $\mathcal{E}(A)$. We first note the family $\mathcal{E}(A)$ is commutative. Indeed, suppose E_1 and E_2 are in $\mathcal{E}(A)$. Then since A commutes with A, E_1 and E_2 commute with A. Thus $E_1E_2 = E_2E_1$. The family $\mathcal{E}(A)$ is sometimes called the resolution of the identity for A.

A simple consequence of this theorem is the following:

Theorem 6.40. *Let A be a bounded self adjoint operator and $\epsilon > 0$. Then there exists real numbers $\lambda_1, \dots, \lambda_n$ in the interval $[-||A||, ||A||]$ and pair-wise orthogonal projections $E_1, E_2, \dots, E_n$ in $\mathcal{E}(A)$ such that*

$$||A - \sum \lambda_j E_j|| < \epsilon.$$

Proof. We can choose complex scalars $c_1, \dots, c_m$ and $E_1, \dots, E_m \in \mathcal{E}(A)$ such that

$$||A - \sum c_j E_j|| \leqslant \epsilon.$$

Since the E_j's commute, we may form projections $E_s = E_{1,s_1} \cdots E_{m,s_m}$ where $s \in \{+,-\}^{\{1,2,\dots,m\}}$ and $E_{i,+} = E_j$ and $E_{j,-} = I - E_j$. The E_s are orthogonal projections and $E_j = \sum_{s(j)=+} E_s$. This implies we may assume $E_i E_j = 0$ for $i \neq j$.

Note we may also take the c_j with $|c_j| \leqslant ||A||$. In fact, if $|c_j| > ||A||$, then $||Av - c_j v|| \leqslant \epsilon ||v||$ for $v \in E_j \mathcal{H}$ implies $|c_j|\,||v|| \leqslant ||c_j v - Av|| + ||Av|| \leqslant (\epsilon + ||A||)||v||$. So $||A|| < |c_j| \leqslant \epsilon + ||A||$. This implies $|c_j - \frac{||A||}{|c_j|} c_j| = ||c_j| - ||A||\,| \leqslant \epsilon$. So if $\mu_j = ||c_j^{-1} A|| c_j$ and $v \in E_j \mathcal{H}$, then

$$||Av - \mu_j v|| \leqslant ||Av - c_j v|| + |c_j - \mu_j|\,||v|| \leqslant 2\epsilon ||v||.$$

Replacing c_j by μ_j for each E_j where $|c_j| > ||A||$ gives

$$||AE_j - c_j E_j|| \leqslant 2\epsilon$$

for all j. The orthogonality of the ranges of the E_j and of the ranges of the $E_j A$ gives $||A - \sum_j c_j E_j|| \leqslant 2\epsilon$.

Thus we may assume the E_j are orthogonal, the scalars c_j satisfy $|c_j| \leqslant ||A||$, and $||A - \sum_j c_j E_j|| \leqslant \epsilon$. Now let $c_j = \lambda_j + \mu_j i$ where λ_j and μ_j are real. We claim $||A - \sum \lambda_j E_j|| \leqslant \epsilon$. In fact,

$$\begin{aligned}
\epsilon^2 &\geqslant ||Av - \sum c_j E_i v||^2 = (Av - \sum_j c_j E_j v, Av - \sum_j c_j E_j v) \\
&= (Av, Av) - \sum_j [\bar{c}_j (Av, E_j v) + c_j (E_j v, Av)] + (\sum_j c_j E_j v, \sum_j c_j E_j v) \\
&= (Av, Av) - \sum_j [\bar{c}_j (Av, E_j v) + c_j (E_j v, Av)] + \sum_j |c_j|^2 ||E_j v||^2 \\
&\geqslant (Av, Av) - \sum_j [\bar{c}_j (Av, E_j v) + c_j (E_j v, Av)] + \sum_j \lambda_j^2 ||E_j v||^2.
\end{aligned}$$

Now using the self adjointness of A and the E_j and the commutativity of the E_j with A, one has

$$\begin{aligned}\sum_j [\bar{c}_j(Av, E_jv) + c_j(E_jv, Av)] &= \sum_j [\bar{c}_j(Av, E_jv) + c_j(AE_jv, v)] \\ &= \sum_j [\bar{c}_j(Av, E_jv) + c_j(E_jA, v)] \\ &= \sum_j [\bar{c}_j(Av, E_jv) + c_j(A, E_jv)] \\ &= \sum_j (c_j + \bar{c}_j)(Av, E_jv) \\ &= 2\sum_j \lambda_j(Av, E_jv).\end{aligned}$$

Hence

$$\begin{aligned}\epsilon^2 &\geqslant (Av, Av) - 2\sum_j \lambda_j(Av, E_jv) + \sum_j \lambda_j^2 ||E_jv||^2 \\ &= (Av, Av) - \sum_j [(Av, \lambda_jE_jv) + (\lambda_jE_j, Av)] + \sum_j \lambda_j^2||E_jv||^2 \\ &= (Av - \sum_j \lambda_jE_jv, Av - \sum_j \lambda_jE_jv) \\ &= ||Av - \sum_j \lambda_jE_jv||^2.\end{aligned}$$

Note $|\lambda_j| = |\mathrm{Re}\, c_j| \leqslant |c_j| \leqslant ||A||$ for each j. □

Corollary 6.41. *Let A be a positive self adjoint operator. Let $\epsilon > 0$. Then there exist orthogonal $E_1, E_2, \ldots, E_n \in \mathcal{E}(A)$ and positive $\lambda_1, \lambda_2, \ldots, \lambda_n$ with*

$$||A - \sum \lambda_i E_i|| < \epsilon.$$

Proof. Choose λ_i real with $||A - \sum \lambda_i E_i|| < \epsilon$. Then note:

$$\begin{aligned}||Av - \sum \lambda_i E_i v||^2 &= (Av, Av) - 2\sum \lambda_i(AE_iv, E_iv) + \sum \lambda_i^2(E_iv, E_iv) \\ &\geqslant (Av, Av) - 2\sum_{\lambda_i > 0} \lambda_i(AE_i, E_iv) + \sum_{\lambda_i > 0} \lambda_i^2(E_iv, E_iv) \\ &= ||Av - \sum_{\lambda_i > 0} \lambda_iE_iv||^2.\end{aligned}$$

□

Theorem 6.42. *Let A be a bounded positive linear operator on a complex Hilbert space $\mathcal{H}$. Then there exists a unique bounded positive linear operator B with $B^2 = A$.*

Proof. We first show we can find a $\delta > 0$ such that if $||\sum \lambda_i E_i - \sum \mu_j F_j|| < \delta$ where E_i and F_j are pairwise orthogonal finite sets in $\mathcal{E}(A)$ and $||A|| + 1 \geqslant \lambda_i \geqslant 0$ and $||A|| + 1 \geqslant \mu_j \geqslant 0$, then $||\sum \sqrt{\lambda_i} E_i - \sum \sqrt{\mu_j} F_j|| < \epsilon$.

Indeed, first note $f(\lambda) = \sqrt{\lambda}$ is uniformly continuous for $0 \leqslant \lambda \leqslant ||A|| + 1$. Hence we choose $\delta > 0$ such that $|\sqrt{\lambda} - \sqrt{\mu}| < \epsilon$ if $|\lambda - \mu| < \delta$ and $0 \leqslant \lambda \leqslant 1 + ||A||$, $0 \leqslant \mu \leqslant 1 + ||A||$.

Now by refining the E_i and F_i, i.e. taking projections $E_{\pm,i} F_{\pm,j}$ where $E_+ = E$ and $E_- = I - E$, we may suppose we have situation $||\sum \lambda_i E_i - \sum \mu_i E_i|| < \delta$. Hence $|\lambda_i - \mu_i| < \delta$ for each i where $E_i \neq 0$. This implies $|\sqrt{\lambda_i} - \sqrt{\mu_i}| < \epsilon$ and thus

$$||\sum \sqrt{\lambda_i} E_i - \sum \sqrt{\mu_i} F_i|| < \epsilon.$$

Now choose $\lambda_{n,i} > 0$, $E_{n,i} \in \mathcal{E}(A)$ with $||A - \sum \lambda_{n,i} E_{n,i}|| < \frac{1}{n}$. Define $B_n = \sum_i \sqrt{\lambda_{n,i}} E_{n,i}$. Then $||A - B_n^2|| < \frac{1}{n}$. We claim B_n is Cauchy in norm. Let $\epsilon > 0$. Pick $\delta > 0$ as above. Choose N so that $n \geqslant N$ implies $\frac{1}{n} < \frac{\delta}{2}$. Thus for $m, n \geqslant N$,

$$||\sum \lambda_{m,i} E_{m,i} - \sum \lambda_{n,i} E_{n,i}|| = ||B_m^2 - B_n^2|| \leqslant ||B_m^2 - A|| + ||A - B_n^2|| < \delta$$

for $n \geqslant N$. Thus $||B_m - B_n|| < \epsilon$. Consequently, the sequence B_n converges in norm to a bounded positive linear operator B. Since $B_n^2 \to A$, we see $B^2 = A$.

Finally we show uniqueness. First we note by the above construction that B commutes with every bounded linear operator commuting with A. Let C be another positive bounded linear operator with $C^2 = A$. Then $CA = C^3 = C^2 C = AC$. Thus $BC = CB$. Consequently, $B^2 - C^2 = (B + C)(B - C) = (B - C)(B + C)$. Thus $B - C = 0$ on the closed subspace $\overline{(B + C)(\mathcal{H})}$. We also note $B - C = 0$ on the kernel of $B + C$. Indeed, if $(B + C)v = 0$, then $(Bv, v) + (Cv, v) = 0$. Since $(Bv, v) \geqslant 0$ and $(Cv, v) \geqslant 0$, we have $(Bv, v) = 0$ and $(Cv, v) = 0$. But as seen above these operators have positive square roots D and E. So $(Bv, v) = (D^2 v, v) = (Dv, Dv) = 0$. Thus $Dv = 0$ and so $Bv = D^2 v = 0$. Similarly, $Cv = 0$. Thus $(B - C)v = 0$. Now $\overline{(B + C)(\mathcal{H})} \oplus \ker(B + C) = \mathcal{H}$ for $((B + C)\mathcal{H})^\perp = \ker(B + C)^*$. So $B = C$. $\square$

Definition 6.43. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A linear transformation U from $\mathcal{H}$ to $\mathcal{K}$ is said to be a partial isometry if U restricted to $(\ker U)^\perp$ is an isometry into $\mathcal{K}$.*

Let $\mathcal{H}_0 = (\ker U)^\perp$ and $\mathcal{K}_0 = U(\mathcal{H}_0) = U(\mathcal{H})$. Then $U|_{\mathcal{H}_0}$ is a unitary transformation of $\mathcal{H}_0$ onto $\mathcal{K}_0$.

Theorem 6.44 (Polar Decomposition)**.** *Let T be a bounded linear operator from Hilbert space $\mathcal{H}$ into Hilbert space $\mathcal{K}$. Then $T = UP$ where U is* *a partial isometry from* $(\ker T)^\perp$ *onto the range closure* $\overline{T(\mathcal{H})}$ *of* T *and* $P = \sqrt{T^*T}$.

Proof. Set $A = T^*T$. Then $A^* = A$ and $(Av, v) = (Tv, Tv) \geqslant 0$ for all $v \in \mathcal{H}$. Thus A is a positive bounded linear operator on $\mathcal{H}$. Hence P is defined. Define $U = 0$ on $\ker P$ and set $UPv = Tv$ on the range of P. This is well defined for if $Pv = Pv'$, then $T^*T(v - v') = P^2(v - v') = 0$, and we see $Tv = Tv'$ for $(T(v - v'), T(v - v')) = 0$.

Since $(\ker P)^\perp = \overline{P^*(\mathcal{H})} = \overline{P(\mathcal{H})}$, we have:

$$\ker P \oplus \overline{P(\mathcal{H})} = \mathcal{H}.$$

We note:

$$(UPv, UPv) = (Tv, Tv) = (T^*Tv, v) = (P^2v, v) = (Pv, Pv).$$

Thus U is an isometry of $P(\mathcal{H})$ onto $T(\mathcal{H})$. It thus has a unique linear extension U to an isometry of $\overline{P(\mathcal{H})}$ onto $\overline{T(\mathcal{H})}$. Since $\ker P \oplus \overline{P(\mathcal{H})} = \mathcal{H}$, we see U is defined on all of $\mathcal{H}$, has kernel $\ker P$ and is an isometry of $\overline{P(\mathcal{H})}$ onto $\overline{T(\mathcal{H})}$. Since $Pv = 0$ if and only if $(Pv, Pv) = 0$ if and only if $(P^2v, v) = 0$ if and only if $(T^*Tv, v) = 0$ if and only if $(Tv, Tv) = 0$ if and only if $Tv = 0$, we see $\ker P = \ker T$. Finally, $UPv = Tv$ for all v. $\square$

Definition 6.45. *Let* π *and* ρ *be two representations of an algebra or a group on topological vector spaces* V *and* W. *An intertwining operator is a continuous linear transformation* $T : V \to W$ *satisfying*

$$T\pi(x) = \rho(x)T \textit{ for all } x.$$

The collection consisting of all intertwining operators from π to ρ will be denoted by $\mathrm{Hom}(\pi, \rho)$. At times when we are dealing with representations of a group G or an algebra $\mathcal{A}$, we shall use $\mathrm{Hom}_G(\pi, \rho)$ and $\mathrm{Hom}_{\mathcal{A}}(\pi, \rho)$ for $\mathrm{Hom}(\pi, \rho)$.

Lemma 6.46. *Let* π, ρ, *and* σ *be Hilbert space representations. Then*

(a) $\mathrm{Hom}(\pi, \rho)$ *is a norm closed vector space in* $\mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_\rho)$;

(b) *If* $T \in \mathrm{Hom}(\pi, \rho)$ *and* $S \in \mathrm{Hom}(\rho, \sigma)$, *then* $ST \in \mathrm{Hom}(\pi, \sigma)$. *In particular,* $\mathrm{Hom}(\pi, \pi)$ *is a normed closed algebra in* $\mathcal{B}(\mathcal{H}_\pi)$.

(c) *If* π *and* ρ *are unitary representations of a group or* $*$ *representations of a* $*$ *algebra and* $T \in \mathrm{Hom}(\pi, \rho)$, *then* $T^* \in \mathrm{Hom}(\rho, \pi)$. *Thus* $\mathrm{Hom}(\pi, \pi)$ *is a norm closed* $*$ *subalgebra of* $\mathcal{B}(\mathcal{H}_\pi)$.

Proof. Clearly $\mathrm{Hom}(\pi, \rho)$ is a vector space. Now if $A_n \to A$ in norm where $A_n \in \mathrm{Hom}(\pi, \rho)$, then $A_n\pi(x) = \rho(x)A_n$ for all n. Since multiplication is norm continuous, we have $A\pi(x) = \rho(x)A$ for all x and thus $A \in \mathrm{Hom}(\pi, \rho)$. For (b), note $ST\pi(x) = S\rho(x)T = \sigma(x)ST$. To see (c), from $T\pi(x) =$

$\rho(x)T$ we obtain $\pi(x)^*T^* = T^*\rho(x)^*$. Now set $s(x) = x^{-1}$ if we have a group and $s(x) = x^*$ if we have a $*$ algebra. Then $\pi(x)^* = \pi(s(x))$ and $\rho(x)^* = \rho(s(x))$ if we are dealing with unitary or $*$ representations. Hence $T^*\rho(s(x)) = \pi(s(x))T^*$ for all x and thus $T^* \in \mathrm{Hom}(\rho, \pi)$. $\square$

Let $T \in \mathrm{Hom}(\pi, \rho)$. Note if $v \in \ker T$, then $T\pi(x)v = \rho(x)Tv = 0$. Hence $\ker T$ is a closed invariant subspace of the representation π. One also notes $\rho(x)Tv = T(\pi(x)v)$. Thus the range of T is an invariant subspace of ρ. Taking its closure gives a closed invariant subspace for the representation ρ. Hence, existence of intertwiners is connected closely with reducibility or the existence of proper closed invariant subspaces. For unitary or $*$ representations the situation is even nicer.

Theorem 6.47. *Let π and ρ be unitary or $*$ representations on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Let $T \in Hom(\pi, \rho)$. Then:*

(a) $(\ker T)^\perp$ *is a closed π-invariant subspace of $\mathcal{H}$ and $\pi_0 = \pi|_{(\ker T)^\perp}$ is unitarily equivalent to the representation ρ_0 obtained by restricting ρ to the closure of the range of T.*

(b) *If T is one-to-one and has dense range, then the representations π and ρ are unitarily equivalent.*

In particular, for unitary or $*$ representations, any two equivalent representations are unitarily equivalent.

Proof. Note (b) follows immediately from (a). For (a), let $T = UP$ be the polar decomposition. Before continuing, we recall from the construction of $P = \sqrt{T^*T}$ and the proof of the polar decomposition theorem that P commutes with all operators commuting with T^*T and $(\ker U)^\perp = \overline{P(\mathcal{H})}$. Thus $U' = U|_{(\ker T)^\perp} = U|_{\overline{P(\mathcal{H})}}$ is a unitary mapping onto $\overline{T(\mathcal{H})}$. Moreover,

$$\begin{aligned} U'\pi_0(x)Pv &= UP\pi(x)v \\ &= T\pi(x)v \\ &= \rho(x)Tv \\ &= \rho_0(x)Tv \\ &= \rho_0(x)U'Pv. \end{aligned}$$

Thus $U'\pi_0(x)w = \rho_0(x)U'w$ for all $w \in \overline{P(\mathcal{H})} = (\ker T)^\perp$. $\square$

Proposition 6.48 (Schur's Lemma)**.** *Let π be a unitary or a $*$ representation. Then π is irreducible if and only if* $\mathrm{Hom}(\pi, \pi) = \{cI \mid c \in \mathbb{C}\}$. *Moreover, if π and ρ are irreducible unitary or $*$ representations, then* $\dim Hom(\pi, \rho)$ *is either* 0 *or* 1.

Proof. We first show $\mathrm{Hom}(\pi,\pi) = \mathbb{C}I$ if and only if π is irreducible. Clearly $\mathbb{C}I \subseteq \mathrm{Hom}(\pi,\pi)$. Suppose π is irreducible. Let A be a self adjoint operator in $\mathrm{Hom}(\pi,\pi)$. Then $\pi(x)A = A\pi(x)$ for all x. Thus $\pi(x)E = E\pi(x)$ for all $E \in \mathcal{E}(A)$. This implies $\mathcal{H}_0 = E(\mathcal{H})$ is a closed invariant subspace. But then $\mathcal{H}_0 = \mathcal{H}$ or $\mathcal{H}_0 = \{0\}$. Hence $\mathcal{E}(A) = \{0, I\}$. But then the spectral theorem implies $A = \lambda I$ for some real scalar λ. Now if $T \in \mathrm{Hom}(\pi,\pi)$, then $T = A+iB$ where $A = \frac{1}{2}(T+T^*)$ and $B = \frac{-i}{2}(T-T^*)$ are both self adjoint. By Lemma 6.46, $T^* \in \mathrm{Hom}(\pi,\pi)$. Thus both A and B are in $\mathrm{Hom}(\pi,\pi)$. So $A = \lambda I$ and $B = \mu I$. Thus $T = (\lambda+\mu i)I$ and we see $\mathrm{Hom}(\pi,\pi) = \mathbb{C}I$. Now if π is not irreducible, we can find a closed proper invariant subspace $\mathcal{H}_0$. Let P_0 be the orthogonal projection onto $\mathcal{H}_0$. By Lemma 6.36, $\mathcal{H}_0^\perp$ is also invariant. Now if $v = v_0 + v_0^\perp$, $\pi(x)Pv = \pi(x)v_0 = P(\pi(x)v_0 + \pi(x)v_0^\perp) = P\pi(x)v$. Thus $P \in \mathrm{Hom}(\pi,\pi)$ and so $\mathrm{Hom}(\pi,\pi) \neq \mathbb{C}I$.

Now assume π and ρ are irreducible and $\mathrm{Hom}(\pi,\rho)$ is nonzero. By Theorem 6.47, we may choose a unitary operator U in $\mathrm{Hom}(\pi,\rho)$. Let $S \in \mathrm{Hom}(\pi,\rho)$. By Lemma 6.46, $U^*S \in \mathrm{Hom}(\pi,\pi)$. So $U^*S = cI$ for some scalar c. Hence $S = UU^*S = cU$ and we see $\mathrm{Hom}(\pi,\rho) = \mathbb{C}U$. □

We next give a version of Schur's Lemma for closed operators. Recall a linear operator A from a linear subspace D of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ is closed if whenever $v_n \to v$ and $\{Av_n\}$ converges, then $v \in D$ and $Av = \lim_n Av_n$.

Proposition 6.49 (Strong Schur's Lemma)**.** *Let π be an irreducible unitary representation of group G on the Hilbert space $\mathcal{H}$. Suppose ρ is a unitary representation of G on Hilbert space $\mathcal{K}$ and $\theta : D \to \mathcal{K}$ is a closed operator from a π invariant linear subspace D of $\mathcal{H}$ into $\mathcal{K}$ satisfying*

$$\theta\pi(g)v = \rho(g)\theta v \text{ for } v \in D \text{ and } g \in G.$$

Then either $D = \{0\}$ or $D = \mathcal{H}$, θ is bounded, and $\theta^*\theta = c^2 I$ where $c \geqslant 0$. Moreover, if $\rho = \pi$, then there is a scalar λ such that $\theta = \lambda I$.

Proof. Let $\widetilde{D} = \{(v, \theta v) \mid v \in D\}$, the graph of θ. This is a vector space. If $v \in D$, let $\tilde{v}$ denote the pair $(v, \theta v)$ in $\widetilde{D}$.

Define an inner product on $\widetilde{D}$ by

$$(\tilde{v}_1, \tilde{v}_2) = (v_1, v_2) + (\theta v_1, \theta v_2).$$

Note $||\tilde{v}||^2 = ||v||^2 + ||\theta v||^2$. Also $\widetilde{D}$ is a complete inner product space for

$$||\tilde{v}_k - \tilde{v}_l||^2 = ||v_k - v_l||^2 + ||\theta v_k - \theta v_l||^2.$$

Thus if $\tilde{v}_k$ is Cauchy, v_k is Cauchy in $\mathcal{H}$ and θv_k is Cauchy in $\mathcal{K}$. Consequently, $v_k \to v$ and $\theta v_k \to w$. Since θ is closed, $\theta v = w$. So $\tilde{v}_k \to \tilde{v}$ in $\widetilde{D}$.

Define a representation $\tilde{\pi}$ of G on Hilbert space $\widetilde{D}$ by

$$\tilde{\pi}(g)\tilde{v} = (\widetilde{\pi(g)v}) = (\pi(g)v, \theta\pi(g)v) = (\pi(g)v, \rho(g)\theta v).$$

The unitarity and strong continuity of both π and ρ imply the representation $\tilde{\pi}$ is strongly continuous and unitary. Set $A\tilde{v} = v$. Note A is a bounded operator on the Hilbert space $\widetilde{D}$ onto the linear subspace D of $\mathcal{H}$. Also

$$A\tilde{\pi}(g)\tilde{v} = A(\widetilde{\pi(g)v}) = \pi(g)v = \pi(g)A\tilde{v}.$$

Thus

$$\tilde{\pi}(g^{-1})A^* = A^*\pi(g^{-1})v.$$

Combining these we have:

$$AA^*\pi(g)v = A\tilde{\pi}(g)A^*v = \pi(g)AA^*v.$$

Thus AA^* is a bounded linear operator on $\mathcal{H}$ commuting with the irreducible unitary representation π. By the usual Schur's Lemma, $AA^* = C^2I$ for some C. This implies $(A^*v, A^*v) = C^2(v, v)$. So $C^2 \geqslant 0$. If $C = 0$, $A^* = 0$ and thus $A = (A^*)^* = 0$. This gives $D = 0$. Thus we may assume $C > 0$ and $D \neq \{0\}$. We thus have

$$||A^*v||^2 = C^2||v||^2$$

where $C > 0$. This implies the range of A is closed. Indeed, if $Av_k \to w$ as $k \to \infty$, then $AA^*Av_k = C^2Av_k$ converges to C^2w. But A^*Av_k is Cauchy for

$$\begin{aligned}
||A^*A(v_k - v_l)||^2 &= (A^*AA^*A(v_k - v_l), v_k - v_l) \\
&= C^2(A^*A(v_k - v_l), v_k - v_l) \\
&= C^2(A(v_k - v_l), A(v_k - v_l))
\end{aligned}$$

converges to 0 as $k, l \to \infty$. Using completeness, there is a v with $A^*Av_k \to v$ and we see $w = \frac{1}{C^2}\lim_k AA^*Av_k = A(\frac{1}{C^2}v)$. It is also invariant under π for $\pi(g)A\tilde{v} = \pi(g)v = A\tilde{\pi}(g)\tilde{v}$. Since π is irreducible, the range of A is $\mathcal{H}$. But $A\widetilde{D} = D$. So $D = \mathcal{H}$.

Now θ is a closed linear operator from $\mathcal{H}$ into $\mathcal{K}$. By the closed graph theorem, θ is bounded. Finally $\theta\pi(g) = \rho(g)\theta$ for all g implies $\pi(g^{-1})\theta^* = \theta^*\rho(g^{-1})$ for all g. Thus $\theta^*\theta\pi(g) = \pi(g)\theta^*\theta$. By the standard Schur's Lemma, $\theta^*\theta = c^2I$ for some $c \geqslant 0$.

The last statement follows from Schur's Lemma for bounded operators. □

For a topological group G, the **unitary dual** $\hat{G}$ is the collection of all equivalence classes of irreducible unitary representations of G. Thus $\hat{G}$ is the collection of all $[\pi]$ where π is an irreducible unitary representation of G and $[\pi] = \{\rho \mid \rho \text{ is an irreducible unitary representation equivalent to } \pi\}$.

This definition may not be set theoretically correct. However, we shall not use this terminology in any meaningfully objective way.

A one dimensional unitary representation of a group G is often referred to as a **character**. Because this terminology is also used in other contexts, e.g. see Definition 6.120; we shall usually call them one-dimensional characters. In the case of abelian groups, we see in the following corollaries that the one-dimensional characters are all the irreducible unitary representations of G and in this case $\hat{G}$ is called the **character group** of G. For more on this, see Exercise 6.5.7.

Corollary 6.50. *Let $Z(G)$ denote the center of G. If π is an irreducible unitary representation of G, then there exists a continuous homomorphism $\chi_\pi : Z(G) \to \mathbb{T}$ such that $\pi(z) = \chi_\pi(z)I$ for all $z \in Z(G)$. The character χ_π is called the **central character** of ρ.*

Corollary 6.51. *Let π be an irreducible unitary representation of an abelian group G. Then π is one-dimensional.*

Proof. By Schur's Lemma, $\pi(g) = \chi(g)I$ for some $\chi(g) \in \mathbb{T}$ for each g. This implies every subspace of $\mathcal{H}_\pi$ is invariant. Since π is irreducible, there are only two closed invariant subspaces. Consequently $\dim \mathcal{H}_\pi = 1$. □

Definition 6.52. *Let π be a unitary representation or a $*$ representation on a Hilbert space $\mathcal{H}$ and let some index set I have cardinality n. Set $n\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}$ where $|I| = n$ and define*

$$n\pi = \bigoplus_{i \in I} \pi.$$

We know $n\pi$ is a unitary representation of a group or a $*$ representation of a $*$ algebra on $n\mathcal{H}$.

Theorem 6.53. *Let π be an irreducible unitary representation of G or an irreducible $*$ representation of a $*$ algebra $\mathcal{A}$, and let ρ be a unitary representation of G or a $*$ representation of a $\mathcal{A}$. Then there is a unique orthogonal projection $P \in Hom(\rho, \rho)$ such that $\rho|_{P(\mathcal{H}_\rho)} \cong n\pi$ and $\rho|_{P^\perp \mathcal{H}_\rho}$ has no subrepresentation unitarily equivalent to π. Moreover, $n \leqslant \dim(Hom(\pi, \rho))$ with equality holding if $Hom(\pi, \rho)$ is finite dimensional.*

Proof. Let U_α be a maximal collection of isometries in $\mathrm{Hom}(\pi, \rho)$ such that the closed spaces $U_\alpha \mathcal{H}_\pi$ are perpendicular. Clearly, $U_\alpha : \mathcal{H}_\pi \to U\mathcal{H}_\pi$ is a unitary equivalence of π with $\rho|_{\mathcal{H}_\alpha}$ where $\mathcal{H}_\alpha = U_\alpha \mathcal{H}_\pi$. Set $P_\alpha = U_\alpha U_\alpha^*$. By Exercise 6.4.1, P_α is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\alpha$. We set $P = \sum P_\alpha$ and shall show P is the orthogonal projection onto the closure of the linear span of the ranges $A\mathcal{H}_\pi$ where $A \in \mathrm{Hom}(\pi, \rho)$.

Let $A \in \mathrm{Hom}(\pi, \rho)$. We show $PA = A$. Suppose not. Set $B = (I - P)A$. Since $P \in \mathrm{Hom}(\rho, \rho)$, $B \in \mathrm{Hom}(\pi, \rho)$ and $PB = 0$. Thus $B\mathcal{H}_\pi \perp P\mathcal{H}$. If $B \neq 0$, then $B^*B \in \mathrm{Hom}(\pi, \pi) = \mathbb{C}I$. Using Schur's Lemma, one has $B^*B = bI$ where $b > 0$. Set $U = \frac{1}{\sqrt{b}}B$. Then $U^*U = I$. This implies $(Uv, Uw) = (U^*Uv, w) = (v, w)$ and thus $U \in \mathrm{Hom}(\pi, \rho)$ is an isometry. Also $U\mathcal{H}_\pi = B\mathcal{H}_\pi \perp P\mathcal{H}$. So the collection U_α was not maximal. Hence $B = 0$, and we see $A = PA$ for $A \in \mathrm{Hom}(\pi, \rho)$. Since P is the orthogonal projection onto the closure of the sum of the spaces $\mathcal{H}_\alpha = U_\alpha\mathcal{H}_\pi$, we have P is the orthogonal projection onto the closure of the linear span of the ranges of the A's in $\mathrm{Hom}(\pi, \rho)$. So P is unique.

We consider the cardinality of the collection P_α. Clearly, we have one copy of π in $P\mathcal{H}$ for each α. Also the P_α are linearly independent. Hence $\dim \mathrm{Hom}(\pi, \rho) \geqslant n$ where n is the cardinality of the P_α's. Suppose $A \in \mathrm{Hom}(\pi, \rho)$. Then $PA = A$. Thus $A = \sum P_\alpha A$. Now $P_\alpha A \in \mathrm{Hom}(\pi, \rho|_{\mathcal{H}_\alpha})$. By Schur's Lemma, $P_\alpha A = c_\alpha U_\alpha$ for some scalar c_α. Thus the $U_\alpha's$ form a 'strong operator topology' base; i.e. every A in $\mathrm{Hom}(\pi, \rho)$ is a unique infinite linear combination of the operators U_α where the sum converges in the strong operator topology. If $\mathrm{Hom}(\pi, \rho)$ is finite dimensional, then the U_α's form a vector space base of $\mathrm{Hom}(\pi, \rho)$ and thus $n = \dim \mathrm{Hom}(\pi, \rho)$. □

The unique projection P is called the π **primary projection** for ρ.

Corollary 6.54. *The π-primary projection for a unitary representation or a $*$ representation ρ is the orthogonal projection whose range is the closure of the linear span of the union of all $A\mathcal{H}_\pi$ where $A \in Hom(\pi, \rho)$.*

Corollary 6.55. *Let π and π' be inequivalent irreducible unitary or $*$ representations. Suppose ρ is a representation (unitary or $*$) on a Hilbert space $\mathcal{H}$. Let $P(\pi)$ and $P(\pi')$ be the π and π' primary projections for ρ. Then $P(\pi)P(\pi') = 0$.*

Proof. Suppose $P(\pi)P(\pi') \neq 0$. Then $P(\pi)A \neq 0$ for some $A \in \mathrm{Hom}(\pi', \rho)$. Thus $P_\alpha A \neq 0$ for some P_α. Thus $\mathrm{Hom}(\pi', \rho|_{\mathcal{H}_\alpha}) \neq \{0\}$. By Schur's Lemma, π' and $\rho|_{\mathcal{H}_\alpha}$ are equivalent. Since $\rho|_{\mathcal{H}_\alpha}$ and π are equivalent, we see π and π' are equivalent, a contradiction. □

Definition 6.56. *A representation is discretely decomposable if it can be written as an internal orthogonal direct sum of irreducible subrepresentations.*

Remark 6.57. The remaining statements of this section are also true for $*$ representations of $*$ algebras. We, however, only state them for unitary representations.

Proposition 6.58. *Let ρ be a discretely decomposable unitary representation of G on a Hilbert space $\mathcal{K}$. Let $\hat{G}_c$ be a family of irreducible unitary* *representations of G consisting of one representation from each equivalence class in $\hat{G}$. For each $\pi \in \hat{G}_c$, let $P(\pi)$ be the π-primary projection for ρ. Then*

$$\bigoplus_{\pi \in \hat{G}_c} P(\pi) = I.$$

Proof. We already have $P(\pi)P(\pi') = 0$ for $\pi \neq \pi'$ in $\hat{G}_c$. Since ρ is discretely decomposable, there exists an index set B and an orthogonal internal direct sum decomposition $\mathcal{K} = \bigoplus_{\beta \in B} \mathcal{K}_\beta$ where $\mathcal{K}_\beta$ is ρ invariant and irreducible. Let $\beta \in B$. Set $\rho_\beta = \rho|_{\mathcal{K}_\beta}$. Pick $\pi \in \hat{G}_c$ such that ρ_β is unitarily equivalent to π. Then there is $A \in \mathrm{Hom}_G(\pi, \rho)$ with $A\mathcal{H}_\pi = \mathcal{K}_\beta$. Consequently $P(\pi)v = v$ for $v \in \mathcal{K}_\beta$. Thus $\oplus_{\pi'} P(\pi') = I$. $\square$

Definition 6.59. *Let ρ be a discretely decomposable unitary representation of a topological group G on a Hilbert space $\mathcal{H}$. If π is an irreducible unitary representation of G, then the multiplicity $m(\pi, \rho)$ of π in ρ is the cardinality n where $\rho|_{P(\pi)\mathcal{H}}$ is unitarily equivalent to $n\pi$.*

By Theorem 6.53, $m(\pi, \rho)$ is the dimension of the vector space $\mathrm{Hom}_G(\pi, \rho)$ when this is a finite dimensional vector space; otherwise $m(\pi, \rho)$ is an infinite cardinal. We conclude with the following result.

Theorem 6.60. *Let ρ be a discretely decomposable unitary representation of a topological group G. Then:*

$$\rho \cong \bigoplus_{\pi \in \hat{G}_c} m(\pi, \rho)\pi.$$

6. Tensor Products of Representations

In this section we deal with tensor products of representations. Though one can tensor tensor product more general topological vector spaces, we shall be looking at only two cases; first, tensor products of finite dimensional representations, and second tensor products of representations on Hilbert spaces. We start with representations on finite dimensional spaces.

Let V and W be finite dimensional vector spaces. We set $\mathcal{L}(V, W)$ to be the vector space of linear transformations from V to W and W^* to be the dual space of W. The **tensor product** $V \otimes W$ of the vector spaces V and W is then the vector space $\mathcal{L}(W^*, V)$. In this space one has elementary tensors $v \otimes w$ which are defined to be the rank one linear transformations given by

$$(v \otimes w)(f) = f(w)v \text{ where } v \in V,\ w \in W, \text{ and } f \in W^*. \tag{6.5}$$

In particular, $V \otimes V^* = \mathcal{L}(V, V)$ has elementary tensors $v \otimes f$ where

$$(v \otimes f)(v') = f(v')v.$$

Recall if $B \in \mathcal{L}(W, W)$, then the **transpose** B^t of B is the linear transformation of W^* defined by

$$\langle w, B^t f\rangle = B^t f(w) = f(Bw) = \langle Bw, f\rangle$$

where we have used the notation $\langle w, f\rangle$ for the linear functional f evaluated at vector w. Now if A and B are linear transformations of V and W, respectively, $A \otimes B$ is defined to be the linear transformation of $V \otimes W$ given by

$$(A \otimes B)(T) = ATB^t.$$

In particular, if $T = v \otimes w \in \mathcal{L}(W^*, V)$, then

$$(A \otimes B)(v \otimes w) = Av \otimes Bw.$$

Definition 6.61. *Let π_1 and π_2 be finite dimensional representations of groups G_1 and G_2 on vector spaces V_1 and V_2. Then the outer tensor product $\pi_1 \times \pi_2$ is the representation of $G_1 \times G_2$ on $V_1 \otimes V_2$ given by*

$$(\pi_1 \times \pi_2)(g_1, g_2) = \pi_1(g_1) \otimes \pi_2(g_2).$$

It is easy to check the continuity of $\pi_1 \otimes \pi_2$ because of the finite dimensionality of $V_1 \otimes V_2$. The **inner tensor product** of two finite dimensional representations π_1 and π_2 of G is defined by

$$(\pi_1 \otimes \pi_2)(g) = \pi_1(g) \otimes \pi_2(g). \tag{6.6}$$

Definition 6.62 (Contragredient Representation)**.** *Let π be a representation of G on a finite dimensional vector space V. Define $\check{\pi}$ on V^* by*

$$\check{\pi}(g)f(v) = f(\pi(g^{-1})v) = \pi(g^{-1})^t f(v).$$

Note one has

$$\begin{aligned}\check{\pi}(xy)f(v) &= f(\pi(xy)^{-1}v)\\ &= f(\pi(y)^{-1}\pi(x)^{-1}v)\\ &= (\check{\pi}(y)f)(\pi(x)^{-1}v)\\ &= \check{\pi}(x)(\check{\pi}(y)f)(v)\end{aligned}$$

for $x, y \in G$. Clearly $\check{\pi}(e) = I$ and thus $\check{\pi}$ is a homomorphism into $\mathrm{GL}(V)$. Again continuity is easy.

In terms of the alternative bilinear notation, we have

$$\langle v, \check{\pi}(g)f\rangle = \langle \pi(g^{-1})v, f\rangle.$$

Notice the similarity with being unitary.

Finally we mention if π is a representation of G on a finite dimensional complex vector space V, then $\pi \otimes \check{\pi}$ is the representation on $\mathcal{L}(V, V)$ given by

$$\pi \otimes \check{\pi}(g)(T) = \pi(g)T\pi(g^{-1}). \tag{6.7}$$

Next we look at tensor products of representations on Hilbert spaces. In (2.8) of Chapter 2, we used the notation $\mathcal{K}\otimes\overline{\mathcal{H}}$ for the space of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$. To further explain this notation, we start by defining $\overline{\mathcal{H}}$.

Let $\mathcal{H}$ be a Hilbert space. Set $\overline{\mathcal{H}} = \mathcal{H}$. To remind us a vector $v \in \overline{\mathcal{H}}$ is in $\overline{\mathcal{H}}$ rather than in $\mathcal{H}$, we write $\bar{v}$ for v even though they are the same vector. Addition, scalar multiplication, and the inner product on $\overline{\mathcal{H}}$ are defined by:

$$(6.8) \qquad \bar{v} + \bar{w} = \overline{v + w},\ \lambda\bar{v} = \bar{\lambda}v = \overline{\bar{\lambda}v},\ \text{and } (\bar{v},\bar{w})_{\bar{\mathcal{H}}} = (w,v)_{\mathcal{H}} = \overline{(v,w)_{\mathcal{H}}}.$$

Proposition 6.63. *$\overline{\mathcal{H}}$ is a Hilbert space and the mapping $I : \mathcal{H} \to \overline{\mathcal{H}}$ is a conjugate linear isometry from $\mathcal{H}$ onto $\overline{\mathcal{H}}$.*

Proof. It is easy to check that $\overline{\mathcal{H}}$ is a vector space. Next note

$$\begin{aligned}(\bar{v} + \bar{v}', \bar{w})_{\overline{\mathcal{H}}} &= (\overline{v + v'}, \bar{w})_{\bar{\mathcal{H}}} \\ &= (w, v + v')_{\mathcal{H}} \\ &= (w, v)_{\mathcal{H}} + (w, v')_{\mathcal{H}} \\ &= (\bar{v}, \bar{w})_{\overline{\mathcal{H}}} + (\bar{v}', \bar{w})_{\bar{\mathcal{H}}}.\end{aligned}$$

Also $\overline{(\bar{v},\bar{w})_{\overline{\mathcal{H}}}} = \overline{(w,v)} = (\bar{w},\bar{v})_{\overline{\mathcal{H}}}$. We also have $(\lambda\bar{v},\bar{w})_{\overline{\mathcal{H}}} = \overline{(\bar{\lambda}v, w)} = \lambda\overline{(v,w)} = \lambda(\bar{v},\bar{w})_{\overline{\mathcal{H}}}$. Since $(\bar{v},\bar{v})_{\overline{\mathcal{H}}} = (v,v)$, we see have an inner product that defines the same norm as the original inner product. Thus $\overline{\mathcal{H}}$ is a Hilbert space. $\square$

Definition 6.64. *Let $B \in \mathcal{B}(\mathcal{H})$. Then $\bar{B}$ is the operator defined on $\overline{\mathcal{H}}$ by*

$$\bar{B}\bar{v} = \overline{Bv}.$$

Remark 6.65. $\bar{B}$ is a linear transformation of $\overline{\mathcal{H}}$ with the same norm as the operator B. For instance, $\bar{B}(\lambda\bar{v}) = \bar{B}(\overline{\bar{\lambda}v}) = \overline{B(\bar{\lambda}v)} = \overline{\bar{\lambda}(Bv)} = \lambda\overline{Bv} = \lambda\bar{B}\bar{v}$. Moreover, $\overline{B^*} = \bar{B}^*$ for

$$\begin{aligned}(\bar{B}\bar{v},\bar{w})_{\overline{\mathcal{H}}} &= (\overline{Bv},\bar{w})_{\overline{\mathcal{H}}} \\ &= (w, Bv)_{\mathcal{H}} \\ &= (B^*w, v)_{\mathcal{H}} \\ &= (\bar{v}, \overline{B^*w})_{\overline{\mathcal{H}}} \\ &= (\bar{v}, \overline{B^*}\bar{w})_{\overline{\mathcal{H}}}.\end{aligned}$$

One can also check $\overline{AB} = \bar{A}\bar{B}$ and U is unitary if and only if $\overline{U}$ is unitary.

Lemma 6.66. *Let π be a representation of G on a Hilbert space $\mathcal{H}$. Then $\bar{\pi}$ defined by $\bar{\pi}(g)\bar{v} = \overline{\pi(g)v}$ is a representation of G on $\overline{\mathcal{H}}$. It is unitary if π is unitary.*

Proof. Clearly $\bar{\pi}(g)(\bar{v}+\bar{w}) = \bar{\pi}(g)(\bar{v})+\bar{\pi}(g)(\bar{w})$ and $\bar{\pi}(g)(\lambda\bar{v}) = \pi(g)(\bar{\lambda}v) = \bar{\lambda}\pi(g)v = \lambda\bar{\pi}(g)\bar{v}$. Strong continuity follows immediately from $||\bar{v}||_{\bar{\mathcal{H}}} = ||v||_{\mathcal{H}}$. If π is unitary, one has

$$(\bar{\pi}(g)\bar{v}, \bar{\pi}(g)\bar{w})_{\overline{\mathcal{H}}} = (\pi(g)w, \pi(g)v)_{\mathcal{H}} = (w, v)_{\mathcal{H}} = (\bar{v}, \bar{w})_{\overline{\mathcal{H}}}.$$

Thus $\bar{\pi}$ is unitary. □

The representation $\bar{\pi}$ is called the **conjugate representation** to π.

Proposition 6.67. *Let π be a unitary representation on a finite dimensional Hilbert space $\mathcal{H}$. Then the mapping $A : \overline{\mathcal{H}} \to \mathcal{H}^*$ given by*

$$\langle v_1, A(\bar{v}_2)\rangle = (v_1, v_2)$$

is an equivalence between the representations $\bar{\pi}$ and $\check{\pi}$.

Proof. First note A is linear and invertible. Moreover,

$$\begin{aligned}\langle v_1, A\bar{\pi}(g)\bar{v}_2\rangle &= \langle v_1, A(\overline{\pi(g)v_2}\,)\rangle \\ &= (v_1, \pi(g)v_2) \\ &= (\pi(g^{-1})v_1, v_2) \\ &= \langle \pi(g^{-1})v_1, A\bar{v}_2\rangle \\ &= \langle v_1, \pi(g^{-1})^t A\bar{v}_2\rangle \\ &= \langle v_1, \check{\pi}(g)A\bar{v}_2\rangle.\end{aligned}$$

Thus $A\bar{\pi}(g) = \check{\pi}(g)A$. □

In Section 3, we showed the space $\mathcal{B}_2(\mathcal{K}, \mathcal{H})$ consisting of the Hilbert-Schmidt operators from Hilbert space $\mathcal{K}$ into Hilbert space $\mathcal{H}$ is a Hilbert space with inner product given by

$$(R, S)_2 = \sum_{\alpha}(Re_{\alpha}, Se_{\alpha}) = \sum_{\alpha}(S^*Te_{\alpha}, e_{\alpha}) = \mathrm{Tr}(S^*T)$$

where $\{e_{\alpha}\}$ is an orthonormal basis of $\mathcal{K}$. As we have already mentioned, we have denoted this Hilbert space by $\mathcal{H} \otimes \overline{\mathcal{K}}$. In this Hilbert space the rank one operator $v \otimes \bar{w}$ behave nicely with respect to this inner product; i.e., see Proposition 2.42.

Because now we have the conjugate Hilbert space $\overline{\mathcal{K}}$, we can define the tensor product of any two Hilbert spaces. Indeed, we define:

Definition 6.68. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then the tensor product is*

$$\mathcal{H} \otimes \mathcal{K} = \mathcal{B}_2(\overline{\mathcal{K}}, \mathcal{H}).$$

In this Hilbert space each rank one operator has form $v \otimes w$ where

$$(v \otimes w)(\bar{w}') = (\bar{w}', \bar{w})_{\bar{\mathcal{K}}} v = (w, w')_{\mathcal{K}} v \tag{6.9}$$

and the inner product satisfies

$$(v \otimes w, v' \otimes w')_2 = (v, v')_{\mathcal{H}} (w, w')_{\mathcal{K}}. \tag{6.10}$$

In particular,

$$||v \otimes w||_2 = ||v||_{\mathcal{H}} ||w||_{\mathcal{K}}. \tag{6.11}$$

Definition 6.69. *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Then $A \otimes B$ is the operator on $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{B}_2(\overline{\mathcal{H}}_2, \mathcal{H}_1)$ defined by*

$$(A \otimes B)(T) = AT\bar{B}^*.$$

It is called the tensor product of the operators A and B.

Proposition 6.70. *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and let $B \in \mathcal{B}(\mathcal{H}_2)$. Then $A \otimes B \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$; in fact, $||A \otimes B|| = ||A||\, ||B||$. Moreover,*

$$(A \otimes B)(v_1 \otimes v_2) = Av_1 \otimes Bv_2 \text{ for all } v_1 \in \mathcal{H}_1 \text{ and } v_2 \in \mathcal{H}_2.$$

Proof. Clearly $T \mapsto AT\bar{B}^*$ is linear. By Proposition 2.32, we know $||AT\bar{B}^*||_2 \leqslant ||A||\, ||T||_2 ||\bar{B}^*|| = ||A||\, ||B||\, ||T||_2$. Thus $||A \otimes B|| \leqslant ||A||\, ||B||$. Now take $T = v \otimes w$. Then

$$\begin{aligned}
(A \otimes B)(v \otimes w)(\bar{w}') &= A(v \otimes w)\bar{B}^*(\bar{w}') \\
&= (\bar{B}^*(\bar{w}'), \bar{w})_{\overline{\mathcal{H}}_2} Av \\
&= (\bar{w}', \bar{B}\bar{w})_{\overline{\mathcal{H}}_2} Av \\
&= (\bar{w}', \overline{Bw})_{\overline{\mathcal{H}}_2} Av \\
&= (Av \otimes Bw)(\bar{w}').
\end{aligned}$$

Thus

$$(A \otimes B)(v \otimes w) = Av \otimes Bw.$$

Recall $||v \otimes w||_2 = ||v||_{\mathcal{H}_1} ||w||_{\mathcal{H}_2}$. Thus if $||v||_{\mathcal{H}_1} = 1$ and $||w||_{\mathcal{H}_2} = 1$, we see $||A \otimes B||_2 \geqslant ||Av \otimes Bw||_2 = ||Av||_{\mathcal{H}_1} ||Bw||_{\mathcal{H}_2}$. Taking supremums over all such v and w gives $||A \otimes B|| \geqslant ||A||\, ||B||$. □

Definition 6.71. *Let π_1 be a unitary representation of a topological group G_1 on a Hilbert space $\mathcal{H}_1$ and π_2 be a unitary representation of a topological group G_2 on a Hilbert space $\mathcal{H}_2$. Define $\pi_1 \times \pi_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by*

$$(\pi_1 \times \pi_2)(g_1, g_2) = \pi_1(g_1) \otimes \pi_2(g_2).$$

Proposition 6.72. *Give $G_1 \times G_2$ the product topology. Then $\pi_1 \times \pi_2$ is a unitary representation of $G_1 \times G_2$ on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. It is called the* **outer tensor product** *of the representations π_1 and π_2.*

Proof. Set $\pi = \pi_1 \times \pi_2$. Using Corollary 2.41, we see

$$\begin{aligned}
(\pi(g_1,g_2)R, \pi(g_1,g_2)S)_2 &= \mathrm{Tr}((\pi_1(g_1)S\bar{\pi}_2(g_2)^*)^*(\pi_1(g_1)R\bar{\pi}_2(g_2)^*)) \\
&= \mathrm{Tr}(\bar{\pi}_2(g_2)S^*\pi_1(g_1)^*\pi_1(g_1)R\bar{\pi}_2(g_2)^*) \\
&= \mathrm{Tr}(\bar{\pi}_2(g_2)S^*R\bar{\pi}_2(g_2^{-1})) \\
&= \mathrm{Tr}(\bar{\pi}_2(g_2^{-1})\bar{\pi}_2(g_2)S^*R) \\
&= \mathrm{Tr}(S^*R) \\
&= (S,R)_2.
\end{aligned}$$

Since $\pi(g_1,g_2)$ is linear, we see $\pi(g_1,g_2)$ is a inner product preserving bounded linear transformation on $\mathcal{H}_1 \otimes \mathcal{H}_2$ for each $g_1 \in G_1$ and $g_2 \in G_2$. Moreover $\pi(e_1,e_2) = I$ and

$$\begin{aligned}
\pi(g_1g_1', g_2g_2')T &= \pi_1(g_1g_1')T\bar{\pi}_2(g_2g_2')^{-1} \\
&= \pi_1(g_1)\pi_1(g_1')T\bar{\pi}_2(g_2')^{-1}\bar{\pi}_2(g_2)^{-1} \\
&= \pi_1(g_1)(\pi(g_1',g_2')T)\bar{\pi}(g_2)^{-1} \\
&= \pi(g_1,g_2)(\pi(g_1',g_2')T).
\end{aligned}$$

Thus π is a homomorphism of $G_1 \times G_2$ into $\mathcal{U}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, the unitary group of the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

To show π is a representation, it suffices to check strong continuity at the identity (e_1,e_2). For rank one tensors $v_1 \otimes v_2$ one has:

$$\begin{aligned}
&||\pi(g_1,g_2)(v_1 \otimes v_2) - v_1 \otimes v_2||_2 \leqslant ||\pi_1(g_1)v_1 \otimes \pi_2(g_2)v_2 - \pi_1(g_1)v_1 \otimes v_2||_2 \\
&\qquad\qquad + \quad ||\pi_1(g_1)v_1 \otimes v_2 - v_1 \otimes v_2||_2 \\
&= ||\pi_1(g_1)v_1 \otimes (\pi_2(g_2)v_2 - v_2)||_2 + ||(\pi_1(g_1)v_1 - v_1) \otimes v_2||_2 \\
&= ||\pi_1(g_1)v_1||\,||\pi_2(g_2)v_2 - v_2|| + ||\pi_1(g_1)v_1 - v_1||\,||v_2|| \\
&= ||v_1||\,||\pi_2(g_2)v_2 - v_2|| + ||\pi_1(g_1)v_1 - v_1||\,||v_2|| \\
&< \epsilon
\end{aligned}$$

if $(g_1,g_2) \in N_1 \times N_2$, where $N_1 = \{g_1 \mid ||\pi_1(g_1)v_1 - v_1||\,||v_2|| < \frac{\epsilon}{2}\}$ and $N_2 = \{g_2 \mid ||v_1||\,||\pi_2(g_2)v_2 - v_2|| < \frac{\epsilon}{2}\}$.

Now if $T \in \mathcal{B}_2(\overline{\mathcal{H}}_2, \mathcal{H}_1)$ and $\epsilon > 0$, using the density of the linear span of the vectors $v \otimes w$ where $v \in \mathcal{H}_1$ and $w \in \mathcal{H}_2$, we can find an S of form $S = \sum_{k=1}^n v_i \otimes w_i$ with $||T - S||_2 < \frac{\epsilon}{3}$. By continuity for the rank one vector $v_i \otimes w_i$, we can find a neighborhood U_i of (e_1,e_2) with

$$||\pi(g_1,g_2)(v_i \otimes w_i) - v_i \otimes w_i|| < \frac{\epsilon}{3n} \text{ for } (g_1,g_2) \in U_i.$$

Set $U = \cap_{i=1}^n U_i$. Then for $(g_1, g_2) \in U$, we have

$$\begin{aligned}
||\pi(g_1,g_2)T - T||_2 &\leqslant ||\pi(g_1,g_2)T - \pi(g_1,g_2)S||_2 + ||\pi(g_1,g_2)S - S||_2 + ||S-T||_2 \\
&= ||\pi(g_1,g_2)S - S||_2 + 2||S-T||_2 \\
&< ||\sum_{i=1}^n (\pi(g_1,g_2)(v_i \otimes w_i) - v_i \otimes w_i)||_2 + \frac{2\epsilon}{3} \\
&\leqslant \sum_{i=1}^n ||\pi(g_1,g_2)(v_i \otimes w_i) - v_i \otimes w_i||_2 + \frac{2\epsilon}{3} \\
&< \epsilon.
\end{aligned}$$

□

Definition 6.73. *Let π_1 and π_2 be unitary representations of G on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the inner tensor product representation $\pi_1 \otimes \pi_2$ is the representation of G given by $g \mapsto \pi_1(g) \otimes \pi_2(g)$.*

Note the inner tensor product $\pi_1 \otimes \pi_2$ is just the outer tensor product representation $\pi_1 \times \pi_2$ of the group $G \times G$ restricted to the diagonal subgroup $G_d = \{(g,g) \mid g \in G\}$.

Lemma 6.74. *Let π be an irreducible unitary representation of G on Hilbert space $\mathcal{H}$ and let I be the identity representation of G on Hilbert space $\mathcal{K}$. Then $\mathrm{Hom}_G(\pi \otimes I, \pi \otimes I)$ consists of the operators $I_{\mathcal{H}} \otimes B$ where B is in $\mathcal{B}(\mathcal{K})$ and $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$.*

Proof. Let A be a bounded linear operator on $\mathcal{B}_2(\bar{\mathcal{K}}, \mathcal{H})$ such that

$$A(\pi(g) \otimes I) = (\pi(g) \otimes I)A$$

for all g. Fix w_1 and w_2 in $\mathcal{K}$ and define $Dv = A(v \otimes w_1)(\bar{w}_2)$. Clearly D is linear, and note since the Hilbert-Schmidt norm is larger than the operator norm, one has $||A(v \otimes w_1)|| \leqslant ||A(v \otimes w_1)||_2$ and thus

$$\begin{aligned}
||Dv|| &\leqslant ||A(v \otimes w_1)||\,||\bar{w}_2|| \leqslant ||A(v \otimes w_1)||_2 ||w_2|| \\
&\leqslant ||A||\,||v \otimes w_1||_2 ||\bar{w}_2|| = ||A||\,||v||\,||w_1||\,||w_2||.
\end{aligned}$$

Moreover, $D \in \mathrm{Hom}_G(\pi, \pi)$ for

$$\begin{aligned}
D\pi(g)v = A(\pi(g)v \otimes w_1)(\bar{w}_2) &= A(\pi(g) \otimes I)(v \otimes w_1)(\bar{w}_2) \\
&= (\pi(g) \otimes I)(A(v \otimes w_1))(\bar{w}_2) \\
&= \pi(g)A(v \otimes w_1))(\bar{w}_2) = \pi(g)Dv.
\end{aligned}$$

Thus by Schur's Lemma $D = c(w_1, w_2)I$ for some scalar $c(w_1, w_2)$. It is easy to verify that c is a sesquilinear mapping on $\mathcal{K} \times \mathcal{K}$ which satisfies

$$|c(w_1, w_2)| \leqslant ||A||\,||w_1||\,||w_2||$$

for all w_1 and w_2. Thus by Proposition 5.28, there is a unique operator $B \in \mathcal{B}(\mathcal{K})$ such that

$$c(w_1, w_2) = (Bw_1, w_2)_{\mathcal{K}}$$

for all w_1 and w_2 in $\mathcal{K}$. Hence

$$A(v \otimes w_1)(\bar{w}_2) = (Bw_1, w_2)_{\mathcal{K}} v = (\bar{w}_2, \overline{Bw_1})_{\overline{\mathcal{K}}} v = (v \otimes Bw_1)(\bar{w}_2).$$

Thus

$$A(v \otimes w) = v \otimes Bw$$

for all v and w. □

Proposition 6.75. *Let π_1 be an irreducible unitary representation of G_1 and let π_2 be an irreducible unitary representation of G_2. Then $\pi_1 \times \pi_2$ is an irreducible unitary representation of $G_1 \times G_2$.*

Proof. Set $\pi = \pi_1 \times \pi_2$. Let A be a bounded linear operator on $\mathcal{B}_2(\bar{\mathcal{H}}_2, \mathcal{H}_1)$ such that

$$A\pi(g_1, g_2) = \pi(g_1, g_2)A$$

for all g_1, g_2. Since $\pi(g_1, e) = \pi_1(g) \otimes I_2$, we see $A \in \mathrm{Hom}_{G_1}(\pi_1 \otimes I_2, \pi_1 \otimes I_2)$. By Lemma 6.74, there is an operator $B \in \mathcal{B}(\mathcal{H}_2)$ such that $A = I_1 \otimes B$. Thus

$$A(v \otimes w) = v \otimes Bw$$

for all v and w. But $\pi(e, g_2)A = A\pi(e, g_2)$ for all g_2. Hence

$$v \otimes \pi_2(g_2)Bw = v \otimes B\pi_2(g)w.$$

This implies $B \in \mathrm{Hom}_{G_2}(\pi_2, \pi_2) = \mathbb{C}I$. Hence $B = cI$. So $A = cI$ and we see $\mathrm{Hom}_{G_1 \times G_2}(\pi_1 \times \pi_2, \pi_1 \times \pi_2) = \mathbb{C}I$. By Schur's Lemma, $\pi_1 \times \pi_2$ is irreducible. □

Corollary 6.76. *Let π be an irreducible unitary representation of a group G. Then $\bar{\pi}$ and $\pi \times \bar{\pi}$ are irreducible representations.*

Proof. Let S be a closed invariant subspace of $\overline{\mathcal{H}}$ under $\bar{\pi}$. Since $\bar{\pi}(g)\bar{v} = \pi(g)v$ where $\bar{v} = v$, we see S is a closed invariant subspace in $\mathcal{H}$ under π. Thus $S = \{0\}$ or $S = \mathcal{H} = \overline{\mathcal{H}}$. So $\bar{\pi}$ is irreducible. Consequently, $\pi \times \bar{\pi}$ is irreducible. □

Lemma 6.77. *Let π and π' be unitary representations of G on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. Then*

$$\mathrm{Hom}_{G \times G}(\pi \times \bar{\pi}, \pi' \times \bar{\pi}') = \{0\} \text{ if } \mathrm{Hom}_G(\pi, \pi') = \{0\}.$$

Proof. Assume $A \in \mathrm{Hom}_G(\pi \times \bar{\pi}, \pi' \times \bar{\pi}')$ is nonzero. Thus there is a $v_1 \otimes \bar{v}_2 \in \mathcal{H} \otimes \overline{\mathcal{H}}$ with $A(v_1 \otimes \bar{v}_2) \neq 0$. But $A(v_1 \otimes \bar{v}_2) \in \mathcal{B}_2(\mathcal{H}', \mathcal{H}')$. Hence there is a $v' \in \mathcal{H}'$ with

$$A(v_1 \otimes \bar{v}_2)(v') \neq 0.$$

Define $T \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ by

$$Tv = A(v \otimes \bar{v}_2)(v').$$

Clearly T is linear and $||Tv||_{\mathcal{H}'} \leqslant ||A(v \otimes \bar{v}_2)||\,||v'|| \leqslant ||A(v \otimes \bar{v}_2)||_2||v'|| \leqslant ||A||\,||v \otimes \bar{v}_2||_2||v'|| = ||A||\,||v||\,||v_2||\,||v'||$. Thus T is bounded. Moreover,

$$\begin{aligned} T\pi(g)v &= (A(\pi(g)v \otimes \bar{v}_2))(v') \\ &= (A((\pi(g) \otimes \bar{\pi}(e))(v \otimes \bar{v}_2))(v') \\ &= ((\pi'(g) \otimes \bar{\pi}'(e))(A(v \otimes \bar{v}_2))(v') \\ &= \pi'(g)A(v \otimes \bar{v}_2)\pi'(e)^*(v') \\ &= \pi'(g)Tv. \end{aligned}$$

Thus $T \in \mathrm{Hom}_G(\pi, \pi')$; and since $Tv_1 = A(v_1 \otimes \bar{v}_2)(v') \neq 0$, we see that $\mathrm{Hom}_G(\pi, \pi') \neq \{0\}$. □

7. Cyclic Representations

We shall see not all unitary representations or representations of $*$ algebras are discretely decomposable. These representations, however, can be written as direct sums of smaller (usually nonirreducible) subrepresentations.

Definition 6.78. *Let π be a nonzero representation of a group G or an algebra $\mathcal{A}$ on a topological vector space V. Then π is a cyclic representation if there is a vector v such that the smallest closed π invariant subspace of V containing v is V.*

Let S be a subset of a vector space V. By $\langle S\rangle$ we mean the algebraic linear span of S; i.e.,

$$\langle S\rangle = \{\sum_{s \in F} \lambda_s s \mid \lambda_s \in \mathbb{C},\ F \text{ a finite subset of } S\}.$$

Thus π is a cyclic representation if and only if there is a nonzero vector v such that $\langle \pi(G)v\rangle$ is dense in V when dealing with groups G or $\pi(\mathcal{A})v$ is dense in V when dealing with algebras $\mathcal{A}$. Any vector v having this property is said to be a **cyclic vector** for π.

Lemma 6.79. *Let π be a nonzero representation a locally convex topological vector space. Then π is irreducible if and only if every nonzero vector is cyclic.*

Proof. We argue only the case for a group G. Let π be irreducible and let $v \neq 0$. Then $\overline{\langle \pi(G)v\rangle}$ is closed, nonzero, and invariant. Thus $\overline{\langle \pi(G)v\rangle} = V$.

Conversely, suppose every nonzero vector v is cyclic. Let W be a nonzero closed invariant subspace. Choose $w \neq 0$ in W. Then $\mathcal{H} = \overline{\langle \pi(G)w\rangle} \subseteq W$. So $W = \mathcal{H}$ and we see π is irreducible. □

Theorem 6.80. *Let π be a unitary representation of a topological group G or a nonzero representation of a $*$ algebra on Hilbert space $\mathcal{H}$. Then π is an internal orthogonal direct sum of cyclic subrepresentations.*

Proof. We argue the algebra case. Consider the collection of all families of pairwise orthogonal closed invariant subspaces of $\mathcal{H}$ each having a cyclic vector. Order this collection by inclusion. Every linearly ordered subset has an upperbound, namely the union of the families in the subset. Consequently, by Zorn's Lemma, there is a maximal collection $\{\mathcal{H}_\alpha \mid \alpha \in A\}$ where $\mathcal{H}_\alpha$ is invariant and closed and $\pi_\alpha = \pi|_{\mathcal{H}_\alpha}$ is a cyclic representation. To finish, it suffices to show $\oplus\mathcal{H}_\alpha = \mathcal{H}$. If not, Lemma 6.36 implies $\mathcal{K} = (\oplus\mathcal{H}_\alpha)^\perp$ is a nonzero invariant closed subspace. Hence one can choose a nonzero vector $v \in \mathcal{K}$ and set $\mathcal{H}'$ to be the closure of the vector subspace $\pi(\mathcal{A})v = \{\pi(x)v \mid x \in \mathcal{A}\}$. Clearly $\mathcal{H}'$ is nonzero, $\mathcal{H}' \perp \oplus\mathcal{H}_\alpha$, and $\pi' = \pi|_{\mathcal{H}'}$ is cyclic. Thus the collection $\{\mathcal{H}'\} \cup \{\mathcal{H}_\alpha \mid \alpha \in A\}$ is strictly larger than the maximal collection $\{\mathcal{H}_\alpha \mid \alpha \in A\}$, a contradiction. □

Exercise Set 6.4

1. Let U be a bounded linear transformation from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$. Show U is a partial isometry if and only if $E = U^*U$ and $F = UU^*$ are projections. Then show $E\mathcal{H} = (\ker U)^\perp$ and $F\mathcal{K} = U(\mathcal{H})$.

2. Let V and W be finite dimensional vector spaces. Show that every bilinear mapping $B : V \times W \to F$ where F is a vector space can be written uniquely in the form $B(v, w) = T(v \otimes w)$ where T is a linear transformation of $V \otimes W$ into F.

3. Show Schur's Lemma given in Proposition 6.49 also holds if π and ρ are representations of $*$ algebras on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

4. Let π be a finite dimensional representation of a group G on vector space V. Let V_0 be a subspace of V. Define $V_0^\perp$ to be the subspace of V^* consisting of those f for which

$$\langle v_0, f\rangle = 0 \text{ for all } v_0 \in V_0.$$

(a) Using the identification $V^{**} = V$, show $(V_0^\perp)^\perp = V_0$.

(b) Show V_0 is π invariant if and only if $V_0^\perp$ is $\check{\pi}$ invariant.

5. Let $\overline{\mathcal{H}}$ be the conjugate Hilbert space to a finite dimensional Hilbert space $\mathcal{H}$ and let A be a linear transformation of $\mathcal{H}$. Show if $v_1, v_2, \ldots, v_n$ is a basis of $\mathcal{H}$, then the matrix of $\bar{A}$ relative to the basis $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ of $\overline{\mathcal{H}}$ is the conjugate of the matrix of A relative to the basis $v_1, v_2, \ldots, v_n$.

6. Let $A, B, U \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. Show $\bar{A}\bar{B} = \overline{AB}$ and U is unitary if and only if $\overline{U}$ is unitary.

7. Let π be a finite dimensional unitary representation. Show $\pi \otimes \bar{\pi}$ and $\pi \otimes \check{\pi}$ are equivalent.

8. Let $G = \mathrm{SU}(2)$ be the compact group of 2×2 unitary matrices. Thus $G = \left\{ \left[\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix} \right] \mid |a|^2 + |b|^2 = 1 \right\}$. Define π by $\pi(g) = g$ for $g \in G$.

(a) Determine $\bar{\pi}$ as a matrix representation.

(b) Show π and $\bar{\pi}$ are unitarily equivalent.

9. Let π be a unitary representation of G on Hilbert space $\mathcal{H}$. Let I be the identity representation of G defined on Hilbert space $\mathcal{H}'$ whose orthonormal basis has dimension cardinality n. Show $\pi \otimes I \cong n\pi$.

10. Let π be a unitary representation of G. Show $A \mapsto \bar{A}$ is a conjugate linear $*$ algebra isomorphism of $\mathrm{Hom}_G(\pi, \pi)$ onto $\mathrm{Hom}_G(\bar{\pi}, \bar{\pi})$.

11. Let π_1, π_2, π_3, and π_4 be unitary representations of G. Show

$$\mathrm{Hom}_{G \times G}(\pi_1 \times \pi_2, \pi_3 \times \pi_4) = 0$$

if and only if $\mathrm{Hom}_G(\pi_1, \pi_3) = \{0\}$ or $\mathrm{Hom}_G(\pi_2, \pi_4) = \{0\}$.

12. Let G_1 and G_2 be locally compact Hausdorff groups and let π_1 and π_2 be completely decomposable unitary representations of G_1 and G_2, respectively. Show the unitary representation $\pi_1 \times \pi_2$ is completely decomposable and if π_1 and π_2 are irreducible unitary representations of G_1 and G_2, then the primary projection $P(\pi_1 \times \pi_2)$ is given by $P(\pi_1 \times \pi_2) = P(\pi_1) \otimes P(\pi_2)$.

13. A finite dimensional representation π of a group G is said to be **completely reducible** if every invariant subspace has a complementary invariant subspace. Show if π is completely reducible, the contragredient representation $\check{\pi}$ is completely reducible.

14. Let π_1 be an irreducible representation of a group G_1 on a finite dimensional complex vector space V_1 and let I be the identity representation of G_2 on a finite dimensional complex vector space V_2. Show

$$\mathrm{Hom}_{G_1 \times G_2}(\pi_1 \times I, \pi_1 \times I) = \{I \otimes B \mid B \in \mathcal{L}(V_2)\}.$$

15. Let π_1 and π_2 be finite dimensional irreducible complex representations of groups G_1 and G_2. Show $\mathrm{Hom}_{G_1 \times G_2}(\pi_1 \times \pi_2, \pi_1 \times \pi_2) = \mathbb{C}I$.

16. In Exercise 6.4.13, we defined a completely reducible representation. A group G is said to be completely reducible if every complex finite dimensional representation of G is completely reducible. Let G be completely reducible.

(a) Show every finite dimensional complex representation of G is a direct sum of irreducible representations.

(b) Show a complex finite dimensional representation π of G is irreducible if and only if $\mathrm{Hom}_G(\pi, \pi) = \mathbb{C}I$.

17. Let π be a representation on a Hilbert space $\mathcal{H}$ having a cyclic vector v. Let P be a projection of the Hilbert space $\mathcal{H}$ onto a closed π invariant subspace. Show Pv is a cyclic vector for that subrepresentation.

18. Let π be a unitary representation of a topological group G. A vector v in $\mathcal{H}_\pi$ is said to be a separating vector for $\mathrm{Hom}_G(\pi,\pi)$ if $A = 0$ whenever $A \in \mathrm{Hom}_G(\pi,\pi)$ and $Av = 0$. Show v is a separating vector for $\mathrm{Hom}_G(\pi,\pi)$ if and only if v is a cyclic vector for π.

19. Let π and π' be unitary representations of a topological group G with $\mathrm{Hom}_G(\pi,\pi') = \{0\}$. Show $\mathrm{Hom}_G(\pi\oplus\pi', \pi\oplus\pi') \cong \mathrm{Hom}_G(\pi,\pi) + \mathrm{Hom}_G(\pi',\pi')$ as $*$ algebras.

20. Suppose v is a cyclic vector for unitary representation π and v' is a cyclic vector for unitary representation π'. If $\mathrm{Hom}_G(\pi,\pi') = \{0\}$, show $v \oplus v'$ is a cyclic vector for $\pi \oplus \pi'$.

21. Let V be finite dimensional complex vector space with basis $e_1, \ldots, e_n$. Let $\mathrm{GL}(n,\mathbb{C})$ be the group of invertible $n \times n$ matrices under multiplication.

(a) Show there is a one-to-one onto correspondence between the representations π of G on V and the continuous homomorphisms $\Pi : G \to \mathrm{GL}(n,\mathbb{C})$, (determined from the basis $e_1, \ldots, e_n$). Such Π are called matrix representations of G.

(b) Show if V is a Hilbert space and $e_1, \ldots, e_n$ is an orthonormal basis, then π is unitary if and only if $\Pi(G) \subseteq U(n)$, the group of $n \times n$ unitary matrices.

22. Let π be an irreducible unitary representation. Let n be a natural number. Show $\mathrm{Hom}_G(n\pi, n\pi) \cong M_{n\times n}(\mathbb{C})$, the algebra of n by n complex matrices.

23. Let π be an irreducible unitary representation of finite dimension d. Show $n\pi$ is cyclic if and only if $n \leqslant d$.

24. Let $\pi_1, \pi_2, \ldots, \pi_s$ be pairwise inequivalent finite dimensional irreducible unitary representations of topological group G. Show

$$\bigoplus_{j=1}^{s} n_j\pi_j \text{ is cyclic if and only if } n_j \leqslant d(\pi_j) \text{ for } j = 1, 2, \ldots. s.$$

8. The Duals of $\mathbb{R}^n$, $\mathbb{T}^n$, and $\mathbb{Z}^n$

One Dimensional Representations of $(\mathbb{R}, +)$.

A one dimensional representation is a continuous homomorphism e of $\mathbb{R}$ into the group of nonzero complex numbers under multiplication. Choose $\epsilon > 0$ such that $e([-\epsilon,\epsilon]) \subseteq \{c \mid |c-1| < 1\}$. Define $a(x) = \log(e(x))$ for $|x| \leqslant \epsilon$. Note $a(x+y) = \log(e(x+y)) = \log(e(x)e(y)) = \log(e(x)) + \log(e(y)) =$

$a(x) + a(y)$ if $|x + y| \leqslant \epsilon$, $|x| \leqslant \epsilon$, and $|y| \leqslant \epsilon$. This implies $ma(\frac{\epsilon}{n}) = a(\frac{\epsilon}{n}) + a(\frac{\epsilon}{n}) + \cdots + a(\frac{\epsilon}{n}) = a(\frac{m}{n}\epsilon)$ if $|\frac{m}{n}| \leqslant 1$. In particular, $na(\frac{\epsilon}{n}) = a(\epsilon)$ and thus $a(\frac{m}{n}\epsilon) = \frac{m}{n}a(\epsilon)$ for $|\frac{m}{n}| \leqslant 1$. By continuity, $a(x\epsilon) = xa(\epsilon)$ for $|x| \leqslant 1$. Thus $a(y) = \frac{y}{\epsilon}a(\epsilon)$ for $|y| \leqslant \epsilon$. This gives $e(y) = e^{cy}$ where $c = \frac{a(\epsilon)}{\epsilon}$ for $|y| \leqslant \epsilon$. Since e is a homomorphism, $\{y \mid e(y) = e^{cy}\}$ is a subgroup of $\mathbb{R}$ containing $[-\epsilon, \epsilon]$. But any subgroup of $\mathbb{R}$ containing an open subset is open and closed and thus all of $\mathbb{R}$. So $e(y) = e^{cy}$ for all y.

For another argument, we note if $F(x) = \int_0^x e(s)\,ds$, then F is differentiable and $F(x) \neq 0$ for x near 0. Also one has

$$\begin{aligned} F(x+y) = \int_0^{x+y} e(s)\,ds &= \int_0^y e(s)\,ds + \int_y^{x+y} e(s)\,ds \\ &= \int_0^y e(s)\,ds + \int_0^x e(s+y)\,ds \\ &= \int_0^y e(s)\,ds + \int_0^x e(s)e(y)\,ds. \end{aligned}$$

So $F(x + y) = F(y) + e(y)F(x)$. Since F is differentiable, we see by taking x near 0 that e is differentiable and by differentiating with respect to y, one obtains

$$e(x + y) = e(y) + e'(y)F(x).$$

Differentiating this with respect to x gives $e'(x + y) = e'(y)e(x)$. So $e'(x) = e'(0)e(x)$. Take $c = e'(0)$ and let $H(x) = \frac{e^{cx}}{e(x)}$. Note H is constant since $H(x)$ has derivative $\frac{e(x)ce^{cx} - c^{cx}e'(x)}{e(x)^2} = 0$. But $H(0) = 1$. Thus $e(x) = e^{cx}$.

Now e is unitary if $cx \in \mathbb{R}i$ for all $x \in \mathbb{R}$. So $c = 2\pi i\omega$ for some $\omega \in \mathbb{R}$. Hence the one dimensional unitary representations of $\mathbb{R}$ are given by

$$e_\omega(x) = e^{2\pi i\omega x}.$$

One Dimensional Representations of $(\mathbb{T}, +)$.

Suppose $\chi : \mathbb{T} \to \mathbb{C}^*$ is a one dimensional representation of the torus. Then $e(x) = \chi(e^{2\pi ix})$ is a one dimensional representation of $\mathbb{R}$ and thus there is a $\omega \in \mathbb{C}$ such that $\chi(e^{2\pi ix}) = e^{2\pi i\omega x}$. But $\chi(e^{2\pi i}) = 1$ implies $e^{2\pi i\omega} = 1$. This occurs if and only if $\omega \in \mathbb{Z}$. Thus the one dimensional representations of $\mathbb{T}$ are given by

$$e_n(z) = e_n(e^{2\pi ix}) = (e^{2\pi ix})^n = z^n$$

for $n \in \mathbb{Z}$.

One Dimensional Representations of $(\mathbb{Z}, +)$.

Note if e is a homomorphism of $\mathbb{Z}$ into $\mathbb{C}^*$, then $e(n) = e(1+1+\cdots+1) = e(1)^n = z^n$ where $z = e(1) \neq 0$. If e is unitary, $|z| = 1$ and so $z \in \mathbb{T}$. Because of these correspondences, it is customary to write $\widehat{\mathbb{R}} = \mathbb{R}$, $\widehat{\mathbb{T}} = \mathbb{Z}$, and $\widehat{\mathbb{Z}} = \mathbb{T}$.

If $e \in \widehat{\mathbb{R}}^n$, then since $t_j \mapsto e(0, \ldots, t_j, 0, \ldots, 0)$ is a representation of $\mathbb{R}$, we see $e(t_1, t_2, \ldots, t_n) = \prod e^{2\pi i \omega_j t_j} = e^{2\pi i(\omega \cdot t)}$ for some $\omega \in \mathbb{R}^n$. So $\widehat{\mathbb{R}}^n = \mathbb{R}^n$. Similarly, a one dimensional unitary representation of $\mathbb{T}^n$ is given by

$$e_k(z) = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$$

where $k \in \mathbb{Z}^n$ and a one dimensional unitary representation of $\mathbb{Z}^n$ is given by

$$e_z(k) = z_1^{k_1} z_2^{k_2} \cdots a_n^{k_n}$$

where $z \in \mathbb{T}^n$. Of course one writes $\widehat{\mathbb{T}}^n = \mathbb{Z}^n$ and $\widehat{\mathbb{Z}}^n = \mathbb{T}^n$.

Table 1. Duals of $\mathbb{R}^n$, $\mathbb{T}^n$, $\mathbb{Z}^n$

Group	Dual	Identification		
$\mathbb{R}$	$\widehat{\mathbb{R}} = \mathbb{R}$	$e_\omega \leftrightarrow \omega \in \mathbb{R}$	where	$e_\omega(x) = e^{2\pi i \omega x}$
$\mathbb{R}^n$	$\widehat{\mathbb{R}^n} = \mathbb{R}^n$	$e_\omega \leftrightarrow \omega \in \mathbb{R}^n$	where	$e_\omega(x) = e^{2\pi i \omega \cdot x}$
$\mathbb{T}$	$\widehat{\mathbb{T}} = \mathbb{Z}$	$e_k \leftrightarrow k \in \mathbb{Z}$	where	$e_k(z) = z^k$
$\mathbb{T}^n$	$\widehat{\mathbb{T}^n} = \mathbb{Z}^n$	$e_k \leftrightarrow k \in \mathbb{Z}^n$	where	$e_k(z) = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$
$\mathbb{Z}$	$\widehat{\mathbb{Z}} = \mathbb{T}$	$e_z \leftrightarrow z \in \mathbb{T}$	where	$e_z(k) = z^k$
$\mathbb{Z}^n$	$\widehat{\mathbb{Z}^n} = \mathbb{T}^n$	$e_z \leftrightarrow z \in \mathbb{T}^n$	where	$e_z(k) = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$

9. Continuity of Representations of Banach $*$ Algebras

In defining representations of Banach algebras, there is no continuity condition. At least for Banach $*$ algebras, it turns out norm continuity is automatic; i.e., if π is a representation of a Banach $*$ algebra $\mathcal{B}$ on a Hilbert space, then $||\pi(x)|| \leqslant ||x||$ for each $x \in \mathcal{B}$. This turns out to be a consequence of the spectral radius theorem. We review and state this theorem without proof here. All functional analysis texts covering the spectral theorem have some form of this theorem.

Let $\mathcal{B}$ be a complex Banach algebra with a multiplicative identity e. If $x \in \mathcal{B}$, the **spectrum** $\sigma(x)$ is the set of complex numbers λ such that $\lambda e - x$ has no multiplicative inverse in $\mathcal{B}$. Thus there is no $y \in \mathcal{B}$ which satisfies

$$y(\lambda e - x) = (\lambda e - x)y = e.$$

Theorem 6.81 (Spectral Radius Theorem)**.** *Let $\mathcal{B}$ be a Banach algebra with identity and suppose $x \in \mathcal{B}$. Then the spectrum $\sigma(x)$ is a nonempty compact subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \leqslant ||x||\}$. Moreover, the spectral radius $|x|_\sigma = \max\{|\lambda| \mid \lambda \in \sigma(x)\}$ is given by*

$$|x|_\sigma = \lim_{n \to \infty} ||x^n||^{\frac{1}{n}}.$$

Banach $*$ algebras with the additional property $||x^*x|| = ||x||^2$ for all x are known as **C* algebras** and play a central role in operator algebras. An important example is the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. Indeed, bounded linear operators A on $\mathcal{H}$ satisfy $||A^*A|| = ||A||^2$. Another example is the space of complex valued continuous functions on a compact Hausdorff space. We look at these spaces in the next section.

Corollary 6.82. *Let $\mathcal{A}$ be a C^* algebra with identity. Then $|x^*x|_\sigma = ||x||^2$ for each $x \in \mathcal{A}$.*

Proof. Since $\mathcal{A}$ is a C^* algebra, $||x^*x|| = ||x||^2$ for each $x \in \mathcal{A}$. Thus $||(x^*x)^2|| = ||(x^*x)^*(x^*x)|| = ||x^*x||^2 = ||x||^4$. From this we see $||(x^*x)^4|| = ||((x^*x)^*(x^*x))^*((x^*x)^*(x^*x))|| = ||(x^*x)^*(x^*x)||^2 = (||x^*x||^2)^2 = ||x||^8$. Repeating this argument and using induction, one obtains

$$||(x^*x)^{2^n}|| = ||x||^{2^{n+1}}$$

for all x. Thus $|x^*x|_\sigma = \lim ||(x^*x)^{2^n}||^{1/2^n} = ||x||^2$. $\square$

Proposition 6.83. *Let π be a representation of a Banach $*$ algebra $\mathcal{B}$ on a Hilbert space $\mathcal{H}$. Then $||\pi(x)|| \leqslant ||x||$ for each $x \in \mathcal{B}$.*

Proof. By Exercise 6.5.20, if $\mathcal{B}$ does not have an identity, we may extend it to a Banach $*$ algebra with identity, and extend π to be a representation of this extended $*$ algebra. Thus we may assume $\mathcal{B}$ has an identity e. If π is obtained by extending from $\mathcal{B}$ to $\mathcal{B}_e$ as in Exercise 6.5.20, then $\pi(e) = I$. If not, then since $\pi(e)^2 = \pi(e)$ and $\pi(e^*) = \pi(e) = \pi(e)^*$, $\pi(e)$ is an orthogonal projection and since $\pi(y)\pi(e)\mathcal{H} = \pi(e)\pi(y)\mathcal{H} \subseteq \pi(e)\mathcal{H}$, the range of $\pi(e)$ is a closed invariant subspace. In this case, since $||\pi(x)|| = ||\pi(x)|_{\pi(e)(\mathcal{H})}||$ for all x, we may replace π by the subrepresentation $x \mapsto \pi(x)|_{\pi(e)\mathcal{H}}$. This implies in all instances, we may assume $\pi(e) = I$.

Now take $x \in \mathcal{B}$ and let $y = x^*x$. Note if $\lambda \notin \sigma(y)$, then $\lambda \notin \sigma(\pi(y))$. Indeed, if $\lambda \notin \sigma(y)$, then there is $z \in \mathcal{B}$ with $(\lambda e - y)z = z(\lambda e - y) = e$. Consequently, $(\lambda I - \pi(y))\pi(z) = \pi(z)(\lambda I - \pi(y)) = \pi(e) = I$, and thus $\lambda \notin \sigma(\pi(y))$. Thus $\sigma(\pi(y)) \subseteq \sigma(y)$ and so $|\pi(y)|_\sigma \leqslant |y|_\sigma = |x^*x|_\sigma \leqslant ||x^*x||^2 \leqslant ||x^*||\,||x|| = ||x||^2$. But $\pi(x) \in \mathcal{B}(\mathcal{H})$, which is a C^*algebra. By Corollary 6.82, $||\pi(x)^*\pi(x)|| = |\pi(x)^*\pi(x)|_\sigma = |\pi(x^*x)|_\sigma = |\pi(y)|_\sigma$. Thus $||\pi(x)||^2 = ||\pi(x)^*\pi(x)|| \leqslant ||x||^2$. $\square$

10. Representations of $C(X)$

Let X be a compact Hausdorff space. $C(X)$ is a Banach $*$ algebra with norm $|f|_\infty = \max\{|f(x)| \mid x \in X\}$ and adjoint defined by $f^*(x) = \overline{f(x)}$. This examples shares the important property $|f|^2 = |f^*f|$ with the Banach $*$

algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space. This property makes $C(X)$ a C^* algebra.

The Gelfand theory shows that $C(X)$ is the canonical example of commutative C^* algebra having an identity. Our intent here is to obtain the irreducible $*$ representations of $C(X)$. In some sense which we do not explore here, representations of $*$ algebras on Hilbert spaces only reflect the C^* algebra qualities of the algebra. Hence in many ways when dealing with $*$ algebras and their representations, one should restrict oneselves to C^* algebras.

Let π be an irreducible nonzero $*$ representation of $C(X)$. Since we are dealing with complex representations, Schur's Lemma implies π is one dimensional. Hence $\pi : C(X) \to \mathbb{C}$. Furthermore π is positive; i.e., $f \geqslant 0$ implies $\pi(f) = \pi(\sqrt{f}^{\,*}\sqrt{f}) = \overline{\pi(\sqrt{f})}\pi(\sqrt{f}) \geqslant 0$. Thus by the Riesz Theorem 6.1, there is a regular Borel measure on X such that $\pi(f) = \mu(f) = \int f\,d\mu$. Also if K_1 and K_2 are disjoint compact subsets then $\mu(K_1)\mu(K_2) = 0$. Indeed, it is easy (see Exercise 6.5.18) to find disjoint open subsets U_1 and U_2 of X with $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$. Then by Proposition 5.23, there exists $f_j \in C(X)$ with $0 \leqslant f_j \leqslant 1$, $f_j = 1$ on K_j, and $\operatorname{supp} f_j \subseteq U_j$ for $j = 1, 2$. Hence $0 \leqslant \mu(K_1)\mu(K_2) \leqslant (\int f_1\,d\mu)(\int f_2\,d\mu) = \pi(f_1)\pi(f_2) = \pi(f_1 f_2) = \pi(0) = 0$. Next note $\mu(X) = 1$ for $\pi(1)^2 = \pi(1) = \mu(X)$ and if $\pi(1) = 0$, then $\pi(f) = 0$ for all f. We claim if K is a compact subset, then $\mu(K) = 0$ or $\mu(K) = 1$. If $\mu(K) < 1$, by outer regularity, there is an open subset $U \supseteq K$ with $\mu(U) < 1$. Now choose $f \in C(X)$ with $0 \leqslant f \leqslant 1$, $f = 1$ on K, and $\operatorname{supp} f \subseteq U$. Then $\mu(K) \leqslant \pi(f^n) = \pi(f)^n \to 0$ as $n \to \infty$ for $\pi(f) = \int f\,d\mu \leqslant \mu(U) < 1$. So $\mu(K) = 0$.

Next let $\mathcal{K}$ be all the compact subsets K of X with $\mu(K) = 1$. We have $X \in \mathcal{K}$. Also if $K_1, K_2 \in \mathcal{K}$, then $K_1 \cap K_2 \in \mathcal{K}$, for if $\mu(K_1 \cap K_2) = 0$, then $\mu(K_1 - K_1 \cap K_2) = \mu(K_2 - K_1 \cap K_2) = 1$, which implies $\mu(X) \geqslant 2$. Thus $\mathcal{K}$ has the finite intersection property. So $\cap K$ is nonempty. We claim it consists of one point p. Indeed, if p and p_1 are distinct in the intersection, then there are disjoint compact neighborhoods N and N_1 of p and p_1. Since $p \notin N_1$, $N_1 \notin \mathcal{K}$. Thus $\mu(N_1) = 0$. Thus $X - \operatorname{int}(N_1) \in \mathcal{K}$. So p_1 does not belong to $\cap_{K \in \mathcal{K}} K$. Thus the intersection has only member p. We claim $\mu\{p\} = 1$. If not $\mu(X - \{p\}) > 0$ and by regularity there is a compact subset F of $X - \{p\}$ such that $\mu(F) > 0$. So $\mu(F) = 1$, a contradiction. Thus $\mu = \epsilon_p$ is point mass at p. Hence $\pi(p) = f(p)$.

Theorem 6.84. *Let X be a compact Hausdorff space. Then the mapping $p \mapsto \pi_p$ where $\pi_p(f) = f(p)$ is a one-to-one correspondence between X and the irreducible $*$ representations of $C(X)$.*

The collection of irreducible $*$ representations of $C(X)$ might understandably be called the *dual* of $C(X)$. This, however, is not the case. It is almost universally known as the **Gelfand spectrum** of $C(X)$. Indeed, the Gelfand Theorem, which we state next gives a complete description of commutative C^* algebras with identity.

Theorem 6.85 (Gelfand)**.** *Let $\mathcal{A}$ be a commutative C^* algebra with identity e. Then the space Δ of nonzero one-dimensional $*$ representations of $\mathcal{A}$ is a compact Hausdorff subspace of the unit ball of the dual space $\mathcal{A}^*$ in the weak $*$ topology. Moreover, $x \mapsto \hat{x}$ where $\hat{x}(\pi) = \pi(x)$ is a $*$ algebra isometry of $\mathcal{A}$ onto $C(\Delta)$.*

11. Regular and Quasi-regular Representations

In this section we give some examples of unitary representations obtained from the natural symmetries given by actions of the group. Their analysis as to how these representations decompose is a central aspect of harmonic analysis. The most natural is the regular representation which we start with next.

11.1. The regular representation.

Definition 6.86. *Let G be a locally compact Hausdorff group with a left Haar measure m. Let $\mathcal{H} = L^2(G, m)$ and for $g \in G$, set $\lambda(g)f(x) = f(g^{-1}x)$. Then λ is called the left regular representation of G.*

Proposition 6.87. *λ is a unitary representation of G.*

Proof. We know we may take m to be a regular Borel measure. We first note $\lambda(g)$ is a unitary operator for each g. Indeed, since $x \mapsto g^{-1}x$ is a homeomorphism, $\lambda(g)$ maps Borel measurable functions to Borel measurable functions. Also

$$||\lambda(g)f||^2 = \int_G |f(g^{-1}x)|^2\,dx = \int_G |f(x)|^2\,dx$$

and hence $\lambda(g)$ is an isometry for each g. Note $\lambda(e) = I$ and

$$\begin{aligned}\lambda(g_1)\lambda(g_2)f(x) &= \lambda(g_2)f(g_1^{-1}x) = f(g_2^{-1}g_1^{-1}x)\\ &= f((g_1g_2)^{-1}x) = \lambda(g_1g_2)f(x).\end{aligned}$$

Thus $\lambda(g_1g_2) = \lambda(g_1)\lambda(g_2)$ and hence $\lambda(g^{-1})\lambda(g) = \lambda(g)\lambda(g^{-1}) = \lambda(e) = I$. Hence $\lambda(g)^{-1} = \lambda(g^{-1})$, and we see λ is a homomorphism of G into $\mathcal{U}(\mathcal{H})$.

To finish we need to show π is strongly continuous at e. Let $\epsilon > 0$ and let $f \in L^2(G, m)$. Choose a compact G_δ neighborhood N_0 of e and $f_0 \in C_c(G)$ with $|f - f_0|_2 < \frac{\epsilon}{3}$. Now by Lemma 5.24, f_0 is left uniformly continuous. Hence there is a compact neighborhood $N \subseteq N_0$ of e such that $|f_0(x^{-1}y) - f_0(y)| \leqslant \frac{\epsilon}{3m(N_0\mathrm{supp}(f_0))^{1/2}}$ if $x \in N$. Thus if $x \in N$,

$$\begin{aligned}|\lambda(x)f_0 - f_0|_2^2 &\leqslant \int |f_0(x^{-1}y) - f_0(y)|^2\,dy\\ &\leqslant \int_{\mathrm{supp}(f_0)\cup x\mathrm{supp}(f_0)} \frac{\epsilon^2}{9m(N_0\mathrm{supp}(f_0))}\,dy\\ &\leqslant \int_{N_0\mathrm{supp}(f_0)} \frac{\epsilon^2}{9m(N_0\mathrm{supp}(f_0))}\,dy\\ &= \frac{\epsilon^2}{9}.\end{aligned}$$

Thus for $x \in N$,

$$\begin{aligned}|\lambda(x)f - f|_2 &\leqslant |\lambda(x)f - \lambda(x)f_0|_2 + |\lambda(x)f_0 - f_0|_2 + |f_0 - f|_2\\ &< 2|f - f_0|_2 + \frac{\epsilon}{3}\\ &< \epsilon.\end{aligned}$$

So λ is strongly continuous at e. □

Corollary 6.88. *Let U be a Borel subset of G with positive Haar measure. Then UU^{-1} contains an open neighborhood of the identity e.*

Proof. By shrinking U and using inner regularity, we may assume U has positive finite measure. Thus $\chi_U \in L^2(G)$. Since the regular representation λ is strongly continuous, $\chi_U * \chi_{U^{-1}}(y) = \int \chi_U(x)\chi_{U^{-1}}(x^{-1}y)\,dx = (\chi_U, \lambda(y)\chi_U)_2$ is continuous and positive at $y = e$. So it is positive in an open neighborhood V of e. Now it is positive at a point y implies there is an $x \in U$ such that $x^{-1}y \in U^{-1}$. So $y \in xU^{-1} \subseteq UU^{-1}$. □

Proposition 6.89. *Let $\phi : G \to H$ be a Borel homomorphism of a locally compact Hausdorff group G into a second countable topological group H. Then ϕ is continuous.*

Proof. By replacing H by $\phi(G)$ with the relative topology, we may assume the mapping ϕ is onto. Now let W be an open neighborhood of e in H. Pick an open neighborhood V of e in H such that $V^{-1}V \subseteq W$. Since H is second countable, the cover $\{xV \mid x \in H\}$ of H has a countable subcover x_jV for $j = 1, 2, \ldots$ and thus the Borel sets $\phi^{-1}(x_jV)$ cover G. Choose g_j with $\phi(g_j) = x_j$. Then if $g \in \phi^{-1}(x_jV)$, $\phi(g) = x_jv$ and thus $\phi(g_j^{-1}g) = v$. So $g_j^{-1}g \in \phi^{-1}(V)$ and we see $g \in g_j\phi^{-1}(V)$. Hence the Borel sets $g_j\phi^{-1}(V)$ cover G. This implies $\phi^{-1}(V)$ has positive Haar measure. By Corollary 6.88, $\phi^{-1}(V)^{-1}\phi(V)$ contains an open neighborhood U of e in G. This implies $\phi(u) \in V^{-1}V$ if $u \in U$. Thus $\phi(U) \subseteq W$. So ϕ is continuous at e. □

Example 6.90. *In the case $G = \mathbb{R}^n$ with its usual topology, one has*

$$\lambda(y)f(x) = f(x - y)$$

for $f \in L^2(\mathbb{R}^n)$. The measure here is Lebesgue measure λ_n. We use the Fourier transform to study this representation. Recall the Fourier transform $\mathcal{F}$ is a unitary mapping of $L^2(\mathbb{R}^n)$. Thus $\hat{\lambda}$ defined by

$$\hat{\lambda}(x) = \mathcal{F}\lambda(x)\mathcal{F}^{-1}$$

is a unitary representation unitarily equivalent to λ. Note if $f \in \mathcal{S}(\mathbb{R}^n)$, then by (b) of Lemma 3.3, $\mathcal{F}(\lambda(x)\mathcal{F}^{-1}f) = \tau(x)\mathcal{F}(\mathcal{F}^{-1}f) = \tau(x)f$. Thus $\hat{\lambda}(x)f(y) = e^{-2\pi i x\cdot y}f(y)$ for $f \in \mathcal{S}(\mathbb{R}^n)$. Now by Proposition 2.55, $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. By taking limits, one obtains

$$\hat{\lambda}(x)f(y) = e^{-2\pi i x\cdot y}f(y) \text{ for } f \in L^2(\mathbb{R}^n). \tag{6.12}$$

11.2. The biregular representation. Let G be a locally compact Hausdorff group with a left Haar measure m. The left regular representation λ is defined on $L^2(G)$ by $\lambda(g)f(x) = f(g^{-1}x)$ and by Exercise 6.5.28, one obtains a unitarily equivalent representation ρ to λ where ρ is defined by $\rho(g)f(x) = \Delta(g)^{1/2}f(xg)$. The representation ρ is known as the **right regular** representation of G. We note one has

$$\lambda(g_1)\rho(g_2) = \rho(g_2)\lambda(g_1) \tag{6.13}$$

for all g_1 and g_2 in G. Indeed,

$$\begin{aligned}
\lambda(g_1)\rho(g_2)f(x) &= \rho(g_2)f(g_1^{-1}x) \\
&= \Delta(g_2)^{1/2}f(g_1^{-1}xg_2) \\
&= \Delta(g_2)^{1/2}\lambda(g_1)f(xg_2) \\
&= \rho(g_2)\lambda(g_1)f(x).
\end{aligned}$$

This implies $\lambda(g_1)\rho(g_2)\lambda(g_1')\rho(g_2') = \lambda(g_1)\lambda(g_1')\rho(g_2)\rho(g_2') = \lambda(g_1g_2)\rho(g_1'g_2')$; and thus if we define B by

$$\begin{aligned}
B(g_1, g_2)f(x) &= \lambda(g_1)\rho(g_2)f(x) \\
&= \Delta(g_2)^{1/2}f(g_1^{-1}xg_2),
\end{aligned} \tag{6.14}$$

then B is a homomorphism of $G \times G$ into the unitary group of $L^2(G)$. This unitary homomorphism is called the **biregular representation** of G.

Proposition 6.91. *The biregular representation B of G is a unitary representation of $G \times G$.*

Proof. Since B is a homomorphism of $G \times G$ into $\mathcal{U}(L^2(G))$, we need only check the strong continuity of $(g_1, g_2) \mapsto B(g_1, g_2)$ at the identity (e, e). Let $\epsilon > 0$ and $f \in L^2(G)$. Using the strong continuity of λ and ρ, there are neighborhoods N_1 and N_2 of e such that $|\lambda(g)f - f|_2 < \frac{\epsilon}{2}$ if $g \in N_1$ and $|\rho(g)f - f|_2 < \frac{\epsilon}{2}$ if $g \in N_2$. Thus if $(g_1, g_2) \in N_1 \times N_2$, then

$$\begin{aligned} |B(g_1, g_2)f - B(e,e)f|_2 &= |\lambda(g_1)\rho(g_2)f - f|_2 \\ &\leqslant |\lambda(g_1)\rho(g_2)f - \rho(g_2)f|_2 + |\rho(g_2)f - f|_2 \\ &= |\rho(g_2)(\lambda(g_1)f - f)|_2 + |\rho(g_2)f - f|_2 \\ &= |\lambda(g_1)f - f|_2 + |\rho(g_2)f - f|_2 \\ &< \epsilon. \end{aligned}$$

So B is strongly continuous. □

11.3. The quasi-regular representation. Now suppose H is a closed subgroup of G. By Lemma 6.16 and Theorem 6.15, we know there is a continuous rho ρ function on G and a quasi-invariant regular Borel measure μ on G/H satisfying

$$\int f(x^{-1}yH)\,d\mu(yH) = \int \frac{\rho(xy)}{\rho(y)} f(yH)\,d\mu(yH)$$

for nonnegative Borel functions f. Recall ρ satisfies $\rho(x) > 0$ and $\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}$ for all x and h.

Take $\mathcal{H} = L^2(G/H, \mu)$ and define

$$\lambda(x)f(yH) = \sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}}\, f(x^{-1}yH) \tag{6.15}$$

for $f \in \mathcal{H}$ and $x \in G$. This turns out to be a unitary representation of G and is usually referred to as the quasi-regular representation of G on $L^2(G/H)$. The use of the specific word 'the' in this description is somewhat confusing; for the representation seems to depend on the choice of ρ or more specifically the regular quasi-invariant measure on G/H. However, one can show if ν is any quasi-invariant regular Borel measure on G/H and one defines

$$\lambda'(x)f(yH) = \sqrt{\frac{d(x\nu)}{d\nu}(y)}\, f(x^{-1}yH), \tag{6.16}$$

then λ and λ' are unitarily equivalent. Indeed, see Exercise 6.5.33.

Proposition 6.92. *The quasi-regular representation λ is a unitary representation of G.*

Proof. Note by Theorem 6.15,

$$\begin{aligned}
|\lambda(x)f|_2^2 &= \int \frac{\rho(x^{-1}y)}{\rho(y)} |f(x^{-1}yH)|^2 \, d\mu(yH) \\
&= \int \frac{\rho(y)}{\rho(xy)} |f(yH)|^2 \, d\mu(xyH) \\
&= \int \frac{\rho(y)}{\rho(xy)} \frac{\rho(xy)}{\rho(y)} |f(yH)|^2 \, d\mu(yH) \\
&= \int |f(yH)|^2 \, d\mu(yH).
\end{aligned}$$

Thus each $\lambda(x)$ is an isometry.

To see λ is a homomorphism we show $\lambda(g_1 g_2)f = \lambda(g_1)\lambda(g_2)f$ and $\lambda(e) = I$. Clearly one has $\lambda(e) = I$. Also

$$\begin{aligned}
\lambda(g_1)\lambda(g_2)f(yH) &= \sqrt{\frac{\rho(g_1^{-1}y)}{\rho(y)}} (\lambda(g_2)f)(g_1^{-1}yH) \\
&= \sqrt{\frac{\rho(g_1^{-1}y)}{\rho(y)}} \sqrt{\frac{\rho(g_2^{-1}g_1^{-1}y)}{\rho(g_1^{-1}y)}} f(g_2^{-1}g_1^{-1}yH) \\
&= \sqrt{\frac{\rho((g_1g_2)^{-1}y)}{\rho(y)}} f((g_1g_2)^{-1}yH) \\
&= \lambda(g_1g_2)f(yH).
\end{aligned}$$

We again need to show λ is strongly continuous. The argument follows the same type of reasoning as the case for the left regular representation. Start by choosing f_0 to be a continuous function with compact support satisfying $|f - f_0|_2 < \frac{\epsilon}{3}$. Fix a compact neighborhood N of e and let K be the compact support of f_0. Let M be the maximum of $|f_0|$. Pick $\delta > 0$ so that

$$\delta^2 < \min\{\frac{\epsilon^2}{36M^2\mu(N\mathrm{supp}(f_0))}, \frac{\epsilon^2}{36\mu(N\mathrm{supp}(f_0))}\}.$$

The uniform left continuity of f_0 on G/H implies there is a neighborhood $N_1 \subseteq N$ of e such that

$$|f_0(x^{-1}yH) - f_0(yH)| \leqslant \delta$$

for $x \in N_1$ and $yH \in K$.

To handle the rho function, we note $R(x, yH) = \frac{\rho(x^{-1}y)}{\rho(y)} - 1$ is continuous on the compact subset $N \times NK$ and $R(e, yH) = 0$ for all $yH \in NK$. This implies there is a neighborhood N_0 of e contained in N_1 such that

$$|R(x, yH)| \leqslant \delta$$

if $x \in N_0$ and $yH \in NK$.

Now

$$|\lambda(x)f_0 - f_0|_2 \leqslant |\lambda(x)f_0 - F_0|_2 + |F_0 - f_0|_2$$

where $F_0(yH) = f_0(x^{-1}yH)$. But if $x \in N_0$, then

$$\begin{aligned}
|\lambda(x)f_0 - F_0|_2^2 &= \int |\sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}} f_0(x^{-1}yH) - f_0(x^{-1}yH)|^2 \, d\mu(yH) \\
&= \int_{x\mathrm{supp}(f_0)} |f_0(x^{-1}yH)|^2 \left| \sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}} - 1 \right|^2 d\mu(yH) \\
&\leqslant \delta^2 M^2 m(N\mathrm{supp}(f_0)) \\
&< \frac{\epsilon^2}{36}
\end{aligned}$$

and

$$\begin{aligned}
|F_0 - f_0|_2^2 &= \int |f_0(x^{-1}yH) - f_0(yH)|^2 \, dy \\
&\leqslant \int_{\mathrm{supp}(f_0) \cup x\mathrm{supp}(f_0)} \delta^2 \, dy \\
&\leqslant \mu(N\mathrm{supp}(f_0))\delta^2 \\
&< \frac{\epsilon^2}{36}.
\end{aligned}$$

Consequently, for $x \in N_0$ we see $|\lambda(x)f_0 - f_0| \leqslant |\lambda(x)f_0 - F_0|_2 + |F_0 - f_0| < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}$, and thus

$$|\lambda(x)f - f|_2 \leqslant |\lambda(x)f - \lambda(x)f_0|_2 + |\lambda(x)f_0 - f_0|_2 + |f_0 - f|_2 < \epsilon.$$

$\square$

Remark 6.93. The quasi-regular representation λ defined in (6.15) is the *left* quasi-regular representation since it acts on a space of functions on the left coset space G/H of H. There is a corresponding *right quasi-regular representation* ρ of G on $L^2(H\backslash G)$ which is unitarily equivalent to λ. The easiest way to obtain this representation is to define $L^2(H\backslash G)$ to be $WL^2(G/H, \mu)$ where μ is a left quasi-invariant measure obtained from a continuous rho function ϕ for H. Here we understand W to be the transformation $Wf(Hx) = f(x^{-1}H)$ for functions f on G/H. To work directly with the rho function ϕ and functions on G, we recall $\check{f}(x) = f(x^{-1})$ for functions f on G. Then note by Exercise 6.1.6 if ϕ is a (left) rho function, then $\check{\phi}$ satisfies

$$\begin{aligned}\check{\phi}(hx) &= \phi(x^{-1}h^{-1}) = \phi(x^{-1})\frac{\Delta_H(h^{-1})}{\Delta_G(h^{-1})} \\ &= \check{\phi}(x)\frac{\Delta_G(h)}{\Delta_H(h)} = \check{\phi}(x)\frac{\Delta^H(h)}{\Delta^G(h)}\end{aligned} \tag{6.17}$$

where Δ^H and Δ^G are the modular functions for right Haar measures on H and G. A positive Borel function satisfying this is said to be a *right rho* function. Now the measure $W_*\mu$ defined on the Borel subsets of $H\backslash G$ by $W_*\mu(E) = \mu(W^{-1}(E))$ is a right-quasi invariant measure. If dx and dh is are left Haar measures on G and H, then $d_rx = dx^{-1}$ and $d_rh = dh^{-1}$ are right Haar measures on these groups. As in the case for the left coset space one has each continuous function with compact support on $H\backslash G$ has form $f^H(Hx) = \int_H f(hx)\,d_rh = \int_H f(h^{-1}x)\,dh = \int_H \check{f}(x^{-1}h)\,dh = \check{f}_H(x^{-1}H)$ for some $f \in C_c(G)$. Moreover,

$$\begin{aligned}\int f^H(Hx)\,dW_*\mu(Hx) &= \int (\check{f})_H(x^{-1}H)\,dW_*\mu(xH) \\ &= \int (\check{f})_H(Hx)\,d\mu(xH) \\ &= \int \check{f}(x)\phi(x)\,dx \\ &= \int f(x^{-1})\phi(x)dx \\ &= \int f(x)\phi(x^{-1})\,d_rx \\ &= \int f(x)\check{\phi}(x)\,d_rx.\end{aligned}$$

Consequently one has

$$\begin{aligned}\int f^H(Hxy)\,dW_*\mu(Hx) &= \int f(xy)\check{\phi}(x)\,d_rx \\ &= \int f(x)\check{\phi}(x)\,\check{\phi}(x)^{-1}\check{\phi}(xy^{-1})\,d_rx \\ &= \int\int \frac{\check{\phi}(xy^{-1})}{\check{\phi}(x)} f(x)\check{\phi}(x)\,d_rx \\ &= \int \frac{\check{\phi}(xy^{-1})}{\check{\phi}(x)} \int f(hx)\,d_rh\,dW_*\mu(Hx) \\ &= \int \frac{\check{\phi}(xy^{-1})}{\check{\phi}(x)} f^H(Hx)\,dW_*\mu(Hx)\end{aligned}$$

and thus

$$d((W_*\mu)y)(Hx) = dW_*\mu(Hxy^{-1}) = \frac{\check{\phi}(xy^{-1})}{\check{\phi}(x)}\, dW_*\mu(Hx).$$

We summarize this in the following Proposition; we leave the remaining details.

Proposition 6.94. *Let H be a closed subgroup of a σ-compact locally compact Hausdorff group G. If ϕ is a continuous right rho function for H, then there is a unique regular Borel measure ν on $H\backslash G$ such that*

$$\int\int_H f(xh)\, d_r h\, d\nu(Hx) = \int f(x)\phi(x)\, d_r x$$

for $f \in C_c(G)$. Moreover, if $f \in L^2(H\backslash G, \nu)$ and $y \in G$, then $\rho(y)$ defined by

$$\begin{aligned} \rho(y)f(Hx) &= \left(\frac{\phi(xy)}{\phi(x)}\right)^{1/2} f(Hxy) \\ &= \left(\frac{d(\nu y^{-1})}{d\nu}(Hx)\right)^{1/2} f(Hxy) \end{aligned} \tag{6.18}$$

gives a unitary representation of G unitarily equivalent to the left quasi-regular representation.

Example 6.95. The ax + b Group: *Recall the $ax+b$ group G consists of all pairs $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$ with multiplication given by*

$$(a,b)(a',b') = (aa', b + ab').$$

The subgroup $H = \{(a,0) \mid a > 0\}$ is a closed subgroup and $H\backslash G$ is homeomorphic to $\mathbb{R}$ under the mapping $H(a,b) = H(a,0)(1,a^{-1}b) \mapsto a^{-1}b$. The action of G on $\mathbb{R}$ becomes $x \cdot (a,b) = a^{-1}(x+b)$ since $H(1,x)(a,b) = H(a,x+b) = H(a,1)(1,a^{-1}(x+b) = H(1,a^{-1}(x+b))$. We find the right quasi-regular representation of G on $L^2(H\backslash G)$. We start by noting from Exercise 6.1.8 that left Haar measure on G is given by

$$\int f(g)\, dg = \int_0^\infty \int f(a,b)\frac{1}{a^2}\, db\, da$$

and the modular function Δ_G is given by $\Delta_G(a,b) = \frac{1}{a}$.

Since H is abelian, $\Delta_H(a,0) = 1$. Define ϕ on G by $\phi(a,b) = a^{-1}$. Then $\phi((a',0)(a,b)) = (aa')^{-1} = \phi(a,b)a'^{-1} = \phi(a,b)\frac{\Delta_G(a',0)}{\Delta_H(a',0)}$. Thus ϕ is a positive right rho function for H and hence there is a right quasi-invariant measure ν on $H\backslash G = \mathbb{R}$ that is defined in terms of the rho function ϕ. We show using the identification $H(1,b) \leftrightarrow b$ of $H\backslash G$ with $\mathbb{R}$ that this measure is Lebesgue measure.

Indeed, if $f \in C_c(\mathbb{R})$*, and* $g \in C_c(\mathbb{R}^+)$ *satisfies* $\int g(a)\frac{1}{a}\,da = 1$*, then* $F(a,b) = g(a)f(\frac{b}{a})$ *satisfies*

$$F^H(H(1,b)) = \int F((a',0)(1,b))\frac{da'}{a'} = \int F((a',a'b))\frac{da'}{a'} = f(b).$$

Thus by Proposition 6.94,

$$\begin{aligned}
\int f(b)\,d\nu(b) &= \int f(H(a,b))\,d\nu(H(a,b)) \\
&= \int F(a,b)\phi(a,b)\,d_r(a,b) \\
&= \int F((a,b)^{-1})\phi((a,b)^{-1})\frac{da}{a^2}\,db \\
&= \iint F(a^{-1},-a^{-1}b)\phi(a^{-1},-a^{-1}b)\frac{da}{a^2}\,db \\
&= \int_0^\infty g(a^{-1})\frac{1}{a}\,da \int_{-\infty}^\infty f(-b)\,db \\
&= \int_0^\infty g(a)\frac{1}{a}\,da \int_{-\infty}^\infty f(b)\,db \\
&= \int_{-\infty}^\infty f(b)\,db.
\end{aligned}$$

Finally using the identification of $H\backslash G$ *with* $\mathbb{R}$ *and (6.18), we see*

$$\begin{aligned}
\rho(a,b)f(x) &= \sqrt{\frac{\phi((1,x)(a,b))}{\phi(1,x)}}f(x\cdot(a,b)) \\
&= \sqrt{\frac{\phi(a,x+b)}{\phi(1,x)}}f(\frac{x-b}{a}) \qquad (6.19) \\
&= \frac{1}{\sqrt{a}}f(\frac{x+b}{a}).
\end{aligned}$$

We use this representation in Example 6.160 of Section18.

Exercise Set 6.5

1. Schur's Lemma 6.48 shows that for unitary or $*$ representations π one has π is irreducible if and only if $\mathrm{Hom}(\pi,\pi) = \mathbb{C}I$. There is a finite dimensional version of Schur's Lemma.

(a) Show if π is an irreducible complex finite dimensional representation of a group or an algebra, then $\mathrm{Hom}(\pi,\pi) = \mathbb{C}I$.

(b) Show the converse to (a) is false. (**Hint:** Consider the group of invertible upper triangular 2×2 matrices in $\mathrm{GL}(2,\mathbb{R})$.)

2. Let π be a unitary or $*$ representation. Let π_0 be an irreducible unitary or $*$ representation and let P_0 be the π_0 primary projection in $\mathrm{Hom}(\pi, \pi)$. Show P_0 commutes with every operator in $\mathrm{Hom}(\pi, \pi)$. Such a projection is said to be *central*.

3. Let π and ρ be unitary representations of a group G. The representations π and ρ are said to be **disjoint** if $\mathrm{Hom}_G(\pi, \rho) = \{0\}$. Show $\mathrm{Hom}_G(\pi, \rho) = \{0\}$ if and only if π and ρ have no unitarily equivalent subrepresentations.

4. Let π and ρ be unitary representations of a topological group G. Show if π and ρ_1 and π and ρ_2 are disjoint, then π and $\rho_1 \oplus \rho_2$ are disjoint.

5. Let ρ be a discretely decomposable unitary representation. Show if $\mathrm{Hom}(\rho, \rho)$ is commutative, then the multiplicity of any irreducible unitary representation π in ρ is either 1 or 0.

6. Let ρ be a discretely decomposable unitary representation. Show if the center of the algebra $\mathrm{Hom}(\rho, \rho)$ consists of only the operators cI where c is complex, then ρ is unitarily equivalent to $n\pi$ for some unique π in $\hat{G}_c$ and some unique cardinal n.

7. Let $\hat{G}_1$ be the one dimensional unitary representations of G; i.e., the collection of one-dimensional characters of G. For $\chi, \chi' \in \hat{G}_1$, define $(\chi\chi')(g) = \chi(g)\chi'(g)$. Show with this multiplication $\hat{G}_1$ is a group. Next define a topology on $\hat{G}_1$ by defining a nonempty subset U of $\hat{G}_1$ to be open if for each χ_0 in U, there is a compact subset K of G and an $\epsilon > 0$ such that $\{x \mid |\chi(g) - \chi_0(g)| < \epsilon \text{ for } g \in K\} \subseteq U$. Show with this topology $\hat{G}_1$ is a Hausdorff topological group.

8. Show the topologies given in Exercise 6.5.7 for the dual groups of $G = \mathbb{R}$, $G = \mathbb{T}$, and $G = \mathbb{Z}$ where these groups have their usual topologies are the usual topologies on $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{T}$, respectively.

9. Let $\mathbb{R}^*$ be the nonzero reals under multiplication and with relative topology. Determine the character group of this abelian group.

10. Determine the character group of the nonzero complex numbers under multiplication.

11. Let G be the finite cyclic group $\mathbb{Z}/m\mathbb{Z}$. Find the character group of G.

12. Let G be a finite group with center $Z(G)$.

(a) Show the regular representation is faithful; i.e., $\lambda(x) = I$ if and only if $x = e$.

(b) Show the central characters on G separate the points of $Z(G)$; that is show if $g \neq e$ is in the center, there is an irreducible unitary representation π of G such that the central character χ_π for π satisfies $\chi_\pi(g) \neq 1$.

13. Let $\mathcal{H}$ be a Hilbert space. For $A \in \mathcal{B}(\mathcal{H})$, define $\pi(A) = A$. Show π is an irreducible representation.

14. Show all finite dimensional unitary representations of a group G are discretely decomposable.

15. Recall the $ax + b$ group consists of all pairs $(a, b) \in \mathbb{R}^2$ with $a > 0$, the relative topology, and multiplication defined by $(a, b)(a', b') = (aa', b + ab')$. Show every finite dimensional unitary representation of this group is one dimensional. **Hint:** Use the discrete decomposability of a finite dimensional unitary representation on the subgroup $x + b = \{(1, b) \mid b \in \mathbb{R}\}$.

16. Show every complex irreducible finite dimensional representation of the $ax+b$ group is one dimensional and is given by $\pi(a, x) = a^c$ for some complex number c. In particular, determine the one-dimensional characters for this group.

17. Let D_4 be the dihedral group. Thus D_4 has eight elements and two generators a and b where $a^2 = b^4 = e$ and $aba = b^3$. Find the center of D_4 and then determine the irreducible unitary representations of D_4 which have trivial central character.

18. Let X be a Hausdorff topological space with disjoint compact subsets K_1 and K_2. Show there are disjoint open subsets V_1 and V_2 with $K_1 \subseteq V_1$ and $K_2 \subseteq V_2$.

19. Let A be a bounded linear operator on $\mathcal{H}$. Show $||A^*A|| = ||A||^2$.

20. Let $\mathcal{B}$ be a Banach $*$ algebra with no identity.

(a) Define $\mathcal{B}_e$ to be the set of all expressions of form $\lambda e + x$ for $\lambda \in \mathbb{C}$ and $x \in \mathcal{B}$**.** Define addition, multiplication, involution, and a norm on $\mathcal{B}_e$ by $(\lambda e + x) + (\mu e + y) = (\lambda + \mu)e + (x + y)$, $(\lambda e + x)(\mu e + y) = \lambda\mu e + (\lambda y + \mu x + xy)$, $(\lambda e + x)^* = \bar{\lambda} e + x^*$, and $||\lambda e + x|| = |\lambda| + ||x||$. Show $\mathcal{B}_e$ is a Banach $*$ algebra with an identity.

(b) Let π be a representation of the $*$ algebra $\mathcal{B}$ on a Hilbert space $\mathcal{H}$. Show $\tilde{\pi}$ defined by $\tilde{\pi}(\lambda e + x) = \lambda I + \pi(x)$ is a representation of the $*$ algebra $\mathcal{B}_e$.

21. Let X be a compact Hausdorff space and let μ be a regular Borel measure on X. Define $\rho(f)(\phi) = f(x)\phi(x)$ for $f \in C(X)$ and $\phi \in L^2(X, \mu)$.

(a) Show ρ is a representation of the $*$ algebra $C(X)$.

(b) Show the algebra $\mathrm{Hom}(\rho, \rho)$ is commutative.

(c) Let $p \in X$ and set $\pi(f) = f(p)$ for $f \in C(X)$. Determine the primary projection $P(\pi)$; and in particular, determine when $P(\pi) \neq 0$.

22. Show the regular representation λ of $\mathbb{T}$ is discretely decomposable and determine all the primary projections for λ. **Hint:** Use Theorem 1.20.

23. Show the regular representation λ of $\mathbb{R}$ is totally continuous; i.e. $P(\pi) = 0$ for each irreducible unitary representation of $\mathbb{R}$. **Hint:** Use the Fourier transform.

24. Let X be a locally compact Hausdorff space and let $C_0(X)$ be the space of continuous functions f such that $\{x \mid |f(x)| \geqslant \epsilon\}$ is compact for each $\epsilon > 0$. With pointwise multiplication and addition and complex complexification for the adjoint, show $C_0(X)$ is a commutative C^* algebra when equipped with norm $|f| = \max\{|f(x)| \mid x \in X\}$. Then determine the irreducible representations of this $*$ algebra. Note $C_0(X)$ has no identity when X is not compact.

25. Let X be a locally compact Hausdorff space with a regular Borel measure μ. Show $L^\infty(\mu)$ is a commutative C^* algebra with identity. Next define a representation π of $L^\infty(\mu)$ on $L^2(\mu)$ by $\pi(f)(\phi)(x) = f(x)\phi(x)$ if $f \in L^\infty$ and $\phi \in L^2$. Show $\phi(L^\infty(\mu)) = \mathrm{Hom}(\pi, \pi)$.

26. Let μ be a regular Borel measure on a locally compact Hausdorff space X. Let $M(X, \mu)$ be the measure algebra given by μ. Thus $M(X, \mu)$ is the collection of equivalence classes of the equivalence relation defined on the Borel subsets of X by $E_1 \sim E_2$ if and only if $\mu(E_1 + E_2) = \mu(E_1 - E_2) + \mu(E_2 - E_1) = 0$.

(a) Show $M(X, \mu)$ is a complete metric space under the metric
$$d([E_1], [E_2]) = \mu(E_1 + E_2).$$

(b) Define a filter $\mathcal{F}$ on $M(X)$ to be a nonempty subfamily of subsets of $M(X)$ with the properties $0 \notin \mathcal{F}$, $E_1 \cap E_2 \in \mathcal{F}$ if E_1 and E_2 are in $\mathcal{F}$, and $E_2 \in \mathcal{F}$ whenever $E_1 \in \mathcal{F}$, $E_2 \in M(X)$, and $E_1 \subseteq E_2$. Show every filter $\mathcal{F}$ on $M(X)$ is contained in a hyperfilter; i.e., every filter is contained in a maximal filter.

(c) Let $\mathcal{F}$ be a hyperfilter. Define ν on $\mathcal{F}$ by $\nu(E) = 0$ if $E \in M(X) - \mathcal{F}$ and $\nu(E) = 1$ if $E \in \mathcal{F}$. Show ν is a finitely additive measure on $M(X)$ with the property $\nu(E_1 \cap E_2) = \nu(E_1)\nu(E_2)$.

(d) Let ν be the measure defined in (c). Show there is a unique one-dimensional representation π of $L^\infty(\mu)$ such that $\pi(\chi_E) = \nu(E)$ for $E \in M(X)$.

(e) Show every representation π in the Gelfand spectrum Δ of $L^\infty(\mu)$ comes from a measure ν defined in terms of a hyperfilter $\mathcal{F}$.

(f) If you know the definition of an ultra filter from point set topology, what is a hyperfilter?

27. Find the three one-dimensional characters of the symmetric group S_3.

28. Let ρ be the right regular representation on G. Thus $\rho(a)f(x) = \Delta(a)^{1/2}f(xa)$ for $f \in L^2(G)$. Show that the map $L^2(G) \to L^2(G)$, $f \mapsto \Delta^{-1/2}\check{f}$, is an unitary equivalence between the left regular representation λ and the right regular representation ρ.

29. Let G be a unimodular locally compact Hausdorff group. Let λ and ρ be the left and right regular representations of G on $L^2(G, m)$ where m is a Haar measure on G. Show that the representation L is irreducible if and only if $G = \{e\}$.

30. Let μ be a measure on space X. Show the mapping $f \mapsto \bar{f}$ is a unitary isomorphism of $L^2(X, \mu)$ onto the conjugate Hilbert space $\overline{L^2(X, \mu)}$.

31. Show the left regular represenation λ is unitarily equivalent to $\bar{\lambda}$.

32. Let π_1 and π_2 be unitary representations of topological groups G_1 and G_2 on the same Hilbert space $\mathcal{H}$. Assume $\pi_1(g_1)\pi_2(g_2) = \pi_2(g_2)\pi_1(g_1)$ for all g_1, $g_2 \in G$. Define $\pi_1 \cdot \pi_2$ on $G_1 \times G_2$ by

$$(\pi_1 \cdot \pi_2)(g_1, g_2) = \pi_1(g_1)\pi_2(g_2).$$

Show $\pi_1 \cdot \pi_2$ is a unitary representation of $G_1 \times G_2$.

33. Use Exercise 6.1.34 and the exercises preceding it to show if ν is a quasi-invariant regular Borel measure on G/H where G is a locally compact Hausdorff space and H is a closed subgroup, then λ' defined in (6.16) is a unitary representation which is unitarily equivalent to the quasi-regular representation λ defined in (6.15).

34. Find the Fourier transform of the representation L given in (6.19) of Example 6.95. That is find the representation $(a, b) \mapsto \mathcal{F}L(a, b)\mathcal{F}^{-1}$ of the $ax + b$ group.

12. The Involutive Banach Algebra $M(G)$

In Section 7 of Chapter 2 we defined a Banach $*$ algebra and showed under convolution and the adjoint $*$ that $L^1(\mathbb{R}^n)$ is a Banach $*$ algebra. This can be done in general for locally compact Hausdorff groups G and more generally for the space of complex measures on G. Furthermore, Exercise 2.6.5 shows the space of complex Borel measures on $\mathbb{R}$ becomes a Banach $*$ algebra. In this section this exercise is seen to be a specific case of a general construction.

Before specifics, let us review some basic properties of complex measures. For references, see [**39**, Rudin] or [**23**, Hewitt and Stromberg] or or other texts on real analysis.

Let X be a locally compact Hausdorff space. A complex Baire measure μ on X is a countably additive function μ from the Baire subsets of X into the complex numbers $\mathbb{C}$. The variation $|\mu|$ defined by

$$|\mu|(E) = \sup\{\sum |\mu(E_i)| \mid E_1, E_2, \ldots, E_k \text{ are disjoint Borel}, \cup E_i = E\}$$

is a finite Baire measure. Moreover, $\mu = \mathrm{Re}\mu + i\mathrm{Im}\mu$ where $\mathrm{Re}\,\mu$ and $\mathrm{Im}\,\mu$ are signed measure having Hahn decompositions $\mathrm{Re}\mu = \mu_1 - \mu_2$, $\mathrm{Im}\,\mu = \mu_3 - \mu_4$. We say μ is a Radon measure if $|\mu|$ is inner regular and thus $\mu(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ is a compact } G_\delta\}$. This is equivalent to each of the measures μ_j being Radon.

We now set $M(X)$ to be the space of complex Radon measures on X. Clearly $M(X)$ is a vector space. We define $||\mu|| = |\mu|(X)$. It is easy to check $M(X)$ is a vector space with addition and scalar multiplication defined by $(\mu_1 + \mu_2)(E) = \mu_1(E) + \mu_2(E)$ and $(c\mu)(E) = c\mu(E)$ for Baire sets E and complex scalars c. Clearly $||\mu|| = 0$ if and only if $\mu = 0$ and $||c\mu|| = |c|\,||\mu||$. Now if $X = E_1 \cup E_2 \cup \cdots \cup E_k$ where the E_j are disjoint Baire subsets of X, then

$$\sum |(\mu_1 + \mu_2)|(E_j) \leqslant \sum_{j=1}^{k} |\mu_1(E_j)| + \sum_{j=1}^{k} |\mu_2(E_j)| \leqslant |\mu_1|(X) + |\mu_2|(X)$$

implies

$$||\mu_1 + \mu_2|| \leqslant ||\mu_1|| + ||\mu_2||.$$

So $||\cdot||$ is a norm.

Proposition 6.96. *$M(X)$ with norm $||\cdot||$ is a Banach space.*

Proof. Let μ_n be Cauchy in $M(X)$. Since $|\mu|(E) \leqslant |\mu|(X) = ||\mu||$ for every Baire subset E of X and complex measure μ, one has $||\mu_m|(E) - |\mu_n|(E)| \leqslant ||\mu_m - \mu_n|| \to 0$ for all Baire subsets E. Thus $|\mu_n|(E) \to |\mu|(E) \in \mathbb{C}$. We need to show $|\mu|$ is a measure. Clearly $|\mu|(\varnothing) = 0$. We show $|\mu|$ is countably additive. Let E_k be a sequence of disjoint Baire sets. Then $|\mu|(E) \geqslant \lim_n |\mu_n|(\cup_{j=1}^k E_j) = \lim_n \sum_{j=1}^k |\mu_n|(E_j) \geqslant \sum_{j=1}^k |\mu|(E_j)$ for all k. So $|\mu|(E) \geqslant \sum_{j=1}^\infty |\mu|(E_j)$. Now $|\mu|(E) = \lim_n |\mu_n|(\cup E_j) = \lim_n \sum_{j=1}^\infty |\mu_n|(E_j) = \lim_n \sum_{j=1}^k |\mu_n|(E_j) + |\mu_n|(\cup_{j=k+1}^\infty E_j)$. Take $\epsilon > 0$. Pick N with $||\mu_n - \mu_N|| \leqslant \epsilon$ for $n \geqslant N$. Then for $n \geqslant N$, $|\mu_n|(\cup_{j=k+1}^\infty E_j) \leqslant |\mu_N|(\cup_{j=k+1}^\infty E_j) + \epsilon$. Now take k large so that $|\mu_N|(\cup_{j=k+1}^\infty E_j) < \epsilon$. Thus $|\mu|(E) \leqslant \sum_{j=1}^k |\mu|(E_j) + \mu_N(\cup_{j=k+1}^\infty E_j) + \epsilon \leqslant \sum_{j=1}^\infty |\mu|(E_j) + 2\epsilon$. So $|\mu|(E) \leqslant \sum |\mu|(E_j)$ and we see $|\mu|$ is a measure.

We also have $|\mu_m(E) - \mu_n(E)| \leqslant ||\mu_m - \mu_n||$ and so $\mu_n(E) \to \mu(E) \in \mathbb{C}$. To see μ is a complex measure, note if E_j are disjoint, then

$$\begin{aligned} |\mu(\cup E_j) - \sum_{j=1}^{k} \mu(E_j)| &= |\mu(\cup_{j=1}^{\infty} E_j) - \mu(\cup_{j=1}^{k} E_j)| \\ &= |\mu(\cup_{j=k+1}^{\infty} E_j)| \\ &= \lim_n |\mu_n(\cup_{j=k+1}^{\infty} E_j)| \\ &\leqslant \lim_n |\mu_n|(\cup_{j=k+1}^{\infty} E_j) \\ &= |\mu|(\cup_{j=k+1}^{\infty} E_j) \to 0 \end{aligned}$$

as $k \to \infty$ for $|\mu|$ is a finite measure. Thus μ is a measure. Now if $E_1, E_2, \ldots, E_k$ are disjoint Baire sets with union X, then $\sum_{j=1}^{k} |(\mu_m - \mu)(E_j)| = \lim_n \sum_{j=1}^{k} |(\mu_m - \mu_n)(E_j)| \leqslant \limsup_n ||\mu_m - \mu_n||$. This implies $||\mu_m - \mu|| \leqslant \limsup_n ||\mu_m - \mu_n||$ and since μ_m is Cauchy, we have $||\mu_m - \mu|| \to 0$ as $m \to \infty$.

Finally, we need to show the finite measure $|\mu|$ is inner regular. Suppose E is a Baire set and let $\epsilon > 0$. Choose n so that $||\, |\mu_n| - |\mu| \,|| < \frac{\epsilon}{4}$. Thus $|\mu_n|(W) - \frac{\epsilon}{4} < |\mu|(W) < |\mu_n|(W) + \frac{\epsilon}{4}$ for all Baire subsets W of G. Since $|\mu_n|$ is inner regular, we can choose a compact G_δ subset K of E with $|\mu_n|(E) < |\mu_n|(K) + \frac{\epsilon}{2}$. Then

$$\begin{aligned} |\mu|(E) - |\mu|(K) &< |\mu_n|(E) + \frac{\epsilon}{4} - |\mu_n|(K) + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2} + |\mu_n|(E) - |\mu_n|(K) \\ &< \epsilon. \end{aligned}$$

Thus $|\mu|(E) - |\mu|(K) < \epsilon$.

□

The following theorem can be derived from the Riesz Theorem 6.1; see Exercises 6.6.5 to 6.6.8. It can also be found in some of the references mentioned earlier. If X is a locally compact Hausdorff space, we shall use both $|f|$ and $||f||_\infty$ for the norm of a function f in $C_c(X)$. Thus $|f| = ||f||_\infty = \max\{|f(x)| \mid x \in X\}$.

Theorem 6.97. *Let X be a locally compact Hausdorff space. Then the dual of the normed linear space $C_c(X)$ is isometrically isomorphic to $M(X)$; i.e., $\mu \mapsto I_\mu$ where $I_\mu(f) = \int f \, d\mu$ for $f \in C_c(X)$ is a vector space isomorphism of $M(X)$ onto $C_c(X)^*$ and $||I_\mu|| = ||\mu|| = |\mu(X)|$ for each μ.*

Now let G be a locally compact Hausdorff topological group. All measures will be assumed to be Radon and thus are inner regular Baire measures.

We fix a left Haar measure λ. Let μ and ν be in $M(G)$. We would like to define I on $C_c(G)$ by

$$I(f) = \iint_{G\times G} f(xy)\, d(\mu \times \nu)(x,y). \tag{6.20}$$

This can be done but one would have to continually work out measure theoretic dexifficulties; namely one could go to countable unions X and Y of G_δ compact subsets which are conull in G with respect to the measures $|\mu|$ and $|\nu|$. To avoid this, we will for the most part be assuming our locally compact Hausdorff groups are σ-compact.

To deal with Baire sets in the σ-compact case, we begin with some notation and recall how product spaces are well behaved in this situation.

If X is a locally compact Hausdorff space, let $\mathcal{BA}(X)$ denote the σ-algebra of Baire subsets of X. Using Proposition 6.12, we know if $X \times Y$ has the product topology and either X or Y is σ-compact, then $\mathcal{BA}(X \times Y) = \mathcal{BA}(X) \times \mathcal{BA}(Y)$. Furthermore, by Proposition 6.7, if X is σ-compact, any complex measure or any measure which is finite on the compact G_δ subsets of X is inner regular and thus is a Radon measure.

Lemma 6.98. *Let G be a σ-compact locally compact Hausdorff group. Let μ and ν be in $M(G)$. Then:*

(a) $\mu \times \nu \in M(G \times G)$*;*

(b) $(x,y) \mapsto f(xy)$ *is Baire measurable on $G \times G$ for each Baire measurable function f on G;*

(c) *If E is a Baire measurable subset of G, then $\{(x,y) \mid xy \in E\}$ is a Baire measurable subset of $G \times G$, and $x^{-1}E$ and Ex^{-1} are Baire subsets of G for all x. Moreover, $x \mapsto \nu(x^{-1}E)$ and $y \mapsto \mu(Ey^{-1})$ are Baire measurable, and*
$(\mu \times \nu)\{(x,y) \mid xy \in E\} = \int \nu(x^{-1}E)\, d\mu(x) = \int \mu(Ey^{-1})\, d\nu(y)$.

Proof. We know $\mathcal{BA}(G \times G) = \mathcal{BA}(G) \times \mathcal{BA}(G)$. So $\mu \times \nu$ is a complex Baire measure and since $G\times G$ is σ-compact, we know $|\mu\times\nu|$ is inner regular and thus Radon. This gives (a). By Lemma 6.13, we know $P : (x,y) \mapsto xy$ is Baire measurable. Thus if U is a Borel subset of $\mathbb{C}$, one has $(f \circ P)^{-1}(U) = P^{-1}(f^{-1}(U)) \in \mathrm{Ba}(G \times G)$ for $f^{-1}(U) \in \mathcal{BA}(G)$. So we have (b). Now (c) follows from this and Fubini's Theorem. Specifically $\chi_E \circ P$ is Baire measurable on $G \times G$. Thus $\{(x,y) \mid xy \in E\} = P^{-1}(E) = (\chi_E \circ P)^{-1}(1)$ is a Baire subset of $G \times G$. Moreover, by Fubini's Theorem:

- For each x and y, the functions $y \mapsto \chi_E \circ P(x,y)$ and $x \mapsto \chi_E \circ P(x,y)$ are Baire measurable.
- The functions $x \mapsto \int \chi_E \circ P(x,y)\, d\nu(y)$ and $y \mapsto \int \chi_E \circ P(x,y)\, d\mu(x)$ are Baire measurable.

- $\iint \chi_E \circ P(x,y)\,d\nu(y)\,d\mu(x) = \iint \chi_E \circ P(x,y)\,d\mu(x)\,d\nu(y) = \iint \chi_E \circ P(x,y)\,d(\mu \times \nu)(x,y)$.

Putting these together with

$$(\chi_E \circ P)(x,y) = \chi_{x^{-1}E}(y) = \chi_{Ey^{-1}}(x) = \chi_{P^{-1}(E)}(x,y)$$

gives the result. □

Now if G is σ-compact, using this lemma we see the integral given in (6.20) is defined for $f \in C_c(G)$ for by Lemma 6.11, continuous functions are Baire measurable. Moreover, I is a continuous linear functional for $|I(f)| \leqslant |f| \int\int d|\mu|\,d|\nu| \leqslant |f|\,|\mu|(G)\,|\nu|(G) = ||\mu||\,||\nu||\,|f|$. Thus by Theorem 6.97, there is a unique measure $\mu * \nu$ in $M(G)$ with $||\mu * \nu|| \leqslant ||\mu||\,||\nu||$ such that

$$\begin{aligned} \int f\,d(\mu * \nu) &= \iint\limits_{G\times G} f(xy)\,d(\mu \times \nu)(x,y) \\ &= \int_G \int_G f(xy)\,d\mu(x)\,d\nu(y) \\ &= \int_G \int_G f(xy)\,d\nu(y)\,d\mu(x) \end{aligned} \tag{6.21}$$

for $f \in C_c(G)$. This measure $\mu * \nu$ is called the **convolution** of the measures μ and ν.

Corollary 6.99. *Let G be a σ-compact locally compact Hausdorff group and suppose $\mu, \nu \in M(G)$. Then the measure $\mu * \nu$ satisfies*

$$\mu * \nu(E) = \int_G \nu(x^{-1}E)\,d\mu(x) = \int_G \mu(Ey^{-1})\,d\nu(y)$$

for all Baire subsets E of G. Moreover, if f is any bounded Baire function or more specifically, any bounded continuous function, then

$$\int_G f\,d(\mu * \nu) = \iint f(xy)\ d(\mu \times \nu)(x,y).$$

Proof. Let $P(x,y) = xy$. Define a complex measure ρ on $\mathcal{BA}(G)$ by $\rho(E) = (\mu \times \nu)(P^{-1}(E))$. Using the Baire measurability of P given in Lemma 6.13, ρ is a complex Baire measure. By the σ-compactness of G, we have $\rho \in M(G)$. Moreover, $\rho(E) = (\mu \times \nu)(E)$ is equivalent to $\iint \chi_E(xy)\,d(\mu \times \nu) = \int \chi_E(g)\,d\rho(g)$ for all Baire subsets E of G. Hence $\iint f(xy)\,d(\mu \times \nu)(x,y) = \int f(g)\,d\rho(g)$ for all simple Baire functions f. Since $\mu \times \nu$ is complex and hence its component are all finite measures, this implies

$$\iint f(xy)\,d(\mu \times \nu)(x,y) = \int f(g)\,d\rho(g)$$

for all bounded Baire measurable functions f. Since continuous functions on G are Baire, we obtain $\iint f(xy)\, d(\mu\times\nu)(x,y) = \int f(g)\, d\rho(g)$ for all $f \in C_c(G)$. Consequently, $\rho = \mu * \nu$. $\square$

Throughout the rest of this section, we assume G to be a σ-compact locally compact Hausdorff group. Recall for a measure μ on G, we have defined $(x\mu)(E) = \mu(x^{-1}E)$ and $(\mu x)(E) = \mu(Ex^{-1})$. Then $x\mu$ and μx are in $M(G)$ when $\mu \in M(G)$ and $(xy)\mu = x(y\mu)$ and $\mu(xy) = (\mu x)y$. These measures are called the **left** and **right translates** of the measure μ. Also for $a \in G$, the point mass ε_a at a is the measure satisfying $\varepsilon_a(f) = \int f(x)\, d\varepsilon_a(x) = f(a)$ for $f \in C_c(G)$. Thus $\varepsilon_a(E) = \chi_a(E)$ which is 1 if $a \in E$ and 0 if $a \notin E$.

Proposition 6.100. *The mapping*

$$M(G) \times M(G) \ni (\mu, \nu) \mapsto \mu * \nu \in M(G)$$

is bilinear, satisfies $||\mu * \nu|| \leqslant ||\mu||\, ||\nu||$, *and has the following properties:*

(a) $(\mu * \nu) * \sigma = \mu * (\nu * \sigma)$

(b) *Let ε_a be the point mass at a. Then $\varepsilon_a * \nu = a\nu$ and $\nu * \varepsilon_a = \nu a$. In particular, $\varepsilon_a * \varepsilon_b = \varepsilon_{ab}$ and $\varepsilon_e * \nu = \nu * \varepsilon_e = \nu$.*

(c) *If μ and ν are real, then $\mu * \nu$ is real.*

(d) *If both μ and ν are positive, then $\mu * \nu$ is positive.*

Proof. For the first part we need only show bilinearity. If suffices thus to show $(\mu, \nu) \to I_{\mu*\nu}$ is bilinear from $M(G) \times M(G)$ into $C_c(G)^*$. But

$$\begin{aligned} I_{(a\mu_1+b\mu_2)*\nu}(f) &= \iint f(xy)\, d(a\mu_1 + b\mu_2)(x)\, d\nu(y) \\ &= a \iint f(xy)\, d\mu_1(x)\, d\nu(y) + b \iint f(xy) d\mu_2(x)\, d\nu(y) \\ &= aI_{\mu_1*\nu}(f) + bI_{\mu_2*\nu}(f), \end{aligned}$$

and similarly $I_{\mu*(a\nu_1+b\nu_2)} = aI_{\mu*\nu_1} + bI_{\mu*\nu_2}$.

Now (a) follows by Fubini's Theorem for

$$\begin{aligned}
I_{(\mu*\nu)*\sigma}(f) &= \iint f(xz)\, d(\mu*\nu)(x)\, d\sigma(z) \\
&= \int\int\int f(xyz)\, d\mu(x)\, d\nu(y)\, d\sigma(z) \\
&= \int \left(\int\int f(xyz)\, d\nu(y)\, d\sigma(z) \right) d\mu(x) \\
&= \int\int f(xw)\, d(\nu*\sigma)(w)\, d\mu(x) \\
&= \int\int f(xw)\, d\mu(x)\, d(\nu*\sigma)(w) \\
&= I_{\mu*(\nu*\sigma)}(f).
\end{aligned}$$

For (b), note

$$\begin{aligned}
\varepsilon_a * \mu(f) &= \int\int f(xy)\, d\varepsilon_a(x)\, d\mu(y) = \int f(ay)\, d\mu(y) \\
&= \int f(y)\, d\mu(a^{-1}y) = \int f(y)\, d(a\mu)(y).
\end{aligned}$$

So $\varepsilon_a * \mu = a\mu$. Similarly, $\mu * \varepsilon_a = \mu a$. Consequently, $\varepsilon_a * \varepsilon_b = a\varepsilon_b = \varepsilon_{ab}$. Finally (c) and (d) follow immediately from the definition. □

Definition 6.101. *Let $\mu \in M(G)$. Then the adjoint μ^* of the measure μ is the measure in $M(G)$ satisfying*

$$\mu^*(f) := \int f(x)\, d\mu^*(x) = \overline{\int \bar{f}(x^{-1})\, d\mu(x)}$$

for $f \in C_c(G)$.

Lemma 6.102. *Let $\mu, \nu \in M(G)$. Then $\mu^* \in M(G)$ and the following holds:*

(a) $\mu^{**} = \mu$

(b) $||\mu^*|| = ||\mu||$,

(c) $\mu \mapsto \mu^*$ *is conjugate linear.*

(d) $(\mu * \nu)^* = \nu^* * \mu^*$;

(e) $\mu^*(E) = \overline{\mu(E^{-1})}$ *for all Baire subsets E of G.*

Proof. Obviously $\mu^{**} = \mu$ and $\mu \mapsto \mu^*$ is complex linear. Furthermore μ^* is linear and $|\mu^*(f)| \leqslant ||f||_\infty ||\mu||$. Hence $\mu^* \in M(G)$ and $||\mu^*|| \leqslant ||\mu||$. As $\mu^{**} = \mu$ it follows that $||\mu^*|| = ||\mu||$. Let $f \in C_c(G)$. Then using Fubini's

Theorem, one has

$$\begin{aligned}
(\mu * \nu)^*(f) &= \overline{\int \overline{f(u^{-1})}\, d(\mu * \nu)(u)} = \overline{\iint \overline{f((xy)^{-1})}\, d\mu(x)\, d\nu(y)} \\
&= \overline{\iint \overline{f(y^{-1}x^{-1})}\, d\mu(x)\, d\nu(y)} = \int \overline{\int \overline{f(y^{-1}x^{-1})}\, d\mu(x)}\, d\bar{\nu}(y) \\
&= \int \int f(y^{-1}x)\, d\mu^*(x)\, d\bar{\nu}(y) = \int \int f(y^{-1}x)\, d\bar{\nu}(y)\, d\mu^*(x) \\
&= \int \overline{\int \overline{f(y^{-1}x)}\, d\nu(y)}\, d\mu^*(x) = \int \int f(yx)\, d\nu^*(y)\, d\mu^*(x) \\
&= \int f(u)\, d(\nu^* * \mu^*)(u)
\end{aligned}$$

Finally, define a complex Baire measure ρ by $\rho(E) = \overline{\mu(E^{-1})}$. Since G is σ-compact, $|\rho|$ is inner regular and thus $\rho \in M(G)$. Moreover, it is easy to check $\int f\, d\rho = \overline{\int \overline{f(x^{-1})}\, d\mu(x)}$ for simple Baire functions f and hence by taking limits for all bounded complex Baire functions. Since continuous functions with compact support are Baire, we see ρ must be μ^*. □

We end this section about $M(G)$ with the following simple consequence of (b) from Proposition 6.100.

Proposition 6.103. *Let G be a locally compact Hausdorff group. Then $M(G)$ is abelian if and only if G is abelian.*

Proof. Assume first that G is abelian. Then $f(xy) = f(yx)$ for all $f \in C_c(G)$ and $x, y \in G$. This implies that $\mu * \nu = \nu * \mu$ for all $\mu, \nu \in M(G)$. Assume now that $M(G)$ is abelian. Then $\varepsilon_{xy} = \varepsilon_x * \varepsilon_y = \varepsilon_y * \varepsilon_x = \varepsilon_{yx}$. If G is not commutative we can find $x, y \in G$ such that $xy \neq yx$. But $\{xy\}$ is compact. Hence there is a $f \in C_c(G)$ such that $f(xy) \neq f(yx)$ which implies that $\varepsilon_{xy}(f) \neq \varepsilon_{yx}(f)$. Thus G is abelian. □

Exercise Set 6.6

1. Let $\mu = \mathrm{Re}\mu + i\mathrm{Im}\mu$ be a complex Borel measure on a space X where $\mathrm{Re}\,\mu$ and $\mathrm{Im}\,\mu$ are signed measure having Hahn decompositions $\mathrm{Re}\mu = \mu_1 - \mu_2$ and $\mathrm{Im}\,\mu = \mu_3 - \mu_4$. Show $|\mu|$ is regular if and only if each of the measures μ_j are regular.

2. Let X be a locally compact Hausdorff space. Exercise 6.1.11 shows $C_0(X)$, the space of complex valued continuous functions on X vanishing at ∞ is a Banach $*$ algebra (actually a C^* algebra). Show $C_0(X)$ is the completion of $C_c(X)$.

3. Let X be a locally compact Hausdorff space. Show every function $f \in C_0(X)$ is Baire measurable on X.

4. Use Lemma 6.4 and Proposition 6.7 to show that every complex Borel measure on a second countable locally compact Hausdorff space is regular.

5. Let X be a locally compact Hausdorff space. Let I be a positive bounded integral on $C_c(X)$. Show the unique Radon measure μ on X with $I(f) = \int f\,d\mu$ is finite and $||I|| = \mu(X)$.

6. Let X be a locally compact Hausdorff space and let $I : C_c(X) \to \mathbb{C}$ be a bounded linear functional which is real valued on real valued functions.

(a) Define I^+ on the positive functions f in $C_c(X)$ by

$$I^+(f) = \sup_{0 \leqslant g \leqslant f} I(f).$$

Show $I^+(cf) = cI^+(f)$ and $I^+(f_1 + f_2) = I^+(f_1) + I^+(f)$ when $c \geqslant 0$ and f, f_1 and f_2 are all nonnegative.

(b) Define $I^+(f) = I^+(f^+) - I^+(f^-)$ for f a real valued function on X. Show I^+ is positive and linear and bounded on the real valued functions.

(c) Let f be real valued. Define $I^-(f) = -(I(f) - I(f^+))$. Then $I = I^+ - I^-$. Show I^- is positive and linear and $||I|| = ||I^+|| + ||I^-||$.

7. Show if μ_1 and μ_2 are finite positive measures on a locally compact Hausdorff space X and I defined on $C_c(X)$ by $I(f) = \int f\,d\mu_1 - \int f\,d\mu_2$ has norm $||I|| = \mu_1(X) + \mu_2(X)$, then the measures μ_1 and μ_2 are disjoint and thus $\mu = \mu_1 - \mu_2$ is the Hahn decomposition of μ.

8. Let X be a locally compact Hausdorff space. Using the previous exercises, show the dual of $C_c(X)$ is $M(X)$; i.e., every bounded complex linear functional I on $C_c(X)$ is given by a unique complex Radon measure μ by

$$I(f) = \int f\,d\mu.$$

Moreover, $||I|| = ||\mu|| = |\mu|(X)$.

9. Let G be a locally compact Hausdorff group. Show the mappings $\mu \mapsto x\mu$ and $\mu \mapsto \mu x$ are linear isometries of $M(G)$.

10. Show $x \mapsto x\mu$ and $x \mapsto \mu x$ are continuous from G into $M(G)$ where $M(G) = C_c(G)^*$ has the weak $*$ topology.

11. Show if μ and ν are regular complex Borel measures on G, and E is a countable union of compact G_δ sets, then $x \mapsto \nu(x^{-1}E)$ and $y \mapsto \mu(Ey^{-1})$

are Borel measurable and

$$\mu * \nu(E) = \int \nu(x^{-1}E)\, d\mu(x) = \int \mu(Ey^{-1})\, d\nu(y).$$

12. Show if μ is a regular complex Borel measure on a locally compact Hausdorff group, then the measure $E \mapsto \overline{\mu(E^{-1})}$ is a regular Borel measure.

13. Let X be a locally compact Hausdorff space. Let $\mathcal{F}(X)$ be the vector space of all bounded complex Borel functions on X and define $||f|| = \sup\{|f(x)| \mid x \in X\}$ for $f \in \mathcal{F}(X)$.

(a) Show $\mathcal{F}(X)$ is a Banach space.

(b) Show the dual space of $\mathcal{F}(X)$ is the space of all finitely additive complex Borel measures on X; i.e., show each continuous linear functional I on $\mathcal{F}(X)$ has form $I(f) = \int f\, d\mu$ for a unique finitely additive measure. (Here one needs to define $\int f\, d\mu$ appropriately.)

(c) Show if $I(f) = \int f\, d\mu$ where μ is finitely additive, then $||I|| = \sup\{\sum_{k=1}^{m} |\mu(E_k)| \mid E_1, E_2, \ldots, E_m$ are disjoint and have union $X\}$.

13. The Banach Algebra $L^1(G)$

In this section G will remain a locally compact Hausdorff topological group which is σ-compact and has a fixed left Haar measure λ. For $f \in L^1(G, \lambda)$, define a complex Baire measure λ_f by

$$\lambda_f(E) = \int_E f\, d\lambda. \tag{6.22}$$

Since G is σ-compact, we know by Proposition 6.7 that λ_f is a Radon measure and so $\lambda_f \in M(G)$. We let Λ be the linear mapping $f \mapsto \lambda_f$ from $L^1(G, \lambda)$ into $M(G)$. Since $\lambda_{f_1} = \lambda_{f_2}$ implies $f_1 = f_2$, the mapping Λ is one-to-one. In fact, $||\Lambda(f)|| = ||\lambda_f|| = |\lambda_f|(G) = \int |f(x)|\, d\lambda(x)$ and thus Λ is an isometry. We now show $L^1(G)$ has a convolution and an involution with the properties $\Lambda(f * g) = \Lambda(f) * \Lambda(g)$ and $\Lambda(f^*) = \Lambda(f)^*$. In the case when G is $\mathbb{R}$ with Lebesgue measure, one obtains the convolution and involution discussed in Section 7 of Chapter 2. We shall see the existence of these operations depends on the invariance properties of the left Haar measure λ. Because of their use in these arguments we list them here. They follow from (c), (e), and (g) of Proposition 6.3.

$$\int f(xy)\, d\lambda(y) = \int f(y)\, d(x\lambda)(y) = \int f(y)\, d\lambda(y) \tag{6.23}$$

$$\int f(yx)\, d\lambda(y) = \int f(y)\, d(\lambda x)(y) = \Delta(x^{-1}) \int f(y)\, d\lambda(y) \tag{6.24}$$

$$\int f(y)\, d\lambda(y) = \int f(y^{-1})\Delta(y^{-1})\, d\lambda(y) \tag{6.25}$$

where the function f is either λ-integrable or nonnegative and Baire measurable.

Proposition 6.104. *Let $\mu \in M(G)$ and $f \in L^1(\lambda)$. Then $\mu * \lambda_f$, $\lambda_f * \mu$, and $(\lambda_f)^*$ are in $\Lambda(L^1(\lambda))$. More specifically the functions $\mu * f$, $f * \mu$, and f^* defined by*

$$\begin{aligned} \mu * f(x) &= \int f(y)\, d\mu(y^{-1}x) = \int f(y^{-1}x)\, d\mu(y) \\ f * \mu(x) &= \int f(xy^{-1})\Delta(y^{-1})\, d\mu(y) \ \textit{and} \\ f^*(x) &= \overline{f(x^{-1})}\, \Delta(x^{-1}) \end{aligned}$$

are in $L^1(\lambda)$ and

$$\begin{aligned} \mu * \lambda_f &= \lambda_{\mu * f}, \\ \lambda_f * \mu &= \lambda_{f*\mu} \ \textit{and} \\ (\lambda_f)^* &= \lambda_{f^*}. \end{aligned}$$

Proof. Let $f \in L^1(\lambda)$ and $\phi \in C_c(G)$. Note $\iint |\phi(yx)f(x)|\, d(|\mu| \times \lambda)(y,x) \leqslant ||\phi||_\infty \int |f(x)|\, d\lambda(x) \int d|\mu|(y) \leqslant ||\phi||_\infty |f|_1 ||\mu||$. Thus by Fubini and (6.23), we have

$$\begin{aligned} \mu * \lambda_f(\phi) &= \int \int \phi(yx)\, d\mu(y)\, f(x)\, d\lambda(x) \\ &= \int \int \phi(x) f(y^{-1}x)\, d\lambda(x)\, d\mu(y) \\ &= \int \phi(x) \int f(y^{-1}x)\, d\mu(y)\, d\lambda(x) \end{aligned}$$

and $\mu * f$ defined by $\mu * f(x) = \int f(y^{-1}x)\, d\mu(y)$ is finite for λ a.e. x. Moreover, since $\iint |f(y^{-1}x)\, d\mu(y)|\, dx \leqslant \iint |f(y^{-1}x)|\, dx\, d|\mu|(y)| = |f|_1 ||\mu||$, $\mu * f$ is an integrable Baire function. Thus $\mu * \lambda_f(\phi) = \lambda_{\mu * f}(\phi)$ for all $\phi \in C_c(G)$.

Similarly, $(x,y) \mapsto \phi(xy)f(x)$ is $\lambda \times |\mu|$ integrable. Moreover, using Fubini and (6.24)

$$\begin{aligned} \lambda_f * \mu(\phi) &= \iint \phi(xy)\, d\lambda_f(x)\, d\mu(y) = \iint \phi(xy) f(x)\, d\lambda(x)\, d\mu(y) \\ &= \iint \phi(x) f(xy^{-1})\Delta(y^{-1})\, d\lambda(x)\, d\mu(y) \\ &= \int \phi(x) \int f(xy^{-1})\Delta(y^{-1})\, d\mu(y)\, d\lambda(x), \end{aligned}$$

we have $f * \mu$ defined by $f * \mu(x) = \int f(xy^{-1})\Delta(y^{-1})\, d\mu(y)$ is Baire and integrable and

$$\lambda_f * \mu(\phi) = \lambda_{f*\mu}(\phi) \text{ for } \phi \in C_c(G).$$

Finally, by (6.24), f^* defined by $f^*(x) = \overline{f(x^{-1})}\,\Delta(x^{-1})$ is integrable and for $\phi \in C_c(G)$, one has

$$\begin{aligned}
(\lambda_f)^*(\phi) &= \overline{\int \overline{\phi(x^{-1})}\,d\lambda_f(x)} \\
&= \overline{\int \overline{\phi(x^{-1})}\,f(x)\,d\lambda(x)} \\
&= \overline{\int \overline{\phi(x)} f(x^{-1})\Delta(x^{-1})\,d\lambda(x)} \\
&= \int \phi(x)\overline{f(x^{-1})}\,\Delta(x^{-1})\,d\lambda(x) \\
&= \int \phi(x) f^*(x)\,d\lambda(x) \\
&= \lambda_{f^*}(\phi).
\end{aligned}$$

□

For $f \in L^1(\lambda)$ and $\mu \in M(G)$, the L^1 functions $f * \mu$ and $\mu * f$ are called the convolutions of f and μ. An immediate consequence of Proposition 6.104 is that for $f, g \in L^1(G)$, one has $f * \lambda_g$ and $\lambda_f * g$ are in $L^1(\lambda)$ and

$$\lambda_f * \lambda_g = \lambda_{f*\lambda_g} = \lambda_{\lambda_f * g}.$$

We hence define the convolution of two $L^1(\lambda)$ functions f and g by $f * g = f * \lambda_g = \lambda_f * g$. We then have $\lambda_{f*g} = \lambda_f * \lambda_g$ and $|f * g|_1 = ||\lambda_f * \lambda_g|| \leqslant ||\lambda_f||\,||\lambda_g|| = |f|_1 |g|_1$.

From Proposition 6.104 we see $\lambda_f * g(x) = \int g(y^{-1}x)\,d\lambda_f(y)$ and thus convolution is given by:

$$f * g(x) = \int f(y) g(y^{-1}x)\,dy. \tag{6.26}$$

One could also use $f * g(x) = f * \lambda_g(x) = \int f(xy^{-1})\Delta(y^{-1})g(y)\,dy$. This suggests the following general definition.

Definition 6.105. *Suppose f and g are complex Baire measurable functions or more generally Borel functions such that $y \mapsto f(y)g(y^{-1}x)$ is integrable for λ a.e. x. Then $f * g$ defined a.e. by $f * g(x) = \int f(y)g(y^{-1}x)\,d\lambda(y)$ is the convolution of f and g.*

With this definition, convolution has many of the properties as convolution on $\mathbb{R}^n$. These include (b) of Lemma 2.75 and Lemmas 2.76 and 2.77 from Chapter 2 and make up some of the exercises at the end of this section.

Returning to the case where f and g are integrable we note

$$\begin{aligned}\lambda_{(f*g)^*} &= (\lambda_{f*g})^* = (\lambda_f * \lambda_g)^* = (\lambda_g)^* * (\lambda_f)^* \\ &= \lambda_{g^*} * \lambda_{f^*} \\ &= \lambda_{g^* * f^*}.\end{aligned}$$

Hence $(f*g)^* = (g^*)*(f^*)$. Furthermore, involution satisfies $(f^*)^* = f$ and $|f^*|_1 = |f|$. Thus the mapping $(f,g) \mapsto f*g$ is bilinear on $L^1(\lambda) \times L^1(\lambda)$, $f \mapsto f^*$ is conjugate linear on $L^1(\lambda)$ and for $f, g \in L^1(\lambda)$, one has

$$|f * g|_1 \leqslant |f|_1 |g|_1 \tag{6.27}$$

$$|f^*|_1 = |f|_1 \tag{6.28}$$

$$(f * g)^* = g^* * f^*. \tag{6.29}$$

Hence in the σ-compact case we have shown:

Proposition 6.106. *With convolution and involution, $L^1(G,\lambda)$ is a Banach $*$ algebra. Moreover, the mapping $f \mapsto \lambda_f$ is a involutive isometric homomorphism of $L^1(G)$ into $M(G)$ whose range is an ideal.*

Exercise Set 6.7

In the following exercises G will always denote a σ-compact locally compact Hausdorff group and λ will be a left-invariant Haar measure on the locally compact Hausdorff topological group G.

1. Let $1 \leqslant p < \infty$. If $f \in L^p(G)$ and $\mu \in M(G)$ then $f * \mu$, $\mu * f \in L^p(G)$ and

$$||f * \mu||_p \leqslant ||f||_p ||\mu|| \text{ and } ||\mu * f||_p \leqslant ||f||_p ||\mu||\,.$$

2. Let f, g, and h be complex Borel functions on G. Show if $f * g$ and $(f*g)*h$ are defined a.e. λ, then $f*(g*h)$ is defined a.e. λ and $(f*g)*h = f*(g*h)$ a.e. λ.

3. Suppose $f * g$ and $f * h$ are defined a.e. λ and a and b are complex numbers. Show $f * (ag + bh) = a(f * g) + b(f * h)$.

4. Let $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show if $f \in L^p(\lambda)$ and $g \in L^q(\lambda)$, then $f * g$ is a continuous function and satisfies $|(f * g)(x)| \leqslant |f|_p |g|_q$ for all x.

5. Suppose $f \in L^1_{\text{loc}}(\lambda)$ and $g \in C_c(G)$. Show $f * g \in C(G)$.

6. Let f and g be complex Borel functions such that $f * g(w)$ exists for λ a.e. w. Let $L_x f(y) = f(x^{-1}y)$. Show $(L_x f) * g(w) = L_x(f * g)(w)$ a.e. w.

7. Let H be a closed subgroup of G and let $X = G/H$. Assume that μ is a positive, G-invariant measure on X. Show that if $f \in L^1(G)$ and

$g \in L^1(X, \mu)$ then

$$G \ni a \mapsto f(a)g(a^{-1}x) \in \mathbb{C}$$

is integrable with respect to λ for μ-almost all $x \in X$. Furthermore

(a) $X \ni y \mapsto f * g(x) := \int f(a)g(a^{-1}x)\, d\lambda$ is integrable with respect to μ

(b) $||f * g||_1 \leqslant ||f||_1 ||g||_1$.

8. Let $1 \leqslant p < \infty$. Show that the map $L : G \to B(L^p(G))$, $L_a f(b) := f(a^{-1}b)$ is strongly continuous, i.e.,

$$\lim_{a \to b} ||L_a f - L_b f||_p = 0$$

for all $f \in L^p(G)$.

9. Let $f, g \in L^2(G)$. Show $f * g$ is defined and $f * g \in C_0(G)$.

10. Assume that there exists a sequence of open set $U_n \in \mathcal{N}(e)$, $n \in \mathbb{N}$, such that $\cap U_n = \{e\}$. We can then assume that $U_{n+1} \subset U_n$. Show the following:

(a) There exists a sequence of compactly supported functions $\{f_n\}_{n\in\mathbb{N}}$ such that $\mathrm{supp}(f) \subset U_n$ and $\int f_n \, d\lambda = 1$.

(b) Let $f \in L^1(G)$ then

$$\lim_{n \to \infty} f_n * f = f\,.$$

14. The Representations of $L^1(G)$

Again we continue to let G be a σ-compact locally compact Hausdorff group. We shall use dg for a fixed left Haar measure on G and return to using λ for the left regular representation of G. Our goal in this section is to show the unitary representations of G can be used to obtain all the representations of the $*$ algebra $L^1(G)$. We start by showing each unitary representation *integrates* to give a representation of the $*$ algebra $M(G)$. First we present some preliminaries.

A representation π of an algebra $\mathcal{A}$ on a topological vector space V is said to be **nondegenerate** if the linear span of the vectors $\pi(a)v$ where $a \in \mathcal{A}$ and $v \in V$ is dense in V. If π is a $*$ representation of a $*$ algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, Exercise 6.8.1 shows π is nondegenerate if and only if the only vector v with $\pi(a)v = 0$ for all $a \in \mathcal{A}$ is 0.

Theorem 6.107. *Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. Then there is a representation Π of the $*$ algebra $M(G)$ on $\mathcal{H}$ satisfying*

$$(\Pi(\mu)v, w)_{\mathcal{H}} = \int (\pi(g)v, w)_{\mathcal{H}}\, d\mu(g)$$

for all $v, w \in \mathcal{H}$. Furthermore, $||\Pi(\mu)|| \leqslant ||\mu||$ for $\mu \in M(G)$.

Proof. The σ-compactness of G and Lemma 6.11 imply the continuous functions $g \mapsto (\pi(g)v, w)_{\mathcal{H}}$ are Baire measurable. Since $|(\pi(g)v, w)_{\mathcal{H}}| \leqslant ||v||\,||w||$ for all g, the sesquilinear form $B_\mu(v, w) := \int(\pi(g)v, w)_{\mathcal{H}}\, d\mu(g)$ is finite and satisfies

$$|B_\mu(v, w)| \leqslant \sup\{|(\pi(g)v, w)_{\mathcal{H}}| \mid g \in G\}||\mu|| \leqslant ||v||_{\mathcal{H}}||w||_{\mathcal{H}}\,||\mu||.$$

By the Riesz Theorem, there exists a unique linear operator $\Pi(\mu)$ on $\mathcal{H}$ with $||\Pi(\mu)|| \leqslant ||\mu||$ and

$$(\Pi(\mu)v, w)_{\mathcal{H}} = \int(\pi(g)v, w)_{\mathcal{H}}\, d\mu(g)$$

for all v and w in $\mathcal{H}$. Since $\mu \mapsto B_\mu$ is linear, $\mu \mapsto \Pi(\mu)$ is linear. To finish we need to show $\Pi(\mu * \nu) = \Pi(\mu)\Pi(\nu)$ and $\Pi(\mu^*) = \Pi(\mu)^*$. Note by Fubini's Theorem and Lemma 6.99

$$\begin{aligned}
(\Pi(\mu * \nu)v, w)_{\mathcal{H}} &= \int(\pi(g)v, w)_{\mathcal{H}}\ d(\mu * \nu)(g) \\
&= \iint(\pi(xy)v, w)_{\mathcal{H}}\, d(\mu \times \nu)(x, y) \\
&= \iint(\pi(y)v, \pi(x^{-1})w)_{\mathcal{H}}\, d\nu(y)\, d\mu(x) \\
&= \int(\Pi(\nu)v, \pi(x^{-1})w)_{\mathcal{H}}\, d\mu(x) \\
&= \int(\pi(x)\Pi(\nu)v, w)_{\mathcal{H}}\, d\mu(x) \\
&= (\Pi(\mu)\Pi(\nu)v, w)_{\mathcal{H}}
\end{aligned}$$

for all v and w. Hence $\Pi(\mu * \nu) = \Pi(\mu)\Pi(\nu)$. Also using Definition 6.101, one has

$$\begin{aligned}
(\Pi(\mu^*)v, w)_{\mathcal{H}} &= \int(\pi(g)v, w)_{\mathcal{H}}\, d\mu^*(g) \\
&= \overline{\int \overline{(\pi(g^{-1})v, w)}_{\mathcal{H}}\, d\mu(g)} \\
&= \overline{\int \overline{(v, \pi(g)w)}_{\mathcal{H}}\, d\mu(g)} \\
&= \overline{\int(\pi(g)w, v)_{\mathcal{H}}\, d\mu(g)} \\
&= \overline{(\Pi(\mu)w, v)}_{\mathcal{H}} \\
&= (v, \Pi(\mu)w)_{\mathcal{H}}.
\end{aligned}$$

This implies $\Pi(\mu^*) = \Pi(\mu)$. $\square$

The representation Π is called the integrated representation on $M(G)$ obtained from π. It is customary to use π instead of Π for the integrated representation. One also uses the shorthand

$$\int \pi(g)\, d\mu(g)$$

to denote the operator $\Pi(\mu)$. Thus $\pi(\mu) = \int \pi(g)\; d\mu(g)$ and

$$(\pi(\mu)v, w)_{\mathcal{H}} = \int (\pi(g)v, w)_{\mathcal{H}}\, d\mu(g).$$

Corollary 6.108. *Let π be a unitary representation of G on Hilbert space $\mathcal{H}$. Then $\pi : L^1(G) \to \mathcal{B}(\mathcal{H})$ defined by $\pi(f) = \Pi(\lambda_f)$ is a nondegenerate $*$ representation of $L^1(G)$ satisfying $||\pi(f)|| \leqslant |f|_1$ for each $f \in L^1(G)$. In particular, $\pi(f)$ is the unique linear operator satisfying*

$$(\pi(f)v, w)_{\mathcal{H}} = \int f(g)(\pi(g)v, w)_{\mathcal{H}}\, dg$$

for $v, w \in \mathcal{H}$.

Proof. By Proposition 6.106, $f \mapsto \lambda_f$ is a $*$ isometric isomorphism of $L^1(G)$ into $M(G)$. This implies $f \mapsto \pi(\lambda_f)$ is a $*$ representation of $L^1(G)$ and $||\pi(f)|| \leqslant |f|_1$. Note

$$(\pi(f)v, w)_{\mathcal{H}} = (\Pi(\lambda_f)v, w)_{\mathcal{H}} = \int (\pi(g)v, w)_{\mathcal{H}}\, d\lambda_f(g) = \int f(g)(\pi(g)v, w)_{\mathcal{H}}\, dg.$$

Thus to finish, we need only show π is nondegenerate. Assume $\pi(f)v = 0$ for all $f \in L^1(G)$. Then $\int f(g)(\pi(g)v, v)_{\mathcal{H}}\, dg = 0$ for all $f \in L^1(G)$. Since π is strongly continuous, there is a neighborhood U of e such that $\mathrm{Re}(\pi(g)v, v)_{\mathcal{H}} > 0$ for $g \in U$. Now choose a nonnegative continuous function f with compact support in U satisfying $f(e) = 1$. Define F by $F(g) = \mathrm{Re}(f(g)(\pi(g)v, v)_{\mathcal{H}})$. Note $F \geqslant 0$ and if $F(e) > 0$, then (a) of Proposition 6.3 gives $\int F(g)\, dg > 0$. But $\int F(g)\, dg = \mathrm{Re}\left(\int f(g)(\pi(g)v, v)_{\mathcal{H}}\, dg\right) = 0$. So $F(e) = (v, v)_{\mathcal{H}} = 0$. $\square$

This corollary has a converse.

Theorem 6.109. *Let Π be a nondegenerate representation of the $*$ algebra $L^1(G)$ on a Hilbert $\mathcal{H}$. Then there is a unitary representation π of G on $\mathcal{H}$ such that $\Pi(f) = \int f(g)\pi(g)\, dg$ for all $f \in L^1(G)$.*

Proof. Define $\pi(x)$ on $\langle\{\Pi(f)v \mid f \in L^1(G),\, v \in \mathcal{H}\rangle$ by

$$\pi(x) \sum_{k=1}^{m} \Pi(f_k)v_k = \sum_{k=1}^{m} \Pi(L(x)f_k)v_k$$

where $L(x)f(y) = f(x^{-1}y)$. We need to show $\pi(x)$ is well defined. Since $\pi(x)$ would be linear if it is well defined, it suffices to show

$$\pi(x)\sum \Pi(f_k)v_k = 0 \text{ whenever } \sum \Pi(f_k)v_k = 0.$$

Now

$$\begin{aligned}(\sum_{k=1}^{m}(\Pi(L(x)f_k)v_k, \sum_{j=1}^{m}\Pi(L(x)f_j)v_j) &= \sum_{j,k}(v_k, \Pi((L(x)f_k)^*\Pi((L(x)f_j)v_j) \\ &= \sum_k\sum_j(v_k, \Pi((L(x)f_k)^* * (L(x)f_j)v_j).\end{aligned}$$

But

$$\begin{aligned}(L(x)f_k)^* * (L(x)f_j)(y) &= \int (L(x)f_k)^*(u)(L(x)f_j)(u^{-1}y)\,du \\ &= \int \Delta(u^{-1})\overline{L(x)f_k(u^{-1})}\,f_j(x^{-1}u^{-1}y)\,du \\ &= \int \Delta(u^{-1})\overline{f_k(x^{-1}u^{-1})}\,f_j(x^{-1}u^{-1}y)\,du \\ &= \int \overline{f_k(x^{-1}u)}f_j(x^{-1}uy)\,du \\ &= \int \overline{f_k(u)}f_j(uy)\,du \\ &= \int \Delta(u^{-1})\overline{f_k(u^{-1})}f_j(u^{-1}y)\,du \\ &= \int f_k^*(u)f_j(u^{-1}y)\,du \\ &= f_k^* * f_j(y).\end{aligned}$$

Thus

$$\begin{aligned}(\sum_{k=1}^{m}(\Pi(L(x)f_k)v_k, \sum_{j=1}^{m}\Pi(L(x)f_j)v_j) &= \sum_{k,j}(v_k, \Pi((L(x)f_k)^* * (L(x)f_j)v_j) \\ &= \sum_{k,j}(v_k, \Pi(f_k^* * f_j)v_j) \\ &= \sum_{k,j}(\Pi(f_k)v_k, \Pi(f_j)v_j) \\ &= (\sum_k \Pi(f_k)v_k, \sum_j \Pi(f_j)v_j).\end{aligned}$$

This implies $||\sum_k \Pi(L(x)f_k)v_k||^2 = ||\sum_k \Pi(f_k)v_k||^2$. So $\pi(x)$ is well defined and is an isometry. By the nondegeneracy of Π, the domain of definition of $\pi(x)$ is dense in $\mathcal{H}$. Thus $\pi(x)$ extends to a linear isometry of $\mathcal{H}$ into $\mathcal{H}$. Clearly $\pi(e) = I$. Now $\pi(xy)\Pi(f)v = \Pi(L(xy)f)v = \Pi(L(x)L(y)f)v =$

$\pi(x)\Pi(L(y)f)v = \pi(x)\pi(y)\Pi(f)v$ implies $\pi(xy) = \pi(x)\pi(y)$. Consequently $\pi(x^{-1})\pi(x) = \pi(x)\pi(x^{-1}) = \pi(e) = I$ and we see $\pi(x)$ is onto and hence is a unitary operator for each $x \in G$.

By Proposition 6.83, $||\Pi(f)|| \leqslant |f|_1$ for $f \in L^1(G)$. Hence $||\Pi(f)v||_{\mathcal{H}} \leqslant |f|_1||v||_{\mathcal{H}}$ for each $v \in \mathcal{H}$ and $f \in L^1(G)$. Since by Exercise 6.7.8, $x \mapsto L(x)f$ is continuous from G into $L^1(G)$, we see that if $\epsilon > 0$ and $w \in \mathcal{H}$ has form $w = \sum_{k=1}^m \Pi(f_k)v_k$, we can choose a neighborhood N of e with $||\Pi(L(x)f_k)v_k - \Pi(f_k)v_k|| \leqslant \frac{\epsilon}{m}$ if $x \in N$ and $k = 1, 2, \ldots, m$. Thus

$$||\pi(x)\sum_{k=1}^m \Pi(f_k)v_k - \sum_{k=1}^m \Pi(f_k)v_k|| \leqslant \sum_{k=1}^m ||\Pi(L(x)\Pi(f_k)v_k - \Pi(f_k)v_k|| \leqslant \epsilon$$

if $x \in N$. Since the space of such w is dense in $\mathcal{H}$, if $v \in \mathcal{H}$ and $\epsilon > 0$, we can pick a w of form $\sum_{k=1}^m \Pi(f_k)v_k$ with $||v - w||_{\mathcal{H}} < \frac{\epsilon}{3}$. For this w, pick a neighborhood N of e with $||\pi(x)w - w||_{\mathcal{H}} \leqslant \frac{\epsilon}{3}$ if $x \in N$. Thus for $x \in N$,

$$\begin{aligned} ||\pi(x)v - v||_{\mathcal{H}} &\leqslant ||\pi(x)v - \pi(x)w||_{\mathcal{H}} + ||\pi(x)w - w||_{\mathcal{H}} + ||w - v||_{\mathcal{H}} \\ &\leqslant ||v - w||_{\mathcal{H}} + \frac{\epsilon}{3} + ||w - v||_{\mathcal{H}} \\ &< \epsilon. \end{aligned}$$

So the homomorphism π is strongly continuous at e. We hence see that π is a unitary representation.

Assume $f, h \in C_c(G)$. Now $f * h$ is an L^1 limit of sums of terms of form $f(x_j)L(x_jh)\lambda(E_j)$ for $f * h(y) = \int f(x)L(x)h(y)\,dx$. Thus $\Pi(f * h)$ is a norm limit of sums of form $\sum f(x_j)\pi(x_j)\Pi(h)\lambda(E_j)$ and these converge in $\mathcal{B}(\mathcal{H})$ to $\int f(x)\pi(x)\Pi(h)\,dx$. This implies $(\Pi(f*h)v, w)_{\mathcal{H}} = \int f(x)(\pi(x)\Pi(h)v, w)_{\mathcal{H}}\,dx$ for all v and w. So $(\Pi(f)\Pi(h)v, w)_{\mathcal{H}} = (\pi(f)\Pi(h)v, w)_{\mathcal{H}}$ for all v and w. Thus $\Pi(f)\Pi(h) = \pi(f\Pi(h)$. By the continuity of π and Π from $L^1(G)$ into $\mathcal{B}(\mathcal{H})$ and the density of $C_c(G)$ in $L^1(G)$, we see $\Pi(f)\Pi(h) = \pi(f)\Pi(h)$ for all $f, h \in L^1(G)$. Since the linear span of the ranges of all $\Pi(h)$ for $h \in L^1(G)$ is dense in $\mathcal{H}$, we have $\Pi(f) = \pi(f)$ for all $f \in L^1(G)$. $\square$

In the case when $\mathcal{H}$ is a separable Hilbert space, one can show the integral $\int \pi(x)v\,d\mu(x)$ exists as a limit in $\mathcal{H}$ of finite sums of form $\sum \pi(x_i)v\,\mu(E_i)$.

Corollary 6.110. $\mathrm{Hom}_G(\pi, \pi) = \mathrm{Hom}_{L^1(G)}(\Pi, \Pi)$.

Proof. $A \in \mathrm{Hom}_G(\pi, \pi)$ if and only if $A\pi(x) = \pi(x)A$ for all $x \in G$. But this occurs if and only if $(\pi(x)v, A^*w)_{\mathcal{H}} = (A\pi(x)v, w)_{\mathcal{H}} = (\pi(x)Av, w)_{\mathcal{H}}$ for all $x \in G$ and $v, w \in \mathcal{H}$. But since $x \mapsto (\pi(x)Av, w)_{\mathcal{H}}$ and $x \mapsto (\pi(x)v, A^*w)_{\mathcal{H}}$ are continuous and bounded, we see by Exercise 6.8.3 that this occurs if and only if $\int f(x)(\pi(x)v, A^*w)_{\mathcal{H}}\,dx = \int f(x)(\pi(x)Av, w)_{\mathcal{H}}\,dx$ for all $f \in C_c(G)$ and $v, w \in \mathcal{H}$. This is equivalent to $(\Pi(f)v, A^*w)_{\mathcal{H}} = (\Pi(f)Av, w)$ for all v and w. So $A \in \mathrm{Hom}_G(\pi, \pi)$ if and only if $A\Pi(f) = \Pi(f)A$ for all $f \in C_c(G)$.

But since $C_c(G)$ is dense in $L^1(G)$ and $||\Pi(f)|| \leqslant |f|_1$ for $f \in L^1(G)$, we see by taking limits that $\mathrm{Hom}_G(\pi,\pi) = \mathrm{Hom}_{L^1(G)}(\Pi,\Pi)$. □

Corollary 6.111. *A closed subspace of $\mathcal{H}$ is invariant under π if and only if it is invariant under Π. Consequently, π is irreducible if and only if Π is irreducible.*

Proof. Note a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ is invariant under π if and only if the orthogonal projection P of $\mathcal{H}$ onto $\mathcal{H}_0$ is in $\mathrm{Hom}_G(\pi,\pi)$ if and only if P is in $\mathrm{Hom}_{L^1(G)}(\Pi,\Pi)$ if and only if the range $\mathcal{H}_0$ of P is invariant under the representation Π. □

Example 6.112. *For $\omega \in \mathbb{R}^n$, let e_ω be the one dimensional unitary representation given by $e_\omega(x) = e^{2\pi i x\cdot\omega}$. Then e_ω integrates to $L^1(\mathbb{R}^n)$. Indeed, operators on the Hilbert space $\mathbb{C}$ are just complex numbers and if $f \in L^1(\mathbb{R}^n)$, then*

$$\begin{aligned} e_\omega(f) &= \int_{\mathbb{R}^n} f(x)e_\omega(x)\,dx \\ &= \int_{\mathbb{R}^n} f(x)e^{2\pi i x\cdot\omega}\,dx \\ &= \hat{f}(-\omega). \end{aligned} \tag{6.30}$$

Example 6.113. *Let λ be the left regular representation of G and suppose $f \in L^1(G)$. Then for $g, h \in L^2(G)$, one can show the integrability of the function $(x,y) \mapsto f(x)g(x^{-1}y)\overline{h(y)}$ and thus by Fubini's Theorem one has*

$$\begin{aligned} (\lambda(f)g,h)_2 &= \int f(x)(\lambda(x)g,h)_2\,dx \\ &= \int f(x)\int \lambda(x)g(y)\overline{h(y)}\,dy\,dx \\ &= \int\int f(x)g(x^{-1}y)\,\overline{h(y)}\,dy\,dx \\ &= \int\left(\int f(x)g(x^{-1}y)\,dx\right)\overline{h(y)}\,dy \\ &= \int(f*g)(y)\,\overline{h(y)}\,dy \\ &= (f*g,h)_2 \end{aligned}$$

when $f \in L^1(G)$. Thus $\lambda(f)(g) = f*g$.

In order to determine $\lambda(\mu)$ for $\mu \in M(G)$, we first note if $f \in C_c(G)$, then $\mu * f$ is defined and is given in Proposition 6.104 by

$$\mu * f(x) = \int f(y^{-1}x)\,d\mu(y).$$

Thus for $f, h \in C_c(G)$, Fubini's Theorem can be used to show:

$$\begin{aligned}(\lambda(\mu)f, h)_2 &= \int (\lambda(y)f, h)_2 \, d\mu(y) \\ &= \int\int f(y^{-1}x)\overline{h(x)}\, dx\, d\mu(y) \\ &= \int\int f(y^{-1}x)\, d\mu(y)\overline{h(x)}\, dy \\ &= \int (\mu * f)(x)\overline{h(x)}\, dy.\end{aligned}$$

So we see $\lambda(\mu)f = \mu * f$ if $f \in C_c(G)$. Now $||\lambda(\mu)|| \leqslant ||\mu||$ and thus $||\mu * f||_2 \leqslant ||\mu||\, ||f||_2$ for $f \in C_c(G)$. Consequently, if $f_n \in C_c(G)$ converges to f in $L^2(G)$, we see $\mu * f_n$ is Cauchy in $L^2(G)$. Thus one can define $\mu * f$ by taking a sequence $f_n \in C_c(G)$ converging to f in $L^2(G)$ and setting:

$$\mu * f := \lim_{n\to\infty} \mu * f_n \text{ in } L^2(G). \tag{6.31}$$

Thus with this definition one has:

$$\lambda(\mu)f = \mu * f \text{ for } \mu \in M(G) \text{ and } f \in L^2(G).$$

15. Invariant Subspaces of the Regular Representation of $\mathbb{R}^n$

In the case of $\mathbb{R}^n$, we showed in Example 6.90 that the representation $\hat{\lambda}$ defined by $\hat{\lambda}(x) = \mathcal{F}\lambda(x)\mathcal{F}^{-1}$ is given on $L^2(\mathbb{R}^n)$ by $\hat{\lambda}(x)f(y) = e^{-2\pi i x\cdot y}f(y)$. Now we have just shown $\lambda(h)f = h * f$ for $h \in L^1(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$. We now determine $\hat{\lambda}(h)$ for $h \in L^1(\mathbb{R}^n)$. One can do this two ways. First, one has $\hat{\lambda}(h) = \mathcal{F}\lambda(h)\mathcal{F}^{-1}$. Thus $\hat{\lambda}(h)f = \mathcal{F}(\lambda(h)\mathcal{F}^{-1}f) = \mathcal{F}(h * \mathcal{F}^{-1}f)$. Now using Exercise 3.3.2, $\mathcal{F}(h * \mathcal{F}^{-1}f) = \hat{h}f$. So $\hat{\lambda}(h)f = \hat{h}f$. We can also determine $\hat{\lambda}(h)$ by using Corollary 6.108. Namely, using the integrability of $(x, y) \mapsto h(x)e^{-2\pi i x\cdot y}f(y)\overline{g(y)}$ and Fubini's Theorem, one has

$$\begin{aligned}(\hat{\lambda}(h)f, g)_2 &= \int h(x)(\hat{\lambda}(x)f, g)_2\, dx \\ &= \int h(x)\int e^{-2\pi i x\cdot y} f(y)\overline{g(y)}\, dy\, dx \\ &= \int \left(\int h(x)e^{-2\pi i x\cdot y}\, dx\right) f(y)\overline{g(y)}\, dy \qquad (6.32) \\ &= \int \hat{h}(y)f(y)\overline{g(y)}\, dy \\ &= (\hat{h}f, g)_2.\end{aligned}$$

Thus $\hat{\lambda}(h)f = \hat{h}f$ for $h \in L^1(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$.

We now determine $\mathrm{Hom}(\hat{\lambda}, \hat{\lambda})$. First note if $A \in \mathrm{Hom}_{L^1(\mathbb{R}^n)}(\hat{\lambda}, \hat{\lambda})$, then by (6.32) and Theorem 3.10, that $A(hf) = hA(f)$ for all $h \in \mathcal{S}(\mathbb{R}^n)$. Now by (2.10) of Chapter 2, if Q is a rectangle in $\mathbb{R}^n$, there is a Schwartz function h of compact support with $0 \leqslant h \leqslant 1$ and $h^{-1}(1) = Q$. Thus $h^n \to \chi_Q$ pointwise. Consequently $h^n f \to \chi_Q f$ and $h^n A(f) \to \chi_Q A(f)$ in L^2. This implies $A(h^n f) \to A(\chi_Q f)$ and thus $A(\chi_Q f) = \chi_Q Af$ for $f \in L^2(\mathbb{R}^n)$. Thus $A\chi_Q f = \chi_Q Af$ for every step function f; i.e., for every function f of form $\sum_{k=1}^m a_k \chi_{Q_k}$ where Q_k is a bounded rectangle.

Let $Q = [0,1)^n$. Define h on $Q + k$ where $k \in \mathbb{Z}^n$ by $h(x) = A(\chi_{Q+k})(x)$ for $x \in Q + k$. Thus if f is a step function, $Af = \sum_{k \in \mathbb{Z}^n} A(\chi_{Q+k} f) = \sum \chi_{Q+k} A(\chi_{Q+k} f) = \sum f \chi_{Q+k} A(\chi_{Q+k})$. So $Af = fh$ whenever f is a step function. Since the step functions are dense in $L^2(\mathbb{R}^n)$ and A is continuous, by taking limits of step functions which converge both pointwise a.e. and in L^2, we see $Af = hf$ for all L^2 functions f. We claim $|h|_\infty \leqslant ||A||$. If not, there is an $M > ||A||$ such that $\{x \mid |h(x)| \geqslant M\}$ has a subset E of finite positive measure. Set $f = \frac{h}{|h|} \lambda_n(E)^{-1/2} \chi_E$. Note $|f|_2 = 1$ and $|Af|_2^2 = \int |h| \lambda_n(E)^{-1} \chi_E(x)\, d\lambda_n(x) \geqslant M$. So $||A|| \geqslant M > ||A||$, a contradiction.

Define for $h \in L^\infty(\mathbb{R}^n)$, the bounded linear operator M_h on $L^2(\mathbb{R}^n)$ by

$$M_h f = hf.$$

Note the prior argument shows $||M_h|| \leqslant |h|_\infty$ and by Exercise 6.8.6, $L^\infty(\mathbb{R}^n)$ is a C^* algebra and the mapping $h \mapsto M_h$ is a $*$ algebraic isometry of $L^\infty(\mathbb{R}^n)$ onto its range.

Theorem 6.114. *Let λ be the left regular representation of $\mathbb{R}^n$. Then the C^* algebras $L^\infty(\mathbb{R}^n)$ and $\mathrm{Hom}(\lambda, \lambda)$ are $*$ algebraically isometrically isomorphic under the correspondence $h \mapsto \mathcal{F}^{-1} M_h \mathcal{F}$.*

Proof. We have just seen if $A \in \mathrm{Hom}(\hat{\lambda}, \hat{\lambda})$, then there is an $h \in L^\infty(\mathbb{R}^n)$ such that $A = M_h$. Now $\hat{\lambda}(x) = \mathcal{F}\lambda(x)\mathcal{F}^{-1}$. This implies $\mathrm{Hom}(\hat{\lambda}, \hat{\lambda}) = \mathcal{F}\mathrm{Hom}(\lambda, \lambda)\mathcal{F}^{-1}$. Thus $\mathrm{Hom}(\lambda, \lambda) = \mathcal{F}^{-1}\mathrm{Hom}(\hat{\lambda}, \hat{\lambda})\mathcal{F} = \{\mathcal{F}^{-1} M_h \mathcal{F} \mid h \in L^\infty(\mathbb{R}^n)\}$. □

Corollary 6.115. *The invariant subspaces of the left regular representation λ of $\mathbb{R}^n$ are precisely the subspaces $\mathcal{F}^{-1}(M_{\chi_A} L^2(\mathbb{R}^n))$ where A is a measurable subset of $\mathbb{R}^n$.*

Proof. A closed subspace $\mathcal{H}_0$ of $L^2(\mathbb{R}^n)$ is invariant under λ if and only if $\lambda(x)\mathcal{H}_0 \subseteq \mathcal{H}_0$ for all x if and only if $\mathcal{F}\lambda(x)\mathcal{F}^{-1}\mathcal{F}\mathcal{H}_0 \subseteq \mathcal{F}\mathcal{H}_0$ for all x if and only if $\hat{\lambda}(x)\mathcal{F}\mathcal{H}_0 \subseteq \mathcal{F}\mathcal{H}_0$ for all x if and only if the orthogonal projection P onto $\mathcal{F}\mathcal{H}_0$ is in $\mathrm{Hom}(\hat{\lambda}, \hat{\lambda})$. But each P in $\mathrm{Hom}(\hat{\lambda}, \hat{\lambda})$ has form M_h for $h \in L^\infty(\mathbb{R}^n)$. Now $M_h^* = M_{\bar{h}}$ and $M_{h^2} = M_h M_h$ implies M_h is an orthogonal projection if and only if h is real valued and $h = h^2$. Thus $h = 0$ or $h = 1$ a.e. x. So $h = \chi_A$ for some measurable set A. We hence see $\mathcal{H}_0$ is invariant under λ if and only if $\mathcal{F}\mathcal{H}_0 = M_{\chi_A}(L^2(\mathbb{R}^n))$ for some measurable subset A of $\mathbb{R}^n$. □

We thus have seen P is an orthogonal projection onto an invariant subspace of λ if and only if there is a Borel subset A such that

$$Pf = \mathcal{F}^{-1}(\chi_A \mathcal{F}(f)). \tag{6.33}$$

Corollary 6.116. *Let A be a Borel subset of $\mathbb{R}^n$ with positive Lebesgue measure and let λ_A be the subrepresentation of the left regular representation λ obtained by restricting λ to $\mathcal{F}^{-1}(\chi_A L^2(\mathbb{R}^n))$. Then a function $f \in \mathcal{F}^{-1}(\chi_A L^2(\mathbb{R}^n))$ is a cyclic vector for λ_A if and only if $\hat{f}(y) \neq 0$ a.e. $y \in A$.*

Proof. Set $L^2_A(\mathbb{R}^n) = \mathcal{F}^{-1}(\chi_A L^2(\mathbb{R}^n))$. Assume $f \in L^2_A(\mathbb{R}^n)$. Then f is cyclic for λ_A if and only if there is no nonzero $g \in L^2_A(\mathbb{R}^n)$ with $(\lambda_A(x)f, g)_2 = 0$ for all $x \in \mathbb{R}^n$. This occurs if and only if there is no nonzero $g \in L^2_A(\mathbb{R}^n)$ with $((\mathcal{F}\lambda(x)\mathcal{F}^{-1})\mathcal{F}f, \mathcal{F}g)_2 = 0$ for all x, and hence if and only if there is no nonzero $g \in L^2_A(\mathbb{R}^n)$ with $(\hat{\lambda}(x)\mathcal{F}f, \mathcal{F}g)_2 = 0$ for all $x \in \mathbb{R}^n$. Now $\hat{f}(y)$ and $\hat{g}(y)$ are 0 a.e. off A. Thus f is cyclic for λ_A if and only if there is no nonzero function g in $L^2(\mathbb{R}^n)$ whose Fourier transform vanishes off A such that

$$\mathcal{F}(\hat{f}\hat{g})(x) = \int e^{-2\pi i x\cdot y}\hat{f}(y)\hat{g}(y)\,dy = 0.$$

But by Corollary 3.8, the Fourier transform is one-to-one on $L^1(\mathbb{R}^n)$ and since $\hat{f}\hat{g} \in L^1(\mathbb{R}^n)$, we see $\hat{f}$ is cyclic for λ_A if and only if there is no nonzero $g \in L^2$ with $\hat{g}$ vanishing off A such that $\hat{f}\hat{g} = 0$ in L^2. But this occurs if and only if $\hat{f}(y)$ is not 0 a.e. $y \in A$. □

Exercise Set 6.8

In the following exercises G unless otherwise stated will be a σ-compact locally compact Hausdorff group.

1. Show a representation π on a $*$ algebra $\mathcal{A}$ is nondegenerate if and only if there is no nonzero vector v with $\pi(x)v = 0$ for all $x \in \mathcal{A}$.

2. Let π be a nondegenerate representation of a Banach $*$ algebra having an identity e. Show $\pi(e) = I$.

3. Suppose $U(x)$ and $V(x)$ are bounded continuous functions and μ is a Radon measure on a locally compact Hausdorff space X with the property $\mu(O) > 0$ for every nonempty open Baire subset O of X. Show if

$\int f(x)U(x)\,d\mu(x) = \int f(x)V(x)\,d\mu(x)$ for every $f \in C_c(X)$, then $U(x) = V(x)$ for all x.

4. Let $f \in L^1(G)$ and for $a, b \in G$, define ${}_af_b$ by ${}_af_b(x) = f(a^{-1}xb)\Delta(b)$ where Δ is the modular function for G. Show

(a) $f \mapsto {}_af_b$ is a linear isometry of $L^1(G)$ onto $L^1(G)$.

(b) Show $({}_af_b)^* = {}_b(f^*)_a$ and $({}_af_b) * ({}_bg_c) = {}_a(f * g)_c$ for $a, b, c \in G$ and $f, g \in L^1(G)$.

(c) Show if π is a unitary representation of G, then

$$\pi({}_af_b) = \pi(a)\pi(f)\pi(b^{-1})$$

5. Let G be a nondiscrete locally compact Hausdorff group. Show there is a representation of $M(G)$ which is nonzero and whose restriction to $L^1(G)$ is trivial. (Hint: Show the atom free measures in $M(G)$ form an ideal in $M(G)$.)

6. Show $L^\infty(\mathbb{R}^n)$ is a C^* algebra under pointwise multiplication and addition, adjoint $h \mapsto \bar{h}$, and norm $||h|| = |h|_\infty$. Then show the mapping $h \mapsto M_h$ is an isometric $*$ isomorphism of the algebra $L^\infty(\mathbb{R}^n)$ onto a norm closed $*$ subalgebra of $\mathcal{B}(L^2(\mathbb{R}^n))$.

In the following 5 exercises μ and ν are regular Borel measures on $\mathbb{R}^n$ and λ_n is Lebesgue measure on $\mathbb{R}^n$.

7. Define for each x, an operator $\pi_\mu(x)$ on $L^2(\mathbb{R}^n, \mu)$ by $\pi_\mu(x)f(y) = e^{-2\pi i x\cdot y}f(y)$.

(a) Show π_μ is a unitary representation of $\mathbb{R}^n$.

(b) Find $\pi_\mu(f)$ for $f \in L^1(\mathbb{R}^n)$.

(c) Show π_μ is unitarily equivalent to π_ν if and only if μ and ν are equivalent measures.

8. Show π_μ is unitarily equivalent to a subrepresentation of the left regular representation of $\mathbb{R}^n$ if and only if $\mu \prec \lambda_n$.

9. Show π_μ is always a cyclic representation. (**Hint:** Use Exercise 3.1.3.)

10. Show $\mathrm{Hom}(\pi_\mu, \pi_\mu) = \{M_f \mid f \in L^\infty(\mu)\}$ where $M_f h = fh$ for $h \in L^2(\mu)$.

11. Show $\mathrm{Hom}(\pi_\mu, \pi_\mu)$ is maximal abelian, i.e., show if $A \in \mathcal{B}(L^2(\mu))$ and $AB = BA$ for all $B \in \mathrm{Hom}(\pi_\mu, \pi_\mu)$, then $B \in \mathrm{Hom}(\pi_\mu, \pi_\mu)$.

12. A function $\varphi : G \to \mathbb{C}$ is called **positive definite** if for all $x_1, \ldots, x_n \in G$ the matrix $(\varphi(x_i^{-1}x_j))$ is positive semi-definite. Show the following:

(a) The function φ is positive definite if and only if $\sum_{i,j=1}^n c_i\bar{c}_j\varphi(x_i^{-1}x_j) \geqslant 0$ for all $x_1, \ldots, x_n \in G$ and all $c_1, \ldots, c_n \in \mathbb{C}$.

(b) Let E be the space of functions $f : G \to \mathbb{C}$ of the form $f = \sum_{j=1}^n c_j L(x_j)\varphi$. Define a bilinear form $(\cdot\,,\cdot)$ on E by

$$(\sum_{i=1}^n c_i L(x_i)\varphi, \sum_{j=1}^k d_j L(y_j)\varphi) := \sum_{i,j} c_i \bar{d}_j \varphi(y_j^{-1}x_i)\,.$$

Show that $(f,f) \geqslant 0$ for all $f \in E$.

(c) Let $N = \{f \in E \mid (f,f) = 0\}$. Then $(\cdot,\cdot)$ defines an inner product on E/N. Let $||\cdot||$ be the corresponding norm on E/N.

(d) For $f \in E$ and $x \in G$ let $\pi(x)f(y) = f(x^{-1}y)$. Then $\pi(G)N \subset N$.

(e) Define $\pi_\varphi(x) : E/N \to E/N$ by $\pi_\varphi(x)[f] := [\pi(x)f]$ where $[f]$ denotes the equivalence class of f. Then $||\pi_\varphi(x)f|| = ||f||$. In particular π_φ extends to a continuous map on V, the completion of E/N in the norm $||\cdot||$, to V. Show if φ is continuous, then π_φ is a unitary representation of G.

16. Central Functions

In this section G is a locally compact Hausdorff group with left Haar measure m.

Definition 6.117. *A complex function f on a group G is central if $f(xy) = f(yx)$ for all x and y in G.*

We note f is central if and only if f is constant on conjugacy classes; i.e., $f(yxy^{-1}) = f(x)$ for all x and y. Many times we will have a measurable function which is "almost central". We would like to replace it by a measurable central function. The following lemma addresses this situation. To do this we will need to know the measurability of the functions $(x,y) \mapsto xy$ and $(x,y) \mapsto yx$. This was established in Lemma 6.13 when G is σ-compact.

Lemma 6.118. *Assume G is a σ-compact locally compact Hausdorff group. Let f be a complex valued Baire measurable function f such that $f(xy) = f(yx)$ for $m \times m$ a.e. (x,y). Then f equals a central Baire measurable function almost everywhere.*

Proof. We recall since G is σ-compact, by Proposition 6.12, $\mathcal{BA}(G \times G) = \mathcal{BA}(G) \times \mathcal{BA}(G)$ and $\mathcal{BA}(G \times G \times G) = \mathcal{BA}(G) \times \mathcal{BA}(G) \times \mathcal{BA}(G)$ and one can apply Fubini's Theorem to the measures $m \times m$ and $m \times m \times m$ on these σ-algebras.

Let $E = \{(x,y) \mid f(xy) = f(yx)\}$. By Lemma 6.13 and the Baire measurability of f, E is a Baire subset of $G \times G$ and we are assuming $(m \times m)(E^c) = 0$.

Define $\Phi : G \times G \to G \times G$ by $\Phi(x, y) = (xy, y)$. Then Φ and Φ^{-1} given by $\Phi^{-1}(x, y) = (xy^{-1}, y)$ are Baire measurable functions. Thus $\Phi^{-1}(E) = \{(x, y) \mid f(x) = f(yxy^{-1})\}$ and $\Phi^{-1}(E^c) = \{(x, y) \mid f(x) \neq f(yxy^{-1})\}$ are Baire subsets of $G \times G$. Moreover, $\Phi^{-1}(E^c)$ has measure 0. Indeed, $(m \times m)(\Phi^{-1}(E^c)) = \int m\{y \mid f(x) \neq f(yxy^{-1})\}\, dm(x) = 0$ if and only if $m\{x \mid f(x) \neq f(yxy^{-1}\} = 0$ for m a.e. y if and only if $m\{xy \mid f(xy) = f(yx)\} = 0$ for m a.e. y if and only if $(m \times m)(E^c) = 0$. Thus $W = \{(x, y) \mid f(yxy^{-1}) = f(x)\}$ is a Baire subset whose complement has measure 0.

Now consider $V = \{(x, y_1, y_2) \mid f(y_1xy_1^{-1}) = f(y_2xy_2^{-1})\}$. Then V is a Baire subset of $G \times G \times G$. We claim its complement relative to $m \times m \times m$ has measure 0. It suffices to show for m a.e. x, the set V_x^c given by $V_x^c = \{(y_1, y_2) \mid f(y_1xy_1^{-1}) \neq f(y_2xy_2^{-1})\}$ has $m \times m$ measure 0. But for a.e. x, $f(y_1xy_1^{-1}) = f(x)$ a.e. y_1 and for a.e. x, $f(y_2xy_2^{-1}) = f(x)$ a.e. y_2. This implies for a.e. x, $f(y_1xy_1^{-1}) = f(y_2xy_2^{-1})$ for $m \times m$ a.e. (y_1, y_2). Consequently, V_x^c has measure 0. By Fubini's Theorem, we then obtain $(m \times m \times m)(V^c) = \int (m \times m)(V_x^c)\, dm(x) = 0$. Now let U be the set of x such that $(m \times m)(V_x^c) = 0$. Again, using Fubini, $x \mapsto (m \times m)(V_x^c)$ is Baire measurable, and thus U is a Baire subset of G whose complement has measure 0. We note U is invariant under conjugation for right translates of sets of Haar measure 0 in G have Haar measure 0; and thus if $f(y_1xy_1^{-1}) = f(y_2xy_2^{-1})$ a.e. (y_1, y_2), then given any y, $f(y_1yxy^{-1}y_1^{-1}) = f(y_2xy_2^{-1})$ a.e. (y_1, y_2) and again right translating, $f(y_1yxy^{-1}y_1^{-1}) = f(y_2yxy^{-1}y_2^{-1})$ a.e. (y_1, y_2).

Define a function F by $F(x) = 0$ if $x \in U^c$ and F is the a.e. constant of the function $(y_1, y_2) \mapsto f(y_1xy_1^{-1}) = f(y_2xy_2^{-1})$. Again using right translates of sets of Haar measure have Haar measure 0, one sees $F(yxy^{-1}) = F(x)$ for $x \in U$ and since U^c is invariant under inner conjugation $F(yxy^{-1}) = F(x) = 0$ for $x \in U^c$.

To see F is Baire measurable, we take a probability measure μ on $\mathcal{BA}(G)$ which is equivalent to Haar measure. For $x \in U$, we note $F(x) = \int f(y_1xy_1^{-1})\, d\mu(y_1)$. The function F is Baire measurable by Fubini.

Finally, since for a.e. x, $f(y_1xy_1^{-1}) = f(x)$ a.e. y_1 and $f(y_2^{-1}xy_2^{-1}) = f(x)$ a.e. y_2, we see $F(x) = f(x)$ for a.e. $x \in G$. $\square$

For measurable functions, being central will mean being equal almost everywhere to a central measurable function.

Lemma 6.119. *Let G be σ-compact. An L^1 function on a unimodular group is central if and only if $\pi(f) \in \mathrm{Hom}_G(\pi, \pi)$ for every unitary representation π of G.*

Proof. Let f be central. Then

$$\begin{aligned}(\pi(f)\pi(g)v,w) &= \int f(x)(\pi(x)\pi(g)v,w)\,dx \\ &= \int f(x)(\pi(xg)v,w)\,dx \\ &= \int f(xg^{-1})(\pi(x)v,w)\,dx \\ &= \int f(g^{-1}x)(\pi(x)v,w)\,dx \\ &= \int f(x)(\pi(gx)v,w)\,dx \\ &= \int f(x)(\pi(x)v,\pi(g^{-1})w)\,dx \\ &= (\pi(f)v,\pi(g^{-1})w) \\ &= (\pi(g)\pi(f)v,w)\end{aligned}$$

for all v, w and g. Thus $\pi(f)\pi(g) = \pi(g)\pi(f)$ for all g.

Conversely, let $\pi(f) \in \mathrm{Hom}_G(\pi,\pi)$ for all π. Then $\lambda(f) \in \mathrm{Hom}_G(\lambda,\lambda)$ where λ is the left regular representation. This implies as seen above working from both sides to the middle that

$$\int f(xg^{-1})(\lambda(x)\phi,\psi)_2\,dx = \int f(g^{-1}x)(\lambda(x)\phi,\psi)_2\,dx$$

for any ϕ and ψ in $L^2(G)$. Thus:

$$\int\int f(xg^{-1})\phi(x^{-1}y)\bar{\psi}(y)\,dy\,dx = \int\int f(g^{-1}x)\phi(x^{-1}y)\bar{\psi}(y)\,dy\,dx.$$

Hence:

$$\int\int f(yxg^{-1})\phi(x^{-1})\bar{\psi}(y)\,dx\,dy = \int\int f(g^{-1}yx)\phi(x^{-1})\bar{\psi}(y)\,dy\,dx.$$

Replacing $\phi(x^{-1})$ by $\phi(x)$ and taking $\phi(x) = \chi_E$ and $\psi = \chi_F$ where E and F have finite Haar measure, one sees

$$\iint\limits_{E\times F} f(yxg^{-1})\,d(m\times m)(x,y) = \iint\limits_{E\times F} f(g^{-1}xy)\,d(m\times m)(x,y).$$

This yields $f(yxg^{-1}) = f(g^{-1}yx)$ for $m \times m$ a.e. (x,y) for each g. Consequently, $f(xg) = f(xg)$ a.e. x and g. Now apply Lemma 6.118. $\square$

Definition 6.120. *Let π be a finite dimensional representation of a group G. Then the character χ_π is the function on G given by*

$$\chi_\pi(g) = \mathrm{Tr}(\pi(g)).$$

Note since $\mathrm{Tr}(\pi(xy)) = \mathrm{Tr}(\pi(x)\pi(y)) = \mathrm{Tr}(\pi(y)\pi(x)) = \mathrm{Tr}(\pi(yx))$, the character χ_π is a central continuous function on G. We note if π and ρ are equivalent finite dimensional representations of G, then $\chi_\pi = \chi_\rho$ for if $A : V_\pi \to V_\rho$ gives the equivalence, then $A\pi(g)A^{-1} = \rho(g)$ for each $g \in G$. Consequently,

$$x_\rho(g) = \mathrm{Tr}(A\pi(g)A^{-1}) = \mathrm{Tr}(\pi(g)) = \chi_\pi(g).$$

We show a partial converse holds.

Theorem 6.121 (Burnside)**.** *Let π be a finite dimensional unitary representation of a group G on a Hilbert space $\mathcal{H}$. Then the linear span of the operators $\pi(g)$ for $g \in G$ is the space of all linear transformations on $\mathcal{H}$.*

Proof. Note if we give G the discrete topology, then π is still irreducible. Moreover, by Corollary 6.76, $\pi \times \bar{\pi}$ on $\mathcal{B}(\mathcal{H}) = \mathcal{H} \otimes \overline{\mathcal{H}}$ is irreducible. Let $\mathcal{A}$ be the subspace of $\mathcal{B}(\mathcal{H})$ spanned by the operators $\pi(g)$ for $g \in G$. We note $\mathcal{A}$ is finite dimensional and is nonzero. Moreover, since $(\pi \times \bar{\pi})(g_1, g_2)A = \pi(g_1)A\pi(g_2^{-1})$ for $A \in \mathcal{H} \otimes \overline{\mathcal{H}}$ and $\pi(g_1)\pi(g)\pi(g_2^{-1}) = \pi(g_1 g g_2^{-1})$, we see $\mathcal{A}$ is invariant. Since $\mathcal{A}$ is closed and $\pi \times \bar{\pi}$ is irreducible, we conclude $\mathcal{A} = \mathcal{H} \otimes \overline{\mathcal{H}}$. $\square$

Lemma 6.122. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and suppose Φ is a $*$ algebra isomorphism of $\mathcal{B}(\mathcal{H})$ satisfying $\mathrm{Tr}(\Phi(A)) = \mathrm{Tr}(A)$ for $A \in \mathcal{B}(\mathcal{H})$. Then there is an unitary transformation U of $\mathcal{H}$ with*

$$\Phi(T) = UTU^* \text{ for } T \in \mathcal{B}(\mathcal{H}).$$

Proof. Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of $\mathcal{H}$. Now $T = e_1 \otimes \bar{e}_1$ is an orthogonal rank one projection. Since $\Phi(T) = \Phi(T^2) = \Phi(T)^2$ and $\Phi(T)^* = \Phi(T^*) = \Phi(T)$, we see $\Phi(T)$ is an orthogonal projection. Moreover, $\mathrm{Tr}(\Phi(T)) = \mathrm{Tr}(T) = 1$. Consequently, $\Phi(T)$ is an orthogonal rank one projection. Hence we can choose f_1, a unit vector, with $\Phi(e_1 \otimes \bar{e}_1) = f_1 \otimes \bar{f}_1$. We next claim for $1 < j \leqslant n$, $\Phi(e_1 \otimes \bar{e}_j) = f_1 \otimes \bar{f}_j$ for a unique vector f_j. Indeed, the operator $\Phi(e_1 \otimes \bar{e}_j)$ is nonzero and since
$\Phi(e_1 \otimes \bar{e}_j)\Phi(e_1 \otimes \bar{e}_j)^* = \Phi(e_1 \otimes \bar{e}_1) = f_1 \otimes \bar{f}_1$ and the range of $\Phi(e_1 \otimes \bar{e}_j)^*$ is the orthogonal complement of the kernel of $\Phi(e_1 \otimes \bar{e}_j)$, we see $\Phi(e_1 \otimes \bar{e}_j)$ has range the linear span of the vector f_1. Using the Riesz representation theorem, this implies $\Phi(e_1 \otimes \bar{e}_j)$ has form $f_1 \otimes \bar{f}_j$ for a unique vector f_j in $\mathcal{H}$. We claim the vectors $f_1, f_2, \ldots, f_n$ form an orthonormal basis of $\mathcal{H}$. Indeed,

note

$$\begin{aligned}(f_i, f_j) &= (f_i, f_j)(f_1, f_1) = (f_i \otimes \bar{f}_1, f_j \otimes \bar{f}_1)_2 \\ &= \mathrm{Tr}((f_i \otimes \bar{f}_1)(f_j \otimes \bar{f}_1)^*) = \mathrm{Tr}((f_1 \otimes \bar{f}_i)^*(f_1 \otimes \bar{f}_j)) \\ &= \mathrm{Tr}(\Phi(e_1 \otimes \bar{e}_i)^*\Phi(e_1 \otimes \bar{e}_j)) = \mathrm{Tr}(\Phi(e_i \otimes \bar{e}_1)\Phi(e_1 \otimes \bar{e}_j)) \\ &= \mathrm{Tr}(\Phi(e_1 \otimes \bar{e}_j) \circ (e_i \otimes \bar{e}_1))) = \mathrm{Tr}((e_i, e_j)e_1 \otimes \bar{e}_1) \\ &= (e_i, e_j) = \delta_{i,j}.\end{aligned}$$

Define U by $Ue_i = f_i$ for $i = 1, 2, \ldots, n$. Then U is unitary. Also

$$\begin{aligned}\Phi(e_i \otimes \bar{e}_j) &= \Phi((e_1 \otimes \bar{e}_i)^* \circ (e_1 \otimes \bar{e}_j)) \\ &= (f_1 \otimes \bar{f}_i)^* \circ (f_1 \otimes \bar{f}_j) \\ &= (f_i \otimes \bar{f}_1) \circ (f_1 \otimes \bar{f}_j) \\ &= (f_1, f_1) f_i \otimes \bar{f}_j \\ &= f_i \otimes \bar{f}_j \\ &= Ue_i \otimes \overline{Ue_j} \\ &= U(e_i \otimes \bar{e}_j)U^*.\end{aligned}$$

Since the $e_i \otimes \bar{e}_j$ for $1 \leqslant i, j \leqslant n$ form a basis for $\mathcal{B}(\mathcal{H})$, we see $\Phi(T) = UTU^*$ for all T in $\mathcal{B}(\mathcal{H})$. □

Theorem 6.123. *Let π and ρ be finite dimensional irreducible unitary representations of a topological Hausdorff group G having equal characters. Then π and ρ are unitarily equivalent.*

Proof. Note $\dim \mathcal{H}_\pi = \chi_\pi(e) = \chi_\rho(e) = \dim \mathcal{H}_\rho$. Consequently, we may assume $\mathcal{H}_\pi = \mathcal{H}_\rho = \mathcal{H}$ for Hilbert spaces of the same dimension are unitarily isomorphic. Let $n = \dim \mathcal{H}$. Next note $\chi_\pi(g) = n$ if and only if $\mathrm{Tr}(\pi(g)) = n$. But the only unitary $n \times n$ matrix with trace n is I. Thus $\chi_\pi(g) = n$ if and only if $\pi(g) = I$. Since $\chi_\pi(g) = \chi_\rho(g)$ for all g, we see the $\ker \pi = \ker \rho$. Let N be this common kernel. Then π and ρ factor to G/N to give faithful irreducible unitary representations π' and ρ' of G/N on Hilbert space $\mathcal{H}$.

Consider the ranges $\mathcal{A}_1 = \pi'(G/N)$ and $\mathcal{A}_2 = \rho'(G/N)$. Define a mapping $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ by $\Phi(\pi'(gN)) = \rho'(gN)$. Note Φ satisfies $\Phi(A_1A_2) = A_1A_2$ and $\Phi(A_1^*) = \Phi(A_1)^*$. Also we have $\mathrm{Tr}(\Phi(A)) = \mathrm{Tr}(A)$ for each $A \in \mathcal{A}_1$. We extend Φ to the linear space $\langle \mathcal{A}_1 \rangle$ by $\Phi(\sum c_i A_i) = \sum c_i \Phi(A_i)$. To see Φ is well defined, note if $B = \sum_{i=1}^k c_i A_i = 0$, then $B^*B = 0$. Hence $\mathrm{Tr}(B^*B) = \sum_{i,j} \bar{c}_i c_j \mathrm{Tr}(A_i^* A_j) = 0$. So $\sum_{i,j} \bar{c}_i c_j \mathrm{Tr}(\Phi(A_i)^*\Phi(A_j)) = 0$. We thence have

$$\mathrm{Tr}\left(\left[\sum_{i=1}^k c_i \Phi(A_i)\right]\left[\sum_{i=1}^k c_i \Phi(A_i)\right]^*\right) = 0$$

and so $\sum c_i \Phi(A_i) = 0$. Note since π and ρ are irreducible, the Burnside Theorem 6.121 gives $\mathcal{B}(\mathcal{H}) = \langle \mathcal{A}_1 \rangle = \langle \mathcal{A}_2 \rangle$. Thus Φ is a $*$ algebra isomorphism of $\mathcal{B}(\mathcal{H})$ satisfying $\mathrm{Tr}(\Phi(T)) = \mathrm{Tr}(T)$ for $T \in \mathcal{B}(\mathcal{H})$. By Lemma 6.122, we see there is a unitary transformation U of $\mathcal{H}$ such that $\Phi(T) = UTU$ for $T \in \mathcal{B}(\mathcal{H})$. Hence

$$\rho(g) = \rho'(gN) = \Phi(\pi'(gN)) = U\pi'(gN)U^* = U\pi(g)U^*$$

and see π and ρ are unitarily equivalent. □

Definition 6.124. *Let π be a unitary representation of a locally compact σ-compact Hausdorff group G on a Hilbert space $\mathcal{H}$. Integrate π to $L^1(G)$ to obtain a $*$ representation on $\mathcal{H}$. Define $L^1(G)_T$ to be the set of all $f \in L^1(G)$ such that $\pi(f)$ is a trace class operator. Define the character Θ_π of π to be the linear functional on $L^1(G)_T$ defined by*

$$\Theta_\pi(f) = \mathrm{Tr}(\pi(f)).$$

Then Θ_π is called the character of the representation π.

Exercise Set 6.9

1. Let $\mathcal{H} \otimes \overline{\mathcal{H}}$ be the $*$ algebra of Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$. Show if Φ is a $*$ algebra onto isomorphism of $\mathcal{H} \otimes \overline{\mathcal{H}}$ which preserves inner products, then there is a unitary operator U on $\mathcal{H}$ such that $\Phi(T) = UTU^*$ for $T \in \mathcal{H} \otimes \bar{\mathcal{H}}$.

2. Show $L^1(G)_T$ is a $*$ ideal in $L^1(G)$.

3. Show if π is finite dimensional, then

$$\Theta_\pi(f) = \int_G f(x)\chi_\pi(x)\, dx$$

where χ_π is the character function for π.

4. Burnside Theorem: Let G be a completely reducible group (see Exercise 4.16) with an irreducible representation π on a finite dimensional complex vector space V. Show the linear span of the $\pi(g)$ for $g \in G$ is $\mathcal{L}(V)$. (**Hint:** Use Exercise 6.4.15 and the irreducibility of $\check{\pi}$.)

5. Assume π and ρ are unitary representations such that $\pi(f)$ and $\rho(f)$ are Hilbert-Schmidt for all $f \in C_c(G)$. Show if $\Theta_\pi(f^* * f) = \Theta_\rho(f^* * f)$ for $f \in C_c(G)$, then $\mathrm{Hom}_G(\pi, \rho) \neq \{0\}$. In particular, if π and ρ are irreducible, then they are unitarily equivalent.

17. Induced Representations

The notion of an induced representation was invented by Frobenius in about 1897 for finite groups and then extended to infinite locally compact groups by

Mackey in the 1950's. In algebraic language, one starts with a group algebra module V for a subgroup H and one forms the tensor module $\mathbb{C}(G) \otimes_{\mathbb{C}(H)} V$ which is a module for the group algebra $\mathbb{C}(G)$. We follow this construction for compact groups and then use that for a model in the general case.

We start with a compact Hausdorff group G and a unitary representation π of a closed subgroup H of G on a Hilbert space $\mathcal{H}$. We take Haar measure m so that $m(G) = 1$. We then have the left regular and right regular representations λ and ρ of G on $L^2(G)$ given by

$$\lambda(x)f(y) = f(x^{-1}y) \text{ and } \rho(x)f(y) = f(yx).$$

As seen in (6.13), $\lambda(x)\rho(y) = \rho(y)\lambda(x)$ for all x and y in G. Now $L^2(G) \otimes \mathcal{H}$ is the Hilbert space of Hilbert-Schmidt operators from $\bar{\mathcal{H}}$ to $L^2(G)$ and $\lambda \otimes I$ is a unitary representation of G on this Hilbert space. Consider the subspace $\mathcal{S}$ spanned by the rank 2 operators $\rho(h)f \otimes v - f \otimes \pi(h^{-1})v$ where h is in H and $v \in \mathcal{H}$ and $f \in L^2(G)$. This subspace is invariant under $\lambda \otimes I$ for

$$\begin{aligned}(\lambda(g) \otimes I)(\rho(h)f \otimes v - f \otimes \pi(h^{-1})v) &= (\lambda(g)\rho(h)f \otimes v - \lambda(g)f \otimes \pi(h^{-1})v \\ &= (\rho(h)\lambda(g)f \otimes v - \lambda(g)f \otimes \pi(h^{-1})v.\end{aligned}$$

Since $\lambda \otimes I$ is a unitary representation that leaves $\mathcal{S}$ invariant, it leaves invariant the closed subspace $\mathcal{S}^{\perp}$. The restriction of $\lambda \otimes I$ to $\mathcal{S}^{\perp}$ is the representation induced by π. It will be denoted by π^G or by $\operatorname{ind}_H^G \pi$. This definition will not work in the general case because in the non compact case $\mathcal{S}^{\perp}$ may be $\{0\}$. We start by describing this representation. It is easiest to do in the case when the Hilbert space $\mathcal{H}$ is separable.

Proposition 6.125. *Let $\mathcal{H}$ be a separable Hilbert space and X be a locally compact Hausdorff space with a regular Borel measure μ. Set $L^2(X, \mathcal{H})$ to be the space of Borel functions f from X into $\mathcal{H}$ such that $\int ||f(x)||^2_{\mathcal{H}}\, d\mu(x) < \infty$; any two identified if they are equal a.e. μ. Then $L^2(X, \mathcal{H})$ is a Hilbert space unitarily isomorphic to $L^2(X) \otimes \mathcal{H}$ under a mapping that sends $fv \mapsto f \otimes v$ for $f \in L^2(G)$ and $v \in \mathcal{H}$. Moreover, the inner product on $L^2(X, \mathcal{H})$ is given by*

$$(f_1, f_2)_2 = \int_X (f_1(x), f_2(x))_{\mathcal{H}}\, d\mu(x).$$

Proof. It is clear that $L^2(X, \mathcal{H})$ is a vector space and if

$$|f|_2^2 = \int ||f(x)||^2_{\mathcal{H}}\, d\mu(x),$$

then $|f|_2 = 0$ if and only if $f = 0$ a.e. μ, $|cf|_2 = |c|\,|f|_2$, and $|f + g|_2 \leqslant |f|_2 + |g|_2$. Now since $\mathcal{H}$ is separable, then there is a orthonormal basis $\{e_k\}_{k=1}^{\infty}$, and we see

$$(f_1(x), f_2(x))_{\mathcal{H}} = \sum_k (f_1(x), e_k)_{\mathcal{H}}(e_k, f_2(x))_{\mathcal{H}} \tag{6.34}$$

is Borel function. Hence $(f_1, f_2)_2$ is defined for

$$|(f_1(x)f_2(x))_{\mathcal{H}}| \leqslant ||f_1(x)||_{\mathcal{H}}||f_2(x)||_{\mathcal{H}}$$

which by Cauchy-Schwarz is integrable in x. We hence see that $L^2(X, \mathcal{H})$ is an inner product space.

We claim it is complete. Let f_n be Cauchy in $L^2(X, \mathcal{H})$. Then $(f_n, e_k)_{\mathcal{H}}$ is Cauchy in $L^2(X)$ for each k. Indeed, $\int |(f_m(x), e_k)_{\mathcal{H}} - (f_n(x), e_k)_{\mathcal{H}}|^2 \, d\mu(x) \leqslant \int |f_m(x) - f_n(x)|^2_{\mathcal{H}} \, d\mu(x)$. Since $L^2(X)$ is complete, for each k there is a Borel measurable complex function $a_k(\cdot)$ such that $(f_n(x), e_k)_{\mathcal{H}} \to a_k(x)$ in $L^2(X)$. Define $f(x)$ by $f(x) = \sum_k a_k(x)e_k$. We show this exists in $L^2(X, \mathcal{H})$ and $f_n \to f$. We note since f_n is Cauchy, there is an $M > 0$ such that $\int ||f_n(x)||^2_{\mathcal{H}} \, d\mu(x) \leqslant M^2$ for all n. Thus $\int \sum_k (f_n(x), e_k)_{\mathcal{H}}|^2 \, d\mu(x) \leqslant M^2$ for all n. Using Fatou's Lemma, and letting $n \to \infty$, gives $\int \sum |a_k(x)|^2 \, d\mu(x) \leqslant M^2$. Thus $\sum_k |a_k(x)|^2 < \infty$ for μ a.e. x and we see $f(x) \in \mathcal{H}$ for μ a.e. x and we have $\int ||f(x)||^2_{\mathcal{H}} \, d\mu(x) \leqslant M^2$. So $f \in L^2(X, \mu)$.

Next we show $f_n \to f$ in $L^2(X, \mathcal{H})$. In fact if $\epsilon > 0$, we can choose N such that for $m, n \geqslant N$, then $\int ||f_m(x) - f_n(x)||^2_{\mathcal{H}} \, d\mu(x) \leqslant \epsilon$. Thus if $m \geqslant N$, again by Fatou's Lemma, $\int ||f_m(x) - f(x)||^2_{\mathcal{H}} \, d\mu(x) \leqslant \liminf_{n\to\infty} \int ||f_m(x) - f_n(x)||^2_{\mathcal{H}} \, d\mu(x) \leqslant \epsilon$. Thus $f_m \to f$ in $L^2(X, \mathcal{H})$ as $m \to \infty$.

Now for each $f \in L^2(X, \mu)$, we define an operator $A_f : \bar{\mathcal{H}} \to L^2(X)$ by

$$A_f(\bar{v})(x) = (f(x), v)_{\mathcal{H}}. \tag{6.35}$$

It is easy to see this is a bounded operator from $\bar{\mathcal{H}}$ into $L^2(X)$. It is also Hilbert-Schmidt. Indeed, using the orthonormal basis $\{e_k\}$ of $\mathcal{H}$, we have

$$\begin{aligned}
\sum_k |A_f(\bar{e}_k)|_2^2 &= \sum_k \int |(f(x), e_k)_{\mathcal{H}}|^2 \, d\mu(x) \\
&= \int \sum_k |(f(x), e_k)_{\mathcal{H}}|^2 \, d\mu(x) \\
&= \int ||f(x)||^2_{\mathcal{H}} \, d\mu(x) \\
&= |f|_2^2
\end{aligned}$$

and thus $f \mapsto A_f$ is an isometry from $L^2(X, \mathcal{H})$ into $L^2(X) \otimes \mathcal{H}$. It is onto for if $f \in L^2(X)$ and $v \in \mathcal{H}$, then

$$A_{fv}(\bar{w})(x) = (f(x)v, w)_{\mathcal{H}} = f(x)(v, w)_{\mathcal{H}} \text{ for a.e. } x.$$

So

$$A_{fv}(\bar{w}) = (f \otimes v)(\bar{w})$$

and we see the range contains the finite rank operators and thus is dense in the space of Hilbert-Schmidt operators. □

Using this proposition, we see that if $\mathcal{H}$ is separable and G is compact, then $L^2(G,\mathcal{H})$ is unitarily isomorphic to $L^2(G) \otimes \mathcal{H}$ under the mapping $f \mapsto A_f \in \mathcal{B}_2(\bar{\mathcal{H}}, L^2(G))$. Moreover, if $\tilde{\lambda}$ is left translation on $L^2(G,\mathcal{H})$; i.e., $\tilde{\lambda}(x)f(y) = f(x^{-1}y)$, then

$$A_{\tilde{\lambda}(x)f} = (\lambda(x) \otimes I(x))A_f \text{ for } f \in L^2(G,\mathcal{H}).$$

Theorem 6.126. *Let π be a unitary representation of a closed subgroup H of a compact Hausdorff group G on a separable Hilbert space $\mathcal{H}$. Under the unitary isomorphism $f \mapsto A_f$ from $L^2(G,\mathcal{H})$ onto $L^2(G) \otimes \mathcal{H}$, $A^{-1}(\mathcal{S}^\perp) = \{f \in L^2(G,\mathcal{H}) \mid f(gh) = \pi(h^{-1})f(g)$ a.e. g for each $h\}$. Hence π^G is unitarily equivalent to the representation $\tilde{\pi}^G$ given on $L^2_\pi(G,\mathcal{H}) = \{f \in L^2(G,\mathcal{H}) \mid f(xh) = \pi(h^{-1})f(x)$ for a.e. x for each $h \in H\}$ by*

$$\tilde{\pi}(g)f(x) = f(g^{-1}x).$$

Proof. If $\mathcal{H}$ is separable, then using A to identify $L^2(G,\mathcal{H})$ and $L^2(G)\otimes\mathcal{H}$, the tensor $f \otimes \pi(h^{-1})v - \rho(h)f \otimes v$ is the function $S(x) = f(x)\pi(h^{-1})v - \rho(h)f(x)v = f(x)\pi(h^{-1})v - f(xh)v$. Thus a function $F \in L^2(G,\mathcal{H})$ is perpendicular to S if

$$\int f(x)(\pi(h^{-1})v, F(x))_{\mathcal{H}}\, dx = \int f(xh)(v, F(x))_{\mathcal{H}}\, dx.$$

This is equivalent to

$$\begin{aligned}\int f(x)(v, \pi(h)F(x))\, dx &= \int f(x)(v, F(xh^{-1}))\, dx \\ &= \int f(x)(v, F(xh^{-1}))\, dx.\end{aligned}$$

This will hold for all S if and only if $\pi(h)F(x) = F(xh^{-1})$ a.e. x for each h. Thus $F(xh) = \pi(h^{-1})F(x)$ a.e. $x \in G$ for each $h \in H$. Also under the correspondence, $f \mapsto A_f$, the operator $\tilde{\lambda}(x)$ is mapped to the operator $\lambda(x) \otimes I$. Thus we see π^G is unitarily equivalent to the representation $\tilde{\pi}^G$ on $L^2_\pi(G)$ defined by

$$\tilde{\pi}^G(x)f(y) = f(y^{-1}x).$$

□

We thus make the following definition. Note that we replaced $f(xh) = \pi(h^{-1})f(x)$ a.e. x for each h with $f(x) = \pi(h^{-1})f(x)$ for all h a.e. x. This change can be made if one can use Fubini's Theorem; for instance when G is σ-compact or second countable.

Definition 6.127. *Let π be a unitary representation of a closed subgroup of a compact group G on a separable Hilbert space $\mathcal{H}$. Then the representation π^G or $\mathrm{ind}_H^G\pi$ defined on $L^2_\pi(G) = \{f \in L^2(G,\mathcal{H}) \mid f(gh) = \pi(h^{-1})f(g)$ for all $h \in H$ a.e. $g\}$ by*

$$\pi^G(x)f(y) = f(x^{-1}y)$$

is the representation induced by π from H to G.

As we have seen, π^G is a unitary representation of the compact group G. Any representation unitarily equivalent to π^G will also be loosely denoted as the representation of G induced from π.

When $\mathcal{H}$ is nonseparable, we can still make $L^2(X,\mathcal{H})$ into a Hilbert space. We only sketch the process and use the density of $C_c(X,\mathcal{H})$ in $L^2(X,\mathcal{H})$. Let $F \in C_c(X,\mathcal{H})$. Then the range of F is a compact subset of $\mathcal{H}$ and thus is separable. Now let $F \in L^2(X,\mathcal{H})$. We can choose $F_n \in C_c(X,\mathcal{H})$ such that $F_n \to F$ in $L^2(X,\mathcal{H})$. Now the range of each F_n is contained in a separable subspace $\mathcal{H}_n$ of $\mathcal{H}$. The smallest closed subspace $\mathcal{H}_\infty$ containing all the subspaces $\mathcal{H}_n$ is then separable. Thus any $F \in L^2(X,\mathcal{H})$ may be assumed to have range in a separable Hilbert subspace of $\mathcal{H}$.

Now let $F_1, F_2 \in L^2(X,\mathcal{H})$. As we have seen we may suppose the ranges of F_1 and F_2 are contained in a separable closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{H}$. Then the smallest closed subspace $\mathcal{H}_0$ containing both $\mathcal{H}_1$ and $\mathcal{H}_2$ is separable, and using a countable orthonormal basis as in (6.34), one then has $x \mapsto (F_1(x), F_2(x))_{\mathcal{H}_0}$ is measurable. So we can define $(F_1,F_2)_2 = \int (F_1(x), F_2(x))\, d\mu(x)$. One can then redo the arguments in the proofs of Proposition 6.125 and Theorem 6.126 and show they hold without the assumption that $\mathcal{H}$ is separable. Because of this one can remove the separability assumption in Definition 6.127. Thus if π is a unitary representation of a closed subgroup H of a compact Hausdorff group G, then π^G is unitary representation defined by

$$\pi^G(x)f(y) = f(x^{-1}y) \tag{6.36}$$

where f is in the space of Borel measurable functions from G into $\mathcal{H}$ satisfying $f(xh) = \pi(h^{-1})f(x)$ for all $h \in H$ a.e. x and $\int ||f(x)||^2_{\mathcal{H}}\, d\mu(x) < \infty$.

The noncompact case. To discuss induced representation for noncompact groups, we need to generalize Lemma 6.14 to the case where we are dealing with continuous functions $f : G \to \mathcal{H}$ satisfying $f(xh) = \pi(h^{-1})f(x)$ for $x \in G$ and $h \in H$. Before stating this generalization, we set up its context. We start with a locally compact Hausdorff group G, a closed subgroup H, and a unitary representation π of H on a Hilbert space $\mathcal{H}$. The mapping κ is the open continuous mapping $x \mapsto xH$ from G onto G/H.

If W is a compact subset of G/H, we define $C_W(G,\pi)$ to be the space of all continuous functions $f : G \to \mathcal{H}$ with the following properties:

$$\begin{aligned} &f(xh) = \pi(h^{-1})f(x) \text{ for all } x \in G \text{ and } h \in H, \\ &f(x) = 0 \text{ if } x \notin \kappa^{-1}(W). \end{aligned} \tag{6.37}$$

For a compact subset K of G, $C_K(G,\mathcal{H})$ is taken to be the space of all continuous functions $f : G \to \mathcal{H}$ such that $f(x) = 0$ for $x \notin K$.

Lemma 6.128. *The mapping $f \mapsto f_H$ where*

$$f_H(x) = \int \pi(h)f(xh)\,dh$$

maps $C_c(G,\mathcal{H})$ onto the space of all the continuous functions $F : G \to \mathcal{H}$ satisfying $F(xh) = \pi(h^{-1})F(x)$ for $x \in G$ and $h \in H$ and F vanishes off $\kappa^{-1}(W)$ for some compact subset W of G/H. Moreover, if W is a compact subset of G/H, there is a compact subset $\widetilde{W}$ of G with $W \subseteq \kappa(\widetilde{W})$ and a linear mapping T_W from $C_W(G,\pi)$ into $C_{\widetilde{W}}(G,\mathcal{H})$ such that $f(x) = \int_H \pi(h)T_Wf(xh)\,dh$ for $f \in C_W(G,\pi)$.

Proof. Let m be a left Haar measure on H and let $f \in C_c(G,\mathcal{H})$. We show f_H on G defined by $f_H(x) = \int_H \pi(h)f(xh)\,dh$ is continuous.

Let $\epsilon > 0$ and fix a compact neighborhood of e. By left uniform continuity of f, we can choose a neighborhood N' of e contained in N such that

$$|f(ny) - f(y)| \leqslant \frac{\epsilon}{m(H \cap x^{-1}N^{-1}\mathrm{supp}\, f)} \text{ for all } y \in G \text{ for } n \in N'.$$

Hence if $n \in N'$, then $f(nxh) = 0$ and $f(xh) = 0$ for $h \notin H \cap x^{-1}N^{-1}\mathrm{supp}\, f$ and we see:

$$\begin{aligned} ||f_H(nx) - f_H(x)||_{\mathcal{H}} &\leqslant \int_H ||\pi(h)f(nxh) - \pi(h)f(xh)||_{\mathcal{H}}\,dh \\ &\leqslant \int_{H\cap x^{-1}N^{-1}\mathrm{supp}\, f} \frac{\epsilon}{m(H \cap x^{-1}N^{-1}\mathrm{supp}\, f)}\,dh \\ &= \epsilon. \end{aligned}$$

Therefore, f_H is continuous. It clearly vanishes off $(\mathrm{supp} f)H$.

Now fix a compact subset W of G/H. Then there is an open set V with $\bar{V}$ compact and $\kappa(V) \supseteq W$. We set $\widetilde{W} = \bar{V}$. By Proposition 5.23, we can find a function $t \in C_c(G)$ such that $t \geqslant 0$ and $t = 1$ on V. Following Lemma 6.14 and its proof, we take $\tilde{t}(xH) = \int t(xh)\,dh$ and for $f \in C_W(G,\pi)$ we define $T_Wf(x) = \frac{t(x)}{\tilde{t}(xH)}f(x)$ where $\frac{0}{0} = 0$. The same argument used in the proof of Lemma 6.14 shows $T_W f$ is continuous and

$$\begin{aligned}(T_W f)_H(x) &= \int \pi(h)\frac{t(xh)}{\tilde{t}(xH)} f(xh)\,dh \\ &= \int \pi(h)\frac{t(xh)}{\tilde{t}(xH)} \pi(h^{-1})f(x)\,dh \\ &= \frac{f(x)}{\tilde{t}(xH)}\int_H t(xh)\,dh \\ &= f(x).\end{aligned}$$

□

Lemma 6.129. *Let f be a bounded Borel function from G into $\mathcal{H}$ satisfying $f(x) = 0$ off $\kappa^{-1}(W)$ for some compact subset W of G/H and $f(xh) = \pi(h^{-1})f(x)$ for all h for a.e. x. Then there is a bounded Borel function F on G into $\mathcal{H}$ which vanishes off a compact subset $\widetilde{W}$ of G such that*

$$F_H(x) = \int \pi(h)F(xh)\,dh \text{ for a.e. } x.$$

Proof. Follow the proof of the second part of Lemma 6.128; i.e., define $F(x) = \frac{t(x)}{\tilde{t}(xH)}f(x)$. This is a Borel function with the desired properties. □

Proposition 6.130. *Let H be a closed subgroup of a σ-compact, locally compact Hausdorff group G and let ρ be a continuous positive rho function on G with corresponding regular quasi-invariant measure μ on G/H. If π is a unitary representation of H on a separable Hilbert space $\mathcal{H}$, define $L^2_\pi(G,\mathcal{H})$ to be the space of Borel functions $f : G \to \mathcal{H}$ such that $f(xh) = \pi(h^{-1})f(x)$ for all $h \in \mathcal{H}$ a.e. $x \in G$ and $\int ||f(x)||^2_{\mathcal{H}}\,d\mu(xH) < \infty$. Then with inner product*

$$(f_1, f_2)_2 = \int_{G/H} (f_1(x), f_2(x))_{\mathcal{H}}\,d\mu(xH),$$

$L^2_\pi(G,\mathcal{H})$ is a Hilbert space and the continuous functions f in $L^2_\pi(G,\mathcal{H})$ such that there is a compact subset K of G/H such that f vanishes off $\kappa^{-1}(K) = KH$ are dense in $L^2_\pi(G,\mathcal{H})$.

Proof. We show $L^2_\pi(G,\mathcal{H})$ is a Hilbert space. We note if $\{e_k\}_{k=1}^\infty$ is an orthonormal basis of $\mathcal{H}$, then $(f_1(x), f_2(x)) = \sum_{k=1}^\infty (f_1(x), e_k)_{\mathcal{H}}(e_k, f_2(x))_{\mathcal{H}}$ is Borel and by following of the first part of the proof of Proposition 6.125, $L^2_\pi(G,\mathcal{H})$ is an inner product space. We show completeness. Let F_n be a sequence with $\sum_n |F_n|_2 < \infty$. We need to show $\sum F_n$ converges in $L^2_\pi(G,\mathcal{H})$. Let $f_n(\kappa(x)) = ||F_n(x)||_{\mathcal{H}}$. We note if $s_n = \sum_{k=1}^n f_k$ and $M = \sum_{k=1}^\infty ||F_k||_2$, then

$$\begin{aligned}|s_n|_2 &= |f_1 + f_2 + \cdots + f_n|_2\\ &\leqslant |f_1|_2 + |f_2| + \cdots + |f_n|_2\\ &= ||F_1||_2 + ||F_2||_2 + \cdots ||F_n||_2\\ &\leqslant M.\end{aligned}$$

Thus $\int s_n^2(xH)\,d\mu(xH) \leqslant M^2$ for all n. By Fatou's Lemma, if $s(xH) = \sum_{k=1}^{\infty} f_k(xH)$, then $\int s(xH)^2\,d\mu(xH) \leqslant M^2$. Thus $s(xH)$ is finite for μ a.e. xH in G/H. By Corollary 6.25, $\sum ||F_n(x)||_{\mathcal{H}} < \infty$ for a.e. x in G. We also note if this sum is finite, so is $\sum ||F_n(xh)||$ for each $h \in H$.

Thus $\sum ||F_n(x)||$ is finite for a.e. x and if it is finite for a particular x, it is finite for all xh where $h \in H$. For those $x \in G$ for which $s(x)$ is finite, we have $\sum_k F_k(x)$ converges in $\mathcal{H}$. Set $S(x)$ to be this sum. For all the remaining x in G, set $S(x) = 0$. Note since $F_n(xh) = \pi(h^{-1})F_n(x)$ for all h for a.e. x, we have $S(xh) = \pi(h^{-1})S(x)$ for all h for a.e. x. Since

$$\begin{aligned}\int_{G/H} ||S(x)||_{\mathcal{H}}^2\,d\mu(xH) &= \int_{G/H} ||\lim_{n\to\infty}\sum_{k=1}^{n} F_k(x)||_{\mathcal{H}}^2\,d\mu(xH)\\ &= \int_{G/H} \lim_{n\to\infty} ||\sum_{k=1}^{n} F_k(x)||_{\mathcal{H}}^2\,d\mu(x)\\ &\leqslant \int \lim_{n\to\infty} |\sum_{k=1}^{n} ||F_k(x)||_{\mathcal{H}}|^2\,d\mu(x)\\ &= \int \lim_{n\to\infty} |\sum_{k=1}^{n} f_k(x)|^2\,d\mu(xH)\\ &\leqslant M^2,\end{aligned}$$

we see $S \in L_\pi^2(G, \mathcal{H})$. Also since s^2 is integrable and for a.e. xH

$$\left(\sum_{k=n+1}^{\infty} f_k(xH)\right)^2 \leqslant s(xH)^2,$$

the Lebesgue dominated convergence theorem implies

$$\begin{aligned}
\lim_{n\to\infty} |S - \sum_{k=1}^{n} F_k|_2^2 &= \lim_{n\to\infty} |\sum_{k=n+1}^{\infty} F_k|_2^2 \\
&= \lim_{n\to\infty} \int || \sum_{k=n+1}^{\infty} F_k(xH)||_{\mathcal{H}}^2 \, d\mu(xH) \\
&\leqslant \lim_{n\to\infty} \int \left(\sum_{k=n+1}^{\infty} f_k(xH) \right)^2 d\mu(xH) \\
&= \int \lim_{n\to\infty} \left(\sum_{k=n+1}^{\infty} f_k(xH) \right)^2 d\mu(xH) \\
&= 0.
\end{aligned}$$

Let $f \in L^2_\pi(G, \mathcal{H})$. We now show how to approximate f. First note by redefining f on a set of measure 0, we may assume $f(xh) = \pi(h^{-1})f(x)$ for all x and h.

Let $\epsilon > 0$. Since $f \in L^2_\pi(G, \mathcal{H})$, $\{xH \mid ||f(xh)||_{\mathcal{H}} \neq 0\}$ is σ-finite relative to the measure μ and hence can be written as a countable union of Borel sets $E_k \subseteq G/H$ of finite measure. By taking finite unions, we may assume $E_k \subseteq E_{k+1}$ for all k. Now $f\chi_{E_k} \circ \kappa \in L^2_\pi(G, \mathcal{H})$ and $f\chi_{E_k} \circ \kappa \to f$ in $L^2_\pi(G, \mathcal{H})$. Since μ is inner regular, for a fixed k, there is an increasing sequence K_n of compact subsets of E_k, such that $\mu(E_k - K_n) \to 0$ as $n \to \infty$. This implies $f\chi_{K_n} \circ \kappa \to f\chi_{E_k}$ in $L^2_\pi(G, \mathcal{H})$. Now for each m, let $S_m = \{x \in G \mid ||f(x)||_{\mathcal{H}} \leqslant m\}$ is a Borel subset of G. Since $f(xh) = \pi(h^{-1})f(x)$, we see $\chi_{S_m}(xh) = \chi_{S_m}(x)$ for all x and h. Consequently, $\chi_{S_m} f\chi_{K_n} \circ \kappa \to f\chi_{K_n} \circ \kappa$ in $L^2_\pi(G, \mathcal{H})$ as $m \to \infty$. Putting all this together, we have $||\chi_{S_m} f\chi_{K_n} \circ \kappa - f||_2 < \frac{\epsilon}{3}$ for some m and n. Thus we have found a bounded Borel function f_0 in $L^2_\pi(G, \mathcal{H})$ which vanishes off $\kappa^{-1}(W)$ for some compact subset W of G/H such that $|f - f_0|_2 < \frac{\epsilon}{3}$. Furthermore, this f_0 satisfies $f(xh) = \pi(h^{-1})f(x)$ for all x and h.

By Lemma 6.129, there is a bounded Borel function $F : G \to \mathcal{H}$ which vanishes off a compact subset $\widetilde{W}$ of G with $F_H = f_0$. Then $a_k = (F, e_k)$ are bounded complex Borel functions on G vanishing off $\widetilde{W}$. Set $F_n = \sum_{k=1}^{n} a_k e_k$. Then since $||F_n(x)||_{\mathcal{H}} \leqslant ||F(x)||_{\mathcal{H}}$, the F_n for $n \geqslant 1$ are uniformly bounded Borel functions and clearly $F_n(x) \to F(x)$ as $n \to \infty$ for all x. So $(F_n)_H \to F_H$ in $L^2_\pi(G, \mathcal{H})$. Indeed, each $(F_n)_H$ vanishes off the compact subset $\kappa(\widetilde{W})$ and if M is an upper bound for both all F_n and F, then if $x \in \widetilde{W}$, one has

$$\begin{aligned}||(F_n)_H(x)|| &\leqslant \int ||F_n(xh)||\,dh \\ &= \int_{x^{-1}\widetilde{W}\cap H} ||F_n(xh)||_{\mathcal{H}}\,dh \\ &\leqslant \int_{\widetilde{W}^{-1}\widetilde{W}\cap H} M\,dh \\ &\leqslant m(\widetilde{W}^{-1}\widetilde{W}\cap H)M.\end{aligned}$$

Set $M' = m(\widetilde{W}^{-1}\widetilde{W} \cap H)$. Since $||(F_n)_H(xh)|| = ||(F_n)_H(x)||$ for $x \in \widetilde{W}$ and and $(F_n)_H(x) = 0$ off $\widetilde{W}H$, we have $||(F_n)_H(x)|| \leqslant M'$ for all x and n. Similarly $||F_H(x)|| \leqslant M'$ for all x. So $xH \mapsto ||(F_n)_H(x) - F_H(x)|| \leqslant 2M'\chi_{\kappa(\widetilde{W})}(xH)$. By the Lebesgue Dominated Convergence Theorem,

$$\int_{G/H} ||(F_n)_H(x) - F_H(x)||^2_{\mathcal{H}}\,d\mu(xH) \to 0 \text{ as } n \to \infty.$$

So can choose n with $|(F_n)_H - F_H|_2 = |(F_n)_H - f_0|_2 < \frac{\epsilon}{3}$. We also have each F_n is bounded by M and $(F_n)_H$ are bounded by M'. So $\sum_{k=1}^n |a_k(x)|^2 = ||F_n(x)||^2_{\mathcal{H}} \leqslant M^2$. So $|a_k(x)| \leqslant M$ for all x and each $k = 1, 2, \ldots, n$.

Now fix a precompact open set V with $V \supseteq \widetilde{W}$. Since a_k vanishes off $\widetilde{W}$ and are bounded by M, we can find continuous functions b_k with supports in V that satisfy $|b_k(x)| \leqslant M$ for all x and

$$\int |a_k(x) - b_k(x)|\,\rho(x)\,dx < \frac{\epsilon}{3n(n+1)Mm(V^{-1}V \cap H)} \text{ for } k = 1, 2, \ldots, n.$$

Set $g(x) = \sum_{k=1}^n b_k(x)e_k$. Then $||g(x)||_{\mathcal{H}} \leqslant nM$ for all x, has compact support inside V, and

$$\begin{aligned}\int ||g(x) - F_n(x)||_{\mathcal{H}}\rho(x)\,dx &\leqslant \sum_{k=1}^n \int |a_k(x) - b_k(x)|\rho(x)\,dx \\ &< \frac{\epsilon}{3(n+1)Mm(V^{-1}V \cap H)}.\end{aligned}$$

Using Hölder's inequality and Theorem 6.15, we see

$$\begin{aligned}
\int ||(g-F_n)_H(x)||^2\, d\mu(xH) &= \int || \int_H (g-F_n)(xh)\, dh||^2\, d\mu(xH) \\
&\leqslant \int \left(\int ||(g-F_n)(xh)||\, dh \right)^2 d\mu(xH) \\
&\leqslant \int \left(\int_{V^{-1}V\cap H} ((n+1)M)^{1/2} ||(g-F_n)(xh)||^{1/2} dh \right)^2 d\mu(xH) \\
&\leqslant \int \int_{V^{-1}V\cap H} (n+1)M\, dh \int_{V^{-1}V\cap H} ||(g-F_n)(xh)||_{\mathcal{H}} dh\, d\mu(xH) \\
&\leqslant m(V^{-1}V\cap H)(n+1)M \int ||(g-F_n)(xh)||_{\mathcal{H}}\, dh\, d\mu(xH) \\
&\leqslant m(V^{-1}V\cap H)(n+1)M \int ||(g-F_n)(x)||_{\mathcal{H}}\, \rho(x)\, dx \\
&< \frac{\epsilon}{3}.
\end{aligned}$$

So $|g_H - f|_2 \leqslant |g_H - (F_n)_H|_2 + |(F_n)_H - F_H|_2 + |f_0 - f|_2 < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ ☐

With some extra care one need not assume G is σ-compact in the statement of Proposition 6.130. Indeed, see Exercise 6.10.1.

In the following theorem, we assume we have the Hilbert space $L^2_\pi(G,\mathcal{H})$ of Proposition 6.130 The measure μ is the regular Borel measure on G/H obtained from the Radon measure given in Theorem 6.15. It satisfies

$$\int \int f(xh)\, dh\, d\mu(xH) = \int f(x)\rho(x)\, dx \tag{6.38}$$

for bounded Borel functions f on G which vanish off a compact subset of G. It is quasi-invariant and satisfies $d\mu(xyH) = \frac{\rho(xy)}{\rho(y)}\, d\mu(yH)$ where ρ is a positive continuous rho function on G.

Theorem 6.131. *Define π^G on $L^2_\pi(G,\mathcal{H})$ by $\pi^G(x)f(y) = \sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}} f(x^{-1}y)$. Then π^G is a unitary representation of G.*

Proof. It is easy to check $\pi^G(x)f(yh) = \pi(h^{-1})\pi^G(x)f(y)$ for all h for a.e. y and one clearly has $\pi^G(e) = I$.

Also

$$\begin{aligned}
\pi^G(x)\pi^G(y)f(z) &= \sqrt{\frac{\rho(x^{-1}z)}{\rho(z)}}\pi^G(y)f(x^{-1}z) \\
&= \sqrt{\frac{\rho(x^{-1}z)\rho(y^{-1}x^{-1}z)}{\rho(z)\rho(x^{-1}z)}}f(y^{-1}x^{-1}z) \\
&= \sqrt{\frac{\rho((xy)^{-1}z)}{\rho(z)}}f((xy)^{-1}z) \\
&= \pi^G(xy)f(z)
\end{aligned}$$

and

$$\begin{aligned}
|\pi^G(x)f|_2^2 &= \int_{G/H} ||\pi^G(x)f(y)||_{\mathcal{H}}^2\, d\mu(yH) \\
&= \int \frac{\rho(x^{-1}y)}{\rho(y)}||f(x^{-1}y)||_{\mathcal{H}}^2\, d\mu(yH) \\
&= \int \frac{\rho(y)}{\rho(xy)}||f(y)||_{\mathcal{H}}^2\, d\mu(xyH) \\
&= \int \frac{\rho(y)}{\rho(xy)}||f(y)||_{\mathcal{H}}^2 \frac{\rho(xy)}{\rho(y)}\, d\mu(yH) \\
&= |f|_2^2.
\end{aligned}$$

Thus π^G is a homomorphism of G into the unitary group of $L^2_\pi(G,\mathcal{H})$. We thus need only show π^G is strongly continuous at e.

Let $f \in L^2_\pi(G,\mathcal{H})$ and suppose $\epsilon > 0$. By Proposition 6.130, we can pick a continuous function f_0 in $L^2_\pi(G,\mathcal{H})$ which vanishes off $\kappa^{-1}(W)$ for some compact subset W of G/H such that $|f - f_0|_2 < \frac{\epsilon}{3}$. Fix a compact symmetric neighborhood N' of e. Now because of π covariance $f_0(zh) = \pi(h^{-1})f_0(z)$ for all h and z, for each $yH \in N'W$, we can pick a neighborhood $N(y)$ of e contained in N' and an open neighborhood $U(y)$ of yH such that $||\sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}}f_0(x^{-1}y) - f_0(y)||_{\mathcal{H}} \leqslant \frac{\epsilon}{3\sqrt{\mu(N'W)}}$ if $x \in N(y)$ and $yH \in U(y)$. Now the $U(y)$ cover the compact set $N'W \subseteq G/H$. Using the compactness of $N'W$, we can find a finite subcover $U(y_1)$, $U(y_2)$, $\dots$, $U(y_n)$ of $N'W$. Set $N = \cap_{k=1}^n N(y_k)$. Now if $yH \in W$ and $x \in N$ we have $||f_0(x^{-1}y) - f_0(y)||_{\mathcal{H}} \leqslant \frac{\epsilon}{3\sqrt{\mu(N'W)}}$; if $x^{-1}yH \in W$, then $xyH \in xW \in N'W$. So $xyH \in U(y_j)$ for some j. Thus $||f_0(x^{-1}xyH) - f_0(xyH)|| \leqslant \frac{\epsilon}{3\sqrt{\mu(N'W)}}$. Finally if both $x^{-1}yH \notin W$ and $yH \notin W$, then $f_0(x^{-1}y) = f_0(y) = 0$. Hence we see $||f_0(x^{-1}y) - f_0(y)||_{\mathcal{H}} \leqslant \frac{\epsilon}{3\sqrt{\mu(N'W)}}$ for any $yH \in W$ and $x \in N$. This implies if $x \in N$, then

$$\begin{aligned}
|\pi^G(x)f_0 - f_0|_2^2 &= \int ||\sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}} f_0(x^{-1}y) - f_0(y)||_{\mathcal{H}}^2 \, d\mu(yH) \\
&= \int_{N'W} ||\sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}} f_0(x^{-1}y) - f(y)||_{\mathcal{H}}^2 \, d\mu(yH) \\
&\leqslant \int_{N'W} \frac{\epsilon^2}{9\mu(N'W)} \, d\mu(yH) \\
&= \frac{\epsilon^2}{9}.
\end{aligned}$$

Thus for $x \in N$,

$$\begin{aligned}
|\pi^G(x)f - f|_2 &\leqslant |\pi^G(x)f - \pi^G(x)f_0|_2 + |\pi^G(x)f_0 - f_0|_2 + |f_0 - f|_2 \\
&= 2|f - f_0|_2 + |\pi^G(x)f_0 - f_0|_2 \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

So π^G is strongly continuous at e and we see π^G is a unitary representation of G. $\square$

Using this theorem, we make the following definition.

Definition 6.132. *Let π be a unitary representation of a closed subgroup H of a σ-compact locally compact Hausdorff group G on a separable Hilbert space $\mathcal{H}$. Let ρ be a positive continuous rho function for H which defines a regular quasi-invariant measure μ on G/H. Then any unitary representation unitarily equivalent to the unitary representation π^G on $L^2_\pi(G, \mathcal{H})$ defined by*

$$\pi^G(x)f(y) = \sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}} f(x^{-1}y)$$

is called the unitary representation of G induced by the representation π of H.

We remark that Exercise 6.10.2 shows if μ is any regular quasi-invariant measure on G/H, then the representation L defined on the Hilbert space of Borel functions $f : G \to \mathcal{H}$ satisfying $f(xh) = \pi(h)^{-1}f(x)$ for all h for a.e. x where $|f|_2 = \left(\int ||f(y)||_2 \, d\mu(yH)\right)^{1/2}$ by

$$L(x)f(y) = \left(\frac{d(x\mu)}{d\mu}(yH)\right)^{1/2} f(x^{-1}y)$$

is unitarily equivalent to π^G and thus is induced from π.

Example 6.133. (Regular Representations) *It is an easy exercise to show if H is a closed subgroup of a locally compact Hausdorff group G, then* $\operatorname{ind}_H^G 1$ *is the left quasi-regular representation of G on G/H. In particular, when $H = \{e\}$, one obtains the left regular representation of G.*

Definition 6.134. *Let G be a topological group containing a closed normal subgroup H and a closed subgroup K. If the mapping $(k, h) \mapsto kh$ is a homeomorphism of $K \times H$ onto G, then G is said to be a semi-direct product of K and H. This is denoted by $G = K \ltimes H$.*

Suppose we have a σ-compact locally compact Hausdorff group G and $G = K \ltimes H$. Since H is normal and closed, Lemmas 5.13 and 5.16 imply G/H is a locally compact Hausdorff group. Moreover, the mapping $k \mapsto kH$ is one-to-one and onto G/H. It is continuous since $g \mapsto gH$ is a continuous mapping. Also if U is an open subset of K, then UH is an open subset of G for the mapping $(k, h) \mapsto kh$ is a homeomorphism. Since κ is an open mapping $UH = (UH)H = \kappa(UH)$ is open. Hence the mapping $k \mapsto kH$ is a group homeomorphism of K onto G/H.

Now note that left Haar measure on G is given by

$$I(f) = \int_K \int_H f(kh)\, dh\, dk$$

for positive Borel functions f. Indeed, note $I(\lambda(k_0)f) = I(f)$ for any $k_0 \in K$. Now for $h_0 \in H$,

$$\begin{aligned} I(\lambda(h_0)f) &= \int_K \int_H f(h_0 kh)\, dh\, dk \\ &= \int_K \int_H f(k(k^{-1}h_0 k)h)\, dh\, dk. \end{aligned}$$

To find a continuous rho function we determine the modular function Δ_G on H. From

$$\int_K \int_H f(khh_0)\, dh\, dk = \int \int \Delta_H(h_0^{-1}) f(kh)\, dh\, dk\,,$$

we see

$$\Delta_G(h_0) = \Delta_H(h_0).$$

Hence the function $\rho(kh) = \Delta_H(h)/\Delta_G(h) = 1$ satisfies

$$\rho(khh') = \rho(kh)\frac{\Delta_H(h')}{\Delta_G(h')}.$$

Hence there is a regular quasi-invariant measure μ on G/H satisfying

$$\int \int f(yh)\, dh\, d\mu(yH) = \int f(x)\, dx$$

for positive Borel functions f. This measure satisfies

$$d\mu(xyH) = d\mu(yH).$$

Now assume $x \in K$ and $yH = kH$, then $d\mu(xyH) = \frac{\rho(xk)}{\rho(x)}\, d\mu(yH) = d\mu(yH)$. This implies the measure μ is left invariant under K. Since mapping $\Phi : K \to G/H$ given by $\Phi(k) = kH$ is topological group isomorphism, the measure ν on K defined by $\nu(E) = \mu(\Phi(E)) = \mu(EH)$ is a left Haar measure on K.

Theorem 6.135. *Let $G = K \ltimes H$ be a σ-compact locally compact Hausdorff semi-direct product group and let π be a unitary representation of H on a separable Hilbert space $\mathcal{H}$. Then π^G is unitarily equivalent to the representation L defined on $L^2(K, \mathcal{H})$ by*

$$L(k_0h_0)f(k) = \pi((k_0^{-1}k)^{-1}h_0(k_0^{-1}k))f(k_0^{-1}k).$$

Proof. Define $T : L^2_\pi(G, \mathcal{H}) \to L^2(K, \mathcal{H})$ by $Tf(k) = f(k)$. It is clearly linear and well defined. Moreover,

$$|Tf|_2^2 = \int ||f(k)||^2_{\mathcal{H}}\, d\nu(k) = \int ||f(k)||^2_{\mathcal{H}}\, d\mu(\Phi(k)),$$

and thus T is an isometry. It is onto for if $f \in L^2(K, \mathcal{H})$, define f_0 defined on G by $f(kh) = \pi(h^{-1})f(k)$. Note $f_0(xh) = \pi(h^{-1})f_0(x)$ for all h a.e. x. Also if $\{e_j\}_{j=1}^\infty$ is an orthonormal basis of $\mathcal{H}$, then $\pi(h^{-1})f(k) = \sum(f(k), \pi(h)e_j)_{\mathcal{H}}e_j$. Thus $x \mapsto f_0(x)$ is Borel if $(k, h) \mapsto (f(k), \pi(h)e_j)_{\mathcal{H}}$ is Borel for each j. But this is Borel if $(k, h) \mapsto (f(k), \pi(h)e_j) \in \mathcal{H} \times \mathcal{H}$ is Borel. Note this is clearly the case. One has $|f_0|_2 = |f|_2$ and $Tf_0 = f$. Thus T is onto.

Finally we calculate $T\pi^G(k_0h_0)T^{-1}$. Namely,

$$\begin{aligned} T\pi^G(k_0h_0)T^{-1}f(k) &= \pi^G(k_0h_0)T^{-1}f(k) \\ &= T^{-1}f(h_0^{-1}k_0^{-1}k) \\ &= T^{-1}f(k_0^{-1}k(k_0^{-1}k)^{-1}h_0^{-1}(k_0^{-1}k)) \\ &= \pi((k_0^{-1}k)^{-1}h_0(k_0^{-1}k))f(k_0^{-1}k). \end{aligned}$$

□

Example 6.136. (The Heisenberg Group) *Recall from Example 5.9 that if V is a finite dimensional real vector space and B is an alternating nondegenerate bilinear form on V then the Heisenberg group is the space $V \times \mathbb{R}$ with multiplication defined by $(v, s)(w, t) = (v + w, t + s + \frac{1}{2}B(v, w))$.*

In Chapter 7, we take $V = \mathbb{R}^n \times \mathbb{R}^n$ and $B((x, y), (x', y')) = \sum(x_iy'_i - y_ix'_i) = x \cdot y' - y \cdot x'$. Then with the product topology, $G = V \times \mathbb{R}$ is a second countable locally compact Hausdorff group and one has $G = K \ltimes H$

where $H = \{((0,y),t) \mid x \in \mathbb{R}^n, t \in \mathbb{R}\}$ *and* $K = \{((x,0),0) \mid x \in \mathbb{R}^n\}$. *One can check* H *is a normal subgroup and the mapping* $(k,h) \mapsto kh$ *is a homeomorphism of* $K \times H$ *onto* G. *Let* π *be the one-dimensional unitary representation of* H *defined by*

$$\pi((0,y),t) = e^{i\lambda t}$$

where λ *is a nonzero real number. Then if* $k_0 = ((x,0),0)$, $k = ((w,0),0)$ *and* $h_0 = ((0,y),t)$, *then* $k_0^{-1}k = ((w-x,0),0)$, *and*

$$\begin{aligned}(k_0^{-1}k)^{-1}h_0(k_0^{-1}k) &= ((x-w,0),0)((0,y),t))((w-x,0),0)\\ &= ((x-w,y),t+\frac{1}{2}((x-w)\cdot y)((w-x,0),0)\\ &= ((0,y),t+\frac{1}{2}((x-w)\cdot y - \frac{1}{2}y\cdot(w-x))\\ &= ((0,y),t+x\cdot y - w\cdot y).\end{aligned}$$

Thus

$$\pi((k_0^{-1}k)^{-1}h_0(k_0^{-1}k))f(k_0^{-1}k) = e^{i\lambda(t+x\cdot y-w\cdot y)}f((w-x,0),0)$$

for $f \in L^2(K)$.

Now K *and* $\mathbb{R}^n$ *are topologically isomorphic groups and thus we may identify* $L^2(K)$ *with* $L^2(\mathbb{R}^n)$ *and since* $k_0h_0 = ((x,0),0)((0,y),t) = ((x,y),t+\frac{1}{2}x\cdot y)$, *we see* π^G *is unitarily equivalent to the unitary representation* π_λ *defined by*

$$\pi_\lambda((x,y),t+\frac{1}{2}x\cdot y)f(w) = e^{i\lambda t}\,e^{i\lambda x\cdot y}e^{-\lambda w\cdot y}f(w-x).$$

By replacing t *with* $t - \frac{1}{2}x\cdot y$, *we obtain the formula:*

$$\pi_\lambda((x,y),t)f(w) = e^{i\lambda t}\,e^{.5\lambda i x\cdot y}e^{-i\lambda w\cdot y}f(w-x).$$

Compare this with formula (7.13) in Chapter 7. The π_λ *turn out to be all the irreducible infinite dimensional unitary representations of the Heisenberg group* G.

Example 6.137. The $\mathbf{ax+b}$ Group: *Recall the* $ax+b$ *group* G *consists of all pairs* $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$ *with multiplication given by*

$$(a,b)(a',b') = (aa', b+ab').$$

The subgroup $H = \{(1,b) \mid b \in \mathbb{R}\}$ *is a closed normal subgroup and* G *is homeomorphic to* $K \times H$ *under the mapping* $((a,0),(1,b)) \mapsto (a,0)(1,b) = (a,ab)$. *Hence* $G = K \ltimes H$. *The group* K *is isomorphic to* $\mathbb{R}^+$ *and by 1.7, we know Haar measure on* $\mathbb{R}^+$ *is given by* $\frac{dx}{x}$. *We can identify* $L^2(K)$ *with*

$L^2(\mathbb{R}^+, \frac{dx}{x})$ by the correspondence $(a,b)K \leftrightarrow a$. Now let π be the representation of H given by $\pi(1,b) = e^{2\pi ib}$. Take $k_0 = (a,0)$, $k = (x,0)$ where $x > 0$, and $h_0 = (1,b)$. Then $k_0h_0 = (a,ab)$ and

$$\begin{aligned}(k_0^{-1}k)^{-1}h_0(k_0^{-1}k) &= (ax^{-1},0)(1,b)(a^{-1}x,0))\\ &= (ax^{-1},ax^{-1}b)(a^{-1}x,0)\\ &= (1,ax^{-1}b).\end{aligned}$$

These imply

$$L((a,0)(1,b))f(x) = \pi(1,ax^{-1}b)f(a^{-1}x) = e^{2\pi iax^{-1}b}f(a^{-1}x).$$

So $L(a,ab)f(x) = e^{iax^{-1}b}f(a^{-1}x)$. Replacing b by b/a gives

$$L(a,b)f(x) = e^{2\pi ix^{-1}b}f(a^{-1}x)$$

where $f \in L^2(\mathbb{R}^+, \frac{dx}{x})$. If one takes the unitary transformation W given by $Wf(x) = f(x^{-1})$ on $L^2(\mathbb{R}^+, \frac{dx}{x})$, then

$$\begin{aligned}WL(a,b)W^{-1}f(x) &= L(a,b)W^{-1}f(x^{-1})\\ &= e^{2\pi ixb}W^{-1}f(a^{-1}x^{-1})\\ &= e^{2\pi ixb}f(ax).\end{aligned} \tag{6.39}$$

If instead of the measure $\frac{dx}{x}$ on $\mathbb{R}^+$, one wants Lebesgue measure, we could take the unitary transformation $T : L^2(\mathbb{R}^+, \frac{dx}{x}) \to L^2(\mathbb{R}^+, dt)$ given by

$$Tf(t) = \frac{1}{\sqrt{t}}f(t).$$

Then $T^{-1}f(x) = \sqrt{x}f(x)$ and so

$$\begin{aligned}TWL(a,b)W^{-1}T^{-1}f(t) &= \frac{1}{\sqrt{t}}WL(a,b)W^{-1}T^{-1}f(t)\\ &= \frac{1}{\sqrt{t}}e^{2\pi ibt}T^{-1}f(at)\\ &= \frac{1}{\sqrt{t}}e^{2\pi ibt}\sqrt{at}f(at)\\ &= \sqrt{a}e^{-\pi ibt}f(at).\end{aligned}$$

Thus the representation $\widehat{\pi^+}$ defined on the $ax+b$ group on $L^2(\mathbb{R}^+, dt)$ by

$$\widehat{\pi^+}(a,b)f(t) = \sqrt{a}e^{2\pi ibt}f(at) \tag{6.40}$$

is induced from the character $(1,b) \mapsto e^{-2\pi ib}$ on the subgroup H. Exercise 6.10.9 shows it is an irreducible representation. If instead of taking the representation π of H to be $\pi(1,b) = e^{2\pi ib}$, one takes $\pi(1,b) = e^{-2\pi ib}$, then

π^G is unitarily equivalent to the unitary representation $\widehat{\pi^-}$ on $L^2(\mathbb{R}^-, dt)$ where

$$\widehat{\pi^-}(a,b)f(t) = \sqrt{a}e^{2\pi ibt}f(at). \tag{6.41}$$

This representation is also irreducible.

We leave the proof of the following proposition as an exercise.

Proposition 6.138. *Let G be a second countable locally compact Hausdorff group with a closed subgroup H. Let ρ be a continuous rho function for H and let μ be the corresponding regular quasi-invariant measure on G/H. Suppose π is a unitary representation of H on a separable Hilbert space $\mathcal{H}$. If there is a Borel mapping $\sigma : G/H \to G$ satisfying $\sigma(xH)H = xH$ for all xH, then the unitary representation π^G is unitarily equivalent to the representation L on $L^2(G/H, \mathcal{H})$ defined by*

$$L(x)f(yH) = \sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}}\pi(\sigma(yH)^{-1}x\sigma(x^{-1}yH))f(x^{-1}yH).$$

Borel mappings $\sigma : G/H \to G$ satisfying $\sigma(xH)H = xH$ for all xH are known as **Borel cross sections**. A result of Mackey (see [**32**, Lemma 1.1]) show they always exist. Each σ selects in a Borel measurable manner a member of the coset xH. We note $x\sigma(x^{-1}yH)H = yH = \sigma(yH)H$ and thus $\sigma(yH)^{-1}x\sigma(x^{-1}yH) \in H$.

Example 6.139. *The $AX + B$ group is an n dimensional analogue of the $ax + b$ group. Our intent here is to give an example directly related to the continuous wavelet transform discussed in Section 12 of Chapter 4. Let G be $(\mathbb{R}^*)^n \times \mathbb{R}^n$ with product topology and multiplication defined by*

$$(a, x)(b, y) = (ab, x + ay)$$

where ab and ay are defined coordinatewise; i.e., $(ab)_i = a_ib_i$ and $(ay)_i = a_iy_i$ for $i = 1, 2, \ldots, n$. We note $G = H \ltimes K$ where $H = (\mathbb{R}^n)^* \times \{0\}$ and $K = \{1\} \times \mathbb{R}^n$ are closed subgroups. We describe the representation $\pi = \mathrm{ind}_H^G 1$. To do this we start by constructing a quasi-invariant measure μ on $\mathbb{R}^n$ which we identify with G/H by $(a, y)H = (1, y)(a, 0)H = (1, y)H$ under the mapping $(a, y)H \leftrightarrow y$. We note one can check Haar measure on H is given by $\frac{da_1 \times da_2 \times \cdots \times da_n}{|a_1a_2\cdots a_n|}$ and being abelian, this group is unimodular. For $a = (a_1, a_2, \ldots, a_n)$, we take $\det a = \prod a_j$ and da to be Lebesgue measure. Let dy be Lebesgue measure on $\mathbb{R}^n$. Then a left Haar measure on G is given by $dg = d(a, y) = \frac{da}{|\det a|^2} \times dy$. The modular function Δ_G can be shown to be $\Delta_G(b, x) = \frac{1}{|\det b|}$ and thus $\rho(a, y) = |\det a|$ is a rho function for H for $\rho((a, y)(b, 0)) = \rho(ab, y) = |\det(ab)| = \rho(a, y)\frac{1}{\Delta_G(b,0)} = \rho(a, y)\frac{\Delta_H(b,1)}{\Delta_G(b,1)}$. Now choose a function $k \in C_c(H)$ with

$k \geqslant 0$ and $\int_H k(h)\,dh = \int_{(\mathbb{R}^*)^n} k(a,0)\frac{da}{|\det a|} = 1$ and suppose $f \in C_c(\mathbb{R}^n)$. Then the function $F(a,y) = k(a,0)f(y)$ is in $C_c(G)$ and $F_H((1,y)H) = \int k(a,0)f(y)\frac{da}{|\det a|} = f(y)$. Consequently, if μ is the left quasi-invariant measure given by ρ on G/H, we have

$$\begin{aligned}
\int f(y)\,d\mu(y) &= \int_G F(g)\rho(g)\,dg \\
&= \int_{(\mathbb{R}^*)^n}\int_{\mathbb{R}^n} k(a,0)f(y)|\det a|\frac{da}{|\det a|^2}\,dy \\
&= \int f(y)\,dy.
\end{aligned}$$

So the measure μ is Lebesgue measure. Now either using the formula (6.15) giving the quasi-regular representation or Proposition 6.138 with Borel section $\sigma : G/H \to G$ given by $\sigma((a,x)H) = (1,x)$, we obtain π is the representation of G on $L^2(\mathbb{R}^n)$ given by

$$\begin{aligned}
\pi(a,x)f(y) &= \sqrt{\frac{\rho((a,x)^{-1}(1,y))}{\rho(1,y)}}f((a,x)^{-1}(1,y)H) \\
&= |\det a|^{-1/2}f((a^{-1},-a^{-1}x)(1,y)H) \\
&= |\det a|^{-1/2}f((a^{-1},-a^{-1}x+a^{-1}y)H) \\
&= |\det a|^{-1/2}f((1,-a^{-1}x+a^{-1}y)H) \\
&= |\det a|^{-1/2}f(a^{-1}(y-x)).
\end{aligned} \tag{6.42}$$

Our next goal will be to show this unitary representation is irreducible. To do this we let $\hat{\pi}(g) = \mathcal{F}\pi(g)\mathcal{F}^{-1}$ where $\mathcal{F}$ is the Fourier transform $\mathcal{F}f(\xi) = \int f(y)e^{-2\pi i\xi\cdot y}\,dy$. We note

$$\begin{aligned}
\mathcal{F}\pi(a,x)f(\xi) &= \int \pi(a,x)f(y)e^{-2\pi i\xi\cdot y}\,dy \\
&= \int |\det a|^{-1/2}f(a^{-1}(y-x))\,e^{-2\pi i\xi\cdot y}\,dy \\
&= \int |\det a|^{-1/2}f(a^{-1}y)e^{-2\pi i\xi\cdot(y+x)}\,dy \\
&= e^{-2\pi i\xi\cdot x}\int |\det a|^{-1/2}f(y)e^{-2\pi i a\xi\cdot y}\,d(ay) \\
&= |\det a|^{1/2}e^{-2\pi i\xi\cdot x}\mathcal{F}f(a\xi).
\end{aligned}$$

This implies $\hat{\pi}$ is defined on $L^2(\mathbb{R}^n)$ by

$$\hat{\pi}(a,x)f(y) = |\det a|^{1/2}e^{-2\pi i x\cdot y}f(ay). \tag{6.43}$$

To see π is irreducible, it suffices to show $\operatorname{Hom}(\hat{\pi},\hat{\pi})$ contains no proper orthogonal projections. But since $\hat{\pi}(1,x)f(y) = e^{-2\pi i x\cdot y}f(y)$, we have seen *in the proof of Corollary 6.115 that if $P \in \mathrm{Hom}(\hat{\pi}, \hat{\pi})$ is a projection whose range is invariant under all $\pi(1, x)$, then there is a Borel subset E of $\mathbb{R}^n$ such that $Pf = \chi_E f$ for all $f \in \mathbb{R}^n$. But since $\hat{\pi}(a, 0)$ commutes with P, we see $P\hat{\pi}(a, 0)f = \hat{\pi}(a, 0)Pf$ or $\chi_E(y)f(ay) = \chi_E(ay)f(ay)$ a.e. y for each a. Now this occurs for all $f \in L^2(\mathbb{R}^n)$ if and only if $\chi_E(ay) = \chi_E(y)$ a.e. y for each $a \in (\mathbb{R}^*)^n$. This implies $a^{-1}E$ and E are the same sets in the measure algebra of Lebesgue measure. Take a probability measure λ_0 on $\mathbb{R}^n$ equivalent to Lebesgue measure. Define $H(x) = \int \chi_E(ax)\, d\lambda_0(a)$. H is a Borel function and for a.e. $x \in E$, we have $H(x) = 1$ a.e. $x \in E$ and $H(x) = 0$ a.e. $x \in E^c$. Thus $E_0 = H^{-1}(1)$ is a Borel set equal to E in the measure algebra. Since $\chi_E(ax) = 1$ a.e. a implies $\chi_E(aa'x) = 1$ a.e. a, we see H has the property, $H(ax) = H(x)$ for all $a \in (\mathbb{R}^*)^n$ and $x \in \mathbb{R}^n$. This implies $aE_0 = E_0$ are the same sets for all a. Now if E_0 has measure 0, $P = 0$ while if E_0 has positive Lebesgue measure, there is a point $x = (x_1, x_2, \ldots, x_n) \in E_0$ with all $x_j \neq 0$. Consequently if $y \in \mathbb{R}^n$ has all $y_j \neq 0$, we see $y = ax$ where $a_j = \frac{y_j}{x_j}$ for $j = 1, 2, \ldots, n$. Thus E_0 contains all such y and thus is conull in $\mathbb{R}^n$. We thus have $P = I$.*

Proposition 6.140. *The unitary representation π defined on $L^2(\mathbb{R}^n)$ by*

$$\pi(a, x)f(y) = |\det a|^{-1/2} f(a^{-1}(y - x))$$

for $a \in (\mathbb{R}^*)^n$ and $x \in \mathbb{R}^n$ is irreducible.

Exercise Set 6.10

1. Show one need not assume G is σ-compact in the Proposition 6.130. (**Hint:** Use Exercises 6.1.31, 6.1.32, and 6.1.34.)

2. Let G be a σ-compact locally compact Hausdorff space with a closed subgroup H. Suppose ρ is a continuous rho function defining a corresponding quasi-invariant measure μ on G/H. Show if π is a unitary representation of H on a separable Hilbert space $\mathcal{H}$ and ν is a regular quasi-invariant measure on G/H, then π^G is unitarily equivalent to the representation L defined by

$$L(x)f(y) = \sqrt{\frac{d(x\mu)}{d\mu}(yH)}\, f(x^{-1}y)$$

on the Hilbert space of Borel functions f on G satisfying $f(xh) = \pi(h^{-1})f(x)$ for all $h \in H$ for a.e. x and $\int ||f(y)||^2_{\mathcal{H}}\, d\nu(yH) < \infty$.

3. Suppose G is a second countable locally compact Hausdorff group with a closed subgroup H and a continuous rho function ρ. Let μ be the corresponding quasi-invariant measure on G/H. Assume there is a Borel function $\sigma : G/H \to G$ satisfying $\sigma(xH)H = xH$ for all x in G. Show if π is a unitary representation of H on a separable Hilbert space $\mathcal{H}$, then π^G is unitarily equivalent to the representation L defined on $L^2(G/H, \mathcal{H})$ by

$$L(x)f(yH) = \sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}}\pi(\sigma(yH)^{-1}x\sigma(x^{-1}yH))f(x^{-1}yH).$$

4. Let G be a second countable locally compact Hausdorff group acting continuously on a second countable locally compact Hausdorff space X with a quasi-invariant measure μ. Assume $[E]$ is in the measure algebra of μ and $[gE] = [E]$ for all g in G. Show there is a G invariant Borel subset W such that $[W] = [E]$. (**Hint:** Take a probability measure λ equivalent to Haar measure on G and consider $H(x) = \int_G \chi_E(gx)\, d\lambda(g)$.)

5. Let E be a Borel subset of $\mathbb{R}$ and suppose one has $a + E = E$ in the measure algebra given by Lebesgue measure for all $a \in \mathbb{R}$. Show E or its complement has measure 0.

6. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism of $\mathbb{R}^n$ and let λ be Lebesgue measure on $\mathbb{R}^n$. Define a Borel measure $\phi_*\lambda$ by $\phi_*\lambda(E) = \lambda(\phi^{-1}(E))$ and a representation π on $L^2(\mathbb{R}^n, \lambda)$ by

$$\pi(x)f(y) = e^{-2\pi i x\cdot\phi(y)}f(y).$$

Assume $\phi_*\lambda$ and λ are equivalent measures.

(a) Show π and $\hat{\lambda}$ are unitarily equivalent.

(b) Show $\mathrm{Hom}(\pi, \pi) = \{M_h \mid h \in L^\infty(\mathbb{R}^n)\}$.

Then determine π when ϕ is given by:

(c) $\phi(y) = -y$;

(d) $\phi(y) = \frac{y}{2\pi}$;

(e) $\phi(y) = y^3$.

7. Let π be a unitary representation of a locally compact Hausdorff group G and let π_0 be a subrepresentation on the invariant subspace $P\mathcal{H}$ where P is an orthogonal projection in $\mathrm{Hom}(\pi, \pi)$. Show $\mathrm{Hom}(\pi_0, \pi_0) = P\mathrm{Hom}(\pi, \pi)P$.

8. Let X be a Borel subset of $\mathbb{R}^n$ with positive Lebesgue measure. Let π be the unitary representation of $\mathbb{R}^n$ on $L^2(X)$ given by $\pi(x)f(y) = e^{ix\cdot y}f(y)$. Show $\mathrm{Hom}(\pi, \pi) = \{M_h \mid h \in L^\infty(X)\}$ where $M_h f = hf$.

9. Show the representation $\widehat{\pi^+}$ given in (6.40) of Example 6.137 of the $ax+b$ group is irreducible.

10. Let G be the $ax + b$ group. Let $\widehat{\pi^+}$ and $\widehat{\pi^-}$ be the unitary representation given in Equations (6.40) and (6.41). Show if in Example 6.137, one takes $\pi(1, b) = e^{i\lambda b}$ where $\lambda \in \mathbb{R}$, then π^G is unitarily equivalent to the representation $\widehat{\pi^+}$ if $\lambda > 0$ and is unitarily equivalent to $\widehat{\pi^-}$ when $\lambda < 0$. (**Hint:** Find a formula for π^G and consider Exercise 6.10.6.)

11. Let G be the $ax+b$ group and consider the quasi-regular representation L given in Equation (6.19) of Example 6.95. Show L and $\widehat{\pi^+} \oplus \widehat{\pi^-}$ are unitarily equivalent. (**Hint:** Use a Fourier transform.)

12. Show the representation $\widehat{\pi^-}$ is the conjugate representation to the representation $\widehat{\pi^+}$ of the $ax+b$ group.

13. Show if λ_1 and λ_2 are nonzero real numbers, the unitary representations π_{λ_1} and π_{λ_2} given in Example 6.136 are not equivalent.

14. Show the unitary representation π_λ of the Heisenberg group given in Example 6.136 is irreducible.

15. Let X be a locally compact Hausdorff space with a regular measure μ and let $\mathcal{H}$ be a separable Hilbert space. A function $x \mapsto T(x)$ from X into $\mathcal{B}(\mathcal{H})$ is strongly Borel if $x \mapsto T(x)v$ is Borel for each $v \in \mathcal{H}$.

(a) Show T is strongly Borel if and only if $x \mapsto (T(x)v, w)_{\mathcal{H}}$ is Borel for any $v, w \in \mathcal{H}$.

(b) Show $x \mapsto ||T(x)||$ is Borel if T is strongly Borel.

(c) If T is strongly Borel, define $||T||_\infty = \inf\{a \geqslant 0 \mid \mu\{x \mid ||T(x)|| > a\} = 0\}$. Show if $L^\infty(X, \mathcal{B}(\mathcal{H}))$ is the space of all strongly Borel functions T with $||T||_\infty < \infty$, then $L^\infty(X, \mathcal{B}(\mathcal{H}))$ with pointwise addition and multiplication, adjoint $T^*(x) = T(x)^*$, and norm $||\cdot||$ is a C^* algebra with identity.

(d) Define a representation ρ of $L^\infty(X, \mathcal{B}(\mathcal{H}))$ on $L^2(X, \mathcal{H})$ by

$$(\rho(T)f)(x) = T(x)f(x).$$

Show ρ is a representation of the C^* algebra $L^\infty(X, \mathcal{B}(\mathcal{H}))$ and the mapping $T \mapsto \rho(T)$ is an isometric $*$ algebra homomorphism onto its range.

16. Let X be a locally compact Hausdorff space with a regular Borel measure μ. Suppose $\mathcal{H}$ is a separable Hilbert space. Let π the representation of $L^\infty(X, \mu)$ on $L^2(X, \mathcal{H})$ given by $\pi(f)h = fh$. Show if $A \in \mathrm{Hom}(\pi, \pi)$, then there is an essentially bounded strongly Borel function $T : X \to \mathcal{B}(\mathcal{H})$ such that $Ah(x) = T(x)f(x)$ a.e. x for each $f \in L^2(X, \mathcal{H})$. (**Hint:** Take a countable dense $\mathbb{Q} + i\mathbb{Q}$ linear subspace V of $\mathcal{H}$ and show $A_{v,w}(x) : L^2(X) \to L^2(X)$ defined a.e. x on $V \times V$ by $A_{v,w}f(x) = (A(fv)(x), w)_{\mathcal{H}}$ extends a.e. from $V \times V$ to a bounded sesquilinear form on $\mathcal{H}$. Then apply Exercise 6.5.25.)

17. Let G be a σ-compact locally compact Hausdorff group and let H be a closed subgroup of G having a unitary representation π on a separable Hilbert space $\mathcal{H}$. Let ρ be a continuous rho function for H and let μ be the corresponding quasi-invariant measure on G/H. Let M be the representation of $L^\infty(G/H,\mu)$ on $L^2_\pi(G,\mathcal{H})$ given by

$$M(h)f = (h\circ\kappa)f \text{ for } h\in L^\infty(G/H) \text{ and } f\in L^2_\pi(G,\mathcal{H}).$$

For each $A\in \mathrm{Hom}(\pi,\pi)$, define $T_A : L^2_\pi(G,\mathcal{H}) \to L^2_\pi(G,\mathcal{H})$ by

$$T_A f(x) = Af(x).$$

Show $A\mapsto T_A$ is a $*$ algebra isomorphism of the C^* algebra $\mathrm{Hom}(\pi,\pi)$ onto the C^* algebra $\mathrm{Hom}(\pi^G,\pi^G)\cap \mathrm{Hom}(M,M)$. Consequently, show π^G is irreducible if and only if π is irreducible and $\mathrm{Hom}(\pi^G,\pi^G)\subseteq \mathrm{Hom}(M,M)$. (**Hint:** Use the argument suggested in the previous exercise to show if $B\in \mathrm{Hom}(M,M)$, then there is a an essentially bounded strongly Borel function $g\mapsto B(g)\in\mathcal{B}(\mathcal{H})$ such that $Bf(g)=B(g)f(g)$ for $f\in L^2_\pi(G,\mathcal{H})$.)

18. Let G be the group $\mathrm{SL}(2,\mathbb{R})$ of 2×2 matrices of determinant 1. Let A be the subgroup of matrices of form $a(t)=\begin{pmatrix} e^t & 0\\ 0 & e^{-t}\end{pmatrix}$ where $t\in\mathbb{R}$ and let N be the subgroup consisting of the matrices $n(x)=\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$ where $x\in\mathbb{R}$. By Corollary 5.26, the mapping $(k,a,n)\mapsto kan$ from $\mathrm{SO}(2)\times A\times N$ to G given by $(k,a,n)\mapsto kan$ is a homeomorphism. Let M be the two element subgroup consisting of the matrices I and $-I$.

(a) Show $P=MAN$ is a closed subgroup of G.

(b) Find a left Haar measure for P and find the modular function for P.

(c) Find a Haar measure for G and show G is unimodular.

(d) Find a rho function for the subgroup P.

(e) Let π be the unitary representation of P defined by $\pi(\pm Ia(t)n(x)) = \epsilon_\pm(\pm I)e^{i\lambda t}$ where $\lambda\in\mathbb{R}$ and $\epsilon_+(I)=\epsilon_+(-I)=\epsilon_-(I)=1$ and $\epsilon_-(-I)=-1$. Find $\rho_{\pm,\lambda}=\pi^G$.

It is known $\rho_{+,\lambda}$ is irreducible for all $\lambda\in\mathbb{R}$ and $\rho_{-,\lambda}$ is irreducible for nonzero $\lambda\in\mathbb{R}$. These representation are known as *the unitary principal series* of $\mathrm{SL}(2,\mathbb{R})$.

18. Square Integrable Representations and Wavelets

In Chapter 4 we considered the windowed Fourier transform and introduced continuous wavelet transforms. We mentioned their basis fits in nicely and is understood best in terms of representation theory. A central concept is the notion of a square integrable representation and indeed, the representations π_λ of the Heisenberg group and π of the general $AX+B$ group given in Proposition 6.140 fit into this category. We give the definition next and then develop the important properties these representations possess.

If G is a locally compact Hausdorff group, we recall Δ_G denotes the modular function for G and dg will be a left Haar measure. One has the following mnemonics which organize the invariance properties for the measure dg.

$$\begin{aligned} d(xg) &= dg \text{ for } x \in G, \\ d(gx) &= \Delta_G(x) dg \text{ for } x \in G, \\ dg^{-1} &= \frac{dg}{\Delta_G(g)} \text{ is right Haar measure.} \end{aligned} \tag{6.44}$$

We recall from Corollary 6.50 that if π is an irreducible unitary representation of G and Z is the center, then π has a central character χ. This is the one-dimensional unitary representation χ of Z satisfying $\pi(\xi) = \chi(\xi)I$ for $\xi \in Z$. We use the locally compact quotient group G/Z. Note since $\Delta_G(\xi) = \Delta_Z(\xi) = 1$ for $\xi \in Z$, $\rho = 1$ is a rho function for Z and left Haar measure $d(gZ)$ on G/Z satisfies

$$\int_{G/Z} f_Z(gZ)\, d(gZ) = \int_G f(g)\, dg \tag{6.45}$$

where $f_Z(gZ) = \int_Z f(g\xi)\, d\xi$ and $d\xi$ is a left Haar measure on Z. We also remark the modular function for G/Z is given by $\Delta_{G/Z}(gH) = \Delta_G(g)$ for $f_Z(gZxZ) = \int_Z f(gx\xi)\, d\xi = \int_Z f(g\xi x)\, d\xi$ and thus $\int_{G/Z} f_Z(gZxZ)\, d(gZ) = \int_G f(gx)\, dg = \Delta_G(x^{-1}) \int_G f(g)\, dg = \Delta_G(x^{-1}) \int_{G/Z} f_Z(gZ)\, d(gZ)$. In particular, G/Z is unimodular if and only if G is unimodular.

Note if π is an irreducible unitary representation on a Hilbert space $\mathcal{H}$ with central character χ, then $|(w, \pi(g\xi)v)_{\mathcal{H}}| = |\chi^{-1}(\xi)|\, |(w, \pi(g)v)_{\mathcal{H}}| = |(w, \pi(g)v)_{\mathcal{H}}|$ and thus defines a continuous function on G/Z.

Definition 6.141. *Let π be a irreducible unitary representation of a locally compact Hausdorff group G on a Hilbert space $\mathcal{H}$. We say π is square-integrable (modulo the center) if there is a nonzero vector v with*

$$\int_{G/Z} |(v, \pi(g)v)_{\mathcal{H}}|^2\, d(gZ) < \infty$$

where $d(gZ)$ is a left Haar measure on G/Z. Any vector for which this is true is said to be admissible.

We shall use the term square integrable in the sense of the above definition. Sometimes, one defines square integrable to mean the square integrability of $g \mapsto (v, \pi(g)v)_{\mathcal{H}}$ on G for some nonzero v. However, the definition we have given is broader as can be seen in Exercise 6.11.9.

Theorem 6.142. *Suppose π is a square integrable unitary representation of a locally compact Hausdorff group G on a Hilbert space $\mathcal{H}$. Let χ be the central character of π, and suppose v is a nonzero admissible vector. Then:*

(a) $\{w \in \mathcal{H} \mid gZ \mapsto (w, \pi(g)v)_{\mathcal{H}}$ *is in* $L^2_\chi(G)\} = \mathcal{H}$*;*

(b) *If* $\theta : \mathcal{H} \to L^2_\chi(G)$ *is defined by* $\theta w(g) = (w, \pi(g)v)_{\mathcal{H}}$*, then* θ *is an intertwining operator between* π *on* $\mathcal{H}$ *and* $\mathrm{ind}_Z^G(\chi)$ *on* $L^2_\chi(G)$*. Moreover,* θ *satisfies* $||\theta w||^2 = A^2||w||^2$ *for all* w *where*

$$A^2 = ||v||^{-2} \int_{G/Z} |(v, \pi(g)v)_{\mathcal{H}}|^2 \, d(gZ).$$

(c) *If* $w, w' \in \mathcal{H}$*, then*

$$(w, w')_{\mathcal{H}} = \frac{1}{A^2} \int_{G/Z} (w, \pi(g)v)_{\mathcal{H}} (\pi(g)v, w')_{\mathcal{H}} \, d(gZ).$$

Proof. Let $D = \{w \in \mathcal{H} \mid \int |(w, \pi(g)v)_{\mathcal{H}}|^2 \, d(gZ) < \infty\}$. Note D is a linear subspace containing the nonzero vector v. Moreover, if $w \in D$ and $x \in G$, then

$$\begin{aligned} \int |(\pi(x)w, \pi(g)v)_{\mathcal{H}}|^2 \, cd(gZ) &= \int |(w, \pi(x^{-1}g)v)_{\mathcal{H}}|^2 \, d(gZ) \\ &= \int |(w, \pi(g)v)_{\mathcal{H}}|^2 \, d(gZ) < \infty. \end{aligned}$$

So D is invariant under π. Since π is irreducible, D is a dense linear subspace of $\mathcal{H}$.

For $w \in D$, set $\theta w(g) = (w, \pi(g)v)_{\mathcal{H}}$. Note $\theta w(g\xi) = (w, \pi(g\xi)v)_{\mathcal{H}} = (w, \pi(g)\pi(\xi)v)_{\mathcal{H}} = \chi(\xi^{-1})(w, \pi(g)v)_{\mathcal{H}}$. This implies $\theta v \in L^2_\chi(G)$. Clearly θ is linear. We claim θ is a closed operator. Indeed, suppose $w_n \to w$ and $f_n = \theta(w_n)$ converges in $L^2_\chi(G)$ to f. By taking a subsequence, we may assume f_n converges pointwise a.e. to f. But $f_n(g) = (w_n, \pi(g)v)_{\mathcal{H}}$ converges pointwise to $(w, \pi(g)v)_{\mathcal{H}}$. Hence $f(g) = (w, \pi(g)v)_{\mathcal{H}}$ a.e. g. So $f = \theta w$ and θ is a closed operator. Furthermore,

$$\begin{aligned} \theta(\pi(x)w)(g) &= (\pi(x)w, \pi(g)v)_{\mathcal{H}} = (w, \pi(x^{-1}g)v)_{\mathcal{H}} \\ &= \theta(w)(x^{-1}g) = \mathrm{ind}_Z^G(\chi)(x)\theta(w)(g). \end{aligned}$$

Hence if $w \in D$, then $\pi(x)w \in D$ and $\theta\pi(x)w = \mathrm{ind}_Z^G(x)\theta w$. So θ intertwines π on D and $\mathrm{ind}\xi$ on $L^2_\xi(G)$. By the strong version of Schur's Lemma (Proposition 6.49), θ is bounded, $D = \mathcal{H}$, and $\theta^*\theta = A^2 I$ where $A > 0$. Since $A^2(v, v)_{\mathcal{H}} = (\theta v, \theta v)_2 = \int |(v, \pi(g)v)_{\mathcal{H}}|^2 \, d(gZ)$, we see

$$\theta^*\theta = \frac{\int |(v, \pi(g)v)_{\mathcal{H}}|^2 \, d(gZ)}{||v||^2} I.$$

Hence $\frac{||\theta v||_2^2}{||v||^2}(w, w)_{\mathcal{H}} = (\theta^*\theta w, w)_{\mathcal{H}} = (\theta w, \theta w)_2$ for all $w \in \mathcal{H}$. We thus have (a) and (b).

To see (c), note since $(\frac{1}{A}\theta w, \frac{1}{A}\theta w')_2 = \frac{1}{A^2}(\theta^*\theta w, w')_{\mathcal{H}} = (w, w')_{\mathcal{H}}$, one has

$$\frac{1}{A^2}\int (w, \pi(g)v)_{\mathcal{H}}(\pi(g)v, w')_{\mathcal{H}}\, d(gZ) = (w, w')_{\mathcal{H}}.$$

□

Corollary 6.143. *Let π be an irreducible unitary representation on a Hilbert space $\mathcal{H}$. Then the set of admissible vectors is a linear invariant subspace of $\mathcal{H}$. Thus the space of admissible vectors is either $\{0\}$ or is dense in $\mathcal{H}$.*

Proof. Let v be admissible. Then

$$\begin{aligned}\int |(\pi(x)v, \pi(g)\pi(x)v)_{\mathcal{H}}|^2\, d(gZ) &= \int |(v, \pi(x^{-1}gx)v)_{\mathcal{H}}|^2\, d(gZ) \\ &= \int \Delta_{G/Z}(x^{-1}Z)|(v, \pi(g)v)_{\mathcal{H}}|^2\, d(gZ) \\ &< \infty.\end{aligned}$$

Thus the set of admissible vectors is invariant under π. It is also clear cv is admissible if $c \in \mathbb{C}$ and v is admissible. Finally, if v_1 and v_2 are admissible, then $g \mapsto (v_2, \pi(g)v_1)_{\mathcal{H}}$ and $g \mapsto (v_1, \pi(g)v_2)_{\mathcal{H}}$ are in $L^2_\chi(G)$. Thus $g \mapsto (v_1+v_2, \pi(g)(v_1+v_2))_{\mathcal{H}}$ is in $L^2_\chi(G)$ and we see the set of admissible vectors is an invariant linear subspace of $\mathcal{H}$. If this space is nonzero, then irreducibility of π implies its closure is $\mathcal{H}$. □

We mention that Exercises 6.11.2 and 6.11.3 show if G is unimodular and π is square integrable, then all vectors are admissible.

If v is a nonzero admissible vector, then the mapping $\theta : \mathcal{H}_\pi \to L^2_\chi(G)$ given by

$$\theta w(g) = (w, \pi(g)v)_{\mathcal{H}}$$

is in $\mathrm{Hom}(\pi, \mathrm{ind}\chi)$ and satisfies

$$\theta^*\theta = A^2 I \text{ where } A^2 = ||\theta^*\theta|| = \frac{1}{||v||^2}\int_{G/Z} |(v, \pi(g)v)_{\mathcal{H}}|^2\, d(gZ).$$

So

$$A^2||w||^2 = ||\theta w||^2 = \int_{G/Z} |(w, \pi(g)v)_{\mathcal{H}}|^2\, d(gZ).$$

One has the following weak formula version for $\theta^*\theta$.

$$\begin{aligned}\theta^*\theta(w) &= \int_{G/Z} (\pi(g)w, v)_{\mathcal{H}}\, \pi(g)v\, d(gZ) \\ &= \int_{G/Z} \pi(g)v \otimes \overline{\pi(g)v}\,(w)\, d(gZ) \qquad (6.46) \\ &= A^2 w\end{aligned}$$

where by this formula one means

$$(A^2 w, w')_{\mathcal{H}} = (\theta^* \theta w, w')_{\mathcal{H}} = (\theta w, \theta w')_2 = \int_{G/Z} (w, \pi(g)v)_{\mathcal{H}} (\pi(g)v, w')_{\mathcal{H}} \, d(gZ)$$

holds for all $w, w' \in \mathcal{H}$.

Proposition 6.144. *Let $\mathcal{H}_a$ be the dense linear subspace of admissible vectors in $\mathcal{H}$ for the irreducible square integrable unitary representation π. Then there is a sesquilinear form B on $\mathcal{H}_a$ such that*

$$\int_{G/Z} (w_1, \pi(g)v_2)_{\mathcal{H}} (\pi(g)v_1, w_2)_{\mathcal{H}} \, d(gZ) = B(v_1, v_2)(w_1, w_2)_{\mathcal{H}}.$$

Proof. Let v_1 and v_2 be admissible vectors. By Theorem 6.142, we know the mappings θ_1 and θ_2 defined by $\theta_j(w) = (w, \pi(g)v_j)_{\mathcal{H}}$ are in $\mathrm{Hom}(\pi, \mathrm{ind}\chi)$. Thus $\theta_1^* \theta_2 \in \mathrm{Hom}(\pi, \pi)$ and by Schur's Lemma, there is a scalar $B(v_1, v_2)$ such that $\theta_1^* \theta_2(w) = B(v_1, v_2)w$. This implies

$$\begin{aligned} B(v_1, v_2)(w_1, w_2)_{\mathcal{H}} &= (\theta_1^* \theta_2 w_1, w_2)_{\mathcal{H}} \\ &= (\theta_2 w_1, \theta_1 w_2)_2 \\ &= \int_{G/Z} \theta_2 w_1(g) \, \overline{\theta_1 w_2(g)} \, d(gZ) \\ &= \int_{G/Z} (w_1, \pi(g)v_2)_{\mathcal{H}} \overline{(w_2, \pi(g)v_1)}_{\mathcal{H}} \, d(gZ) \\ &= \int_{G/Z} (w_1, \pi(g)v_2)(\pi(g)v_1, w_2)_{\mathcal{H}} \, d(gZ). \end{aligned}$$

The sesquilinearity of B follows from this formula. □

Remark 6.145. If G is unimodular, then Exercises 6.11.3 and 6.11.4 show $\mathcal{H}_a = \mathcal{H}$ and B is given by $B(v_1, v_2) = \frac{1}{d}(v_1, v_2)_{\mathcal{H}}$ where d is known as the *formal degree* of π. When $\mathcal{H}$ is finite dimensional, Exercise 6.11.5 shows d is the dimension of $\mathcal{H}$.

If G is not unimodular, the sesquilinear form is not continuous. It was shown in a paper by Duflo-Moore (see [**13**]) that there is a closed invertible linear operator D on a dense linear subspace of $\mathcal{H}$ having the property that a vector v is admissible if and only if v is in the domain of $D^{-1/2}$ and then

$$B(v_1, v_2) = (D^{-1/2} v_1, D^{-1/2} v_2)_{\mathcal{H}}.$$

D is called the formal degree operator.

Remark 6.146. The formula

$$\int_{G/Z} (w_1, \pi(g)v_2)_{\mathcal{H}} \overline{(w_2, \pi(g)v_1)}_{\mathcal{H}} \, d(gZ) = B(v_1, v_2)(w_1, w_2)_{\mathcal{H}}.$$

can be interpreted as a weak integral decomposition of the operator T_{v_1,v_2} where $T_{v_1,v_2} = \theta_1^*\theta_2$. Namely

$$T_{v_1,v_2} = \int_{G/Z} \pi(g)v_1 \otimes \overline{\pi(g)v_2}\, d(gZ) = B(v_1, v_2)I$$

weakly; i.e.,

$$\int_{G/Z} (\pi(g)v_1 \otimes \overline{\pi(g)v_2}\,)\, w\, d(gZ) = \int_{G/Z} (w, \pi(g)v_2)\pi(g)v_1\, d(gZ) \tag{6.47}$$
$$= B(v_1, v_2)w \text{ weakly}$$

or more precisely

$$\begin{aligned}(\theta_1^*\theta_2 w, w')_{\mathcal{H}} &= \int_{G/Z} (w, \pi(g)v_2)_{\mathcal{H}}(\pi(g)v_1, w')_{\mathcal{H}}\, d(gZ)\\ &= \int_{G/Z} ((w, \pi(g)v_2)_{\mathcal{H}}\pi(g)v_1, w')_{\mathcal{H}}\, d(gZ)\\ &= \int_{G/Z} ((\pi(g)v_1 \otimes \overline{\pi(g)v_2})(w), w')_{\mathcal{H}}\, d(gZ)\\ &= B(v_1, v_2)(w, w')_{\mathcal{H}}\\ &= (B(v_1, v_2)I(w), w')_{\mathcal{H}}\end{aligned}$$

for all $w, w' \in \mathcal{H}$.

Definition 6.147. *Let π be a square integrable irreducible unitary representation on a Hilbert space $\mathcal{H}$. A* **wavelet** *vector is any nonzero admissible vector v. The* **wavelet transform** *for this wavelet vector is the mapping $W_v : \mathcal{H} \to L^2_\chi(G)$ where $W_v w(g) = (w, \pi(g)v)_{\mathcal{H}}$.*

We thus have W_v is the linear transformation θ given in Theorem 6.142. Summarizing:

$$W_v^* W_v = B(v, v)I = \frac{|W_v v|_2^2}{||v||_{\mathcal{H}}^2} \tag{6.48}$$

$$||W_v|| = B(v, v)^{1/2} = \frac{|W_v|_2}{||v||_{\mathcal{H}}} \tag{6.49}$$

$$W_{v_1}^* W_{v_2} = B(v_1, v_2)I \text{ if } v_1 \text{ and } v_2 \text{ are wavelet vectors.} \tag{6.50}$$

Moreover, if v is a wavelet vector, the range of W_v is a closed $\mathrm{ind}_Z^G\chi$ invariant linear subspace of $L^2_\chi(G)$ consisting of continuous functions.

Definition 6.148. *Let π be a square-integrable unitary representation of G on $\mathcal{H}$. Let $f \in L^2_\chi(G)$. Then if $v \in \mathcal{H}_a$, define $\pi_\chi(f)v$ weakly by*

$$(\pi_\chi(f)v, w)_{\mathcal{H}} = \int_{G/Z} f(g)(\pi(g)v, w)_{\mathcal{H}}\, d(gZ).$$

Note $\pi_\chi(f)v$ for $v \neq 0$ exists by the Riesz Representation Theorem. Indeed, $w \mapsto \int_{G/Z} f(g)(\pi(g)v,w)_{\mathcal{H}}\, d(gZ)$ is continuous since by the Cauchy-Schwarz inequality and (6.49)

$$\begin{aligned}\left|\int_{G/Z} f(g)(\pi(g)v,w)_{\mathcal{H}} d(gZ)\right| &= |(f, W_v w)_2| \\ &\leqslant |f|_2 |W_v w|_2 \\ &\leqslant \frac{|W_v v|_2}{||v||_{\mathcal{H}}} |f|_2 ||w||_{\mathcal{H}}.\end{aligned}$$

Furthermore, $||\pi_\chi(f)v||_{\mathcal{H}} \leqslant \frac{|W_v v|_2}{||v||_{\mathcal{H}}}$. Note by Exercises 6.11.3 and 6.11.4, if G is unimodular, then $\mathcal{H}_a = \mathcal{H}$ and $|W_v v|_2 = \frac{1}{\sqrt{d}}||v||_{\mathcal{H}}$ where d is the formal degree of π. Consequently

$$||\pi_\chi(f)|| \leqslant \frac{1}{\sqrt{d}}|f|_2 \text{ when } G \text{ is unimodular.} \tag{6.51}$$

Proposition 6.149. *Let v be a wavelet vector for an irreducible square integrable unitary representation π with central character χ. Then for $f \in L^2_\chi(G)$, one has*

$$\pi_\chi(f)v = W_v^*(f).$$

Proof.

$$\begin{aligned}(W_v^*(f), w)_{\mathcal{H}} &= (f, W_v w)_2 \\ &= \int_{G/Z} f(g)\overline{(w, \pi(g)v)}_{\mathcal{H}}\, d(gZ) \\ &= \int_{G/Z} f(g)(\pi(g)v, w)_{\mathcal{H}}\, d(gZ) \\ &= (\pi_\chi(f)v, w)_{\mathcal{H}}.\end{aligned}$$

□

We restate many of the prior results in terms of wavelet notation in the following theorem.

Theorem 6.150. *Let π_1 and π_2 be square integrable irreducible unitary representations of a locally compact Hausdorff group G on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and suppose π_1 and π_2 have the same central character χ. Let v_1 and v_2 be wavelet vectors for π_1 and π_2. Then:*

(a) *If π_1 and π_2 are not unitarily equivalent, then $W_{v_1}(\mathcal{H}_1) \perp W_{v_2}(\mathcal{H}_2)$ in $L^2_\chi(G)$.*

(b) *Let $\pi_1 = \pi_2$. Then*

$$(W_{v_1}w_1, W_{v_2}w_2)_2 = B(v_2, v_1)(w_1, w_2)_{\mathcal{H}}$$

for $w_1, w_2 \in \mathcal{H}$.

(c) *The image spaces of wavelet transforms are closed subspaces of* $L^2_\chi(G)$ *consisting of continuous functions.*

(d) *If* $B(v_1, v_2) \neq 0$ *and* $\pi = \pi_1 = \pi_2$, *then*

$$\begin{aligned} w &= \frac{1}{B(v_2, v_1)} \int_{G/Z} (w, \pi(g)v_1)_{\mathcal{H}} \pi(g) v_2 \, d(gZ) \\ &= \frac{1}{B(v_1, v_2)} \int_{G/Z} (w, \pi(g)v_2)_{\mathcal{H}} \pi(g) v_1 \, d(gZ). \end{aligned}$$

Equivalently:

$$w = \frac{1}{B(v_2, v_1)} \pi(W_{v_1}(w)) v_2 = \frac{1}{B(v_1, v_2)} \pi(W_{v_2} w) v_1.$$

Proof. To see (a), let v_1 and v_2 be admissible vectors for π_1 and π_2 and suppose $\mathcal{H}_1$ is the Hilbert space for π_1 and $\mathcal{H}_2$ is the space for π_2. Let θ_j be the linear operators given by $\theta_j w_j = (w_j, \pi_j(g)v_j)_{\mathcal{H}_j}$ for $w_j \in \mathcal{H}_j$ where $j = 1$ or 2. Then $\theta_j \in \mathrm{Hom}(\pi_j, \mathrm{ind}_Z^G \chi)$ and consequently $T = \theta_2^* \theta_1 \in \mathrm{Hom}(\pi_1, \pi_2)$. By Schur's Lemma, $T = 0$ if π_1 and π_2 are inequivalent. This implies $(\theta_2^* \theta_1 v_1, v_2)_{\mathcal{H}_2} = (\theta_1 v_1, \theta_2 v_2)_2 = 0$. So

$$\int_{G/Z} (v_1, \pi_1(g)v_1)_{\mathcal{H}_1} \overline{(v_2, \pi_2(g)v_2)}_{\mathcal{H}} \, d(gZ) = 0.$$

Note (b) is just Proposition 6.144 and (c) is immediate for if θ is the mapping $W_v : \mathcal{H} \to L^2(G)$ where v is a wavelet vector, then

$$\theta w(g) = (w, \pi(g)v)_2$$

and by (b), $\frac{1}{B(v,v)^{1/2}} \theta$ is an isometry of $\mathcal{H}$ into $L^2_\chi(G)$. Finally (d) is just a restatement of Proposition 6.144; the second part being Definition 6.148. □

Remark 6.151. If v is a nonzero admissible vector for an irreducible representation π, we have seen the wavelet transform $\theta : \mathcal{H}_\pi \to L^2_\chi(G)$ defined by

$$\theta w(g) = (w, \pi(g)v) = W_v(w)(g)$$

has the property that $\frac{1}{A}\theta$ where $A = \frac{|\theta v|_2}{||v||_{\mathcal{H}}}$ is an isometry of $\mathcal{H}$ onto a closed subspace of $L^2_\chi(G)$ consisting of continuous functions. Consequently the mapping $\theta^{-1} : \theta\mathcal{H} \to \mathcal{H}$ is continuous. Thus

$$\theta w \mapsto w \mapsto (w, \pi(g)v)_{\mathcal{H}} = \theta w(g)$$

is a composition of continuous functions and thus is continuous. In conclusion, the space $\theta\mathcal{H}$ is a Hilbert space of continuous functions on G satisfying

$$f \mapsto f(g)$$

is a continuous function on $\theta\mathcal{H}$ for each $g \in G$. This makes the Hilbert space $\theta\mathcal{H}$ a *reproducing Hilbert space.*

Definition 6.152. *Let X be a locally compact Hausdorff space. Suppose $\mathcal{H}$ is a Hilbert space where each member is a continuous function from X into $\mathbb{C}$. If for $x \in X$, the evaluation mapping* ev_x *defined by*

$$\mathrm{ev}_x f = f(x)$$

is a continuous continuous on $\mathcal{H}$, then $\mathcal{H}$ is said to be a reproducing kernel Hilbert space.

We thus see if v is a wavelet vector for a square-integrable representation π, the closed subspace $W_v(\mathcal{H}_\pi)$ is a reproducing kernel Hilbert space.

We remark one can extend this definition. Let X be a locally compact Hausdorff space and let $\mathcal{W}$ be a Hilbert space. Then a Hilbert space $\mathcal{H}$ of continuous functions f from X into $\mathcal{W}$ is called a *reproducing Hilbert space* if

$$\mathrm{ev}_x : \mathcal{H} \to \mathcal{W} \text{ where } \mathrm{ev}_x(f) = f(x)$$

is a continuous linear transformation of $\mathcal{H}$ into $\mathcal{W}$ for each $x \in X$.

Theorem 6.153. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of continuous functions on a locally compact Hausdorff space X with values in a Hilbert space $\mathcal{W}$. Then there is a separately continuous function $K : X \times X \to \mathcal{B}(\mathcal{W})$ where $\mathcal{B}(\mathcal{W})$ has the weak operator topology satisfying*

(a) *$K(x, y)^* = K(y, x)$ for all $x, y \in X$;*
(b) *for each $y \in X$ and $w \in \mathcal{H}$, the function $K_y w$ defined by $K_y w(x) = K(x, y)w$ is in $\mathcal{H}$;*
(c) *$(f(y), w)_{\mathcal{W}} = (f, K_y w)_{\mathcal{H}}$ for $f \in \mathcal{H}$, $w \in \mathcal{W}$, and $y \in X$;*
(d) *the linear span of the vectors $K_y w$ where $y \in X$ and $w \in \mathcal{W}$ is dense in $\mathcal{H}$.*

Proof. The linear transformations $\mathrm{ev}_x f = f(x)$ are continuous from $\mathcal{H}$ into $\mathcal{W}$. Thus $\mathrm{ev}_x \mathrm{ev}_y^* : \mathcal{W} \to \mathcal{W}$ is a continuous linear transformation from $\mathcal{W}$ into $\mathcal{W}$ for each pair $(x, y) \in X \times X$. We call this operator $K(x, y)$. We note $K(x, y)^* = K(y, x)$. Thus

$$(K(x, y)w_1, w_2)_{\mathcal{W}} = (\mathrm{ev}_x \mathrm{ev}_y^* w_1, w_2)_{\mathcal{W}} = ((\mathrm{ev}_y)^* w_1(x), w_2)_{\mathcal{W}}$$

is continuous in x for fixed y. So $x \mapsto K(x, y)$ is continuous into the weak operator topology. Similarly, since

$$(K(x, y)w_1, w_2)_{\mathcal{W}} = (w_1, K(y, x)w_2)_{\mathcal{W}} = (w_1, \mathrm{ev}_y \mathrm{ev}_x^* w_2)_{\mathcal{W}} = (w_1, \mathrm{ev}_x^* w_2(y))_{\mathcal{W}},$$

we have continuity in y.

Now note $\mathrm{ev}_y^* w$ is a function on X whose value at x is $\mathrm{ev}_x \mathrm{ev}_y^* w$; i.e., $\mathrm{ev}_y^* w(x) = K(x, y)w$. Thus if $K_y w(x) = K(x, y)w$, we have

$$(f(y), w)_{\mathcal{W}} = (\mathrm{ev}_y f, w)_{\mathcal{W}} = (f, \mathrm{ev}_y^* w)_{\mathcal{W}} = (f, K_y w)_{\mathcal{H}}.$$

Finally, to see the linear span of the $K_y w$ is dense in $\mathcal{H}$, note if $f \in \mathcal{H}$ and $f \perp K_y w$ for all y and w, then

$$(f(y), w)_{\mathcal{W}} = (f, K_y w)_{\mathcal{H}} = 0$$

for all w for each y. So $f(y) = 0$ for all y. Thus $f = 0$. $\square$

In the case when $\mathcal{W} = \mathbb{C}$, K is a function function from $X \times X$ into $\mathbb{C}$ and $f(y) = K_y$ where $K_y(x) = K(x, y)$.

Theorem 6.154. *Let v be a wavelet vector for a square-integrable unitary representation π of G. If χ is the central character of π and $\theta = W_v$ is the wavelet transform of $\mathcal{H}$ into $L^2_\chi(G)$, then the reproducing kernel for $\theta(\mathcal{H})$ is given by*

$$K(g_1, g_2) = \frac{1}{A^2} W_v v(g_2^{-1} g_1) = \frac{1}{A^2} (\pi(g_2)v, \pi(g_1)v)_{\mathcal{H}}$$

where $A^2 = B(v, v) = \frac{|W_v v|_2^2}{||v||^2}$. In particular,

$$W_v w(g) = \int_{G/Z} \theta w(g_1) K(g_1, g)\, d(g_1 Z)$$

for $g \in G$ and each $w \in \mathcal{H}$.

Note K satisfies $K(g_1\xi_1, g_2\xi_2) = \chi(\xi_1\xi_2^{-1})K(g_1, g_2)$ for $g_1, g_2 \in G$ and $\xi_1, \xi_2 \in Z$.

Proof. From (c) of Theorem 6.142, we know

$$\pi(g)v = \frac{1}{A^2} \int_{G/Z} (\pi(g)v, \pi(g_1)v)\pi(g_1)v\, d(g_1 Z)$$

weakly in $\mathcal{H}$. So

$$\begin{aligned}
\theta w(g) &= (w, \pi(g)v)_{\mathcal{H}} \\
&= \frac{1}{A^2}\int_{G/Z}(w, (\pi(g)v, \pi(g_1)v)_{\mathcal{H}}\pi(g_1)v)_{\mathcal{H}}\, d(g_1Z) \\
&= \frac{1}{A^2}\int(\pi(g_1)v, \pi(g)v)_{\mathcal{H}}(w, \pi(g_1)v)_{\mathcal{H}}\, d(g_1Z) \\
&= \frac{1}{A^2}\int_{G/Z}(\pi(g_1)v, \pi(g)v)_{\mathcal{H}}\theta w(g_1)\, dg_1 \\
&= \int K(g, g_1)\theta w(g_1)\, dg_1 \\
&= \int \theta w(g_1)\overline{K(g_1, g)}\, dg_1 \\
&= (\theta w, K_g).
\end{aligned}$$

□

Example 6.155. The Heisenberg group and the windowed Fourier transform.

In Section 11 of Chapter 4, we looked at the windowed Fourier transform S_ψ *defined in (4.21) of that section by*

$$\mathcal{S}_\psi(f)(u,\omega) = \int f(x)\overline{\psi\,(x-u)}e^{-2\pi i\omega\cdot x}\, dx.$$

We show here this is just a wavelet transform for the representation π_λ *where* $\lambda = -2\pi$ *and* π_λ *is the irreducible unitary representation of the Heisenberg group* G *given in Section 6.136 by*

$$\pi_\lambda((x,y),t)f(w) = e^{i\lambda t}e^{.5\lambda i x\cdot y}e^{-i\lambda w\cdot y}f(w-x)$$

for $f \in L^2(\mathbb{R}^n)$*. We recall the Heisenberg group is unimodular with Haar measure* $dg = dx \times dy \times dt$*. We note that* π_λ *is irreducible; see either Exercise 6.10.14 or Theorem 7.11.*

The fact that π_λ *is square-integrable and the sesquilinear form* B *for* π_λ *are consequences of results obtained in Chapter 7. To start, note that* G/Z *can be identified with* $\mathbb{R}^n \times \mathbb{R}^n$ *under the topological group isomorphism* $((x,y),t)Z \leftrightarrow (x,y)$ *and under this mapping Haar measure becomes* $dx \times dy$*. In Chapter 7, instead of* dx *for Lebesgue measure on* $\mathbb{R}^n$*, we use* $d_nx = \frac{1}{(2\pi)^{n/2}}\, dx$*. Thus by Theorem 7.9,*

$$\begin{aligned}
\frac{(2\pi)^n}{|\lambda|^n}(f,f')_2(h',h)_2 &= \int_{G/Z}(f, \pi_\lambda(g)h)_2(\pi_\lambda(g)h', f')_2\, d(gZ) \\
&= \iint (f, \pi_\lambda((x,y),0)h)_2(\pi_\lambda((x,y),0)h', f')_2\, dx \times dy
\end{aligned} \tag{6.52}$$

for $f, f', h, h' \in L^2(\mathbb{R}^n)$*. Thus the sesquilinear form* B *in Proposition 6.144 is given by*

$$B(h,h') = \frac{(2\pi)^n}{|\lambda|^n}(h,h')_2$$

and π_λ *has formal degree* $\frac{|\lambda|^n}{(2\pi)^n}$*. Hence every vector in* $L^2(\mathbb{R}^n)$ *is admissible and for any* ϕ *and* ψ *in* $L^2(\mathbb{R}^n)$ *where* $(\phi,\psi)_2 \neq 0$*, we have*

$$f = \frac{1}{B(\phi,\psi)}\int_{G/Z} W_\psi f(g)\pi_\lambda(g)\phi\, d(gZ)$$

where this integral is understood weakly. This can be rewritten as

$$f = \frac{|\lambda|^n}{(2\pi)^n(\phi,\psi)_2}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} W_\psi(f)(u,\omega)\pi_\lambda((u,\omega),0)\phi\, du\, d\omega.$$

Let ψ *be nonzero in* $\mathcal{H} = L^2(\mathbb{R}^n)$*. The wavelet transform* W_ψ *is given by* $W_\psi f((u,\omega),0) = (f, \pi_\lambda((u,\omega),0)\psi)_2 = \int_{\mathbb{R}^n} f(x)\overline{\pi_\lambda((u,\omega),0)\psi(x)}\, dx$*. So*

$$\begin{aligned} W_\psi f((u,\omega),0) &= \int_{\mathbb{R}^n} f(x)\overline{e^{.5\lambda iu\cdot\omega}e^{-i\lambda x\cdot\omega}\psi(x-u)}\, dx \\ &= e^{-.5\lambda iu\cdot\omega}\int_{\mathbb{R}^n} f(x)e^{i\lambda x\cdot\omega}\overline{\psi(x-u)}\, dx. \end{aligned}$$

For the case when $\lambda = -2\pi$*, we obtain*

$$W_\psi f((u,\omega),0) = e^{\pi iu\cdot\omega}\int_{\mathbb{R}^n} f(x)\overline{\psi(x-u)}e^{-2\pi ix\cdot\omega}\, dx = e^{i\pi u\cdot\omega}S_\psi(f)(u,\omega)$$

where $S_\psi(f)$ *is the windowed Fourier transform of* f*. Next note, if* $\lambda = -2\pi$*, then* $B(h,h') = (h,h')_2$*.*

Writing out (d) of Theorem 6.150 we see

$$f = \frac{1}{B(\phi,\psi)}\int_{G/Z} W_\psi f(u,\omega)\pi_{-2\pi}((u,\omega),0)\phi\, d((u,\omega)Z).$$

So

$$\begin{aligned} f(x) &= \frac{1}{(\phi,\psi)_2}\iint e^{iu\cdot\omega}S_\psi(f)(u,\omega)e^{-iu\cdot\omega}e^{2\pi ix\cdot\omega}\phi(x-u)\, du\, d\omega \\ &= \frac{1}{(\phi,\psi)_2}\iint S_\psi f(u,\omega)e^{2\pi ix\cdot\omega}\phi(x-u)\, du\, d\omega \\ &= \frac{1}{(\phi,\psi)_2}\int \mathcal{F}_2 S_\psi f(u,-\omega)\psi(x-u)\, du \end{aligned}$$

where this holds weakly but can be shown to hold pointwise a.e. x*.*

This is the inversion formula given in Theorem 4.84 for the windowed Fourier transform. We also mention that the Plancherel Formula given in Theorem 4.80 is easily shown to be (6.52) with $\lambda = -2\pi$*.*

Before continuing with our examples and showing the wavelet transform of Section 12 in Chapter 4 is just a transform associated with a square-integrable representation, we show how we can fit them in to an interesting family of induced representations.

Let H be a second countable locally compact Hausdorff group with left Haar measure dh and suppose H acts continuously and linearly on $\mathbb{R}^n$. Thus $\mathbb{R}^n$ is a left H space and $x \mapsto hx$ is a linear transformation of $\mathbb{R}^n$ for each $h \in H$. To simplify the presentation, we add two assumptions:

$$\begin{aligned} hx = x \text{ for all } x &\implies h = 1 \\ hx = x \text{ for all } h &\implies x = 0. \end{aligned} \tag{6.53}$$

We set $G = H \times \mathbb{R}^n$ with product topology and define a multiplication by

$$(h, x)(h', y) = (hh', x + hy). \tag{6.54}$$

With this multiplication and topology G is a semi-direct product of the groups $\{(h, 0) \mid h \in H\}$ and $\{(1, x) \mid x \in \mathbb{R}^n\}$ and we write $G = H \ltimes \mathbb{R}^n$. As each h in H gives a linear transformation of $\mathbb{R}^n$ we define $\det h$ to be the determinant of this transformation and let h^t be its transpose. In particular one has

$$\begin{aligned} d(hx) &= |\det h|\, dx \\ (hx) \cdot y &= x \cdot (h^t y) \end{aligned} \tag{6.55}$$

where dx is Lebesgue measure on $\mathbb{R}^n$. We shall use a and b to denote elements in the group H and x and y to be elements in $\mathbb{R}^n$.

To start, we need to determine the center of G. We note $(a, x)(b, y) = (b, y)(a, x)$ for all b and y if and only if $ab = ba$ and $x + ay = y + bx$ for all b, y. Taking $b = 1$, we see $ay = y$ for all y. Consequently, $x = bx$ for all b. The simplifying conditions (6.53) were made so that we would have $x = 0$ and $a = 1$, and thus the center is trivial.

By Exercises 6.11.11 and 6.11.12, a left Haar measure for G is given by $d(h, x) = |\det h|^{-1}\, dh \times dx$ and the quasi-regular representation $\operatorname{ind}_H^G 1$ can be taken to be the representation ρ on $L^2(\mathbb{R}^n)$ given by:

$$\rho(h, x) f(y) = |\det h|^{-1/2} f(h^{-1}(y + x)) \tag{6.56}$$

If one lets $\mathcal{F}$ be the Fourier transform, then the unitary representation $\hat{\rho}$ defined by $\hat{\rho}(g) = \mathcal{F}\rho(g)\mathcal{F}^{-1}$ on $L^2(\mathbb{R}^n)$ is given by

$$\hat{\rho}(h, x)\hat{f}(y) = |\det h|^{1/2} e^{2\pi i x \cdot y} \hat{f}(h^t y). \tag{6.57}$$

To check this we note if $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$\begin{aligned}
\mathcal{F}\rho(h,x)f(\omega) &= \int \rho(h,x)f(y)e^{-2\pi i y\cdot\omega}\,dy \\
&= \int |\det h|^{-1/2} f(h^{-1}(y+x))e^{-2\pi i y\cdot\omega}\,dy \\
&= \int |\det h|^{-1/2} f(h^{-1}y)e^{-2\pi i (y-x)\cdot\omega}\,dy \\
&= e^{2\pi i x\cdot\omega}|\det h|^{-1/2}\int f(y)e^{-2\pi i hy\cdot\omega}\,d(hy) \\
&= e^{2\pi i x\cdot y}|\det h|^{1/2}\int f(y)e^{-2\pi i y\cdot h^t w}\,dy \\
&= |\det h|^{1/2}e^{2\pi i x\cdot y}\mathcal{F}f(h^t\omega).
\end{aligned}$$

By Exercise 6.11.13, we have the following.

Proposition 6.156. *If P is an orthogonal projection onto a nonzero closed $\hat{\rho}$ invariant subspace of $L^2(\mathbb{R})$, then $Pf = \chi_W f$ for a Borel subset W of $\mathbb{R}^n$ where W has positive Lebesgue measure and satisfies $h^t W = W$ for all $h \in H$. Moreover, the representation $\hat{\rho}_W$ obtained by restricting $\hat{\rho}$ to $PL^2(\mathbb{R}^n)$ is irreducible if and only if whenever U is a Borel subset of W which satisfies $h^t U = U$ in the Lebesgue measure algebra of $\mathbb{R}^n$ for all h in H, then U or its complement in W has measure 0.*

Now let S be a Borel subset of $\mathbb{R}^n$ which is H^t invariant and has positive Lebesgue measure and such that the representation $\hat{\rho}$ on $L^2_S(\mathbb{R}^n)$ is irreducible. We determine conditions for this representation to be square integrable and find this representation's wavelets.

Theorem 6.157. *Let $\hat{\pi}$ be the irreducible representation $\hat{\rho}$ restricted to the invariant subspace $L^2_S(\mathbb{R}^n)$. Then for each vector $\hat{\psi} \in L^2_S(\mathbb{R}^n)$ there is a $B(\hat{\psi},\hat{\psi}) \in [0,\infty]$ such that*

$$B(\hat{\psi},\hat{\psi}) = \int_H |\hat{\psi}(h^t y)|^2\,dh$$

for a.e. $y \in S$.

Moreover, the vector $\hat{\psi}$ is admissible if and only if $B(\hat{\psi},\hat{\psi}) < \infty$. Thus π is square-integrable if and only if there is a nonzero ψ in $L^2_S(\mathbb{R}^n)$ with $0 < B(\hat{\psi},\hat{\psi}) < \infty$. If $\hat{\phi}$ and $\hat{\psi}$ are admissible, the sesquilinear form B for $\hat{\pi}$ is given by

$$B(\hat{\phi},\hat{\psi}) = \int_H \hat{\phi}(h^t y)\overline{\hat{\psi}(h^t y)}\,dh$$

where the expression $\int_H \hat{\phi}(h^t y)\overline{\hat{\psi}(h^t y)}\,dh$ is constant in y a.e. on S.

Proof. Let $\hat{\psi}$ and $\hat{\phi}$ and $\hat{f}$ and $\hat{g}$ all be in $L^2_S(\mathbb{R}^n)$.

Set $F(h,x) = (\hat{f}, \hat{\rho}(h,x)\hat{\psi})_2$ and $G(h,x) = (\hat{g}, \hat{\rho}(h,x)\hat{\phi})_2$. Then:

$$\begin{aligned} F(h,x) = (\hat{f}, \hat{\rho}(h,x)\hat{\psi})_2 &= \int \hat{f}(y)\overline{\hat{\rho}(h,x)\hat{\psi}(y)}\,dy \\ &= \int \hat{f}(y)|\det h|^{1/2} e^{-2\pi i x\cdot y}\overline{\hat{\psi}(h^t y)}\,dy \end{aligned}$$

and similarly

$$G(h,x) = (\hat{g}, \hat{\rho}(h,x)\hat{\phi})_2 = \int \hat{g}(y)|\det h|^{1/2} e^{-2\pi i x\cdot y}\overline{\hat{\phi}(h^t y)}\,dy.$$

So if integrable, by the Plancherel Theorem we have:

$$\begin{aligned} \int F(h,x)\overline{G(h,x)}\,dx = \int |\det h| &\left(\int \hat{f}(y)e^{-2\pi i x\cdot y}\overline{\hat{\psi}(h^t y)}\,dy\right) \\ &\left(\int \overline{\hat{g}(\omega)}e^{2\pi i x\cdot\omega}\hat{\phi}(h^t\omega)\,d\omega\right)dx \\ &= |\det h| \int \hat{F}_h(x)\overline{\hat{G}_h(x)}\,dx \end{aligned} \tag{6.58}$$

where $F_h(y) = \hat{f}(y)\overline{\hat{\psi}(h^t y)}$ and $G_h(y) = \hat{g}(y)\,\overline{\hat{\phi}(h^t y)}$. Thus $F(h,x)\overline{F(h,x)}$ is integrable on G if and only if

$$\int_H |\det h| \int_{\mathbb{R}^n} \overline{\hat{F}_h(x)}\,\hat{F}_h(x)\,dx|\,|\det h|^{-1}\,dh < \infty.$$

This occurs if and only if

$$\int |\int \overline{\hat{F}_h(x)}\hat{F}_h(x)\,dx|\,dh < \infty$$

and thus by the Plancherel Theorem if and only if

$$\int\int \overline{F_h(x)}F_h(x)dx\,dh < \infty$$

which is the same as

$$\int\int |\hat{f}(y)|^2|\hat{\psi}(h^t y)|^2\,dy\,dh < \infty.$$

By Fubini, this is equivalent to

$$\int_S \int_H |\hat{f}(y)|^2 \int |\hat{\psi}(h^t y)|^2\,dh\,dy < \infty.$$

Define a function R on S by $R(y) = \int |\hat{\psi}(h^t y)|^2\,dh$. We note $R(b^t y) = \int |\hat{\psi}((bh)^t y)|^2\,dh = \int |\hat{\psi}(h^t y)|^2\,dh$ by left invariance of the Haar measure dh. Thus for $\hat{\psi}$ to be admissible, we would need $R(y)$ to be finite a.e. y on S. But for c rational and nonnegative, $S_c = \{y \in S \mid R(y) \geqslant c\}$ is H^t invariant and thus has measure 0 or has complement in S with measure 0. If λ is

Lebesgue measure on $\mathbb{R}^n$, this implies $R(y) = \inf\{c > 0 \mid \lambda(S_c) = 0\}$ a.e. y. Hence we obtain $\hat{\psi} \in L^2_S(\mathbb{R}^n)$ is admissible if and only if there is a finite nonnegative constant C^2 with

$$\int_H |\hat{\psi}(h^t y)|^2 \, dh = C^2 \text{ for a.e. } y \in S.$$

In particular, if $\hat{\phi}$ is also admissible, Equation (6.58) implies

$$\begin{aligned}\int\int F(h,x)\overline{G(h,x)} |\det h|^{-1} \, dx |\det h| \, dh &= \int_H \int \hat{F}_h(x)\overline{\hat{G}_h(x)} \, dx \, dh \\ &= \int_H \int_S \hat{f}(y)\overline{\hat{\psi}(h^t y)}\overline{\hat{g}(y)}\hat{\phi}(h^t y) \, dy \, dh \\ &= \int_S \hat{f}(y)\overline{\hat{g}(y)} \int_H \overline{\hat{\psi}(h^t y)}\hat{\phi}(h^t y) \, dh \, dy \\ &= B(\hat{\phi}, \hat{\psi}) \int_S \hat{f}(y) \, \overline{\hat{g}(y)} \, dy\end{aligned}$$

where once again $y \mapsto \int_H \overline{\hat{\psi}(h^t y)}\hat{\phi}(h^t y) \, dh$ is a constant $B(\hat{\phi}, \hat{\psi})$ a.e. y in S. □

Corollary 6.158. *Let $S = H^t y$ have positive Lebesgue measure. Then the irreducible representation π obtained by restricting ρ to $F^{-1}L^2_S(\mathbb{R}^n)$ is square integrable if and only if the stabilizer $H_y = \{h \in H \mid h^t y = y\}$ is a compact subgroup of H. Furthermore, when H_y is compact and for each x, $a(x)H_y$ is the coset where $(a(x)^{-1})^t y = x$, then a nonzero function ϕ is a wavelet function for π if and only if*

$$\int_H |\hat{\phi}(h^t y)|^2 \, dh < \infty$$

which occurs if and only if

$$\int_S |\hat{\phi}(x)|^2 \, \frac{|\det(a(x))|}{\Delta(a(x))} \, dx < \infty.$$

Moreover, with an appropriate normalization of the left Haar measure dh, the sesquilinear form B on the vector space of admissible vectors is given by

$$B(\phi, \psi) = \int_H \hat{\phi}(h^t y)\overline{\hat{\psi}(h^t y)} \, dh = \int_S \hat{\phi}(x)\overline{\hat{\psi}(x)} \, \frac{|\det(a(x))|}{\Delta(a(x))} \, dx.$$

Finally, one has the inversion formula

$$f = \frac{1}{B(\phi, \psi)} \iint\limits_{H \times \mathbb{R}^n} |\det h|^{-1} W_\psi(f)(h,x)\pi(h,x)\phi \, (dh \times dx)$$

weakly if $B(\phi, \psi) \neq 0$.

Proof. Let ρ be a continuous rho function for the subgroup H_y of H and let Δ be the modular function for H and Δ_y be the modular function for H_y. Thus $\rho(a\xi) = \rho(a)\frac{\Delta_y(\xi)}{\Delta(\xi)}$ for $a \in H$ and $\xi \in H_y$ and by Theorem 6.15

$$\int_H |\hat{\phi}(h^t y)|^2 \rho(h)\, dh = \int_{H/H_y} \int_{H_y} |\hat{\phi}((h\xi)^t y)|^2 \, d\xi \, d(hH_y).$$

Thus

$$\begin{aligned}
\int_H |\hat{\phi}(h^t y)|^2 \, dh &= \int \Delta(h) |\hat{\phi}(h^t y)|^2 (\Delta(h^{-1})\, dh) \\
&= \int \Delta(h^{-1}) |\hat{\phi}((h^{-1})^t y)|^2 \, dh \\
&= \int \rho(h) \left(\rho(h)^{-1} \Delta(h^{-1}) |\hat{\phi}((h^{-1})^t y)|^2\right) dh \\
&= \int_{H/H_y} \int_{H_y} \rho(h\xi)^{-1} \Delta(\xi^{-1}h^{-1}) |\hat{\phi}((h^{-1})^t (\xi^{-1})^t y)|^2 \, d\xi \, d(hH_y) \\
&= \int_{H/H_y} \int_{H_y} \rho(h\xi)^{-1} \Delta(\xi^{-1}h^{-1}) |\hat{\phi}((h^{-1})^t y)|^2 \, d\xi \, d(hH_y) \\
&= \int_{H/H_y} |\hat{\phi}((h^{-1})^t y|^2 \int_{H_y} \rho(h\xi)^{-1} \Delta(h\xi)^{-1} \, d\xi \, d(hH_y) \\
&= \int_{H/H_y} |\hat{\phi}((h^{-1})^t y|^2 F(hH_y) \, d(hH_y)
\end{aligned}$$

where $F(hH_y) = \int_{H_y} (\rho\Delta)(h\xi)^{-1} \, d\xi$. Note

$$\begin{aligned}
F(hH_y) &= \int_{H_y} \rho(h)^{-1} \Delta(\xi) \Delta_y(\xi)^{-1} \Delta(h) \Delta(\xi^{-1}) \, d\xi \\
&= \rho(h)^{-1} \Delta(h) \int_{H_y} \Delta_y(\xi^{-1}) \, d\xi.
\end{aligned}$$

So $F(hH_y) = \frac{\Delta(h)}{\rho(h)} m_y(H_y)$. Thus if $m_y(H_y) = \infty$, π is not square integrable while if $m_y(H_y)$ is finite, then by Exercise 6.1.13, H_y is compact. This implies $\Delta_y = 1$ and $\Delta|_{H_y} = 1$. Thus we could have taken $\rho = 1$ and by Theorem 6.15, we see the measure μ_y on H/H_y is invariant under the action of H. We may now assume $m_y(H_y) = 1$. So $F(hH_y) = \Delta(h)^{-1}$ and thus $\int_H |\hat{\phi}(h^t y)|^2 \, dh = \int_{H/H_y} |\hat{\phi}((h^{-1})^t y|^2 \Delta(h)^{-1} \, d(hH_y)$.

Now since $((bh)^{-1})^t y = (b^{-1})^t (h^{-1})^t y$, the mapping from $H/H_y \to S$ given by $hH_y \mapsto (h^{-1})^t y$ is an equivariant continuous one-to-one mapping onto $S = H^t y$. Since both the measure $|\det h|^{-1} d(hH_y)$ and Lebesgue measure dx have the same relative invariance; namely

$$\int f((b^{-1})^t x) \, dx = |\det(b)| \int f(x) \, dx$$

and

$$\begin{aligned}\int f(bhH_y)|\det h|^{-1}\,d(hH_y) &= \int f(hH_y)|\det(b^{-1}h)|^{-1}d(hH_y)\\ &= |\det b|\int f(hH_y)|\det h|^{-1}\,d(hH_y),\end{aligned}$$

it follows by Corollary 6.18 that there is a constant c so that under this correspondence $c|\det h|^{-1}d\mu_y(hH_y) = dx$ on S. Adjusting the left Haar measure dh by multiplying it by $\frac{1}{c}$, we may take $c = 1$. So $\hat{\phi}$ is a wavelet vector if and only if

$$\begin{aligned}\int_H |\hat{\phi}(h^t y)|^2\,dh &= \int_{H/H_y} |\hat{\phi}((h^{-1})^t y)|^2\Delta(h)^{-1}\,d(hH_y)\\ &= \int_{H/H_y} |\hat{\phi}((h^{-1})^t y)|^2\Delta(h)^{-1}|\det h||\det h|^{-1}\,d(hH_y)\\ &= \int |\hat{\phi}(x)|^2\frac{|\det(a(x))|}{\Delta(a(x))}\,dx < \infty\end{aligned}$$

where for each x, $a(x)H_y$ is the element in H/H_y where $(a(x)^{-1})^t y = x$. Since S has positive Lebesgue measure, we can find a nonzero $\hat{\phi}$ which vanishes off a compact subset in S. So we see π has a wavelet vector. Moreover, by doing the same integration for the formula giving B in Theorem 6.157, one sees

$$B(\hat{\phi},\hat{\psi}) = \int_H \hat{\phi}(h^t y)\overline{\hat{\psi}(h^t y)}\,dh = \int \hat{\phi}(x)\overline{\hat{\psi}(x)}\frac{|\det(a(x))|}{\Delta(a(x))}\,dx.$$

□

Remark 6.159. We note for the group $G = H \ltimes \mathbb{R}^n$, the mapping $\alpha : G \mapsto G$ given by $\alpha(a,x) = (a,-x)$ is a topological group isomorphism. Thus if ρ is the unitary representation of G given by (6.56), then $\rho\circ\alpha$ is the representation satisfying

$$\rho\circ\alpha(h,x)f(y) = |\det h|^{-1/2}f(h^{-1}(y-x)) \tag{6.59}$$

for $f \in L^2(\mathbb{R}^n)$ is a unitary representation of G. Moreover ρ and $\rho\circ\alpha$ have the same invariant subspaces. Also, since α preserves Haar measure, we see closed irreducible subspaces have the same admissible vectors and if ϕ and ψ are admissible, then

$$\begin{aligned}B_{\rho\circ\alpha}(\phi,\psi)(f_1,f_2)_2 &= \iint (f_1,\rho(\alpha(g))\psi)_2(\rho(\alpha(g))\phi,f_2)_2\,dg\\ &= \iint (f_1,\rho(g)\psi)_2(\rho(g)\phi,f_2)_2\,dg\\ &= B_\rho(\phi,\psi)\,(f_1,f_2)_2\end{aligned}$$

and thus $B_{\rho\circ\alpha} = B_\rho$. Consequently, Proposition 6.156, Theorem 6.157, and Corollary 6.158 hold as stated with $\rho \circ \alpha$ replacing ρ and $\pi \circ \alpha$ replacing π. In particular, we have

$$\begin{aligned} f(y) &= \frac{1}{B(\phi,\psi)} \iint\limits_{H\times\mathbb{R}^n} |\det h|^{-1} W_\psi(f)(h,x)(\rho\circ\alpha)(h,x)\phi(y)(dh\times dx) \\ &= \frac{1}{B(\phi,\psi)} \iint\limits_{H\times\mathbb{R}^n} |\det h|^{-1} W_\psi(f)(h,x)|\det h|^{1/2}\phi(h^t(y-x))\,dh\times dx \\ &= \frac{1}{B(\phi,\psi)} \iint\limits_{H\times\mathbb{R}^n} W_\psi(f)(h,x)|\det h|\phi(h^t(y-x))\,dh\times dx \end{aligned} \tag{6.60}$$

where

$$\begin{aligned} W_\psi(f)(h,x) &= (f, \rho\circ\alpha(h,x)\psi)_2 \\ &= |\det h|^{1/2}\int_{\mathbb{R}^n} f(y)\overline{\psi(h^t(y-x))}\,dx. \end{aligned}$$

We note the representation $\rho\circ\alpha$ is the (left) quasi-regular representation of G on $L^2(G/H)$ and ρ is the (right) quasi-regular representation of G on $L^2(H\backslash G)$.

Example 6.160. *In Example 6.137 we constructed the irreducible representations $\widehat{\pi^+}$ and $\widehat{\pi^-}$ of the $ax+b$ group. This example fits well with the results just established for if we take $H = \mathbb{R}^+$ under multiplication and define for each $a > 0$ the linear transformation $x \mapsto ax$, then the group $H \ltimes \mathbb{R}$ is the $ax+b$ group. Moreover, there are three orbits under the action $a\cdot x = (a^{-1})^t x$; namely, $\mathbb{R}^+$, $\mathbb{R}^-$, and $\{0\}$. In the first two cases we have stabilizers $H_1 = H_{-1} = \{1\}$ and in the third, we have stabilizer H. Thus by Proposition 6.156 and Theorem 6.157 and its corollary, we know the representations $\hat{\rho}_\pm$ obtained by restricting $\hat{\rho}$ to $L^2(\mathbb{R}^+)$ and $\hat{\rho}$ to $L^2(\mathbb{R}^-)$ are irreducible and square integrable.*

But by the formulas (6.40) and (6.41) we see these are just the representations $\widehat{\pi^+}$ and $\widehat{\pi^-}$ obtained in Example 6.137 by inducing; i.e., $\hat{\rho}_+ = \widehat{\pi^+} = \mathrm{ind}_{\{1\}\times\mathbb{R}}^G\chi_+$ and $\widehat{\rho_-} = \hat{\pi}^- = \mathrm{ind}_{\{1\}\times\mathbb{R}}^G\chi_-$ where $\chi_\pm$ are the one dimensional characters of $\{1\}\times\mathbb{R}$ given by $\chi_\pm(1,b) = e^{\pm 2\pi i b}$.

Let H_2^+ and H_2^- be the classical Hardy spaces of functions $f \in L^2(\mathbb{R})$ where $\hat{f}|_{\mathbb{R}^-} = 0$ and $\hat{f}|_{\mathbb{R}^+} = 0$.

Theorem 6.161. *The subspaces H_+^2 and H_-^2 are invariant and irreducible and the representations π^+ and π^- obtained by restricting ρ to these subspaces are square integrable. Moreover, a function $\phi \in H_2^\pm$ is a wavelet function for $\pi^\pm$ if and only if $\int_{\mathbb{R}^\pm} \frac{|\hat{\phi}(x)|^2}{|x|}dx < \infty$ and the sesquilinear form*

B between two wavelets ϕ and ψ for $\pi^{\pm}$ is given by

$$B_{\pm}(\phi,\psi) = \int_{\mathbb{R}^{\pm}} \hat{\phi}(x)\overline{\hat{\psi}(x)}\,\frac{dx}{|x|}.$$

The wavelet transform $W_\psi f(a,b) = (f, \rho(a,b)\psi)_2 = (\hat{f}, \hat{\rho}(a,b)\hat{\psi})_2$ is given by

$$W_\psi f(a,b) = \sqrt{a}\int_{\mathbb{R}^{\pm}} \hat{f}(x)e^{-2\pi ibx}\overline{\hat{\psi}(ax)}\,dx$$

and one has

$$f = \frac{1}{B_{\pm}(\phi,\psi)}\int_{-\infty}^{\infty}\int_0^{\infty} W_\psi f(a,b)\rho(a,b)\phi\frac{da}{a^2}\,db$$

weakly if $B_{\pm}(\phi,\psi) \neq 0$.

Proof. These results follow from Proposition 6.156, Theorem 6.157 and Corollary 6.158 for $\hat{\rho}(g) = F\rho(g)F^{-1}$ and $FH^2_+ = L^2(\mathbb{R}+)$ and $FH^2_- = L^2(\mathbb{R}^-)$. Thus $\hat{\rho}_+ = F\pi^+F^{-1}$ and $\hat{\rho}_- = F\pi^-F^{-1}$. Since $\hat{\rho}_{\pm}$ are square-integrable, so are π^+ and π^-. Moreover, all conditions for admissibility, etc. for π^+ and π^- follow from the corresponding conditions on the representations $\hat{\rho}_+$ and $\hat{\rho}_-$. Thus by Corollary 6.158, using left $\frac{da}{a}$ as the left Haar measure on $H = \mathbb{R}^+ \times \{0\}$, we see ϕ is a wavelet vector for π^+ if and only if $\int_0^\infty |\hat{\phi}(a^t 1)|^2\,\frac{da}{a} < \infty$. But this is equivalent to $\int_{\mathbb{R}^+} |\hat{\phi}(x)|^2\frac{dx}{|x|} < \infty$. Furthermore, we know if ϕ and ψ are wavelet functions, then

$$B_+(\phi,\psi) = B_{\hat{\rho}_+}(\hat{\phi},\hat{\psi}) = \int_{\mathbb{R}^+} \hat{\phi}(a^t 1)\overline{\hat{\psi}(a^t 1)}\,\frac{da}{a} = \int_{\mathbb{R}^+} \hat{\phi}(x)\overline{\hat{\psi}(x)}\,\frac{dx}{x}$$

and one has the weak inversion formula

$$f = \frac{1}{B_+(\phi,\psi)}\iint\limits_{H\times\mathbb{R}} |\det a|^{-1}W_\psi f(a,x)\pi^+(a,x)\phi\,d(a,x)$$

where $d(a,x) = \frac{da}{a} \times dx$ is a left Haar measure on G. Thus

$$f = \frac{1}{B_{\pm}(\phi,\psi)}\int_{-\infty}^{\infty}\int_0^{\infty} W_\psi f(a,b)\rho(a,b)\phi\frac{da}{a^2}\,db$$

weakly if $B_+(\phi,\psi) \neq 0$. The arguments for π^- are verbatim copies. $\square$

Example 6.162. *The $AX + B$ group G equals $H \ltimes \mathbb{R}^n$ where $H = (\mathbb{R}^*)^n$ acts linearly on $\mathbb{R}^n$ by $(ax)_j = a_j x_j$. So Corollary 6.158 applies. But under the action $a \cdot x = (a^t)^{-1}x = a^{-1}x$, there is only one orbit with positive Lebesgue measure. Namely, $H \cdot y$ where $y = (1, 1, \cdots, 1)$. In this case, $H \cdot y$ is the open set S consisting of all points $x \in \mathbb{R}^n$ where all the coordinates of x are nonzero and the stabilizer H_y of y is trivial and thus by Remark 6.159, the representations $\pi = \rho \circ \alpha$ and $\hat{\pi} = \hat{\rho} \circ \alpha$ restricted to $F^{-1}L^2_S(\mathbb{R}^n)$ and*

$L^2_S(\mathbb{R}^n)$ are irreducible and square integrable and a function $\hat{\phi}$ in $L^2_S(\mathbb{R}^n)$ is admissible for $\hat{\pi}$ if and only if ϕ is admissible for π if and only if

$$\int_H |\hat{\phi}(h^t y)|^2\, dh = \int_{\mathbb{R}^n} |\hat{\phi}(x_1, x_2, \ldots, x_n)|^2 \frac{dx}{|x_1 x_2 \cdots x_n|} < \infty.$$

Moreover, by (6.59), the representation π is the representation given by (6.42) in Example 6.139 and the condition for a function ψ to be a wavelet for π is the same as that described in Section 12 of Chapter 4. Furthermore, the formula (4.28) from Chapter 4 for a wavelet transform is obtained from Definition 6.147. Indeed,

$$\begin{aligned} W_\psi f(a,b) &= (f, \pi(a,b)\psi)_2 = (\hat{f}, \hat{\pi}(a,b)\hat{\psi})_2 \\ &= (\hat{f}, \hat{\rho} \circ \alpha(a,b)\hat{\psi})_2 \\ &= |\det a|^{1/2} \int \hat{f}(\omega)\overline{e^{-2\pi i\omega\cdot b}\hat{\psi}(a\omega)}\, d\omega \\ &= |\det a|^{1/2} \int \hat{f}(\omega) e^{2\pi i \omega\cdot b}\overline{\hat{\psi}(M(a)\omega)}\, d\omega. \end{aligned}$$

Moreover, since

$$B_{\rho\circ\alpha}(\phi, \psi) = \int_{(\mathbb{R}^*)^n} \hat{\phi}(x)\overline{\hat{\psi}(x)}\, \frac{dx}{|x_1 x_2 \cdots x_n|}$$

we see the Plancherel Theorem 4.90 of Chapter 4 is just Proposition 6.144 and Theorem 6.142 and since $\phi_{a,b}(x) = |\det a|^{-1/2}\psi(M(a)^{-1}(x-b)) = \pi(a,b)\psi(x)$, the inversion formula given in Theorem 4.91 is just the inversion formula (6.60).

There are many other examples where one can apply Corollary 6.158. We mention two here, leaving the details of the first to an exercise.

First let $H = \mathbb{R}^+ \times \mathrm{SO}(n)$ be the product group with product topology and define $(a, A)y = aAy$. Then the orbit of any nonzero point y under the action $(a, A)\cdot y = ((a, A)^{-1})^t y = a^{-1}Ay$ is $\mathbb{R}^n\backslash\{0\}$ and the stabilizer H_y of the point $y = (1, 0, \cdots, 0)$ is $\{1\}\times\mathrm{SO}(n-1)$ where $\mathrm{SO}(n-1)$ is the subgroup of $\mathrm{SO}(n)$ consisting of the A with $Ay = y$ and is isomorphic to the group of orthogonal $(n-1)\times(n-1)$ matrices with determinant 1. Exercise 6. 11.14 establishes the following theorem.

Theorem 6.163. *Let $S = \{x \in \mathbb{R}^n \mid x \neq 0\}$ and $H = \mathbb{R}^+ \times \mathrm{SO}(n)$. If π is the unitary representation of $H \ltimes \mathbb{R}^n$ given by*

$$\pi(a, A, b)f(x) = a^{-n/2} f(a^{-1}A^{-1}(x-b))$$

for $f \in L^2(\mathbb{R}^n)$, then π is square integrable. Moreover, if Haar measure dA is chosen on the compact group $\mathrm{SO}(n)$ with $\int_{\mathrm{SO}(n)} 1\, dA = \sigma(S^{n-1})$, the *surface area of the unit sphere in* $\mathbb{R}^n$*, then* ψ *is a wavelet function for* π *if and only if*

$$\iint_S |\hat{\psi}(x)|^2 \frac{1}{|x|^n}\, dx < \infty.$$

Furthermore, if ϕ *and* ψ *are wavelet functions and*

$$W_\psi f(a, A, b) = a^{n/2} \iint_S \hat{f}(\omega) e^{2\pi i b\cdot\omega} \overline{\hat{\psi}(aA^{-1}\omega)}\, d\omega$$

is the wavelet transform and

$$B(\phi, \psi) = \iint_S \hat{\phi}(x)\overline{\hat{\psi}(x)} \frac{dx}{|x|^n} \neq 0,$$

then

$$f(x) = \frac{1}{B(\phi,\psi)} \int_{\mathbb{R}^n} \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^+} W_\psi f(a, A, b) a^{-n/2} \phi(a^{-1}A^{-1}(x - b))\, \frac{da}{a^{n+1}}\, dA\, db$$

weakly in $L^2(\mathbb{R}^n)$.

As our final example, we let $H = \mathbb{R}^+ \times \mathrm{SO}(1, n-1)$ act on $\mathbb{R}^n$ by $(a, A)x = aAx$ and take $y_0 = e_1 = (1, 0, 0, \ldots, 0)$. Under the action

$$(a, A) \cdot x = (a^{-1}A^{-1})^t x,$$

the stabilizer H_0 of y_0 is $\{1\} \times \mathrm{SO}(n-1)$ and thus is compact. Moreover, the orbit S of y_0 under this action is $S = \{(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x > 0,\, x^2 - |y|^2 > 0\}$. This implies Corollary 6.158 applies to the representation π of $G = H \ltimes \mathbb{R}^n$ given by

$$\pi(a, A, b)f(x) = a^{-n/2} f(a^{-1}A^{-1}(x - b))$$

for $f \in \mathcal{F}^{-1}(L^2_S(\mathbb{R}^n))$. Thus we know π is square integrable and a function ψ in $L^2(\mathbb{R}^n)$ is a wavelet if and only if $\int |\hat{\psi}(aA^t y_0)|^2\, (\frac{da}{a} \times dA) < \infty$. Now let (x, y) be in the orbit S. We wish to find an $a(x, y) \in H$ with $a(x, y) \cdot y_0 = (x, y)$. First take $p \in \mathbb{R}^+$ with $p^{-1} = \sqrt{x^2 - |y|^2}$. Then $p(x, y) = (x_1, y_1)$ where $x_1^2 - |y_1|^2 = 1$. Next choose $T \in \mathrm{SO}(n-1)$ so that $T^{-1}y_1 = (0, |y_1|, 0, 0, \cdots, 0)$, and then pick $t \in \mathbb{R}$ with $(\cosh(-t)x_1 - \sinh(-t)\,|y_1|, -\sinh(-t)x_1 + \cosh(-t)|y_1|) = (1, 0)$. Then if we let $a(x, y) = (p, \tilde{T}\tilde{A}(t))$ where $\tilde{T}(x, y) = (x, Ty)$ and $\tilde{A}(t)(x_1, x_2, z) = (\cosh t\, x_1 + \sinh t\, x_2, \sinh t x_1 + \cosh t\, x_2, z)$ for $x_1, x_2 \in \mathbb{R}$ and $z \in \mathbb{R}^{n-2}$, then $a(x, y) \cdot y_0 = (x, y)$. We note $\det(a(x, y)) = p^n = \frac{1}{(x^2 - |y|^2)^{n/2}}$. Thus with an appropriate normalization of Haar measure dA on $\mathrm{SO}(1, n-1)$, one has ψ is a wavelet function if and only if $\iint_S |\hat{\psi}(x, y)|^2 \frac{1}{(x^2 - |y|^2)^{n/2}}\, dx\, dy < \infty$. Concluding, we have:

Theorem 6.164. *Let $H = \mathbb{R}^+ \times \mathrm{SO}(1, n-1)$ and let $S = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^{n-1} \mid x^2 - |y|^2 > 0\}$. Let π be the unitary representation of $H \ltimes \mathbb{R}^n$ given by*

$$\pi(a, A, b)f(x) = a^{-n/2} f(a^{-1}A^{-1}(x - b))$$

for $f \in \mathcal{F}^{-1}L^2_S(\mathbb{R}^n)$. Then π is square integrable and a function ψ is a wavelet function for π if and only if

$$\iint_S |\hat{\psi}(x, y)|^2 \frac{1}{(x^2 - |y|^2)^{n/2}}\, dx\, dy < \infty.$$

Furthermore, if ϕ and ψ are wavelet functions and

$$W_\psi f(a, A, b) = a^{n/2} \iint_S \hat{f}(\omega) e^{2\pi i b\cdot\omega} \overline{\hat{\psi}(aA^t\omega)}\, d\omega$$

is the wavelet transform and

$$B(\phi, \psi) = \iint_S \hat{\phi}(x, y)\overline{\hat{\psi}(x, y)} \frac{dx\, dy}{(x^2 - |y|^2)^{n/2}} \neq 0,$$

then with a proper choice dA for Haar measure on $\mathrm{SO}(1, n-1)$,

$$f(x) = \frac{1}{B(\phi, \psi)} \int_{\mathbb{R}^n} \int_{\mathrm{SO}(1,n)} \int_{\mathbb{R}^+} W_\psi f(a, A, b) a^{-n/2} \phi(a^{-1}A^{-1}(x - b)) \frac{da}{a^{n+1}}\, dA\, db$$

weakly for all f in $\mathcal{F}^{-1}L^2_S(\mathbb{R}^n)$.

Exercise Set 6.11

1. Define admissibility with respect to a right Haar measure and show if a vector is admissible with respect to a left Haar measure, then it is admissible with respect to every right Haar measure.

2. Let π be an irreducible unitary representation of a locally compact Hausdorff space. Show a nonzero vector v is admissible if and only if there is a nonzero vector w such that

$$\int_{G/Z} |(w, \pi(g)v)_{\mathcal{H}}|^2\, d(gZ) < \infty.$$

3. Let G be a unimodular group and let π be an irreducible unitary representation of G. Then the linear subspace of admissible vectors is either $\{0\}$ or $\mathcal{H}$, the Hilbert space for π.

4. Show if G is unimodular and the irreducible unitary representation π is square integrable, then there is a $d > 0$ such that

$$\frac{1}{d}(v_1, v_2)_{\mathcal{H}}(w_1, w_2)_{\mathcal{H}} = \int_{G/Z} (w_1, \pi(g)v_2)_{\mathcal{H}}(\pi(g)v_1, w_2)_{\mathcal{H}}\, d(gZ)$$

for all $v_1, v_2, w_1, w_2 \in \mathcal{H}$, the Hilbert space for π. The positive constant d is called the formal degree of the representation π.

5. Let π be a finite dimensional unitary square integrable representation of a unimodular group G. Show the formal degree of π is its dimension. (**Hint:** Define T by $Tw = \int_{G/Z}(w, \pi(g)v)_{\mathcal{H}}\pi(g)v\, d(gZ)$. Note $T = \theta^*\theta = \int_{G/Z}(\pi(g)v \otimes \overline{\pi(g)v})\, d(gZ)$ and use this to calculate the trace.)

6. Show the modular function Δ for G is integrable if and only if G is compact.

7. Let π be an irreducible square integrable unitary representation. Recall the formal degree operator is a closed invertible positive operator D on $\mathcal{H}_a$ onto $\mathcal{H}_a$ such that

$$(D^{-1}v_1, v_2)_{\mathcal{H}}(w_1, w_2)_{\mathcal{H}} = \int_{G/Z}(w_1, \pi(g)v_2)_{\mathcal{H}}(\pi(g)v_1, w_2)_{\mathcal{H}}\, d(gZ)$$

for all $v_1, v_2 \in \mathcal{H}_a$ and $w_1, w_2 \in \mathcal{H}$. Show $D\pi(g)v = \Delta(g)\pi(g)Dv$ for $v \in \mathcal{H}_a$.

8. Let $f \in L^1_\chi(G)$ and π be a unitary representation of G with central character χ. Show $\pi_\chi(f)w$ exists weakly for all $w \in \mathcal{H}_\pi$, $\pi_\chi(f)$ is a linear operator, and

$$||\pi_\chi(f)|| \leqslant |f|_1 = \int |f(g)|\, d(gZ).$$

9. Let π be an irreducible unitary representation of G. Suppose there is a nonzero vector v such that $g \mapsto (v, \pi(g)v)_{\mathcal{H}}$ is in $L^2(G)$. Show the center Z of G is compact. Now suppose Z is compact. Show

(a) If v is an admissible vector for π, then $g \mapsto (v, \pi(g)v)_{\mathcal{H}}$ is in $L^2(G)$.

(b) Let χ be a one-dimensional character of Z. Show if $d\xi$ is a Haar measure on Z of total mass 1 and Haar measure on G/Z is given as in (6.44), then the inclusion mapping $L^2_\chi(G) \to L^2(G)$ is an isometry.

(c) Show if $\chi_1 \neq \chi_2$, then $L^2_{\chi_1}(G)$ and $L^2_{\chi_2}(G)$ are orthogonal in $L^2(G)$.

(d) Show if v is a wavelet vector for π, then the properties for W_v given in Theorem 6.150 hold for $L^2(G)$; thus all integrals are over G instead of G/Z and use dg instead of $d(gZ)$.

10. Suppose π is a unitary representation of a group G on a Hilbert space $\mathcal{H}$ and there is a one-dimensional character χ of the center Z of G such that $\pi(\xi) = \chi(\xi)I$ for $\xi \in Z$. Show if there is a dense linear subspace $\mathcal{H}_a$ of $\mathcal{H}$ and a sesquilinear form B on $\mathcal{H}_a$ such that

$$\int (w_1, \pi(g)v_1)_{\mathcal{H}}(\pi(g)v_2, w_2)_{\mathcal{H}}\, d(gZ) = B(v_2, v_1)(w_1, w_2)_{\mathcal{H}}$$

for $v_1, v_2 \in \mathcal{H}_a$ and $w_1, w_2 \in \mathcal{H}$, then π is irreducible.

11. Let G be the group $H \ltimes \mathbb{R}^n$ with multiplication defined in (6.54). Show if dh is a left Haar measure for H and dx is Lebesgue measure, then $d(h \times x) = |\det h|\,(dh \times dx)$ is a left Haar measure for G. Then show if Δ is the modular function for H, then the modular function Δ_G for G is given by $\Delta_G(a, y) = \frac{|\det a|}{\Delta(a)}$.

12. Let G be the group $H \ltimes \mathbb{R}^n$ with multiplication defined by (6.54). Show the *right* quasi-regular representation $\mathrm{ind}_H^G 1$ is unitarily equivalent to the representation ρ defined on $L^2(\mathbb{R}^n)$ by

$$\rho(h, x)f(y) = |\det h|^{1/2} f(h^{-1}(y + x)).$$

(**Hint:** Follow the argument in Example 6.95).

13. Let $\hat{\rho}$ be the representation given in (6.57). Follow the argument in Example 6.139 to show:

(a) Every closed $\hat{\rho}$ invariant nonzero subspace of $L^2(\mathbb{R}^n)$ has form

$$L^2_W(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \mid f = 0 \text{ off } W\}$$

where W is a Borel subset of $\mathbb{R}^n$ of positive Lebesgue measure satisfying $h^t W = W$ for all $h \in H$.

(b) Let $\hat{\rho}_W$ be $\hat{\rho}$ restricted to $L^2_W(\mathbb{R}^n)$. Show $\hat{\rho}_W$ is irreducible if and only if every Borel subset E of W satisfying $(h^t E \cap E^c) \cup (E \cap (h^t E)^c)$ has Lebesgue measure 0 for all $h \in H$ has Lebesgue measure 0 or has complement in W with Lebesgue measure 0. (This says the action of H on W is ergodic.)

14. Let H be the product group $\mathbb{R}^+ \times \mathrm{SO}(n)$ with product topology and $G = H \ltimes \mathbb{R}^n$ where H acts on $\mathbb{R}^n$ by $(a, A)x = aAx$. Show the representation π defined on $L^2(\mathbb{R}^n)$ by

$$\pi(a, A, b)f(x) = a^{-n/2} f(a^{-1}A^{-1}(x - b))$$

is square-integrable and a nonzero function ψ is a wavelet for π if and only if $\int_{\mathbb{R}^n} |\hat{\psi}(x)|^2 \frac{dx}{|x|^n} < \infty$. Let

$$W_\psi(f)(a, A, x) = a^{-n/2} \int f(y) e^{2\pi i x \cdot y} \overline{\psi(a^{-1}A^{-1}(y - x))}\, dy$$

be the wavelet transform. Show if ϕ and ψ are wavelet functions and dA is a Haar measure on $\mathrm{SO}(n)$ satisfying $\int_{\mathrm{SO}(n)} 1\, dA = \sigma(S^{n-1})$, the surface area of S^{n-1}, then

$$f(y) = \frac{1}{B(\phi, \psi)} \int\int\int W_\psi(f)(a, A, x) a^{-n/2} \phi(a^{-1}A^{-1}(y - x)) \frac{da}{a^{n+1}}\, dA\, dx$$

weakly provided

$$B(\phi, \psi) = \int_{\mathbb{R}^n} \hat{\phi}(x) \overline{\hat{\psi}(x)} \frac{dx}{|x|^n} \neq 0.$$

15. Let $S = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } x^2 - y^2 > 0\}$. Define a nonzero function ϕ in $\mathcal{F}^{-1}L^2_S(\mathbb{R}^2)$ to be a wavelet function if $B(\phi, \phi) = \iint_S \frac{|\hat{\phi}(x,y)|^2}{x^2-y^2}\, dx\, dy < \infty$. For $t \in \mathbb{R}$, let $A(t)$ be the linear transformation of $\mathbb{R}^2$ given by $A(t)(x_1, x_2) = (\cosh t\, x_1 + \sinh t\, x_2, \sinh t\, x_1 + \cosh t\, x_2)$. Define the wavelet transform $W_\psi f$ for $f \in \mathcal{F}^{-1}L^2_S(\mathbb{R}^2)$ by

$$W_\psi f(a, t, y) = a \iint\limits_S \hat{f}(\omega)\overline{\hat{\phi}(aA(t)\omega)}e^{2\pi i y\cdot\omega}\, d\omega$$

where $a > 0$, $t \in \mathbb{R}$, and $y \in \mathbb{R}^2$. Show one has the weak formula

$$f(x) = \frac{1}{a^4 B(\phi, \phi)} \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} W_\psi f(a, t, b)\phi(a^{-1}A(-t)(x - b))da\, dt\, db.$$

(**Hint:** Use the Jacobian to show the appropriate measures are equal.)

16. Let G be the group $H \ltimes \mathbb{R}^n$ with multiplication defined in (6.54). Assume $y \in \mathbb{R}^n$ and H_y is the closed subgroup stabilizing y under the action $h \cdot x = (h^{-1})^t x$. It is known that the orbit $H \cdot y$ is a Borel subset of $\mathbb{R}^n$ and the mapping ϕ defined by $hH_y \mapsto h \cdot y$ is a Borel isomorphism. That is the Borel subsets of H/H_y map under this mapping in a one-to-one manner onto the Borel subsets of $\mathbb{R}^n$ contained in the orbit $H \cdot y$. Let ρ be a row function for H_y and let μ_y be the corresponding quasi-invariant measure on H/H_y. Let μ be the measure on the Borel subsets of $H \cdot y$ that satisfies

$$\int f(x)\, d\mu(x) = \int_{H/H_y} f(h \cdot y)\, d\mu_y(hH_y)$$

for positive Borel functions f on $H \cdot y$.

(a) Let χ be defined on the closed subgroup $H_y \ltimes \mathbb{R}^n$ of G by $\chi(h, x) = e^{ix\cdot y}$. Show χ is a unitary representation.

(b) Show indχ is unitarily equivalent to the unitary representation $\hat{\pi}$ defined on $L^2(H \cdot y, \mu)$ by

$$\hat{\pi}(h, x')f(x) = \left(\frac{d(h \cdot \mu)}{d\mu}(x')\right)^{1/2} e^{ix'\cdot x} f(h^t x).$$

(c) Show $\hat{\pi}$ is irreducible.

(d) Show $\hat{\pi}$ is square-integrable if and only if H_y is compact.

Chapter 7

The Heisenberg Group

In this chapter Fourier analysis will be used to do harmonic analysis on the Heisenberg group. In Chapter 3, the Fourier transform we used was

$$\mathcal{F}f(y) = \int f(x)e^{-2\pi i x\cdot y}\, dx$$

where dx is Lebesgue measure on $\mathbb{R}^n$. However, from Exercise 3.2.3, one can use any positive multiple of Lebesgue measure and obtain a corresponding Fourier transform. As seen in that exercise, if one uses $d_n x = \frac{1}{(2\pi)^{n/2}}\, dx$ or $\frac{1}{(2\pi)^{n/2}}$ times ordinary Lebesgue measure on $\mathbb{R}^n$, one has Fourier transform

$$\mathcal{F}f(y) = \int f(x)e^{-ix\cdot y}\, d_n x$$

with inverse Fourier transform

$$\mathcal{F}^{-1}f(x) = \int f(y)e^{ix\cdot y}\, d_n y.$$

It will be convenient in this chapter to use the measure $d_n x$ and all L^p norms, all convolutions, and all integrations will be done with respect to this measure. We shall also use the notation f_x for the left translate $\lambda(x)f$ where $\lambda(x)f(y) = f(y-x)$ and e_y for the exponential function $e_y(x) = e^{ix\cdot y}$. We recall the following basic facts in regards to this transform for $f \in \mathcal{S}(\mathbb{R}^n)$.

(1) $\mathcal{F}(f_x) = e_{-x}\mathcal{F}(f)$

(2) $\mathcal{F}(e_x f) = \mathcal{F}(f)_x$

(3) $\mathcal{F}(p(D)f)(y) = p(iy)\mathcal{F}f(y)$

(4) $\mathcal{F}(pf) = p(iD)\mathcal{F}(f)$

where in the above p is a polynomial function $p(x) = \sum_{|\alpha| \leqslant d} c_\alpha x^\alpha$ and $p(iy) = \sum_{|\alpha| \leqslant d} c_\alpha i^{|\alpha|} y^\alpha$ and $p(iD) = \sum_{|\alpha| \leqslant d} c_\alpha i^{|\alpha|} D^\alpha$.

1. Group Structure

As we saw in Example 5.9 of Chapter 5, multiplication on a Heisenberg group involves a symplectic form on a finite dimensional real vector space. We recall the definition here.

Definition 7.1. *Let V be a vector space over $\mathbb{R}$. Then a symplectic form $\langle \cdot, \cdot \rangle$ on V is a bilinear real valued mapping on $V \times V$ satisfying*

(1) $\langle v, w \rangle = -\langle w, v \rangle$ *for all* $v, w \in V$ *(alternating or skew)*

(2) $\langle w, v \rangle = 0$ *for all* v *implies* $w = 0$ *(nondegeneracy)*

Recall a bilinear form on V is alternating if and only if $\langle v, v \rangle = 0$ for all $v \in V$. Indeed, if $\langle \cdot, \cdot \rangle$ is alternating, then $\langle v, v \rangle = -\langle v, v \rangle$ and thus $\langle v, v \rangle = 0$. Conversely if $\langle v, v \rangle = 0$ for all v, then $\langle v + w, v + w \rangle = 0$ for all v and w. Hence $\langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle = \langle v, w \rangle + \langle w, v \rangle = 0$. This gives $\langle v, w \rangle = -\langle w, v \rangle$.

Lemma 5.41 implies the following structure for symplectic forms $\langle \cdot, \cdot \rangle$ on a finite dimensional real vector V.

There is a basis $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n$ of V satisfying

$$\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0 \text{ for all } i \text{ and } j \tag{7.1}$$

$$\langle e_i, f_j \rangle = \delta_{i,j} \text{ for all } i \text{ and } j. \tag{7.2}$$

A basis $e_1, e_2, \ldots, e_n; f_1, f_2, \ldots, f_n$ with properties 7.1 and 7.2 is called a **symplectic basis** while a **symplectic vector space** is a vector space with a symplectic form. We thus see finite dimensional symplectic vector spaces with the same dimension are isomorphic.

Example 7.2. *Let $(x, y), (x', y') \in \mathbb{R}^n \times \mathbb{R}^n$. Define $\langle (x, y), (x', y') \rangle = x \cdot y' - y \cdot x'$. Then $\langle \cdot, \cdot \rangle$ is a symplectic form.*

This symplectic form can be obtained from the usual complex inner product on $\mathbb{C}^n$ under the identification $x + iy$ with $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Indeed

$$\langle (x, y), (x', y') \rangle = -(y \cdot x' - x \cdot y') = -\mathrm{Im}(x + iy, x' + iy')$$

where the inner product on $\mathbb{C}^n$ is given by

$$(z, z') = \sum_{j=1}^{n} z_j \bar{z}'_j.$$

We will use this symplectic form throughout this chapter and will identify in many instances $x + iy$ and the pair (x, y) for x, $y \in \mathbb{R}^n$.

Exercise 7.1.1 shows we may always assume a finite dimensional real symplectic vector space V is a complex vector space with an inner product where the symplectic form satisfies $\langle w, v\rangle = -\mathrm{Im}(w, v)$.

Definition 7.3. *Let $\langle \cdot, \cdot \rangle$ be a symplectic form on $\mathbb{R}^{2n}$. Set $H_n = \mathbb{R}^{2n} \times \mathbb{R}$. Define $(w,t)\cdot(v,s) = (w+v, t+s+\frac{1}{2}\langle w, v\rangle)$. H_n with this multiplication is called the Heisenberg group of dimension $2n+1$.*

On $\mathbb{C}^n \times \mathbb{R}$, the multiplication would be

$$(z,t)(w,s) = (z+w, t+s-\frac{1}{2}\mathrm{Im}(z,w)).$$

Straight forward calculations show this multiplication is associative, $(0,0)$ is the identity and $(w,t)^{-1} = (-w,-t)$.

Remark 7.4. This group is very close to being commutative. Indeed, even though $(v,s)\cdot(w,t) = (v+w, s+t+\frac{1}{2}\langle v,w\rangle) \neq (v+w, s+t-\frac{1}{2}\langle v,w\rangle) = (w,t)(v,s)$ if $\langle v,w\rangle \neq 0$, one may be tempted to write the multiplication as addition, but one needs to remember this addition is not commutative. We thus would have

$$(v,t)+(w,s) = (v+w, t+s+\frac{1}{2}\langle v,w\rangle).$$

In particular $(v,t)+(-v,-t) = (0,0)$.

Since H_n is $\mathbb{R}^{2n+1}$, one has the usual differential operators on $C^\infty(H_n)$ and all the function spaces defined on Euclidean spaces. In particular, H_n is a group in which multiplication and inversion are differentiable mappings; i.e., it is a noncommutative Lie group. Exercise 7.1.2 shows it is a matrix group.

2. Vector Fields and the Lie Algebra of H_n

We do not wish to discuss in any detail the coordinate free definition of vectors and vector fields where these are differential operators. Since we are dealing with the Euclidean space $\mathbb{R}^{2n+1}$ and will be using coordinates $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, t)$ where the entries here are real, one can define a **vector field** on H_n to be a differential operator of form

$$D = \sum_{j=1}^{n}(F_j\frac{\partial}{\partial x_j} + G_j\frac{\partial}{\partial y_j}) + H\frac{\partial}{\partial t}$$

where the F_j, G_j, and H are real valued functions on H_n. Such a vector field is said to be C^∞ if the functions F_j, G_j, and H are C^∞ functions on $\mathbb{R}^{2n+1}$. The **bracket** $[D, D']$ of two C^∞ vector fields D and D' is the operator defined on $C^\infty(H_n)$ by

$$[D, D'] = D \circ D' - D' \circ D.$$

Exercise 7.1.3 shows the bracket of two C^∞ vector fields is again a C^∞ vector field and one has the following three properties:

(1) $$(D, D') \mapsto [D, D'] \text{ is real bilinear}$$

(2) $$[D, D'] = -[D', D] \text{ and}$$

(3) $[D, [D', D'']] + [D', [D'', D]] + [D'', [D, D']] = 0$ (the Jacobi identity).

A vector space with a real bilinear mapping on itself into itself with properties (2) and (3) is a real **Lie algebra**. In particular, the vector space of C^∞ vector fields on H_n is an infinite dimensional Lie algebra.

There is a natural group method to find C^∞ vector fields on H_n. Namely, let $w \in \mathbb{C}^n$ and $c \in \mathbb{R}$. If $f \in C^\infty(H_n)$ and $\Phi_s(z, t) = (z, t)(sw, sc)$, then

$$f \mapsto \frac{d}{ds}|_{s=0} f \circ \Phi_s$$

is a differential operator given by a C^∞ vector field. From Exercise 7.1.4, the mappings $s \mapsto (sw, sc)$ are the one parameter groups in H_n. Essentially what one is doing here is differentiating f along curves obtained by multiplying on the right in the group H_n by a one parameter curve. By Exercise 7.1.5, if $L(h)f(h') = f(h^{-1}h')$, then these give the vector fields D satisfying

$$D(L(h)f)) = L(h)D(f)$$

for $f \in C^\infty(H_n)$ and $h \in H_n$. Vector fields with this property are said to be **left invariant**. If $R(h)$ is defined on functions on H_n by $R(h)f(h') = f(h'h)$, then a vector field D is **right invariant** if $DR(h) = R(h)D$ on $C^\infty(H_n)$.

To see the specific outcome, let $w = (a, b) = a + ib$ where $a, b \in \mathbb{R}^n$. Then $\Phi_s(x, y, t) = (x, y, t)(sa, sb, sc) = (x + sa, y + sb, t + sc + \frac{1}{2}s(x \cdot b - y \cdot a))$ Hence

(7.3) $$\begin{aligned} \frac{d}{ds}|_{s=0} f \circ \Phi_s(x, y) &= \frac{d}{ds}|_{s=0} f(x + sa, y + sb, t + sc + \frac{1}{2}s(x \cdot b - y \cdot a)) \\ &= \sum_{j=1}^n a_j \frac{\partial f}{\partial x_j}(x, y, t) + b_j \frac{\partial f}{\partial y_j}(x, y, t) + (c + \frac{1}{2}\langle (x, y), (a, b)\rangle) \frac{\partial f}{\partial t}(x, y, t) \\ &= \left(\sum_{j=1}^n a_j (\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}) + \sum_{j=1}^n b_j (\frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}) + c \frac{\partial}{\partial t} \right) f(x, y, t). \end{aligned}$$

Define vector fields X_j, Y_j, and Z by

(7.4) $$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}$$

(7.5) $$Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}$$

(7.6) $$Z = \frac{\partial}{\partial t}$$

where j=1, 2,. . . , n. Thus the vector field given in (7.3) is given by

$$\sum a_j X_j + b_j Y_j + cZ.$$

Let $a,\, b \in \mathbb{R}^n$. Define

$$X(a) = \sum_{j=1}^{n} a_j X_j = \sum_{j=1}^{n} a_j\Big(\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}\Big) \tag{7.7}$$

$$Y(b) = \sum_{j=1}^{n} b_j Y_j = \sum_{j=1}^{n} b_j\Big(\frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}\Big). \tag{7.8}$$

In this notation, one has

$$(X(a) + Y(b) + cZ) f(x, y, t) = \frac{d}{ds} f((x, y, t)(sa, sb, sc))|_{s=0}.$$

Proposition 7.5. *These vector fields satisfy the following bracket rules:*

$$[X_j, X_k] = [Y_j, Y_k] = 0$$
$$[X_j, Y_k] = \delta_{j,k} Z = -[Y_k, X_j]$$
$$[X_j, Z] = [Y_j, Z] = 0.$$

In particular,

$$[X(a), Y(b)] = (a \cdot b) Z \text{ and}$$
$$[X(a) + Y(b) + cZ, X(a') + Y(b') + c'Z] = \langle (a, b), (a', b') \rangle Z.$$

We remark that the brackets of the $X(a) + Y(b) + cZ$ behave as the symplectic form $\langle \cdot, \cdot \rangle$ and the X_j and Y_k behave analogously to a symplectic basis.

Proof. The brackets between X_j and X_k and between Y_j and Y_k are 0 for these differential operators commute. Similarly $[X_j, Z] = [Y_j, Z] = 0$ for all j for $\frac{\partial}{\partial t}$ commutes with the operators $\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}$. For $[X_j, Y_k]$, note

$$\begin{aligned} X_j Y_k f &= \Big(\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}\Big)\Big(\frac{\partial}{\partial y_k} + \frac{1}{2} x_k \frac{\partial}{\partial t}\Big) f \\ &= \left(\frac{\partial^2}{\partial x_j \partial y_k} + \frac{1}{2}\delta_{j,k} \frac{\partial}{\partial t} + \frac{1}{2} x_k \frac{\partial^2}{\partial x_j \partial t} - \frac{1}{2} y_j \frac{\partial^2}{\partial t\, \partial y_k} - \frac{1}{4} x_k y_k \frac{\partial^2}{\partial t^2} \right) f \end{aligned}$$

and similarly

$$Y_k X_j f = \left(\frac{\partial^2}{\partial y_k \partial x_j} - \frac{1}{2}\delta_{j,k} \frac{\partial}{\partial t} - \frac{1}{2} y_j \frac{\partial^2}{\partial t\, \partial y_k} + \frac{1}{2} x_k \frac{\partial^2}{\partial t\, \partial x_j} - \frac{1}{4} x_k y_j \frac{\partial^2}{\partial t^2} \right) f$$

Subtracting gives

$$[X_j, Y_k] f = \delta_{j,k} \frac{\partial}{\partial t} f = \delta_{j,k} Z\, f.$$

Hence,

$$\begin{aligned}[X(a), Y(b)] &= \sum_{j,k} a_j b_k [X_j, Y_k] \\ &= \sum_{j,k} \delta_{j,k} a_j b_k Z \\ &= \sum_{j} a_j b_j Z \\ &= (a \cdot b)\, Z\end{aligned}$$

and

$$\begin{aligned}[X(a) + Y(b) + cZ, X(a') + Y(b') + c'Z] &= [X(a), Y(b')] + [Y(b), X(a')] \\ &= [X(a), Y(b')] - [X(a'), Y(b)] \\ &= (a \cdot b' - b \cdot a')Z \\ &= \langle (a,b), (a',b') \rangle Z.\end{aligned}$$

□

Thus the vector space consisting of all vector fields $X(a) + Y(b) + cZ$ where $a, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ is a $2n + 1$ dimensional Lie algebra. It is called the Lie algebra of H_n and will be denoted by $\mathfrak{h}_n$. We make an identification between the vector space $\mathfrak{h}_n$ and $\mathbb{C}^n \oplus \mathbb{R} \cong \mathbb{R}^{2n} \oplus \mathbb{R}^n$ using the correspondence $X(a) + Y(b) + cZ \leftrightarrow (a + ib, c) \leftrightarrow (a, b, c)$ where a, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Under this correspondence, Proposition 7.5 says the Lie bracket on $\mathfrak{h}_n$ satisfies:

$$[(a,b,c), (a',b',c')] = (0, 0, \langle (a,b), (a',b') \rangle). \tag{7.9}$$

Exercise Set 7.1

1. Let $\langle \cdot, \cdot \rangle$ be a symplectic form on a real finite dimensional vector space V. Show that V may be turned into a complex vector space with a complex inner product $(\cdot, \cdot)$ satisfying

$$\langle v, w \rangle = -\mathrm{Im}(v, w) \text{ for all } v, w \in V.$$

2. Let H be the collection of all matrices of form

$$\begin{bmatrix} 1 & x & t \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{bmatrix}$$

where $x, y \in \mathbb{R}^n$, I_n is the $n \times n$ identity matrix and $t \in \mathbb{R}$. Show H is a group and find an isomorphism of H_n onto H.

3. A C^∞ vector field on $\mathbb{R}^n$ is a differential operator of form

$$X = \sum_{j=1}^{n} F_j \frac{\partial}{\partial x_j}$$

where $F_j \in C^\infty(\mathbb{R}^n)$.

(a) Suppose $X = \sum F_j \frac{\partial}{\partial x_j}$ and $X' = \sum F'_j \frac{\partial}{\partial x_j}$ are C^∞ vector fields on $\mathbb{R}^n$. Show the bracket $[X, X'] = XX' - X'X$ is the C^∞ vector field

$$\sum H_k \frac{\partial}{\partial x_k}$$

where $H_k = \sum_j \left(F_j \frac{\partial F'_k}{\partial x_j} - F'_j \frac{\partial F_k}{\partial x_j} \right)$.

(b) Show $(X, X') \mapsto [X, X']$ is real bilinear on the vector space of C^∞ vector fields.

(c) Show $[X, X'] = -[X', X]$ if X and X' are C^∞ vector fields.

(d) Show $[X, [X', X'']] + [X', [X'', X]] + [X'', [X, X']] = 0$ for C^∞ vector fields X, X', and X.

4. A one parameter group in H_n is a continuous homomorphism from $(\mathbb{R}, +)$ into H_n. Show a one parameter group ϕ satisfies

$$\phi(t) = t\phi(1) = (tv, ts)$$

where $\phi(1) = (v, s)$.

5. Show a vector field D on H_n has form

$$D = \sum_j (a_j (\frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}) + b_j (\frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t})) + c \frac{\partial}{\partial t}$$

if and only if it is left invariant on H_n; i.e.,

$$L_h(Df) = D(L_h f)$$

for all $h \in H_n$ and $f \in C^\infty(H_n)$.

6. If D is a C^∞ vector field on H_n, then $\check{D}$ is the vector field defined on H_n by $\check{f} = \check{D}(f) = \widetilde{D(\check{f})}$ for $f \in C^\infty(H_n)$.

(a) Show $\check{D}$ is a C^∞ vector field.

(b) Show $D \mapsto \check{D}$ is linear and $\check{\check{D}} = D$.

(c) Show D is right invariant if and only if the the vector field $\check{D}$ is left invariant.

(b) Show $[\widetilde{D_1, D_2}]$ is $[\check{D}_1, \check{D}_2]$ for any pair D_1, D_2 of C^∞ vector fields on H_n.

(d) Find $\check{Z}$, and $\check{X}_j$ and $\check{Y}_j$ for $j = 1, \ldots, n$.

(e) Show $[\check{X}(a) + \check{Y}(b) + c\check{Z}, \check{X}(a') + \check{Y}(b') + c'\check{Z}] = \langle (a, b), (a', b') \rangle \check{Z}$.

(f) Show if $D = X(a) + Y(b) + cZ$, then

$$\check{D}f(x, y, t) = \frac{d}{ds}|_{s=0} f((-sa, -sb, -sc)(x, y, t)).$$

7. Let $\mathfrak{h}_n = \mathbb{R}^{2n} \times \mathbb{R}$ be the Lie algebra of the Heisenberg group H_n.

(a) Show all triple Lie brackets are 0 in $\mathfrak{h}_n$.

(b) Suppose N is a finite dimensional Lie algebra having the property that all triple Lie brackets are 0. Define a multiplication in N by $x \cdot y = x + y + \frac{1}{2}[x, y]$. Show N is a group.

8. Consider the vector fields $X' = \frac{\partial}{\partial x}$, $Y' = \frac{\partial}{\partial y} + x\frac{\partial}{\partial t}$, and $Z' = \frac{\partial}{\partial t}$ on $\mathbb{R}^3 = \{(x, y, t) \mid x, y, t \in \mathbb{R}\}$. Show X', Y', Z' satisfy the same brackets as $X = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial t}$, $Z = \frac{\partial}{\partial t}$, a basis for $\mathfrak{h}_1$.

Determine a multiplication $\cdot$ on $\mathbb{R}^3$ such that

(a) $X'f(x, y, t) = \frac{d}{ds} f((x, y, t) \cdot (s, 0, 0))|_{s=0}$

(b) $Y'f(x, y, t) = \frac{d}{ds} f((x, y, t) \cdot (0, s, 0))|_{s=0}$

(c) $Z'f(x, y, t) = \frac{d}{ds} f((x, y, t) \cdot (0, 0, s))|_{s=0}$.

Then show this group is isomorphic to the group H_1. How does this relate to Exercise 7.1.2?

9. Let $\mathbb{R}^4 = \{(w, x, y, z) \mid w, x, y, z \in \mathbb{R}\}$. Then consider the C^∞ vector fields $W = \frac{\partial}{\partial w}$, $X = \frac{\partial}{\partial x} + w\frac{\partial}{\partial y} + wx\frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$, and $Z = \frac{\partial}{\partial z}$. Show $[W, X] = Y$, $[X, Y] = Z$, $[W, Y] = 0$, $[W, Z] = 0$, $[X, Z] = 0$, $[Y, Z] = 0$. Thus the linear span of W, X, Y, Z form a four dimensional Lie algebra of C^∞ vector fields on $\mathbb{R}^4 = \{(w, x, y, z) \mid w, x, y, z \in \mathbb{R}\}$. Find a group multiplication on $\mathbb{R}^4$ such that for $f \in C^\infty(\mathbb{R}^4)$ one has

$$\frac{d}{dt}|_{t=0} f((w,x,y,z)\cdot se_k) = \begin{cases} Wf(w,x,y,z) & \text{when } k=1 \\ Xf(w,x,y,z) & \text{when } k=2 \\ Yf(w,x,y,z) & \text{when } k=3 \\ Zf(w,x,y,z) & \text{when } k=4. \end{cases}$$

3. Quantum Mechanics and Representations of H_n

In classical mechanics, a state of a system is determined by the positions and momentums of the particles in the system. The position coordinates are usually called q's and the momentum coordinates p's. Associated with such a system are a Hamiltonian system. We will not go into this here, but when the particles are free of external forces, there is an associated symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$ where n is the number of position coordinates. It is given by $\langle (q,p),(q',p')\rangle = q\cdot p' - p\cdot q'$. To obtain a symplectic basis, let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Then $(e_j,0)$ and $(0,e_j)$ are vectors in $\mathbb{R}^n \times \mathbb{R}^n$ which form a symplectic basis of $\mathbb{R}^{2n}$.

If one uses the coordinates $(q_1, q_2, \cdots, q_n, p_1, p_2, \ldots, p_n, t)$ on $\mathbb{R}^{2n+1}$ instead of the coordinates $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, t)$, one sees using equations 7.4–7.6 that

$$\begin{aligned} X_j &= \frac{\partial}{\partial q_j} - \frac{1}{2} p_j \frac{\partial}{\partial t} \\ Y_j &= \frac{\partial}{\partial p_j} + \frac{1}{2} q_j \frac{\partial}{\partial t} \\ Z &= \frac{\partial}{\partial t} \end{aligned}$$

form a basis of the left invariant vector fields on H_n and the corresponding vectors $X_j \leftrightarrow (e_j,0)$ $Y_j \leftrightarrow (0,e_j)$ for $j = 1,2,\ldots,n$ form a symplectic basis of $\mathbb{R}^n \times \mathbb{R}^n$.

In quantum mechanics, the states of a system of free particles are given by one dimensional vector subspaces of $L^2(\mathbb{R}^n)$ where n is the number of position coordinates. There are self adjoint operators Q_j and P_j corresponding to the position and momentum coordinates of the particles. It is known that they are given by

$$\begin{aligned} Q_j f(x) &= x_j f(x) \\ P_j f(x) &= -ih \frac{\partial}{\partial x_j} f(x) \end{aligned}$$

where h is Planck's constant. The commutators of these two groups of operators are the Heisenberg commutation relations. Namely, $[Q_j, P_k] =$

$\delta_{j,k} ihI$. Indeed,

$$\begin{aligned}
[Q_j, P_k](f)(x) &= Q_j P_k f(x) - P_k Q_j f(x) \\
&= x_j P_k f(x) + ih \frac{\partial}{\partial x_k}(Q_j f(x)) \\
&= -ihx_j \frac{\partial}{\partial x_k} f(x) + ih \frac{\partial}{\partial x_k}(x_j f(x)) \\
&= -ihx_j \frac{\partial}{\partial x_k} f(x) + ih\delta_{j,k} f(x) + ihx_j \frac{\partial}{\partial x_k} f(x) \\
&= ih\delta_{j,k} f(x).
\end{aligned}$$

These operators will be shown to generate an irreducible unitary representation of the Heisenberg group H_n on the Hilbert space $L^2(\mathbb{R}^n)$.

3.1. Obtaining representations. We construct $2n+1$ self adjoint operators (unbounded linear operators) on $L^2(\mathbb{R}^n)$ that essentially satisfy the commutation relations of the Heisenberg Lie algebra; namely $[X_j, Y_k] = \delta_{j,k} Z$. We set

$$\begin{aligned}
P_j f(w) &= i \frac{\partial}{\partial w_j} f(w) = iD^{e_j} f(w) \\
Q_k f(w) &= -\lambda w_k f(w) \\
\\
Rf(w) &= \lambda f(w)
\end{aligned}$$

The operators P_j and Q_j are except for scaling the momentum and position operators in quantum mechanics.

Note

$$\begin{aligned}
[P_j, Q_k] f(w) &= P_j(-\lambda w_k f)(w) - Q_k(i \frac{\partial}{\partial w_j} f(w)) \\
&= -i\lambda \delta_{j,k} f(w) - i\lambda w_k \frac{\partial}{\partial w_j} f(w) + i\lambda w_k \frac{\partial}{\partial w_j} f(w) \\
&= -\delta_{j,k} iR.
\end{aligned}$$

To make them satisfy the bracket formulas for the Heisenberg Lie algebra, we take instead iP_j, iQ_k, and iR. These operators are

$$iP_j f(w) = -\frac{\partial}{\partial w_j} f(w),\ iQ_k f(w) = -i\lambda w_k f(w),\ iRf(w) = i\lambda f(w).$$

Now we have

$$[iP_j, iQ_k] = \delta_{j,k}(iR).$$

To be more general, we define for $a, b \in \mathbb{R}^n$ operators

$$P_a = \sum a_j P_j,\ Q_b = \sum_j b_j Q_j. \tag{7.10}$$

Then

$$\begin{aligned}[P_a, Q_b] &= \sum_{j,k} [a_j P_j, b_k Q_k] \\ &= -\sum_j a_j b_j (iR) \\ &= -a \cdot b(iR)\end{aligned}$$

and similarly

$$[iP_a, iQ_b] = (a \cdot b)(iR).$$

We remark the brackets of these operators behave exactly as the brackets of $\sum a_j X_j$, $\sum b_j Y_j$, and Z.

Our Heisenberg group H_n consists of pairs $(x+iy, t)$ where multiplication is defined by

$$(z, t)(z', t') = (z + z', t + t' - \frac{1}{2}\mathrm{Im}(z, z'))$$

We define operators formally on $L^2(\mathbb{R}^n)$ by

$$\pi_\lambda(x, 0) = \exp(iP_x),\ \pi_\lambda(iy, 0) = \exp(iQ_y),\ \pi_\lambda(0, t) = \exp(itR).$$

We calculate formally:

$$\begin{aligned}\pi_\lambda(x, 0) f(w) &= \sum \frac{1}{k!}(iP_x)^k f(w) \\ &= \sum_k \frac{1}{k!}(-x_1 D_1 - x_2 D_2 - \cdots - x_n D_n)^k f(w) \\ &= f(w - x) \text{ (here we are using a formal Taylor series)} \\ \pi_\lambda(iy, 0) f(w) &= \sum_k \frac{1}{k!}(iQ_y)^k f(w) \\ &= \sum_k \frac{1}{k!}(-i\lambda y \cdot w)^k f(w) \\ &= e^{-i\lambda y \cdot w} f(w)\end{aligned}$$

and finally

$$\pi_\lambda(0, t) f(w) = \sum_k \frac{1}{k!}(itR)^k f(w) = \sum_k \frac{1}{k!}(i\lambda t)^k f(w) = e^{i\lambda t} f(w).$$

The argument for $\pi_\lambda(x,0)$ above can be made using the Fourier transform. Note

$$F(iP_x f)(\xi) = F(\sum_j -x_j \frac{\partial}{\partial x_j} f)(\xi) = \sum_j -x_j(i\xi_j)\hat{f}(\xi) = -i(x\cdot\xi)\hat{f}(\xi).$$

Thus

$$F(\pi(x,0)f)(\xi) = \sum_k \frac{(-ix\cdot\xi)^k}{k!}\hat{f}(\xi) = e^{-ix\cdot\xi}\hat{f}(\xi).$$

Hence

$$\begin{aligned} \pi(x,0)f(w) &= F^{-1}(e_{-x}\hat{f})(w) \\ &= F(e_{-x}\hat{f})(-w) \\ &= (\lambda(-x)\check{f})(-w) \\ &= \check{f}(-w+x) \\ &= f(x-w). \end{aligned}$$

Proposition 7.6. *The operators* $\pi_\lambda(x,0)$, $\pi_\lambda(iy,0)$, *and* $\pi_\lambda(0,t)$ *are unitary operators on* $L^2(\mathbb{R}^n)$. *Moreover,*

$$(x+iy,t) \mapsto \pi_\lambda(x+iy,t) := \pi_\lambda(0,\frac{1}{2}x\cdot y)\pi_\lambda(iy,0)\pi_\lambda(x,0)\pi_\lambda(0,t)$$

is a unitary representation of H_n *on the Hilbert space* $L^2(\mathbb{R}^n)$.

Proof.

$$\pi_\lambda(x+iy,t)f(w) = e^{i\lambda t}e^{\frac{1}{2}i\lambda(x,y)}e^{-i\lambda(y,w)}f(w-x).$$

We check it is a homomorphism. Note $\pi_\lambda(0,0) = I$ and

$$\begin{aligned} \pi_\lambda(x+iy,t)\pi(a+ib,s)f(w) &= e^{i\lambda t}e^{\frac{1}{2}i\lambda(x,y)}e^{-i\lambda(y,w)}\pi(a+ib,s)f(w-x) \\ &= e^{i\lambda t}e^{\frac{1}{2}i\lambda(x,y)}e^{-i\lambda(y,w)}e^{i\lambda s}e^{\frac{1}{2}i\lambda(a,b)}e^{-i\lambda(b,w-x)}f(w-x-a) \\ &= e^{i\lambda(t+s)}e^{\frac{1}{2}i\lambda((x,b)-(y,a))}e^{\frac{1}{2}i\lambda((x,y)+(x,b)+(a,y)+(a,b))} \\ &\qquad \times e^{-i\lambda(y+b,w)}f(w-x-a) \\ &= \pi_\lambda(x+a+i(y+b),t+s+\frac{1}{2}(x\cdot b - y\cdot a))f(w) \\ &= \pi_\lambda((x+iy,t)(a+ib,s))f(w). \end{aligned}$$

Thus π_λ is a homomorphism. To see it is strongly continuous, note if $f_0 \in C_c(\mathbb{R}^n)$, then $\pi_\lambda(x+iy,t)f_0 \to f_0$ in L^2 as $(x+iy,t)\to 0$ by the dominated convergence theorem. In general, if $f \in L^2$ and $\epsilon > 0$, one can choose an $f_0 \in C_c(\mathbb{R}^n)$ such that

$$||f-f_0||_2 < \frac{\epsilon}{3}.$$

Hence

$$\begin{aligned} ||\pi_\lambda(h)f - f|| &\leqslant ||\pi_\lambda(h)f - \pi_\lambda(h)f_0|| + ||\pi_\lambda(h)f_0 - f_0|| + ||f_0 - f|| \\ &< \frac{2\epsilon}{3} + ||\pi_\lambda(h)f_0 - f_0||_2 < \epsilon \end{aligned}$$

if h is close to $(0,0)$ in H_n. Thus π_λ is continuous at the identity. Since π_λ is a homomorphism, Proposition 6.32 and Lemma 5.10 imply π_λ is strongly continuous everywhere on H_n. □

We check that π_λ satisfies:

$$\pi_\lambda(x)\pi_\lambda(iy) = e^{i\lambda(x,y)}\pi_\lambda(iy)\pi_\lambda(x) \tag{7.11}$$

$$\pi(x+iy,t) = e^{i\lambda t}e^{\frac{1}{2}i\lambda(x,y)}\pi_\lambda(iy)\pi_\lambda(x) = e^{i\lambda t}e^{-\frac{1}{2}i\lambda(x,y)}\pi_\lambda(x)\pi_\lambda(iy). \tag{7.12}$$

Indeed, note $\pi_\lambda(x)f(w) = f(w-x)$ and $\pi_\lambda(iy)f(w) = e^{-i\lambda(y,w)}f(w)$. Thus

$$\begin{aligned} \pi_\lambda(x)\pi_\lambda(iy)f(w) &= \pi_\lambda(iy)f(w-x) \\ &= e^{-i\lambda(y,w-x)}f(w-x) \\ &= e^{i\lambda(x,y)}e^{-i\lambda(y,w)}\pi_\lambda(x)f(w) \\ &= e^{i\lambda(x,y)}\pi_\lambda(iy)\pi_\lambda(x)f(w) \end{aligned}$$

and

$$\begin{aligned} \pi_\lambda(x+iy,t)f(w) &= e^{i\lambda t}e^{\frac{1}{2}i\lambda(x,y)}e^{-i\lambda(y,w)}f(w-x) \\ &= e^{i\lambda t}e^{\frac{1}{2}i\lambda(x,y)}\pi_\lambda(iy)\pi_\lambda(x)f(w) \\ &= e^{i\lambda t}e^{-\frac{1}{2}i\lambda(x,y)}\pi_\lambda(x)\pi_\lambda(iy)f(w). \end{aligned} \tag{7.13}$$

We will show π_λ is irreducible.

4. The Orthogonality Relations

Let $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. As seen by Theorem 3.11 and Exercises 3.2.1 and 3.2.2, the partial Fourier transform defined by

$$\mathcal{F}_2(f)(x,\omega) = \int f(x,y)e^{-i\omega\cdot y}\,d_ny$$

is a linear homeomorphism of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with all the same properties as the full Fourier transform.

Let F be a Schwartz function on $\mathbb{R}^n \times \mathbb{R}^n$. Define a linear operator $\pi_\lambda(F)$ on $L^2(\mathbb{R}^n)$ by

$$\pi_\lambda(F)k(x) = \iint F(u,v)\pi_\lambda(u,v,0)k(x)\,d_nu\,d_nv. \tag{7.14}$$

Note this makes sense for

$$\begin{aligned}\pi_\lambda(F)k(x) &= \iint F(u,v)e^{\frac{1}{2}i\lambda u\cdot v}e^{-i\lambda x\cdot v}k(x-u)\,d_nu\,d_nv\\ &= \iint F(u+x,v)e^{\frac{1}{2}i\lambda(u+x)\cdot v}e^{-i\lambda x\cdot v}k(-u)\,d_nu\,d_nv\\ &= \iint F(x-u,v)e^{\frac{1}{2}i\lambda(-x-u)\cdot v}k(u)\,d_nv\,d_nu\\ &= \int \mathcal{F}_2F(x-u,\frac{1}{2}\lambda(x+u))k(u)\,d_nu.\end{aligned}$$

Hence $\pi_\lambda(F)$ is an integral operator with kernel

$$K_{\lambda,F}(x,y) = \int F(x-y,w)e^{-i\frac{\lambda}{2}(x+y,w)}\,d_nw.$$

Consequently,

$$K_{\lambda,F}(x,y) = \mathcal{F}_2F(x-y,\frac{\lambda}{2}(x+y)). \tag{7.15}$$

Hence the kernel of $\pi_\lambda(F)$ is given by

$$K_{\lambda,F}(x,y) = (\mathcal{F}_2F)\circ\Phi(x,y) \text{ where} \tag{7.16}$$

$$\Phi(x,y) = (x-y,\frac{\lambda}{2}(x+y)). \tag{7.17}$$

Note Φ is an invertible linear transformation with inverse given by

$$\Phi^{-1}(x,y) = (\frac{x}{2}+\frac{y}{\lambda},-\frac{x}{2}+\frac{y}{\lambda}). \tag{7.18}$$

Since Φ and Φ^{-1} are invertible linear transformations on $\mathbb{R}^n\times\mathbb{R}^n$ we have

$$\iint F\circ\Phi(x,y)\,d_nx\,d_ny = \frac{1}{|\lambda|^n}\iint F(x,y)\,d_nx\,d_ny \text{ and} \tag{7.19}$$

$$\iint F\circ\Phi^{-1}(x,y)\,d_nx\,d_ny = |\lambda|^n\iint F(x,y)\,d_nx\,d_ny \tag{7.20}$$

for $F\in L^1(\mathbb{R}^n\times\mathbb{R}^n)$. Indeed, this is a direct consequence of Corollary 2.23, or one can argue as in the following sequence of equalities:

$$\begin{aligned}\iint F(\Phi(x,y))\,d_nx\,d_ny &= \iint F(x-y,\frac{\lambda}{2}(x+y))\,d_nx\,d_ny\\ &= \iint F(x,\frac{\lambda}{2}x+\lambda y)\,d_nx\,d_ny\\ &= \frac{1}{|\lambda|^n}\iint F(x,\frac{\lambda}{2}x+y)\,d_ny\,d_nx\\ &= \frac{1}{|\lambda|^n}\iint F(x,y)\,d_ny\,d_nx.\end{aligned}$$

Now from the fact that $\mathcal{F}_2$ carries Schwartz functions to Schwartz functions and since by Proposition 2.62, the composition of Schwartz functions with invertible linear transformations are Schwartz, we see that $\mathcal{F}_2(F) \circ \Phi$ is Schwartz if F is Schwartz. Thus the operator $\pi_\lambda(F)$ has a Schwartz kernel. Moreover,

$$\begin{aligned}
||K_{\lambda,F}||_2^2 &= \int |K_{\lambda,F}(x,y)|^2 \, d_n x \, d_n y \\
&= \iint |(\mathcal{F}_2 F) \circ \Phi(x,y)|^2 \, d_n x \, d_n y \\
&= \frac{1}{|\lambda|^n} \iint |F(x,y)|^2 \, d_n y \, d_n x
\end{aligned}$$

and thus:

$$||K_{\lambda,F}||_2 = \frac{1}{|\lambda|^{n/2}} ||F||_2 \tag{7.21}$$

Since

$$K_{\lambda,F} \circ \Phi^{-1} = \mathcal{F}_2(F),$$

we see

$$F = \mathcal{F}_2^{-1}(K_{\lambda,F} \circ \Phi^{-1})$$

and so

$$F(x,y) = \int K_{\lambda,F}(\frac{x}{2} + \frac{\omega}{\lambda}, -\frac{x}{2} + \frac{\omega}{\lambda}) e^{iy \cdot w} \, d_n w. \tag{7.22}$$

Lemma 7.7. *Let $f, h \in \mathcal{S}(\mathbb{R}^n)$. Then $(x,y) \mapsto (f, \pi_\lambda(x+iy,0)h)_2$ is the Schwartz function on $\mathbb{R}^n \times \mathbb{R}^n$ given by*

$$(f, \pi_\lambda(x+iy,0)h)_2 = \frac{1}{|\lambda|^n} \mathcal{F}_2^{-1} \left((f \otimes \bar{h}) \circ \Phi^{-1} \right)(x,y).$$

Proof. By Theorem 2.85 and Proposition 2.62 we know $(f \otimes \bar{h}) \circ \Phi^{-1}$ is a Schwartz function. Consequently, $\mathcal{F}_2^{-1} \left((f \otimes \bar{h}) \circ \Phi^{-1} \right)$ is also Schwartz.

Since Φ^{-1} is given by $\Phi^{-1}(x,y) = (\frac{x}{2} + \frac{y}{\lambda}, -\frac{x}{2} + \frac{y}{\lambda})$, we see

$$\begin{aligned}
(f, \pi_\lambda(x+iy)h) &= \int e^{-\frac{1}{2} i\lambda(x,y)} e^{i\lambda(y,w)} f(w) \bar{h}(w-x) \, d_n w \\
&= e^{-\frac{1}{2} i\lambda(x,y)} \int f(w + x/2) \bar{h}(w - x/2) \, e^{i\lambda(y, w+x/2)} \, d_n w \\
&= \int f(\frac{x}{2} + w) \bar{h}(-\frac{x}{2} + w) e^{i\lambda(y,w)} \, d_n w \\
&= \frac{1}{|\lambda|^n} \int f(\frac{x}{2} + \frac{w}{\lambda}) \bar{h}(-\frac{x}{2} + \frac{w}{\lambda}) e^{i(y,w)} \, d_n w \\
&= \frac{1}{|\lambda|^n} \mathcal{F}_2^{-1} \left((f \otimes \bar{h}) \circ \Phi^{-1} \right)(x,y).
\end{aligned}$$

□

Corollary 7.8. $(\pi_\lambda(x+iy,0)f,h)_2 = \frac{1}{|\lambda|^n}\mathcal{F}_2^{-1}\left((f\otimes\bar{h})\circ\Phi^{-1}\right)(-x,-y)$.

Proof. $(\pi_\lambda(x+iy,0)f,h)_2 = (f,\pi_\lambda(-x-iy,0)h)_2$. □

The function $z \mapsto (f,\pi_\lambda(z,0)h)$ is a *matrix coefficient* of the representation π_λ. These were used in Chapter 6 and arise frequently in representation theory, and one needs a useful notation for these functions. The notation $M_{f,h}(z) = (f,\pi_\lambda(z,0)h)$ is commonly used. However, because of the behavior of these functions or more precisely, the form of the operators $\pi_\lambda(M_{f,h})$, we shall call these functions $f\otimes_\lambda \bar{h}$. Thus we define

(7.23) $$(f\otimes_\lambda \bar{h})(x,y) = (f,\pi_\lambda(x+iy,0)h)_2 \text{ for } f,\ h\in L^2(\mathbb{R}^n).$$

We also shall make repeated use of integrals of form

$$\int f(w)\pi_\lambda(x+iy,0)h(w)\,d_nw$$

where f and h are in $L^2(\mathbb{R}^n)$. We note these are matrix coefficients of the representation $\pi_{-\lambda}$. Indeed,

(7.24)
$$\begin{aligned}\int f(w)\pi_\lambda(x+iy,0)h(w)\,d_nw &= \int f(w)e^{\frac{i\lambda(x,y)}{2}}e^{-i\lambda(y,w)}h(w-x)\,d_nw\\ &= \int f(w)\overline{e^{\frac{-\lambda i(x,y)}{2}}e^{i\lambda(y,w)}\bar{h}(w-x)}\,d_nw\\ &= (f,\pi_{-\lambda}(x+iy)\bar{h})_2\\ &= (f\otimes_{-\lambda} h)(x,y).\end{aligned}$$

Theorem 7.9 (Orthogonality)**.** *Let f, h, f', and h' belong to $L^2(\mathbb{R}^n)$. Then $f\otimes_\lambda\bar{h}$ and $f'\otimes_\lambda\bar{h}'$ are L^2 functions on $\mathbb{R}^n\times\mathbb{R}^n$ and*

$$\iint (f,\pi_\lambda(x+iy,0)h)_2\overline{(f',\pi_\lambda(x+iy,0)h')}_2\,d_nx\,d_ny = \frac{1}{|\lambda|^n}(f,f')_2(h',h)_2.$$

Proof. We start with $f,h,f',h'\in\mathcal{S}(\mathbb{R}^n)$. Note

$$\pi_\lambda(x+iy,0)f(w) = e^{\frac{1}{2}i\lambda(x,y)}e^{-i\lambda(y,w)}f(w-x).$$

Thus

$$(f,\pi_\lambda(x+iy,0)h)_2 = \int e^{-\frac{1}{2}i\lambda(x,y)}e^{i\lambda(y,w)}f(w)\bar{h}(w-x)\,d_nw.$$

Hence

$$
\begin{aligned}
&\iint (f,\pi_\lambda(x+iy,0)h)_2\overline{(f',\pi_\lambda(x+iy,0)h')}\,d_nx\,d_ny\\
&= \iiiint e^{-\frac{1}{2}i\lambda(x,y)}e^{i\lambda(y,w)}f(w)\bar{h}(w-x)\,e^{\frac{1}{2}i\lambda(x,y)}e^{-i\lambda(y,w')}\bar{f}'(w')h'(w'-x)\,d_nw\,d_nw'\,d_nx\,d_ny\\
&= \iiiint e^{i\lambda(y,w)}e^{-i\lambda(y,w')}f(w)\bar{f}'(w')\bar{h}(w-w'-x)h'(-x)\,d_nw\,d_nw'\,d_nx\,d_ny\\
&= \iiiint e^{i\lambda(y,w+w')}e^{-i\lambda(y,w')}f(w+w')\bar{f}'(w')\bar{h}(w-x)h'(-x)\,d_nw\,d_nw'\,d_nx\,d_ny\\
&= \iiiint e^{i\lambda(y,w)}f(w-w')\bar{f}'(-w')\bar{h}(w-x)h'(-x)\,d_nw\,d_nw'\,d_nx\,d_ny\\
&= \iint e^{i\lambda(y,w)}\bar{\check{f}}'*f(w)\check{h}'*\bar{h}(w)d_nw\,d_ny\\
&= \frac{1}{|\lambda|^n}\iint e^{i(y,w)}\bar{\check{f}}'*f(w)\check{h}'*\bar{h}(w)d_nw\,d_ny\\
&= \frac{1}{|\lambda|^n}\int \mathcal{F}^{-1}((\bar{\check{f}}'*f)(\check{h}'*\bar{h}))(y)\,d_ny\\
&= \frac{1}{|\lambda|^n}(\bar{\check{f}}'*f)(0)\check{h}'*\bar{h}(0)\\
&= \frac{1}{|\lambda|^n}\int \bar{f}'(-w)f(-w)\,d_nw\int h'(-w')\bar{h}(-w')\,d_nw'\\
&= \frac{1}{|\lambda|^n}\int f(w)\bar{f}'(w)\,d_nw\int h'(w')\bar{h}(w')\,d_nw'\\
&= \frac{1}{|\lambda|^n}(f,f')_2(h',h)_2.
\end{aligned}
$$

Extend to the general case by using the density of $\mathcal{S}(\mathbb{R}^n)$ in L^2 and by taking L^2 limits. □

Remark 7.10. We give an alternate method for obtaining the above result. Let $\mathcal{F}_2$ be the partial Fourier transform in the second variable on $\mathbb{R}^n\times\mathbb{R}^n$. Thus $\mathcal{F}_2(F)(x,v)=\int F(x,y)e^{-i(y,v)}\,d_ny$. Then $\mathcal{F}_2$ is unitary isomorphism of L^2 onto L^2. Moreover, as seen by the argument in Lemma 7.7 the matrix coefficient $f\otimes_\lambda\bar{h}(x+iy)=(f,\pi_\lambda(x+iy,0)h)_2$ is given by $f\otimes_\lambda\bar{h}(x+iy)=\frac{1}{|\lambda|^n}\mathcal{F}_2^{-1}\left((f\otimes\bar{h})\circ\Phi^{-1}\right)(x,y)$. Since $\mathcal{F}_2^{-1}$ is a unitary isomorphism of $L^2(\mathbb{R}^{2n})$,

$$
\begin{aligned}
&\left(\frac{1}{|\lambda|^n}\mathcal{F}_2^{-1}((f\otimes\bar{h})\circ\Phi^{-1}),\ \frac{1}{|\lambda|^n}\mathcal{F}_2^{-1}((f'\otimes\bar{h}')\circ\Phi^{-1})\right)_2\\
&\quad= \frac{1}{|\lambda|^{2n}}\iint f(\frac{x}{2}+\frac{w}{\lambda})\bar{h}(-\frac{x}{2}+\frac{w}{\lambda})\bar{f}'(\frac{x}{2}+\frac{w}{\lambda})h'(-\frac{x}{2}+\frac{w}{\lambda})\,d_nx\,d_nw\\
&\quad= \frac{1}{|\lambda|^n}\iint f(\frac{x}{2}+w)\bar{h}(-\frac{x}{2}+w)\bar{f}'(\frac{x}{2}+w)h'(-\frac{x}{2}+w)\,d_nx\,d_nw\\
&\quad= \frac{1}{|\lambda|^n}\iint f(x+w)\bar{f}'(x+w)\bar{h}(w)h'(w)\,d_nx\,d_nw\\
&\quad= \frac{1}{|\lambda|^n}\iint f(x)\bar{f}'(x)\bar{h}(w)h'(w)\,d_nx\,d_nw=\frac{1}{|\lambda|^n}(f,f')_2(h',h)_2.
\end{aligned}
$$

and we have the orthogonality relations for π_λ.

Corollary 7.11. *For $\lambda \neq 0$, the representation π_λ is an irreducible unitary representation.*

Proof. Let $\mathcal{M}$ be an invariant nonzero closed subspace of $L^2(\mathbb{R}^n)$. We show $\mathcal{M} = L^2$. It suffices to show $\mathcal{M}^\perp = \{0\}$. Let $f \in \mathcal{M}^\perp$ and take $h \in \mathcal{M}$ with $h \neq 0$. Then by the orthogonality theorem,

$$0 = \iint (f, \pi_\lambda(x+iy,0)h)_2 \overline{(f, \pi_\lambda(x+iy,0)h)_2}\, d_n x\, d_n y = \frac{1}{|\lambda|^n}(f,f)_2(h,h)_2.$$

The only way this can happen is $(f,f)_2 = 0$ and thus $f = 0$. $\square$

5. The Wigner and Weyl Transforms

The **general Wigner transform** on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$\begin{aligned} W_\lambda(F)(x,y) &= \int e^{i\lambda(y,w)} F(\frac{1}{2}x + w, -\frac{1}{2}x + w)\, d_n w \\ &= \frac{1}{|\lambda|^n}\int e^{i(y,w)} F(\frac{1}{2}x + \frac{w}{\lambda}, -\frac{1}{2}x + \frac{w}{\lambda})\, d_n w \\ &= \frac{1}{|\lambda|^n}\mathcal{F}_2^{-1}(F \circ \Phi^{-1})(x,y) \end{aligned} \tag{7.25}$$

When $\lambda = 1$, it is called the **Wigner transform**. It is important for as we have seen in Lemma 7.7 the matrix coefficient $f \otimes_\lambda \bar{h}$ is given by:

$$f \otimes_\lambda \bar{h} = W_\lambda(f \otimes \bar{h}). \tag{7.26}$$

In Equation 7.14, we defined $\pi_\lambda(F)$ for F a Schwartz function on $\mathbb{R}^n \times \mathbb{R}^n$. Indeed, it is the linear operator with Schwartz kernel $K_{\lambda,F} = (\mathcal{F}_2 F) \circ \Phi$. We shall discard the distinction between the operator and its kernel.

Definition 7.12. *Let $F \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the* **Weyl transform** *$\pi_\lambda(F)$ of F is the L^2 function defined by*

$$\pi_\lambda(F) = (\mathcal{F}_2 F) \circ \Phi.$$

When F is a Schwartz function, $\pi_\lambda(F)$ is thus both a Schwartz function and an operator with Schwartz kernel $\pi_\lambda(F)$. We thus have the misuse of notation

$$\pi_\lambda(F) f(x) = \int_{\mathbb{R}^n} \pi_\lambda(F)(x,y)\, f(y)\, d_n y.$$

Proposition 2.62 and the fact that $\mathcal{F}_2$ is homeomorphism imply the Weyl transform $F \mapsto \pi_\lambda(F) := (\mathcal{F}_2 F) \circ \Phi$ is a linear homeomorphism of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$.

The orthogonality relations for π_λ can be rewritten in terms of Weyl transforms.

Proposition 7.13 (Weyl transform orthogonality formula)**.** *If f and h are nonzero functions in $L^2(\mathbb{R}^n)$, then*

$$\pi_\lambda(f \otimes_\lambda \bar{h}) = \frac{1}{|\lambda|^n} f \otimes \bar{h}$$

both as a function and as a rank one operator.

Proof.

$$\begin{aligned}
(\pi_\lambda(f \otimes_\lambda \bar{h})f', h')_2 &= \iint (f \otimes_\lambda \bar{h})(x,y)(\pi_\lambda(x+iy,0)f', h')_2 \, d_n x \, d_n y \\
&= \iint (f \otimes_\lambda \bar{h})(x,y)\overline{(h', \pi_\lambda(x+iy,0)f')_2} \, d_n x \, d_n y \\
&= \iint (f \otimes_\lambda \bar{h})(x,y)\overline{h' \otimes_\lambda \bar{f}'} \, d_n x \, d_n y \\
&= (f \otimes_\lambda \bar{h}, h' \otimes_\lambda \bar{f}')_2 \\
&= \frac{1}{|\lambda|^n}(f, h')_2 (f', h)_2 \\
&= \frac{1}{|\lambda|^n}((f', h)_2 f, h')_2 \\
&= \frac{1}{|\lambda|^n}((f \otimes \bar{h})(f'), h')_2
\end{aligned}$$

for all f' and h' in $L^2(\mathbb{R}^n)$. $\square$

Using earlier notation, this proposition gives:

$$K_{\lambda, f \otimes_\lambda \bar{h}} = \frac{1}{|\lambda|^n} f \otimes \bar{h}. \tag{7.27}$$

As we have seen, the mapping $\Phi' : \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ given by $\Phi'(F) = F \circ \Phi$ is a linear homeomorphism of $\mathcal{S}(\mathbb{R}^n)$.

Thus $\mathcal{F}_2^{-1}(U \circ \Phi)$ and $\mathcal{F}_2(U) \circ (\Phi^{-1})'$ are tempered distributions for each tempered distribution U on $\mathbb{R}^{2n}$. We thus define the **Wigner and Weyl transforms** of U by

$$W_\lambda(U) = \mathcal{F}_2^{-1}(U \circ \Phi') \tag{7.28}$$

and

$$\pi_\lambda(U) = \frac{1}{|\lambda|^n} \mathcal{F}_2(U) \circ (\Phi^{-1})'. \tag{7.29}$$

Thus

$$W_\lambda(U)(F) = U \circ \Phi'(\mathcal{F}_2^{-1} F) = U((\mathcal{F}_2^{-1} F) \circ \Phi) \tag{7.30}$$

and

$$\pi_\lambda(U)(F) = \frac{1}{|\lambda|^n}\mathcal{F}_2(U)(F \circ \Phi^{-1}) = \frac{1}{|\lambda|^n}U(\mathcal{F}_2(F \circ \Phi^{-1})) \tag{7.31}$$

for $F \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$.

In terms of integral notation for distributions, one has

$$W_\lambda(U)(F) = \iint (\mathcal{F}_2^{-1}F)(\Phi(x,y))\, dU(x,y) \text{ and} \tag{7.32}$$

$$\pi_\lambda(U)(F) = \frac{1}{|\lambda|^n}\iint \mathcal{F}_2(F \circ \Phi^{-1})(x,y)\, dU(x,y). \tag{7.33}$$

That these definitions are consistent with the definitions given in Equation 7.25 and in Definition 7.12 is left as Exercises 7.2.3 and 7.2.4. Indeed if U is a distribution given by an L^2 function F, then $\pi_\lambda(U)$ is the distribution given by $(\mathcal{F}_2F)\circ\Phi$ and $W_\lambda(U)$ is the distribution given by $\frac{1}{|\lambda|^n}\mathcal{F}_2^{-1}(F\circ\Phi^{-1})$.

Remark 7.14. The definition of the distribution $\pi_\lambda(U)$ is motivated by the formula

(7.34)

$$\pi_\lambda(U)(f \otimes h) = U(f \otimes_{-\lambda} h) = \iint \left(\int f(w)\pi_\lambda(x+iy,0)h(w)\, d_nw \right) dU(x,y).$$

Indeed, Exercises 7.2.5 and 7.2.6 show $(f,h) \mapsto U(f \otimes_{-\lambda} h)$ is a continuous bilinear mapping on $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$. The Schwartz kernel theorem states that any continuous bilinear $\mathbb{C}$ valued mapping on $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ has form $(f,h) \mapsto T(f \otimes h)$ for some tempered distribution T on $\mathbb{R}^n \times \mathbb{R}^n$. In the case of $(f,h) \mapsto U(f \otimes_{-\lambda} h)$, we do not need to invoke the Schwartz theorem for we have

$$\pi_\lambda(U) = \frac{1}{|\lambda|^n}(\mathcal{F}_2U) \circ (\Phi^{-1})'.$$

Proposition 7.15. *Let f and h be Schwartz functions on $\mathbb{R}^n$. Then*

$$W_\lambda(U)(f \otimes_{-\lambda} h) = \frac{1}{|\lambda|^n}U(f \otimes h).$$

Proof. By Equation 7.30, we have

$$W_\lambda(U)(f \otimes_{-\lambda} h) = U(\mathcal{F}_2^{-1}(f \otimes_{-\lambda} h) \circ \Phi).$$

Now

$$\begin{aligned} \mathcal{F}_2^{-1}(f \otimes_{-\lambda} h)(x,y) &= \int (f \otimes_{-\lambda} h)(x,w)e^{i(w,y)}\, d_nw \\ &= \int\int f(u)\pi_\lambda(x+iw,0)h(u)e^{i(w,y)}\, d_nu\, d_nw \\ &= \iint f(u)e^{\frac{i\lambda(x,w)}{2}}e^{-i\lambda(w,u)}h(u-x)e^{i(w,y)}\, d_nu\, d_nw. \end{aligned}$$

Thus

$$\begin{aligned}\mathcal{F}_2^{-1}(f\otimes_{-\lambda} h)(\Phi(x,y)) &= \mathcal{F}_2^{-1}(f\otimes_{-\lambda} h)(x-y,\frac{\lambda}{2}(x+y))\\ &= \iint f(u)e^{\frac{i\lambda(x-y,w)}{2}}e^{-i\lambda(w,u)}h(u-x+y)e^{i(w,\frac{\lambda}{2}(x+y))}\,d_nu\,d_nw\\ &= \iint f(u)h(u-x+y)e^{i\lambda(x,w)}e^{-i\lambda(w,u)}\,d_nu\,d_nw.\end{aligned}$$

Fix x and y and define Schwartz function ψ by $\psi(u) = f(u)h(u-x+y)$. Then we see

$$\begin{aligned}\mathcal{F}_2^{-1}(f\otimes_{-\lambda} h)(\Phi(x,y)) &= \iint \psi(u)e^{-i\lambda(w,u)}\,d_nu\,e^{i\lambda(x,w)}\,d_nw\\ &= \int \hat{\psi}(\lambda w)e^{i\lambda(x,w)}\,d_nw\\ &= \frac{1}{|\lambda|^n}\int \hat{\psi}(w)e^{i(x,w)}\,d_nw\\ &= \frac{1}{|\lambda|^n}\psi(x)\\ &= \frac{1}{|\lambda|^n}f(x)h(y)\\ &= \frac{1}{|\lambda|^n}(f\otimes h)(x,y).\end{aligned}$$

Consequently,

$$\begin{aligned}W_\lambda(U)(f\otimes_{-\lambda} h) &= U(\mathcal{F}_2^{-1}(f\otimes_{-\lambda} h)\circ\Phi)\\ &= \frac{1}{|\lambda|^n}U(f\otimes h).\end{aligned}$$

□

We summarize:

Theorem 7.16. *Let U be a tempered distribution on $\mathbb{R}^n\times\mathbb{R}^n$. Then:*

(a) $W_\lambda(f\otimes\bar{h}) = f\otimes_\lambda \bar{h}$ *for* $f,\ h\in L^2(\mathbb{R}^n)$

(b) $W_\lambda(U)(f\otimes\pi_{-\lambda}h) = \frac{1}{|\lambda|^n}U(f\otimes h)$ *if* $f,h\in\mathcal{S}(\mathbb{R}^n)$

(c) $\pi_\lambda(f\otimes_\lambda\bar{h}) = \frac{1}{|\lambda|^n}f\otimes\bar{h}$ *for* $f,\ h\in L^2(\mathbb{R}^n)$

(d) $\pi_\lambda(U)(f\otimes\bar{h}) = U(f\otimes_\lambda\bar{h})$ *if* $f,h\in\mathcal{S}(\mathbb{R}^n)$

(e) $W_\lambda(\pi_\lambda(U)) = \frac{1}{|\lambda|^n}U$

(f) $\pi_\lambda(W_\lambda(U)) = \frac{1}{|\lambda|^n}U$

Proof. We have (a) from Equation 7.26 and (b) and (c) are the results of Propositions 7.15 and 7.13. We stated (d) in Equation 7.34. Its proof is Exercise 7.2.6.

For (e), using Equations 7.30 and 7.31, we see

$$\begin{aligned} W_\lambda(\pi_\lambda(U))(F) &= \pi_\lambda(U)((\mathcal{F}_2^{-1}F)\circ\Phi) \\ &= \frac{1}{|\lambda|^n}U(\mathcal{F}_2((\mathcal{F}_2^{-1}F)\circ\Phi\circ\Phi^{-1}) \\ &= \frac{1}{|\lambda|^n}U(\mathcal{F}_2\mathcal{F}_2^{-1}F) \\ &= \frac{1}{|\lambda|^n}U(F). \end{aligned}$$

The same argument works for (f). □

The bars or conjugates in the above results can all be removed if desired by replacing h by $\bar{h}$.

Exercise Set 7.2

1. Show the representation π_λ where $\lambda \neq 0$ can be obtained from the representation π_1 by $\pi_\lambda(g) = \pi_1(\phi_\lambda(g))$ where ϕ_λ is the group automorphism of H_n defined by

$$\phi_\lambda(x+iy,t) = (x+i\lambda y, \lambda t).$$

2. In Remark 4.81 of Chapter 4, it was stated that the windowed Fourier transform is a matrix coefficient of a representation of the Heisenberg group. Indeed, let ρ be the representation π_λ when $\lambda = -2\pi$. Show if ψ and f are in $L^2(\mathbb{R}^n)$, then the windowed Fourier transform given in Definition 4.4.21 satisfies

$$S_\psi(f)(u,\omega) = \frac{e^{\pi i(u\cdot w)}}{(2\pi)^{n/2}}(f,\rho(u+i\omega,0)\psi)_2.$$

Then show the orthogonality relation given in Theorem 7.9 is the Plancherel formula given in Theorem 4.4.80.

3. Show if $F\in L^2(\mathbb{R}^n\times\mathbb{R}^n)$, the Weyl transform $\pi_\lambda(F)$ as a distribution is the distribution given by the L^2 function $\mathcal{F}_2(F)\circ\Phi$.

4. Let $F\in L^2(\mathbb{R}^n\times\mathbb{R}^n)$. Show $W_\lambda(F)$ as a distribution is given by the function $\frac{1}{|\lambda|^n}\mathcal{F}_2^{-1}(F\circ\Phi^{-1})$.

5. Show the mappings $(f,h)\mapsto f\otimes_\lambda\bar{h}$ and $(f,h)\mapsto f\otimes_{-\lambda}h$ are continuous bilinear mappings of $\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$ whose ranges span Schwartz dense linear subspaces.

6. Show if U is a tempered distribution on $\mathbb{R}^n \times \mathbb{R}^n$, then the Weyl transform $\pi_\lambda(U)$ satisfies

$$\pi_\lambda(U)(f \otimes h) = \iint \left(\int_{\mathbb{R}^n} f(w)\pi_\lambda(x+iy,0)h(w)\, d_n w \right) dU(x,y).$$

6. Twisted Convolution on $L^1(\mathbb{R}^n \times \mathbb{R}^n)$

We now define a natural multiplication on functions on $\mathbb{R}^n \times \mathbb{R}^n$ which is based on multiplication on H_n. It is called twisted convolution and will be denoted by

$$F\#_\lambda H.$$

It should have the property that since π_λ respects multiplication on H_n, π_λ respects twisted convolution on functions. Suppose we have functions F and H which are 0 except on finite sets. We would want

$$(\sum F(z)\pi_\lambda(z,0))(\sum H(z')\pi_\lambda(z',0)) = \sum (F\#_\lambda H)(w)\pi_\lambda(w,0).$$

So then we would have

$$\begin{aligned}\sum_w (F\#_\lambda H)(w)\pi_\lambda(w,0) &= \sum_w \sum_{z+z'=w} F(z)H(z')\pi_\lambda(z+z', \frac{1}{2}\langle z,z'\rangle) \\ &= \sum_w \sum_z F(z)H(w-z)e^{\frac{1}{2}i\lambda\langle z,w-z\rangle}\pi_\lambda(w,0) \\ &= \sum_w \left(\sum_z F(z)H(w-z)e^{\frac{1}{2}i\lambda\langle z,w\rangle} \right) \pi_\lambda(w,0).\end{aligned}$$

This suggests one would define

$$F\#_\lambda H(w) = \sum_z F(z)H(w-z)e^{\frac{1}{2}i\lambda\langle z,w\rangle},$$

and this is precisely what one would do if one were using the discrete topology on H_n.

For the usual topology, we make the following definition.

Definition 7.17. *Let F and H be in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Then the twisted convolution $F\#_\lambda H$ of F and H is defined by*

$$F\#_\lambda H(w) = \int e^{i\frac{\lambda}{2}\langle z,w\rangle} F(z)H(w-z)\, d_{2n}z.$$

We first remark that $z \mapsto F(z)H(w-z)$ is integrable for a.e. w. Indeed, this follows from Fubini's Theorem and the invariance of Lebesgue measure. Moreover, if one takes the usual adjoint F^* defined for $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$F^*(x,y) = \overline{F(-x,-y)},$$

then as can be seen by Exercise 7.3.5, $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ with twisted convolution and this adjoint is a noncommutative Banach $*$ algebra. In particular, one has:

$$|F \#_\lambda H|_1 \leqslant |F|_1 \, |H|_1 \tag{7.35}$$

$$(F \#_\lambda G) \#_\lambda H = F \#_\lambda (G \#_\lambda H) \tag{7.36}$$

$$(F \#_\lambda H)^* = H^* \#_\lambda F^* \tag{7.37}$$

as well as the obvious relations $(F^*)^* = F$ and $|F^*|_1 = |F|_1$ for F, G, and H in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

What we have defined is sometimes called **λ-twisted convolution**. If $\lambda = 1$, we obtain the standard definition of twisted convolution. When $\lambda = 0$, we have ordinary convolution of functions on $\mathbb{R}^{2n}$.

In Section 14 of Chapter 6 we showed how one can integrate unitary representations to representations of $L^1(G)$. We can do a similar construction and integrate representations π of H_n where $\pi(0,t) = e^{i\lambda t} I$ for all t to give representations of $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. We give the weak definition. The arguments for the existence of these integrals are made in the same manner as when one integrated representations in Section 14 of Chapter 6.

Definition 7.18. *Let π be a unitary representation of H_n on a Hilbert space $\mathcal{H}$ satisfying $\pi(0,t) = e^{i\lambda t} I$ for all $t \in \mathbb{R}$. Let $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. For $v \in \mathcal{H}$, define $\pi(F)v$ by*

$$(\pi(F)v, w) = \iint F(x,y)(\pi(x+iy,0)v, w)_{\mathcal{H}} \, d_n x \, d_n y$$

for $w \in \mathcal{H}$.

One can also give a strong definition for $\pi(F)v$ by approximating F by simple functions and then taking a limit in $\mathcal{H}$. Thus we write:

$$\pi(F)v = \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} F(x,y)\pi(x+iy,0)v \, d_n x \, d_n y \tag{7.38}$$

for $v \in \mathcal{H}$.

One would expect

$$\pi(F)\pi(H) = \pi(F \#_\lambda H)$$

where

$$\pi(F) = \int F(z,0)\pi\lambda(z,0) \, d_{2n} z.$$

Also we have $\pi(F)$ is bounded when $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ and

$$||\pi(F)|| \leqslant |F|_1. \tag{7.39}$$

Indeed,

$$\begin{aligned}
|(\pi(F)v, w)_{\mathcal{H}}| &\leqslant \int |F(z)|\,|(\pi(z,0)v, w)_{\mathcal{H}}|\,d_{2n}z \\
&\leqslant \int |F(z)|\,||\pi(z,0)v||_{\mathcal{H}}\,||w||_{\mathcal{H}}\,d_{2n}z \\
&\leqslant |F|_1||v||_{\mathcal{H}}\,||w||_{\mathcal{H}}
\end{aligned}$$

for all v, $w \in \mathcal{H}$. So $|\pi(F)| \leqslant |F|_1$.

Proposition 7.19. *Suppose* π *is a unitary representation on* H_n *satisfying* $\pi(0,t) = e^{i\lambda t}I$ *for all* t. *Then* $\pi(F\#H) = \pi(F)\pi(H)$ *and* $\pi(F^*) = \pi(F)^*$ *for* F, $H \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof. Formally, we have:

$$\begin{aligned}
\pi(F\#_\lambda H) &= \iint F\#H(z)\pi(z)\,d_{2n}z \\
&= \iint \left(\iint e^{\frac{\lambda}{2}i\langle z',z\rangle}F(z')H(z-z')\,d_{2n}z'\right)\pi(z,0)\,d_{2n}z \\
&= \iiiint e^{\frac{\lambda}{2}i\langle z',z+z'\rangle}F(z')H(z)\pi(z+z',0)\,d_{2n}z'\,d_{2n}z \\
&= \iiiint e^{\frac{\lambda}{2}i\langle z',z\rangle}F(z')H(z)e^{-i\frac{\lambda}{2}\langle z',z\rangle}\pi(z',0)\pi(z,0)\,d_{2n}z'\,d_{2n}z \\
&= \iint F(z')\pi(z',0)d_{2n}z' \iint H(z)\pi(z,0)d_{2n}z \\
&= \pi(F)\pi(H)
\end{aligned}$$

and

$$\begin{aligned}
(\pi(F)v, w)_{\mathcal{H}} &= \iint F(z)(\pi(z,0)v, w)_{\mathcal{H}}\,d_{2n}z \\
&= \iint F(z)(v, \pi(-z,0)w)_{\mathcal{H}}\,d_{2n}z \\
&= \iint F(-z)(v, \pi(z,0)w)_{\mathcal{H}}\,d_{2n}z \\
&= (v, \iint \overline{F(-z)}\pi(z,0)w\,d_{2n}z)_{\mathcal{H}} \\
&= (v, \pi(F^*)w)_{\mathcal{H}}.
\end{aligned}$$

□

7. Twisted Convolution on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$

We also note we can define the twisted convolution of any two L^2 functions F and H by

$$F\#_\lambda H(w) = \int e^{i\frac{\lambda}{2}\langle z,w\rangle} F(z)H(w-z)\,d_{2n}z. \tag{7.40}$$

Note this integral exists for each w for $z \mapsto F(z)$ and $z \mapsto H(w-z)$ are L^2 functions on $\mathbb{R}^n \times \mathbb{R}^n$ for all w. Indeed Exercise 7.3.1 shows $F\#_\lambda H$ is in fact a continuous function when F and H are in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, when $\lambda = 0$, $F\#_0 H(w) = (F, L(w)H)_2$ where L is the left regular representation $\mathbb{R}^{2n}$.

In Definition 7.12 we defined $\pi_\lambda(F)$ when F is an L^2 function. There it was a distribution given by the L^2 function $\mathcal{F}_2(F) \circ \Phi = \pi_\lambda(F)$. Thus it can be thought of as a Hilbert-Schmidt operator with an L^2 kernel. Our intent in this section is to show why this is an appropriate interpretation.

Recall that $\pi_\lambda(U)$ is defined for every tempered distribution U on $\mathbb{R}^n \times \mathbb{R}^n$ and that if F is a Schwartz function, then $\pi_\lambda(F)$ is the distribution given by the Schwartz function $(\mathcal{F}_2 F) \circ \Phi$. Furthermore, as seen in Exercise 7.2.3, if $F \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, then the distribution $\pi_\lambda(F)$ is given by the L^2 function $(\mathcal{F}_2 F) \circ \Phi$. This makes Definition 7.12 consistent with the definition given in Equation 7.29. Moreover, when F is Schwartz, $\pi_\lambda(F)$ thought of as an operator on $L^2(\mathbb{R}^n)$ has Schwartz kernel $(\mathcal{F}_2 F) \circ \Phi$. If F is L^2, we should be able to interpret $\pi_\lambda(F)$ as a Hilbert-Schmidt operator with L^2 kernel $(\mathcal{F}_2 F) \circ \Phi$. The operator $\pi_\lambda(F)$ should be defined by

$$\pi_\lambda(F)f(x) = \iint F(u,v)\pi_\lambda(u+iv,0)f(x)\,d_n u\,d_n v.$$

That is the main content of the following theorem.

Theorem 7.20. *Let $\lambda \neq 0$. If F is an L^1 and L^2 function on $\mathbb{R}^n \times \mathbb{R}^n$ and if $f \in L^2(\mathbb{R}^n)$, then $F(u,v)\pi_\lambda(u+iv,0)f(x)$ is integrable in u and v for a.e. x and the operator $\pi_\lambda(F)$ defined in Definition 7.18 is given by*

$$\pi_\lambda(F)f(x) = \iint F(u,v)\pi_\lambda(u+iv,0)f(x)\,d_n u\,d_n v.$$

Furthermore π_λ on $L^1 \cap L^2$ extends to a linear mapping from $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ onto the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. Also,

(a) *for $F \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, $\pi_\lambda(F)$ has L^2 kernel $\mathcal{F}_2(F) \circ \Phi$ and*

$$||\pi_\lambda(F)||_2 = \frac{1}{|\lambda|^{n/2}}|F|_2;$$

(b) *if* $F, H \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$*, then* $F\#_\lambda H \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ *and*

$$|F\#_\lambda H|_2 \leqslant \frac{1}{|\lambda|^{n/2}} |F|_2 \, |H|_2,$$
$$\pi_\lambda(F\#_\lambda H) = \pi_\lambda(F)\pi_\lambda(H);$$

(c) $\pi_\lambda(F^*) = \pi_\lambda(F)^*$ *for* $F \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof. We claim $F(u,v)\pi_\lambda(u+iv,0)f(x) = F(u,v)e^{\frac{1}{2}i\lambda(u,v)}e^{-i\lambda(v,x)}f(x-u)$ is integrable in u and v for a.e. x. First take $T(u) = \int |F(u,v)| \, d_n v$. By Fubini's Theorem, $\int T(u) \, d_n u = |F|_1$. Thus $|T|_1 = |F|_1$. Furthermore, by Lemma 2.77 $u \mapsto T(u)f(x-u)$ is integrable a.e. x and

$$\int \left(\int T(u)|f(x-u)| \, d_n u \right)^2 d_n x \leqslant |T|_1^2 \, |f|_2^2.$$

From this we see

$$\int \left(\iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} |F(u,v)f(x-u)| \, d_n u \, d_n v \right)^2 d_n x \leqslant |F|_1^2 |f|_2^2.$$

In particular, $\iint F(u,v)\pi_\lambda(u+iv,0)f(x) \, d_n u \, d_n v$ exists for a.e. x and defines an L^2 function on $\mathbb{R}^n$. Consequently, A defined by

$$A(f)(x) = \iint F(u,v)\pi_\lambda(u+iv,0)f(x) \, d_n u \, d_n v$$

is a bounded linear operator on $L^2(\mathbb{R}^n)$. We shall show $A = \pi_\lambda(F)$.

We start by finding a kernel for A. Recall we know $F(u,v)f(x-u)$ is integrable in u and v for a.e. x. Also since $F \in L^2$, one has for a.e. x and u, $F(x-u,v)$ is integrable in v and therefore:

$$\begin{aligned}
Af(x) &= \iint F(u,v)\pi_\lambda(u+iv,0)f(x) \, d_n u \, d_n v \\
&= \iint F(u,v)e^{\frac{1}{2}i\lambda(u,v)}e^{-i\lambda(v,x)}f(x-u) \, d_n u \, d_n v \\
&= \iint F(x+u,v)e^{\frac{1}{2}i\lambda(x+u,v)}e^{-i\lambda(v,x)}f(-u) \, d_n u \, d_n v \\
&= \iint F(x-u,v)e^{-\frac{1}{2}i\lambda(x+u,v)}f(u) \, d_n u \, d_n v \\
&= \int \mathcal{F}_2 F(x-u, \frac{1}{2}\lambda(x+u))f(u) \, d_n u \\
&= \int (\mathcal{F}_2 F) \circ \Phi(x,u)f(u) \, d_n u.
\end{aligned}$$

Thus A has L^2 kernel $(\mathcal{F}_2 F) \circ \Phi$ and so by Theorem 2.34 is a Hilbert-Schmidt operator.

To see $A = \pi_\lambda(F)$, we first note that for $f, h \in L^2(\mathbb{R}^n)$, Hölder's inequality, the formula for Φ^{-1} given in Equation 7.18, and Corollary 7.8 give the following:

$$(x,y) \mapsto (\mathcal{F}_2 F) \circ \Phi(x,y)\,(\bar{h} \otimes f)(x,y) \text{ is integrable} \tag{7.41}$$

$$(\bar{h} \otimes f) \circ \Phi^{-1}(x,y) = \bar{h}(\frac{x}{2} + \frac{y}{\lambda}) f(-\frac{x}{2} + \frac{y}{\lambda}) = (f \otimes \bar{h}) \circ \Phi^{-1}(-x,y) \tag{7.42}$$

$$(\pi_\lambda(x+iy,0)f, h)_2 = \frac{1}{|\lambda|^n} (\mathcal{F}_2^{-1}(f \otimes \bar{h}) \circ \Phi^{-1})(-x,-y) \tag{7.43}$$

Thus using Equation 7.19,

$$\begin{aligned}
\int Af(x)\,\bar{h}(x)\,d_n x &= \iint \mathcal{F}_2(F) \circ \Phi(x,y) f(y)\,d_n y\,\bar{h}(x)\,d_n x \\
&= \iint \mathcal{F}_2(F) \circ \Phi(x,y)(\bar{h} \otimes f)(x,y)\,d_n x\;d_n y \\
&= \frac{1}{|\lambda|^n} \iint \mathcal{F}_2 F(x,y)(\bar{h} \otimes f)(\Phi^{-1}(x,y))\,d_n x\,d_n y \\
&= \frac{1}{|\lambda|^n} \iint \mathcal{F}_2 F(x,y)(f \otimes \bar{h})(\Phi^{-1}(-x,y))\,d_n x\,d_n y \\
&= \frac{1}{|\lambda|^n} \iint F(x,y) \mathcal{F}_2((f \otimes \bar{h}) \circ \Phi^{-1})(-x,y)\,d_n x\,d_n y \\
&= \frac{1}{|\lambda|^n} \iint F(x,y) \mathcal{F}_2^{-1}((\bar{h} \otimes f) \circ \Phi^{-1})(-x,-y)\,d_n x\,d_n y \\
&= \iint F(x,y)(\pi_\lambda(x+iy,0)f,h)_2\,d_n x\,d_n y \\
&= (\pi_\lambda(F)f, h)_2.
\end{aligned}$$

Therefore, $Af = \pi_\lambda(F)f$. Thus $\pi_\lambda(F)$ has L^2 kernel $(\mathcal{F}_2 F) \circ \Phi$ and thus is a Hilbert-Schmidt operator.

Now the same argument for finding the L^2 norm of the kernel $K_{\lambda,F} = (\mathcal{F}_2 F) \circ \Phi$ in Equation 7.21 shows

$$|(\mathcal{F}_2 F) \circ \Phi|_2 = \frac{1}{|\lambda|^{n/2}} |F|_2.$$

Consequently, the Hilbert-Schmidt norm of $\pi_\lambda(F)$ is $\frac{1}{|\lambda|^{n/2}}|F|_2$ and since every Schwartz function has form $(\mathcal{F}_2 F) \circ \Phi$ for a Schwartz function F, we see π_λ mapping $L^1 \cap L^2$ into $\mathcal{B}_2(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ is bounded, has dense range and has a bounded inverse. Therefore, π_λ extends to a linear bijection of $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ onto the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. Since $\pi_\lambda(F)$ has Hilbert Schmidt norm $\frac{1}{|\lambda|^{n/2}}|F|_2$, we see convergence in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is the same as convergence in Hilbert-Schmidt norm. Hence we obtain (a).

To prove (b), note from Proposition 7.19 we know from that $\pi_\lambda(F\#_\lambda H) = \pi_\lambda(F)\pi_\lambda(H)$ if F and H are L^1 and L^2 functions on $\mathbb{R}^n \times \mathbb{R}^n$. Thus using Equation 2.2 and Proposition 2.32 we see $\frac{1}{|\lambda|^{n/2}}|F\#_\lambda H|_2 = ||\pi_\lambda(F\#_\lambda H)||_2 \leqslant ||\pi_\lambda(F)||_2\, ||\pi_\lambda(H)||_2 = \frac{1}{|\lambda|^n}|F|_2\,|H|_2$. Thus

$$|F\#_\lambda H|_2 \leqslant \frac{1}{|\lambda|^{n/2}}|F|_2|H|_2$$

for such functions F and H. Note by Exercise 7.3.1 if F and H are in L^2 and F_k and H_k are sequences of functions which are both L^1 and L^2 and converge in L^2 to F and H, then $F_k\#_\lambda H_k$ converges uniformly to $F\#_\lambda H$. Thus by Fatou's Lemma,

$$\begin{aligned}\iint |F\#_\lambda H(x,y)|^2\, d_nx\, d_ny &\leqslant \lim \int |F_k\#_\lambda H_k(x,y)|^2\, d_nx\, d_ny\\ &\leqslant \lim \frac{1}{|\lambda|^n}|F_k|_2^2|H_k|_2^2\\ &= \frac{1}{|\lambda|^n}|F|_2^2\,|H|_2^2.\end{aligned}$$

Thus

$$|F\#_\lambda H|_2 \leqslant \frac{1}{|\lambda|^{n/2}}|F|_2\,|H|_2 \text{ for any } F,\, H \in L^2(\mathbb{R}^n \times \mathbb{R}^n).$$

This implies $F_k\#_\lambda H_k \to F\#_\lambda H$ in L^2 as $k \to \infty$ and consequently,

$$\begin{aligned}\pi_\lambda(F\#_\lambda H) &= \lim_k \pi_\lambda(F_k\#_\lambda H_k)\\ &= \lim_k \pi_\lambda(F_k)\,\pi_\lambda(H_k)\\ &= \pi_\lambda(F)\,\pi_\lambda(H).\end{aligned}$$

Similarly since $F \mapsto F^*$ is an isometry of L^2 and $\pi_\lambda(F_k^*) = \pi_\lambda(F_k)^*$ for function F_k which are both L^1 and L^2, we have $\pi_\lambda(F^*) = \lim \pi_\lambda(F_k^*) = \lim \pi_\lambda(F_k)^* = \pi_\lambda(F)^*$. □

It is surprising that the twisted convolution of two square integrable functions on $\mathbb{R}^n \times \mathbb{R}^n$ is square integrable. Indeed, Exercise 2.5.15 shows the ordinary convolution of two square integrable functions may not be square integrable.

8. The Unitary Dual

Let π be an irreducible unitary representation of H_n.

Lemma 7.21. *There is a scalar λ such that $\pi(0,t) = e^{i\lambda t}I$.*

Proof. We have $\pi(0,t)\pi(z,s) = \pi(z,s)\pi(0,t)$ for each (z,s). By Schur's Lemma, there is a complex scalar $\chi(t)$ of absolute value 1 with $\pi(0,t) = \chi(t)I$. But $\pi(0,t_1+t_2) = \pi(0,t_1)\pi(0,t_2)$ implies $\chi(t_1+t_2) = \chi(t_1)\chi(t_2)$. By Corollary 6.51, $x(t) = e^{i\lambda t}$ for some real scalar λ. $\square$

Proposition 7.22. *Let π be an irreducible unitary representation of H_n with $\pi(0,t) = I$ for all t. Then π is one dimensional and there is a vector $w \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\pi(z,t) = e^{i\langle z,w\rangle}I$ for all (z,t).*

Proof. Note $z \to \pi(z,0)$ is an irreducible unitary representation of the abelian group $\mathbb{R}^n \times \mathbb{R}^n$. By Corollary 6.51, π is one dimensional. Thus $\pi(z,t) = \chi(z)$ where χ is a continuous homomorphism of $\mathbb{R}^n \times \mathbb{R}^n$ into the complex numbers of modulus 1. From Table 1 in Chapter 6, we know that each irreducible unitary representation of $\mathbb{R}$ is given by $t \mapsto e^{i\mu t}$ for some unique real scalar μ. Hence $\chi((te_j,0)) = e^{i\lambda_j t}$ and $\chi((0,te_j)) = e^{i\mu_j t}$ where λ_j and μ_j are real scalars. Set $w = ((-\mu_1, -\mu_2, \ldots, -\mu_n), (\lambda_1, \lambda_2, \ldots, \lambda_n))$. Then if $z = (t,s)$ where $t,\ s \in \mathbb{R}^n$, then

$$\begin{aligned}
\pi((x,y),t) &= \prod e^{i\lambda_j t_j} e^{i\mu_j s_j} \\
&= e^{i\langle (t,s),(-\mu,\lambda)} \\
&= e^{i\langle z,w\rangle}.
\end{aligned}$$

$\square$

We now suppose $\pi(0,t) = e^{i\lambda t}$ where $\lambda \neq 0$.

Proposition 7.23. *Let $f, h, f', h' \in L^2(\mathbb{R}^n)$. Then*

$$(f \otimes_\lambda \bar{h})\#(f' \otimes_\lambda \bar{h}') = \frac{1}{|\lambda|^n}(f',h)_2(f \otimes_\lambda \bar{h}').$$

Proof. This is essentially a corollary of Theorem 7.20 and Theorem 7.9. Indeed,

$$\begin{aligned}
\pi_\lambda((f \otimes_\lambda \bar{h})\#(f' \otimes_\lambda \bar{h}')) &= \pi_\lambda(f \otimes_\lambda \bar{h})\pi_\lambda(f' \otimes_\lambda \bar{h}') \\
&= \frac{1}{|\lambda|^n}(f \otimes \bar{h}) \circ \frac{1}{|\lambda|^n}(f' \otimes \bar{h}') \\
&= \frac{1}{|\lambda|^n}(f',h)_2 \frac{1}{|\lambda|^n}(f \otimes \bar{h}') \\
&= \frac{1}{|\lambda|^n}(f',h)_2 \pi_\lambda(f \otimes_\lambda \bar{h}') \\
&= \pi_\lambda(\frac{1}{|\lambda|^n}(f',h)_2 f \otimes_\lambda \bar{h}').
\end{aligned}$$

Since π_λ is one-to-one on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, we see

$$(f \otimes_\lambda \bar{h})\#(f' \otimes_\lambda \bar{h}') = \frac{1}{|\lambda|^n}(f', h)_2 f \otimes_\lambda \bar{h}'.$$

□

Recall we are using $F_z(z') = \lambda(z)F(z') = F(z'-z)$ for $z, z' \in \mathbb{C}^n$.

Lemma 7.24. *Let $F \in \mathcal{S}_{2n}$ and suppose π is a unitary representation of H_n with $\pi(0,t) = e^{i\lambda t}I$ for $t \in \mathbb{R}^n$. Then*

$$\pi(F)\pi(z,0) = \pi(e^{\frac{1}{2}i\lambda\langle \cdot, z\rangle}F_z) \text{ and}$$

$$\pi(z,0)\pi(F) = \pi(e^{\frac{1}{2}i\lambda\langle z, \cdot\rangle}F_z).$$

Proof.

$$\begin{aligned}
\pi(z,0)\pi(F) &= \int \pi(z,0)F(z')\pi(z',0)\, d_{2n}z' \\
&= \int F(z')\pi(z+z', \tfrac{1}{2}\langle z, z'\rangle)\, d_{2n}z' \\
&= \int e^{\frac{1}{2}i\lambda\langle z, z'\rangle}F(z')\pi(z+z',0)\, d_{2n}z' \\
&= \int e^{\frac{1}{2}i\lambda\langle z, z'\rangle}F(z'-z)\pi(z',0)\, d_{2n}z' \\
&= \pi(e^{\frac{1}{2}i\lambda\langle z, \cdot\rangle}F_z).
\end{aligned}$$

Similarly

$$\pi(F)\pi(z,0) = \pi(e^{\frac{1}{2}i\lambda\langle \cdot, z\rangle}F_z).$$

□

Theorem 7.25. *Let f_0 be a Schwartz function on $\mathbb{R}^n$ with $|f_0|_2 = 1$. Set $E(z) = (f_0, \pi_\lambda(z,0)f_0)_2$. Then the Schwartz function E satisfies*

(a) $E\#_\lambda E = \frac{1}{|\lambda|^n}E$

(b) $E\#_\lambda(e^{i\frac{\lambda}{2}\langle z,\cdot\rangle}E_z) = \frac{1}{|\lambda|^n}E(-z)E$

(c) $E^* = E$

(d) *If π is a unitary representation of H_n on a Hilbert space $\mathcal{H}$ satisfying $\pi(0,t) = e^{i\lambda t}I$ for $t \in \mathbb{R}$, then $|\lambda|^n\pi(E)$ is an orthogonal projection on $\mathcal{H}$.*

Proof. Note by Proposition 7.23,

$$E\#_\lambda E = (f_0 \otimes_\lambda \bar{f}_0)\#_\lambda(f_0 \otimes_\lambda \bar{f}_0) = \frac{1}{|\lambda|^n}(f_0, f_0)_2 f_0 \otimes_\lambda \bar{f}_0 = \frac{1}{|\lambda|^n}E.$$

Now from Lemma 7.24, $\pi_\lambda(z,0)\pi_\lambda(E) = \pi_\lambda(e^{i\frac{\lambda}{2}\langle z,\cdot\rangle}E_z)$. Hence one has $\pi_\lambda(E)\pi_\lambda(z,0)\pi_\lambda(E) = \pi_\lambda(E\#e^{i\frac{\lambda}{2}\langle z,\cdot\rangle}E_z)$. But by Proposition 7.13, $\pi_\lambda(E) = \frac{1}{|\lambda|^n} f_0 \otimes \bar{f}_0$. Thus

$$\begin{aligned}
\pi_\lambda(E)\pi_\lambda(z,0)\pi_\lambda(E)h &= \frac{1}{|\lambda|^{2n}}(f_0\otimes\bar{f}_0)\,\pi_\lambda(z,0)(h,f_0)_2 f_0 \\
&= \frac{1}{|\lambda|^{2n}}(h,f_0)_2(f_0\otimes\bar{f}_0)\,(\pi_\lambda(z,0)f_0) \\
&= \frac{1}{|\lambda|^{2n}}(h,f_0)_2(\pi_\lambda(z,0)f_0,f_0)_2 f_0 \\
&= \frac{1}{|\lambda|^{2n}}E(-z)(h,f_0)_2 f_0 \\
&= \frac{1}{|\lambda|^{2n}}E(-z)(f_0\otimes\bar{f}_0)(h) \\
&= \frac{1}{|\lambda|^{n}}E(-z)\pi_\lambda(E)(h).
\end{aligned}$$

Thus $\pi_\lambda(E\#_\lambda e^{i\frac{\lambda}{2}\langle z,\cdot\rangle}E_z) = \pi_\lambda(\frac{1}{|\lambda|^n}E(-z)E)$. Since π_λ is one-to-one,

$$E\#_\lambda(e^{i\frac{\lambda}{2}\langle z,\cdot\rangle}E_z) = \frac{1}{|\lambda|^n}E(-z)E.$$

For (c), we have

$$\begin{aligned}
E^{\#}(z) &= \overline{E(-z)} \\
&= \overline{(f_0,\pi_\lambda(-z,0)f_0)} \\
&= (\pi_\lambda(-z,0)f_0,f_0) \\
&= (f_0,\pi_\lambda(z,0)f_0) \\
&= E(z).
\end{aligned}$$

Now $\pi(E) = \pi(E^*) = \pi(E)^*$ and $|\lambda|^n\pi(E)\,|\lambda|^n\pi(E) = |\lambda|^{2n}\pi(E\#E) = |\lambda|^{2n}\pi(\frac{1}{|\lambda|^n}E) = |\lambda|^n\pi(E)$. Thus $|\lambda|^n\pi(E)$ is an orthogonal projection. $\square$

Theorem 7.26 (Stone-Von Neumann)**.** *Let $\lambda \neq 0$. Suppose π is a unitary representation of H_n on a Hilbert space $\mathcal{H}$ where $\pi(0,t) = e^{i\lambda t}I$ for all t. Then $\mathcal{H}$ is a direct sum of orthogonal irreducible invariant subspaces $\mathcal{H}_k$ where each unitary representation $\pi|_{\mathcal{H}_k}$ is unitarily equivalent to π_λ. In particular, if π is irreducible, then there is a unitary operator U of $\mathcal{H}$ onto $L^2(\mathbb{R}^n)$ such that*

$$U\pi((x,y),t)U^*f(w) = e^{i\lambda t}e^{\frac{1}{2}i\lambda(x,y)}e^{-i\lambda(y,w)}f(w-x).$$

Proof. To simplify matters we assume $\lambda = 1$. To handle the general case one can either apply Exercises 7.2.1 and 7.3.8 or make easy modifications in the following argument.

We have by Definition 7.18 the operators $\pi(F) = \int F(z,0)\pi(z,0)\, d_{2n}z$ for $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. We let $F\#H = F\#_1 H$. Proposition 7.19 shows $\pi(F)\pi(H) = \pi(F\#H)$ and $\pi(F^*) = \pi(F)^*$. Let f_0 be a Schwartz function on $\mathbb{R}^n$ such that $|f_0|_2 = 1$. By Proposition 7.25, if E is the matrix coefficient $f_0 \otimes_{\pi_1} \bar{f}_0$, we have $\pi(E)$ is an orthogonal projection onto a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$.

We also note that if $F \neq 0$ and is Schwartz and $\mathcal{M}$ is a closed nonzero invariant subspace of $\mathcal{H}$, then $\pi(F)|_{\mathcal{M}} \neq 0$. Indeed, if $u, v \in \mathcal{M}$, then by Lemma 7.24 we have

$$\begin{aligned}(\pi(z,0)\pi(F)\pi(-z,0)v, w)_{\mathcal{H}} &= (\pi(e^{\frac{1}{2}i\langle z,\cdot\rangle}F_z)\pi(-z,0)v, w)_{\mathcal{H}}\\ &= (\pi(e^{\frac{1}{2}i\langle z,\cdot\rangle}e^{\frac{1}{2}\langle \cdot,-z\rangle}F)v, w)_{\mathcal{H}}\\ &= (\pi(e^{i\langle z,\cdot\rangle}F)u, w)_{\mathcal{H}}.\end{aligned}$$

But

$$(\pi(e^{i\langle z,\cdot\rangle}F)u, v)_{\mathcal{H}} = \int e^{i\langle z,z'\rangle}F(z')(\pi(z',0)u, v)_{\mathcal{H}}\, d_{2n}z'.$$

Hence if $\pi(F) = 0$ on $\mathcal{M}$, we see

$$\int e^{i(x\cdot y' - y\cdot x')}F(x'+iy')(\pi(x'+iy',0)u, v)_{\mathcal{H}}\, d_n x\, d_n y = 0$$

for all x, y. This implies $\mathcal{F}(F(\cdot)(\pi(\cdot,0)u,v))_{\mathcal{H}} \equiv 0$ for each u and $v \in \mathcal{M}$. But then $F(z)(\pi(z)u,v)_{\mathcal{H}} = 0$ for all z and $u, v \in \mathcal{M}$. Taking $v = \pi(z,0)u$, we see this can only occur if $F(z) = 0$ for all z. Using Lemma 7.7, we see E is Schwartz, and hence $\pi(E) \neq 0$ on any closed invariant subspace $\mathcal{M}$ of $\mathcal{H}$.

Now note by (d) of Proposition 7.25, $\pi(E)$ is an orthogonal projection. Hence using Lemma 7.24 again and (b) of Proposition 7.25, we have

$$\begin{aligned}(\pi(z,0)\pi(E)u, \pi(z',0)\pi(E)v)_{\mathcal{H}} &= (\pi(E)\pi(-z',0)\pi(z,0)\pi(E)u, v)_{\mathcal{H}}\\ &= (\pi(E)e^{\frac{1}{2}i\langle z,z'\rangle}\pi(z-z',0)\pi(E)u, v)_{\mathcal{H}}\\ &= (e^{\frac{1}{2}i\langle z,z'\rangle}\pi(E)\pi(e^{\frac{1}{2}i\langle z-z',\cdot\rangle}E_{z-z'})u, v)_{\mathcal{H}}\\ &= (e^{\frac{1}{2}i\langle z,z'\rangle}\pi(E\#(e^{\frac{1}{2}i\langle z-z',\cdot\rangle}E_{z-z'}))u, v)_{\mathcal{H}}\\ &= (e^{\frac{1}{2}i\langle z,z'\rangle}\pi(E(z'-z)E)u, v)_{\mathcal{H}}\\ &= e^{\frac{1}{2}i\langle z,z'\rangle}E(z'-z)(\pi(E)u, \pi(E)v)_{\mathcal{H}}.\end{aligned}$$

Thus if $u, v \in \pi(E)\mathcal{H}$, we have

$$(\pi(z,0)u, \pi(z',0)v)_{\mathcal{H}} = e^{\frac{1}{2}i\langle z,z'\rangle}E(z'-z)(u,v)_{\mathcal{H}}.$$

In particular, if u_k form an orthonormal basis of $\pi(E)\mathcal{H}$, then the vector subspaces $\mathcal{H}_k$ which are the closed linear spans of the vectors $\pi(z,0)u_k$ for

$z \in \mathbb{C}^n$ are orthogonal. We claim $\oplus\mathcal{H}_k = \mathcal{H}$. Suppose not. Let $\mathcal{M} = (\oplus\mathcal{H}_k)^\perp$. Then $\pi(E)\mathcal{M} = 0$. Since $\mathcal{M}$ is π invariant, we see $\mathcal{M} = \{0\}$.

Define a mapping U from $\mathcal{H}_k$ into $L^2(\mathbb{R}^n)$ by

$$U(\sum a_j \pi(z_j, 0)u_k) = \sum a_j \pi_1(z_j, 0) f_0.$$

Clearly U is defined on a dense subspace of $\mathcal{H}_k$. Recall

$$(\pi(z, 0)u_k, \pi(z', 0)u_k)_{\mathcal{H}} = e^{\frac{1}{2}i\langle z, z'\rangle} E(z' - z).$$

We do not know if U is well defined, but we do note

$$\begin{aligned}(U(\sum a_r \pi(z_r, 0)u_k), U(\sum a_s \pi(z_s, 0)u_k))_2 &= \sum_{r,s} a_r \bar{a}_j (\pi_1(z_r, 0)f_0, \pi_1(z_s, 0)f_0)_2 \\ &= \sum_{r,s} a_r \bar{a}_s e^{\frac{1}{2}i\langle z_r, z_s\rangle} E(z_s - z_r) \\ &= \sum_{r,s} a_r \bar{a}_s (\pi(z_r, 0)u_k, \pi(z_s, 0)u_k)_{\mathcal{H}} \\ &= (\sum a_r \pi(z_r, 0)u_k, \sum a_s \pi(z_s, 0)u_k)_{\mathcal{H}}.\end{aligned}$$

Hence U is a linear isometry. In particular U is well defined for if $\sum a_r \pi(z_r, 0)u_k = \sum b_s \pi(z'_s, 0)u_k$, then

$$||U(\sum a_r \pi(z_r, 0)u_k - \sum b_s \pi(z'_s, 0)u_k)||_2 = 0$$

and we see

$$\sum a_r \pi_1(z_r, 0)f_0 = \sum b_s \pi_1(z'_s, 0)f_0.$$

Hence U extends to a unitary operator of $\mathcal{H}_k$ onto a closed subspace of $L^2(\mathbb{R}^n)$. Moreover, since

$$\begin{aligned}\pi_1(z, 0)U(\sum a_r \pi(z_r, 0)u_k) &= \sum a_r \pi_1(z, 0)\pi_1(z_r, 0)f_0 \\ &= \sum a_r \pi_1(z + z_r, \frac{1}{2}\langle z, z_r\rangle)f_0 \\ &= \sum e^{\frac{1}{2}i\langle z, z_r\rangle} a_r \pi_1(z + z_r, 0)f_0 \\ &= U(\sum e^{\frac{1}{2}i\langle z, z_r\rangle} a_r \pi(z + z_r, 0)u_k) \\ &= U\pi(z, 0)(\sum a_r \pi(z_r, 0)u_k),\end{aligned}$$

we have

$$\pi_1(z, 0)U(v) = U(\pi(z, 0)v)$$

for all $v \in \mathcal{H}_k$. Thus the range of U is a nonzero closed invariant subspace of π_1. Since π_1 is irreducible, $U(\mathcal{H}_k) = L^2(\mathbb{R}^n)$, and since

$$U\pi(z, t) = e^{it}U\pi(z, 0) = e^{it}\pi_1(z, 0)U = \pi_1(z, t)U,$$

we see π on the Hilbert space $\mathcal{H}_k$ is unitarily equivalent to π_1. □

Exercise Set 7.3

1. Let F and H be in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Show $F\#_\lambda H$ is a continuous function on $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, show if F_k and H_k are sequences in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ converging in norm to F and H, then $F_k\#_\lambda H_k$ converges uniformly to $F\#_\lambda H$ as $k \to \infty$.

2. Let $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Show if H is Schwartz, then

$$\iint \int |H(x,y)F(x,v)|\, d_n x\, d_n y\, d_n v < \infty.$$

3. Show if F is in $L^1(\mathbb{R}^n)$ and $\lambda \neq 0$, then the Weyl transform $\pi_\lambda(F)$ as a distribution is given by the function $\mathcal{F}_2(F) \circ \Phi$

4. Show if F is in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ and $\lambda \neq 0$, then the bounded linear operator $\pi_\lambda(F)$ on $L^2(\mathbb{R}^n)$ is given on Schwartz functions f by

$$\pi_\lambda(F)(f)(x) = \int (\mathcal{F}_2 F) \circ \Phi(x,y) f(y)\, d_n y$$

a.e. x. **Hint:** Show $(\mathcal{F}_2 F) \circ \Phi(x,y) f(y)$ is integrable in y for a.e. x.

5. Show $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ is a noncommutative Banach $*$ algebra. That is show twisted convolution is associative, $||F\#_\lambda H||_1 \leqslant ||F||_1 ||H||_1$, $F \mapsto F^*$ is a norm preserving conjugate linear isomorphism and $(F\#_\lambda H)^* = H^*\#_\lambda F^*$.

6. Show a unitary representation π of H_n on a Hilbert space $\mathcal{H}$ with $\pi(0,t) = e^{i\lambda t} I$ for all t where λ is a nonzero real number is irreducible if and only if the only closed invariant subspaces under all the operators $\pi(F) = \iint F(x,y)\pi(x+iy,0)\, d_n x\, d_n y$ for $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ are $\{0\}$ and $\mathcal{H}$.

7. As seen in Exercise 2.5.16, the space of Hilbert-Schmidt operators is an example of a Hilbert algebra. Let $d_\lambda z$ be the measure on $\mathbb{R}^n \times \mathbb{R}^n$ given by $d_\lambda z = |\lambda|^n\, d_{2n} z = |\lambda|^n\, d_n x \times d_n y$. Show the Weyl transform π_λ is a Hilbert algebra isometry of $L^2(\mathbb{R}^n \times \mathbb{R}^n, d_\lambda z)$ under λ-twisted convolution and function adjoints onto the Hilbert algebra of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ under composition and operator adjoints.

8. Suppose λ is a nonzero real number. Let π be a unitary representation of the Heisenberg group H_n on Hilbert space $\mathcal{H}$ satisfying $\pi(0,t) = e^{i\lambda t} I$ for all t. Define π' on H_n by $\pi'(x+iy,t) = \pi(x + \frac{iy}{\lambda}, \frac{t}{\lambda})$. Show π' is a unitary representation of H_n satisfying $\pi'(0,t) = e^{i\lambda t} I$ for all t.

9. Let U be an $n \times n$ unitary matrix. Show the representations π_λ and $(z,t) \mapsto \pi(zU,t)$ are unitarily equivalent.

10. Let $\tilde{\pi}_\lambda(z,t) = \pi_\lambda(iz,t)$. Show the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary equivalence of π_1 with $\tilde{\pi}_1$.

11. Show $\int_{\mathbb{R}^n} e^{-i(z,x)} e^{-\frac{1}{2}|x|^2}\, d_n x = e^{-\frac{1}{2}z^2}$ where $z^2 = \sum z_j^2$. (Hint: It suffices to do the one-dimensional case.)

9. The Plancherel Measure

Proposition 7.27. *Let f be a Schwartz function on H_n. Then if $\lambda \neq 0$, $\pi_\lambda(f)$ is a trace class operator on $L^2(\mathbb{R}^n, d_n x)$ and*

$$\operatorname{Tr}(\pi_\lambda(f)) = \frac{1}{|\lambda|^n} \mathcal{F}_c^{-1} f(0, \lambda)$$

where $\mathcal{F}_c^{-1}$ is the inverse Fourier transform over the center; i.e.

$$\mathcal{F}_c^{-1} f(z, \lambda) = \int f(z, t) e^{i\lambda t}\, d_1 t.$$

Proof.

$$\begin{aligned}
\pi_\lambda(f) &= \iint f(z,t)\pi_\lambda(z,t)\, d_{2n}z\, d_1 t \\
&= \iint f(z,t)e^{i\lambda t}\, d_1 t\, \pi_\lambda(z,0)\, d_{2n} z \\
&= \int F_\lambda(z)\pi_\lambda(z,0)\, d_{2n} z \\
&= \pi_\lambda(F_\lambda)
\end{aligned}$$

where

$$F_\lambda(z) = \mathcal{F}_c f(z, -\lambda)$$

and $\mathcal{F}_c$ is the Fourier transform on H_n in the central coordinate t. Hence F_λ is Schwartz. In this case we know the operator $\pi_\lambda(F_\lambda)$ has Schwartz kernel the Weyl transform $\pi_\lambda(F_\lambda) = \mathcal{F}_2(F_\lambda) \circ \Phi$. Hence by Theorem 4.74 and Formulas 7.16 and 7.17,

$$\begin{aligned}
\operatorname{Tr}(\pi_\lambda(f)) &= \operatorname{Tr}(\pi_\lambda(F_\lambda)) \\
&= \int \mathcal{F}_2(F_\lambda) \circ \Phi(x,x)\, d_n x \\
&= \int \mathcal{F}_2(F_\lambda)(0, \lambda x)\, d_n x \\
&= \iint F_\lambda(0,w) e^{i(\lambda x, w)}\, d_n w\, d_n x \\
&= \frac{1}{|\lambda|^n} \int e^{-i(0,x)} \int \mathcal{F}_c f((0,w), -\lambda)\, e^{i(x,w)}\, d_n w\, d_n x \\
&= \frac{1}{|\lambda|^n} \mathcal{F}_c f(0, -\lambda).
\end{aligned}$$

$\square$

Note

$$\begin{aligned}\pi_\lambda(g)\pi_\lambda(f) &= \int f(g')\pi_\lambda(g)\pi_\lambda(g')\,dm(g')\\ &= \int f(g')\pi_\lambda(gg')\,dm(g')\\ &= \int f(g^{-1}g')\pi_\lambda(g')\,dm(g')\\ &= \int L(g)f(g')\pi_\lambda(g')\,dm(g')\\ &= \pi_\lambda(L(g)f)\end{aligned}$$

where $f \in L^1(H_n)$.

Recall the left regular representation of a locally compact group G with a left Haar measure m is the representation L defined on $L^2(G)$ by $L(g)f(x) = f(g^{-1}x)$ and the right regular representation of G is defined on $L^2(G)$ by $R(g)f(x) = \Delta(g)^{1/2}f(xg)$. Moreover, the biregular representation is the representation B of $G \times G$ given by $B(g_1, g_2)f = L(g_1)R(g_2)f$.

We have denoted the Hilbert space of Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$ by $\mathcal{B}_2(\mathcal{H}, \mathcal{H})$ and also by $\mathcal{H}\bar{\otimes}\bar{\mathcal{H}}$. To simplify notation, we use $\mathcal{H}_2$ to denote the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n, d_nx)$. Thus

$$\mathcal{H}_2 = L^2(\mathbb{R}^n, d_nx) \otimes \overline{L^2(\mathbb{R}^n, d_nx)}.$$

Let (X, μ) be a σ-finite measure space. A function $T : X \to \mathcal{H}_2$ is said to be **strongly measurable** if $x \mapsto T(x)f$ is measurable from X into the Hilbert space $L^2(\mathbb{R}^n, d_nx)$ for each f in $L^2(\mathbb{R}^n, d_nx)$. As seen by Exercise 7.4.2, T is strongly measurable if and only if T is a measurable function from X into $\mathcal{H}_2$ and the space $L^2(X, \mathcal{H}_2)$ with inner product given by

$$(T_1, T_2) = \int_X (T_1(x), T_2(x))_2\,d\mu(x)$$

is a Hilbert space.

In Theorem 7.20, we see if $\lambda \neq 0$, there is a bicontinuous onto linear transformation $\pi_\lambda : L^2(\mathbb{R}^n \times \mathbb{R}^n, d_{2n}z) \to \mathcal{H}_2$ given by

$$\pi_\lambda(F)(h) = \int \mathcal{F}_2(F) \circ \Phi(x, y)\,h(y)\,d_ny.$$

It satisfies

$$\pi_\lambda(F)(h) = \iint F(x, y)\pi_\lambda(x + iy, 0)h\,d_nx\,d_ny$$

when F is in $L^1 \cap L^2$. Now if $f \in L^2(H_n)$, $\pi_\lambda(f)$ is defined a.e. λ. Indeed, $f = \mathcal{F}_c(\mathcal{F}_c^{-1}f)$ where $\mathcal{F}_c^{-1}f \in L^2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$. Thus for almost all λ,

$\mathcal{F}_c^{-1} f(\cdot, \lambda)$ is in $L^2(\mathbb{R}^n \times \mathbb{R}^n,\, d_n x \times\, d_n y)$. We define

$$\pi_\lambda(f) = \pi_\lambda(F_\lambda)$$

when $F_\lambda(x, y) = \mathcal{F}_c^{-1} f(x, y, \lambda)$ is in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. This agrees with our earlier definition of $\pi_\lambda(f)$ when $f \in L^1(H_n) \cap L^2(H_n)$. Indeed,

$$\begin{aligned}
\pi_\lambda(f)(h)(x) &= \iiint f(u+iv, t)\pi_\lambda(u+iv, t)h(x)\ d_1 t\, d_n u\, d_n v \\
&= \iiint f(u+iv, t)e^{i\lambda t}\pi_\lambda(u+iv, 0)h(x)\, d_1 t\, d_n u\, d_n v \\
&= \iint \mathcal{F}_c^{-1} f(u+iv, \lambda)\pi_\lambda(u+iv, 0)h(x)\, d_n u\, d_n v \\
&= \pi_\lambda(F_\lambda)h(x).
\end{aligned}$$

To see $\lambda \mapsto \pi_\lambda(f)$ is strongly measurable, it suffices by Exercise 7.4.3 to show $\lambda \mapsto (h_1, \pi_\lambda(f)h_2)_2$ is measurable for each pair h_1 and h_2 of Schwartz functions on $\mathbb{R}^n$. By Exercise 7.2.3, the Weyl transform $\pi_\lambda(F_\lambda)$ of the distribution F_λ is given by the function $\mathcal{F}_2(F_\lambda) \circ \Phi$. Putting this together using Theorem 7.16, we see

$$\begin{aligned}
\iint \mathcal{F}_2(F_\lambda) \circ \Phi(x, y)\bar{h}_1 \otimes h_2(x, y)\, d_n x\, d_n y &= \pi_\lambda(F_\lambda)(\bar{h}_1 \otimes h_2) \\
&= \iint \bar{h}_1 \otimes_\lambda h_2(x, y)\, F_\lambda(x, y)\, d_n x\, d_n y.
\end{aligned}$$

Now $(x, y, \lambda) \mapsto \bar{h}_1 \otimes_\lambda h_2(x, y)$ is measurable. Using the measurability of $\mathcal{F}_c^{-1} f$ and Fubini's Theorem, we see

$$\begin{aligned}
(h_1, \pi_\lambda(F_\lambda)h_2)_2 &= \int h_1(u)\overline{\pi_\lambda(F_\lambda)h_2(u)}\, d_n u \\
&= \int h_1(u)\overline{\int \mathcal{F}_2(F_\lambda) \circ \Phi(u, v)h_2(v)\, d_n v}\ d_n u \\
&= \overline{\int \bar{h}_1(u) \int \mathcal{F}_2(F_\lambda) \circ \Phi(u, v)h_2(v)\, d_n v\ d_n u} \\
&= \overline{\iint \mathcal{F}_2(F_\lambda) \circ \Phi(u, v)(\bar{h}_1 \otimes h_2)(u, v)\, d_n u\, d_n v} \\
&= \overline{\pi_\lambda(F_\lambda)(\bar{h}_1 \otimes h_2)} \\
&= \iint \bar{h}_1 \otimes_\lambda h_2(x, y)\, F_\lambda(x, y)\, d_n u\, d_n v \\
&= \iint \bar{h}_1 \otimes_\lambda h_2(x, y)\, \mathcal{F}_c^{-1} f(x, y, \lambda)\, d_n x\, d_n y
\end{aligned}$$

is measurable in λ.

Theorem 7.28 (Plancherel Theorem). *The mapping $f \mapsto \hat{f}$ where $\hat{f}(\lambda) = \pi_\lambda(f)$ is a unitary mapping from $L^2(H_n, d_{2n}z \times d_1 t)$ onto the Hilbert space $L^2(\mathbb{R}, \mathcal{H}_2, |\lambda|^n \, d_1\lambda)$. Moreover, if f is Schwartz or is a finite linear combination of functions of form $f_1 * f_2$ where f_1 and f_2 are in $L^1(H_n) \cap L^2(H_n)$, then*

$$f(z,t) = \int \mathrm{Tr}(\pi_\lambda(-z,-t)\pi_\lambda(f))|\lambda|^n \, d_1\lambda.$$

Proof. We show $f \mapsto \hat{f}$ where $\hat{f}(\lambda) = \pi_\lambda(f)$ is a well defined unitary isomorphism of $L^2(H_n)$ onto $L^2(\mathbb{R}, \mathcal{H}_2, |\lambda|^n \, d_1\lambda)$. Note by Theorem 7.20 that the mapping $F \mapsto \pi_\lambda(F)$ from $L^2(\mathbb{R}^n \times \mathbb{R}^n, d_{2n}z)$ into $\mathcal{H}_2$ is one-to-one and onto and satisfies

$$||\pi_\lambda(F)||_2 = \frac{1}{|\lambda|^{n/2}}|F|_2$$
$$\pi_\lambda(F_1 \#_\lambda F_2) = \pi_\lambda(F_1)\pi_\lambda(F_2) \text{ and}$$
$$\pi_\lambda(F^*) = \pi_\lambda(F)^*.$$

Hence $|\hat{f}(\lambda)|_2 = \frac{1}{|\lambda|^{n/2}}|F_\lambda|_2$ where $F_\lambda(z,\lambda) = \mathcal{F}_c^{-1} f(z,\lambda)$. and we see

$$\begin{aligned}
\int |\hat{f}(\lambda)|_2^2 \, |\lambda|^n \, d_1\lambda &= \int |F_\lambda|_2^2 \, d_1\lambda \\
&= \iint |\mathcal{F}_c^{-1} f(z,\lambda)|^2 \, d_{2n}z \, d_1\lambda \\
&= \iint |\mathcal{F}_c^{-1} f(z,\lambda)|^2 \, d_1\lambda \, d_{2n}z \\
&= \iint |f(z,t)|^2 \, d_1 t \, d_{2n}z \\
&= |f|_2^2.
\end{aligned}$$

To see that it is onto, we start by recalling from Theorem 7.20 that for $F \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, one has $\pi_\lambda(F)$ has kernel $K_{\lambda,F} = \mathcal{F}_2(F) \circ \Phi$ where $\mathcal{F}_2 F(x,y) = \int F(x,w) e^{-i(y,w)} \, d_n w$ and $\Phi(x,y) = (x-y, \frac{\lambda}{2}(x+y))$.

Suppose $\lambda \mapsto F(\lambda)$ is measurable and satisfies $\int |F(\lambda)|_2^2 \, |\lambda|^n \, d_1\lambda < \infty$. By Exercise 7.4.5, $F(\lambda) \circ \Phi^{-1}$ is measurable in λ and satisfies $\int |F(\lambda) \circ \Phi^{-1}|_2^2 \, d_1\lambda < \infty$. Using Exercise 7.4.5, there is a measurable function $\widetilde{F}$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ satisfying

$$\textbf{(a)} \qquad F(\lambda) \circ \Phi^{-1}(x,y) = \widetilde{F}(x,y,\lambda)$$

for a.e. (x,y) for a.e. λ. In particular $\widetilde{F}$ is $L^2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ and we can define f in $L^2(H_n)$ by

$$f = \mathcal{F}_c \mathcal{F}_2^{-1} \widetilde{F}.$$

We note

$$\hat{f}(\lambda) = \pi_\lambda(f) = \pi_\lambda(F_\lambda)$$

for those λ for which $(x, y) \mapsto F_\lambda(x, y) = \mathcal{F}_c^{-1} f(x, y, \lambda)$ is in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. But since this holds a.e. λ and $\mathcal{F}_c^{-1} f = \mathcal{F}_2^{-1} \widetilde{F}$, we see $F_\lambda(x, y) = \mathcal{F}_2^{-1} \widetilde{F}(x, y, \lambda)$ a.e. (x, y) for a.e. λ. Thus

$$\hat{f}(\lambda) = \pi_\lambda(\mathcal{F}_2^{-1} \widetilde{F}(\cdot, \cdot, \lambda)) \text{ for a.e. } \lambda.$$

This implies $\hat{f}(\lambda)$ is the Hilbert-Schmidt operator with kernel

$$K_\lambda(x, y) = \mathcal{F}_2(\mathcal{F}_2^{-1} \widetilde{F})(\Phi(x, y), \lambda) = \widetilde{F}(\Phi(x, y), \lambda).$$

for a.e. λ. But from Equation (a), $\widetilde{F}(\Phi(x, y), \lambda) = F(\lambda)(x, y)$ a.e. (x, y) for a.e. λ. This gives $\hat{f}(\lambda) = F(\lambda)$ for a.e. λ and we see $f \mapsto \hat{f}$ maps $L^2(H_n)$ onto $L^2(\mathbb{R}, \mathcal{H}_2, |\lambda|^n \, d_1\lambda)$.

Let f be Schwartz on H_n. Set $g = (z, t) = (x + iy, t) = (x, y, t)$ be in H_n. We calculate $\operatorname{Tr}(\pi_\lambda(g)^{-1} \pi_\lambda(f))$. Using Exercise 6.8.4 or by an easy computation, we know $\pi_\lambda(g^{-1}) \pi_\lambda(f) = \pi_\lambda(L(g^{-1}) f)$; and from Proposition 7.27 we have

$$\operatorname{Tr}(\pi_\lambda(L(g^{-1}) f)) = \frac{1}{|\lambda|^n} \mathcal{F}_c^{-1}(L(g^{-1}) f)(0, \lambda).$$

These together with the Fourier inversion formula give

$$\begin{aligned} f(g) &= L(g^{-1}) f((0,0), 0) \\ &= \int \mathcal{F}_c^{-1}(L(g^{-1}) f)((0,0), \lambda) e^{-i\lambda \cdot 0} \, d_1\lambda \\ &= \int \operatorname{Tr}(\pi_\lambda(L(g^{-1}) f)) |\lambda|^n e^{-i\lambda \cdot 0} \, d_1\lambda \\ &= \int \operatorname{Tr}(\pi_\lambda(g)^{-1} \pi_\lambda(f)) \, |\lambda|^n \, d_1\lambda. \end{aligned}$$

Now suppose f_1 and f_2 are integrable and square integrable on H_n. Note

$$f_1 * f_2(g) = (f_1, L(g^{-1}) f_2^*)_2.$$

Indeed,

$$\begin{aligned} f_1 * f_2(g) &= \int f_1(g_1) f_2(g_1^{-1} g)\, dg_1 \\ &= \int f_1(g_1) f_2((g^{-1} g_1)^{-1})\, dg_1 \\ &= \int f_1(g_1) \overline{f_2^*(g^{-1} g_1)}\, dg_1 \\ &= \int f_1(g_1) \overline{L(g) f_2^*(g_1)}\, dg_1 \\ &= (f_1, L(g) f_2^*)_2. \end{aligned}$$

Recall $\pi_\lambda(L(g)f_2^*) = \pi_\lambda(g)\pi_\lambda(f_2^*)$, $\pi_\lambda(f_2^*)^* = \pi_\lambda(f_2)$, and $\pi_\lambda(g)^* = \pi_\lambda(g^{-1})$. Since the mapping $f \mapsto \hat{f}$ is a unitary mapping on $L^2(H_n)$, we then have

$$\begin{aligned} f_1 * f_2(g) &= \int (\pi_\lambda(f_1),\, \pi_\lambda(L(g) f_2^*))_2\, |\lambda|^n\, d_1\lambda \\ &= \int \mathrm{Tr}(\pi_\lambda(f_1)\pi_\lambda(L(g) f_2^*)^*)\, |\lambda|^n\, d_1\lambda \\ &= \int \mathrm{Tr}(\pi_\lambda(f_1)(\pi_\lambda(g)\pi_\lambda(f_2^*))^*)\, |\lambda|^n\, d_1\lambda \\ &= \int \mathrm{Tr}(\pi_\lambda(f_1)\pi_\lambda(f_2)\pi_\lambda(g^{-1}))\, |\lambda|^n\, d_1\lambda \\ &= \int \mathrm{Tr}(\pi_\lambda(g^{-1})\pi_\lambda(f_1 * f_2)\, |\lambda|^n\, d_1\lambda. \end{aligned}$$

□

Remark 7.29. The Hilbert-space $L^2(\mathbb{R}, \mathcal{H}_2, |\lambda|^n\, d_1\lambda)$ is a simple example of a direct integral of Hilbert spaces. In this case $L^2(\mathbb{R}^n, \mathcal{H}_2, |\lambda|^n\, d_1\lambda)$ is $\int_{\mathbb{R}}^{\oplus} \mathcal{H}_2\, d\mu(\lambda)$ where $d\mu(\lambda)$ is the measure $d\mu(\lambda) = |\lambda|^n\, d_1\lambda$. In general, if $\lambda \mapsto \mathcal{H}_\lambda$ is a measurable (appropriately defined) mapping from a measure space (S, μ), then $\int^{\oplus} \mathcal{H}_\lambda\, d\mu(\lambda)$ is the collection of all measurable (again this needs to be defined) functions f on S with $f(\lambda) \in \mathcal{H}_\lambda$ a.e. λ and for which $\int |f(\lambda)|^2_{\mathcal{H}_\lambda}\, d\mu(\lambda) < \infty$.

Exercise Set 7.4

1. Let f be a Schwartz function on H_n. Show $f * f^*(0,0) = |f|_2^2$. Use the inversion formula $\phi(0,0) = \int \mathrm{Tr}(\hat{\phi}(\lambda))\, |\lambda|^n\, d_1\lambda$ to show

$$|f|_2^2 = \int |\hat{f}(\lambda)|_2^2\, |\lambda|^n\, d_1\lambda.$$

2. Let T, T_1 and T_2 be strongly measurable functions from $\mathbb{R}^n$ into $\mathcal{H}_2$, the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. Show:

(a) $x \mapsto (T_1(x)f_1, T_2(x)f_2)_2$ is measurable for each pair of functions f_1 and f_2 in $L^2(\mathbb{R}^n, d_n x)$.

(b) Show $x \mapsto ||T(x)||_2$ is measurable.

(c) Show $x \mapsto (T_1(x), T_2(x))_2$ is measurable.

(d) Show $x \mapsto T(x)$ is strongly measurable if and only if it is measurable; i.e., $T^{-1}(V)$ is measurable in $\mathbb{R}^n$ for any open subset U of $\mathcal{H}_2$.

(e) Define $L^2(\mathbb{R}^n, \mathcal{H}_2)$ to be the vector space of all measurable functions $T : \mathbb{R}^n \to \mathcal{H}_2$ satisfying

$$\int_{\mathbb{R}^n} ||T(x)||_2^2 \, d_n x < \infty$$

identifying them if they are equal almost everywhere. Show

$$(T_1, T_2) = \int (T_1(x), T_2(x))_2 \, d_n x.$$

defines an inner product on $L^2(\mathbb{R}^n, \mathcal{H}_2)$ which makes it a Hilbert space.

3. Let X be a measurable space. Show a mapping $T : X \to \mathcal{H}_2$ is strongly measurable if and only if $x \mapsto (h_1, T(x)h_2)_2$ is measurable for each pair h_1 and h_2 of Schwartz functions on $\mathbb{R}^n$.

4. Suppose $\lambda \mapsto F(\lambda)$ is a strongly measurable function from $\mathbb{R}$ into $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying $\int |F(\lambda)|_2^2 \, |\lambda|^n \, d_1\lambda < \infty$. Show $\lambda \mapsto F(\lambda) \circ \Phi^{-1}$ is measurable from $\mathbb{R}$ into $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying

$$\int |F(\lambda) \circ \Phi^{-1}|_2^2 \, d_1\lambda = \int |F(\lambda)|_2^2 |\lambda|^n \, d_1\lambda.$$

5. Show if $\lambda \mapsto F(\lambda)$ is measurable from $\mathbb{R}$ into $\mathcal{H}_2 = L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and $\int |F(\lambda)|_2^2 \, d_1\lambda < \infty$, then there is a function $\widetilde{F} \in L^2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ satisfying

$$F(\lambda)(x, y) = \widetilde{F}(x, y, \lambda)$$

a.e. (x, y) for a.e. λ.

6. Let U be the unitary mapping from $L^2(H_n)$ onto $L^2(\mathbb{R}, \mathcal{H}_2, |\lambda|^n d_1\lambda)$ given by $U(f)(\lambda) = \hat{f}(\lambda)$. Let B be the biregular representation of H_n. Show

$$U(B(g_1, g_2)f)(\lambda) = \pi_\lambda(g_1) U f(\lambda) \pi_\lambda(g_2)^{-1} = \pi_\lambda \times \bar{\pi}_\lambda(g_1, g_2)(U(f)(\lambda))$$

a.e. λ for each $f \in L^2(H_n)$ and any g_1 and g_2 in H_n.

7. Use the Plancherel Theorem to show if f_1 and f_2 are in $L^2(H_n)$, then the continuous function $f_1 * f_2^*$ on H_n is given at each point by

$$f_1 * f_2^*(z,t) = \int \mathrm{Tr}(\pi_\lambda(-z,-t)\pi_\lambda(f_1)\pi_\lambda(f_2)^*)\, |\lambda|^n\, d_1\lambda.$$

Remark 7.30. Exercise 7.4.6 above shows $f \mapsto \hat{f}$ carries the biregular representation B on H_n to the representation $\widehat{B}$ on $L^2(\mathbb{R}, \mathcal{H}_2, |\lambda|^n d_1\lambda)$ given by

$$\widehat{B}(g_1,g_2)f)(\lambda) = \pi_\lambda \otimes \bar{\pi}_\lambda(g_1,g_2)\hat{f}(\lambda).$$

In direct integral terminology, the transform $f \mapsto \hat{f}$ carries the biregular representation to the direct integral representation

$$\int_{\mathbb{R}}^{\oplus} \pi_\lambda \otimes \bar{\pi}_\lambda(g_1,g_2)\ d\mu(\lambda)$$

where $d\mu(\lambda) = |\lambda|^n\, d_1\lambda$.

Remark 7.31 (Abstract Plancherel Theorem)**.** A type I group is a second countable locally compact Hausdorff group whose unitary representations are all type I. We do not go into this here, but remark that abelian, connected nilpotent and semi-simple Lie groups, and compact groups are type I. Let G be a unimodular locally compact type I group. Then if m is a Haar measure on G, there is a unique measure μ (called the Plancherel measure) on $\hat{G}$, the space of equivalence classes of irreducible unitary representations of G, such that

$$\int |f(g)|^2\, dm(g) = \int_{\hat{G}} \mathrm{Tr}(\pi(f)\pi(f)^*)\, d\mu(\pi).$$

For H_n, we have $\int |f(z,t)|^2\, d_{2n}z\, d_1t = \int ||\pi_\lambda(f)||_2^2\, |\lambda|^n\, d_1\lambda$. However, Proposition 2.42 gives $||\pi_\lambda(f)||_2^2 = (\pi_\lambda(f), \pi_\lambda(f))_2 = \mathrm{Tr}(\pi_\lambda(f)\pi_\lambda(f)^*)$. Moreover, the mapping $f \mapsto \hat{f}$ where $\hat{f}(\pi) = \pi(f)$ is a unitary isomorphism of $L^2(G)$ onto $\int_{\hat{G}} \mathcal{H}_2(\mathcal{H}_\pi)\, d\mu(\pi)$ which carries the biregular representation of G onto $\int_{\hat{G}}^{\oplus} \pi \otimes \bar{\pi}\, d\mu(\pi)$. There is also an inversion formula. Namely, if $f \in C_c(G) * C_c(G)$, then

$$f(g) = \int \mathrm{Tr}(\pi(g^{-1})\pi(f))\, d\mu(\pi).$$

Remark 7.32. For $G = \mathbb{R}^n$ and $dm(x) = d_nx$, one has $\hat{G} = \mathbb{R}^n$ and $d\mu(y) = d_ny$. Namely, with $y \in \mathbb{R}^n$ we identify the unique irreducible unitary representation $e_y(x) = e^{-i(x,y)}$. Then $\hat{f}(y)$ is the Fourier transform of f and

$$\int |\hat{f}(y)|^2\, d_ny = \int \mathrm{Tr}(e_y(f)e_y(f)^*)\, d_ny = \int |f(x)|^2\, d_nx.$$

Thus with d_nx as Haar measure on $\mathbb{R}^n$, the corresponding Plancherel measure is d_ny.

Chapter 8

Compact Groups

1. Representations of Compact Groups

Theorem 8.1. *Let π be a nonunitary representation of a compact Hausdorff group G on a Hilbert space $\mathcal{H}$ with inner product $(\cdot\,,\cdot)$. Then there is an inner product $(\cdot\,,\cdot)'$ on $\mathcal{H}$ defining an equivalent norm such that π is unitary in this inner product.*

Proof. We first note the collection of operators $\pi(g)$ where $g \in G$ is pointwise bounded. Indeed $g \mapsto ||\pi(g)v||$ is continuous on G. Hence it has a maximum value. Thus for each $v \in \mathcal{H}$, there is an $M_v < \infty$ with $||\pi(g)v|| \leqslant M_v$ for all g. By the Principle of Uniform Boundedness, the operators $\pi(g)$ are uniformly bounded; i.e., there is an $0 < M < \infty$ with $||\pi(g)|| \leqslant M$ for all g. Hence

$$\frac{1}{M^2}(v,v) = \frac{1}{M^2}||\pi(g^{-1})\pi(g)v||^2 \leqslant ||\pi(g)v||^2 \leqslant M^2||v||^2 = M^2(v,v)$$

for all g.

Define inner product $(\cdot\,,\cdot)'$ by

$$(v,w)' = \int (\pi(g)v, \pi(g)w)\,dg$$

where dg is the Haar measure on G with $\int dg = 1$. This is a positive semidefinite complex inner product on $\mathcal{H}$. Moreover,

$$\frac{1}{M^2}(v,v) \leqslant \int (\pi(g)v, \pi(g)v)\,dg \leqslant M^2(v,v).$$

Consequently $\frac{1}{M}||v|| \leqslant ||v||' \leqslant M||v||$, and we see one has an inner product giving an equivalent norm. Thus $\mathcal{H}$ with inner product $(\cdot\,,\cdot)'$ is still a Hilbert space and since

$$(\pi(g)v, \pi(g)w)' = \int (\pi(g'g)v, \pi(g'g)w)\, dg' = \int (\pi(g')v, \pi(g')w)\, dg' = (v, w)',$$

we see the operators $\pi(g)$ are all unitary. □

Lemma 8.2. *Let G be a compact Hausdorff group and let π and π' be unitary representations of G on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. Let T be a bounded linear operator from $\mathcal{H}$ into $\mathcal{H}'$. Then for each $v \in \mathcal{H}$, the integral*

$$\int_G \pi'(g)T\pi(g^{-1})v\, dx$$

exists in the Riemann sense and $v \mapsto \int_G \pi'(g)T\pi(g^{-1})v\, dg$ is a bounded linear operator on $\mathcal{H}$ into $\mathcal{H}'$ which intertwines the representations π and π'. This operator is called $\int_G \pi'(g)T\pi(g^{-1})\, dg$.

Proof. Set $F(g) = \pi'(g)T\pi(g^{-1})v$. This is a continuous function from G into $\mathcal{H}'$. Its range is a compact subset of the metric space $\mathcal{H}'$ and thus is separable. This implies F is a uniform limit of simple function s_n. Indeed, let $\epsilon > 0$. Choose $v'_1, \ldots, v'_m$ in $F(G)$ such that if $g \in G$, then $||F(g) - v'_k|| \leqslant \epsilon$ for some k. Define $G_k = \{g \in G \mid ||F(g) - v'_k|| \leqslant \epsilon,\ ||F(g) - v'_i|| > \epsilon \text{ for } i < k\}$. The G_k partition G into measurable sets and the simple function $s(g) = \sum_{k=1}^m \chi_{G_k}(g)v'_k$ satisfies $||s(g) - F(g)|| \leqslant \epsilon$ for all g. We define $\int s(g)\, dg = \sum m(G_k)v'_k$. If follows easily if s_n converges uniformly to F, then $\int s_n(g)\, dg$ converges in $\mathcal{H}'$. Thus $\int F(g)\, dg$ exists.

Since $|| \int s(g)\, dg|| \leqslant \int ||s(g)||\, dg$ for simple functions, we have

$$\begin{aligned} || \int \pi'(g)T\pi(g^{-1})v\, dg|| &\leqslant \int ||\pi'(g)T\pi(g^{-1})v||\, dg \\ &\leqslant \int_G ||T||\, ||v||\, dg \\ &= ||T||\, ||v||. \end{aligned}$$

Thus the operator $\int \pi'(g)T\pi(g^{-1})\, dg$ is bounded. To see it is intertwining, note

$$\begin{aligned} \int \pi'(g)T\pi(g^{-1})\pi(x)v\, dg &= \int \pi'(g)T\pi((x^{-1}g)^{-1})v\, dg \\ &= \int \pi'(xg)T\pi(g^{-1})v\, dg \\ &= \int \pi'(x)\pi'(g)T\pi(g^{-1})v\, dg \\ &= \pi'(x) \int \pi'(g)T\pi(g^{-1})v\, dg. \end{aligned}$$

□

Lemma 8.3. *Let G be a compact Hausdorff group and let π be a unitary representation of G. Let $v \neq 0$ in $\mathcal{H}$. Then there is a continuous central function f with $\pi(f)v \neq 0$.*

Proof. Suppose $\pi(f)v = 0$ for all continuous central functions f. For $h \in C(G)$, set $h^c(x) = \int h(gxg^{-1})\,dg$. Note h^c is continuous. Indeed, if $\epsilon > 0$, by left uniform continuity, we can choose an open neighborhood U of e so that

$$|h(ux) - h(x)| < \epsilon \text{ for all } x \in G \text{ and } u \in U.$$

For each g pick a symmetric open neighborhood $N(g)$ of e with $gN(g)^3g^{-1} \subseteq U$. The open sets $gN(g)$ cover G and thus by compactness, we can choose $g_1, \ldots, g_k$ where $\cup g_iN(g_i) = G$. Set $N = \cap_{i=1}^k N(g_i)$. Let $a \in N$ and $g \in G$. Then $g = g_in_i$ for some i and $n_i \in N(g_i)$. Thus $gag^{-1} = g_in_ian_i^{-1}g_i^{-1} \in g_iN(g_i)^3g_i^{-1} \subseteq U$ and we see

$$|f(gaxg^{-1}) - f(gxg^{-1})| = |f(gag^{-1}(gxg^{-1})) - f(gxg^{-1})| < \epsilon.$$

Consequently,

$$|h^c(ax) - h^c(x)| \leqslant \int |h(gaxg^{-1}) - h(gxg^{-1})|\,dg < \epsilon \text{ for } a \in N.$$

Moreover, h^c is central for each $h \in C(G)$. Indeed,

$$h^c(yxy^{-1}) = \int h(gyx(gy)^{-1})\,dg = \int h(gxg^{-1})\,dg = h^c(x)$$

for all x and y. Now if $\pi(f)v = 0$ for all continuous central functions f, we would have $\pi(h^c)v = 0$ for all h. Hence

$$\begin{aligned} 0 &= \int h^c(x)(\pi(x)v, v)\ dx \\ &= \int\int h(g^{-1}xg)(\pi(x)v, v)\,dg\,dx \\ &= \int\int h(x)(\pi(gxg^{-1})v, v)\,dx\,dg \\ &= \int h(x)\int(\pi(gxg^{-1})v, v)\,dg\,dx \end{aligned}$$

for all $h \in C(G)$. Thus $\int(\pi(gxg^{-1})v, v)\,dg = 0$ for almost all x. But as $x \mapsto \int(\pi(gxg^{-1})v, v)\,dg$ is continuous (see above), we can by taking $x = e$ conclude that $||v||^2 = 0$, a contradiction. $\square$

Lemma 8.4. *Let G be a compact Hausdorff group and let π be a unitary cyclic representation of G. Then π is unitarily equivalent to a subrepresentation of the left regular representation. Moreover, $\pi(f)$ is a Hilbert-Schmidt operator for every $f \in L^2(G)$.*

Proof. Let v be a cyclic vector for ρ. Define $Tw(y) = (w, \pi(y)v)$ for $w \in \mathcal{H}_\pi$ and $y \in G$. Since $Tw \in C(G)$, $Tw \in L^2(G)$ for all w. Clearly T is linear and $||Tw||^2 = \int |(w, \pi(g)v)|^2 \, dm(g) \leqslant \int ||w||^2 ||v||^2 \, dm(g) = ||v||^2 ||w||^2$. Thus $T \in \mathcal{B}(\mathcal{H}_\pi, L^2(G))$. Moreover,

$$\begin{aligned} T(\pi(x)w)(y) &= (\pi(x)w, \pi(y)v) \\ &= (w, \pi(x^{-1}y)v) \\ &= Tw(x^{-1}y) \\ &= \lambda(x)(Tw)(y). \end{aligned}$$

Thus $T \in \mathrm{Hom}_G(\pi, \lambda)$. We claim T is one-to-one. Indeed, $Tw = 0$ implies $(w, \pi(y)v) = 0$ and thus $(w, v') = 0$ for all $v' \in \overline{\langle \pi(G)v \rangle} = \mathcal{H}_\pi$. Thus $w = 0$. By Theorem 6.47, there is a unitary equivalence of π with λ restricted to the closure of the range of T.

Since $U\pi(f)U^{-1} = \lambda(f)|_{U(\mathcal{H}_\pi)}$, to show $\pi(f)$ is compact, we need only show $\lambda(f)$ is compact for every $f \in L^2(G)$. But if $\phi, \psi \in L^2(G)$, one has

$$\begin{aligned} (\lambda(f)(\phi), \psi)_2 &= \int f(y)(\lambda(y)\phi, \psi)_2 \, dm(y) \\ &= \int f(y) \int \lambda(y)\phi(x)\bar{\psi}(x) \, dm(x) \, dm(y) \\ &= \int \int f(y)\phi(y^{-1}x)\bar{\psi}(x) \, dm(y) \, dm(x) \\ &= \int \int f(xy)\phi(y^{-1})\bar{\psi}(x) \, dm(y) \, dm(x) \\ &= \int \int f(xy^{-1})\phi(y) \, dm(y) \, \bar{\psi}(x) \, dm(x) \\ &= (T_K f, \psi)_2 \end{aligned}$$

where T_K is the integral operator with kernel $K(x, y) = f(xy^{-1})$. Since $\iint |f(xy^{-1})|^2 \, dm(x) \, dm(y) = \iint |f(x)|^2 \, dm(x) \, dm(y) = |f|_2^2$, we see the kernel is L^2. By Theorem 2.34, the operator T_K is Hilbert-Schmidt. $\square$

Theorem 8.5. *Every unitary representation of a compact Hausdorff group is an internal orthogonal direct sum of finite dimensional irreducible unitary subrepresentations.*

Proof. Let π be a unitary representation on Hilbert space $\mathcal{H}$. Take a maximal collection $\{\mathcal{H}_\alpha \mid \alpha \in A\}$ of orthogonal finite dimensional invariant irreducible subspaces of $\mathcal{H}$. Set $\mathcal{H}_0 = \bigoplus \mathcal{H}_\alpha$. Then $\mathcal{H}_0$ is a closed linear subspace of $\mathcal{H}$ and is invariant under π. Assume $\mathcal{H}_0 \neq \mathcal{H}$. By Lemma 6.36, $\mathcal{H}_0^\perp$ is nonzero and invariant. Choose nonzero vector $v \in \mathcal{H}_0^\perp$ and set $\mathcal{K} = \overline{\langle \pi(G)v \rangle}$. Then $\mathcal{K} \subseteq \mathcal{H}_0^\perp$. Let $\pi' = \pi|_{\mathcal{K}}$. Then π' is unitary and has cyclic vector v.

By Lemma 8.3, there is a central function $f \in C(G)$ with $\pi'(f)v \neq 0$. Using Lemma 8.4 and Corollary 2.49, we see $\pi'(f)$ is a compact operator. Moreover, since f is central, $\pi'(f) \in \mathrm{Hom}_G(\pi', \pi')$. Set $T = \pi'(f)^*\pi'(f)$. Then T is nonzero, positive, and compact and by Lemma 6.46, $T \in \mathrm{Hom}_G(\pi', \pi')$. By the spectral theorem for compact operators, i.e., Theorem 2.51, there is a nonzero eigenvalue λ for T. By Proposition 2.47, $\mathcal{K}_\lambda = \{w \in \mathcal{K} \mid Tw = \lambda w\}$ is finite dimensional and since $T\pi'(g) = \pi'(g)T$ for all $g \in G$, $\mathcal{K}_\lambda$ is invariant. Indeed, note $T\pi'(g)w = \pi'(g)Tw = \pi'(g)(\lambda w) = \lambda\pi'(g)w$ if $w \in \mathcal{K}_\lambda$. Now there exists a π invariant subspace $\mathcal{K}_0$ of $\mathcal{K}_\lambda$ of smallest positive dimension. Since $\mathcal{K}_0$ is irreducible under π and $\mathcal{K}_0 \perp \mathcal{H}_\alpha$ for all $\alpha \in A$, we see the collection $\{\mathcal{H}_\alpha \mid \alpha \in A\}$ is not maximal. Hence our assumption $\oplus\mathcal{H}_\alpha \neq \mathcal{H}$ was incorrect. □

Corollary 8.6. *Every irreducible representation of a compact Hausdorff group G on a Hilbert space $\mathcal{H}$ is finite dimensional.*

Proof. By Theorem 8.1, we may assume the representation is unitary. □

2. Unitary Dual

Recall the unitary dual $\hat{G}$ of a topological group G is the set of unitary equivalence classes of irreducible unitary representations of G.

If π is an irreducible unitary representation of G, we set $[\pi]$ to be unitary equivalence class of π. To classify the dual of G means to obtain a listing of irreducible unitary representations of G, one from each unitary equivalence class in $\hat{G}$.

Theorem 8.7. *Let G_1 and G_2 be compact Hausdorff groups. Then every irreducible unitary representation of $G_1 \times G_2$ is equivalent to a tensor product representation $\pi_1 \times \pi_2$ where π_1 is an irreducible unitary representation of G_1 and π_2 is an irreducible unitary representation of G_2.*

Proof. Assume π is an irreducible unitary representation of $G_1 \times G_2$ on Hilbert space $\mathcal{H}$. Define $\rho(g_1) = \pi(g_1, e_2)$. Then ρ is a unitary representation of G_1. But by Theorem 8.5, ρ has an irreducible subrepresentation π_1 on a ρ invariant subspace $\mathcal{H}_1$ of $\mathcal{H}$. Let P be the π_1-primary projection for ρ on $\mathcal{H}$. By Exercise 6.4.9, P commutes with every bounded linear operator commuting with ρ. Since $\pi(e_1, g_2)$ commutes with $\pi(g_1, e_2)$ for all g_1, we see $P\pi(e_1, g_2) = \pi(e_1, g_2)P$ for all g_2. But this and $P \in \mathrm{Hom}_G(\rho, \rho)$ implies $P \in \mathrm{Hom}_{G_1 \times G_2}(\pi, \pi) = \mathbb{C}I$. Since $P \neq 0$, $P = I$. From Theorem 6.53 and Exercise 6.4.9, we know $\rho|_{P\mathcal{H}} \cong n\pi_1 \cong \pi_1 \otimes I_2$ where I_2 is the identity representation of G_1 on a Hilbert space $\mathcal{H}_2$ of dimension n. Thus there is a unitary operator $U \in \mathrm{Hom}_{G_1}(\rho, \pi_1 \otimes I_2)$. Define $\pi'(g_1, g_2) = U\pi(g_1, g_2)U^{-1}$. Then π' is unitarily equivalent to π and $\pi'(g_1, e_2) = U\rho(g_1)U^{-1} = (\pi_1 \otimes$

$I_2)(g_1) = \pi_1(g_1) \otimes I_2$. Set $A = \pi'(e_1, g_2)$. Then $A(\pi_1(g_1) \otimes I_2) = (\pi_1(g_1) \otimes I_2)A$ for all $g_1 \in G_1$. By Lemma 6.74, $A = I_1 \otimes \pi_2(g_2)$ for a unique bounded linear operator $\pi_2(g_2)$ on $\mathcal{H}_2$. Hence $\pi'(g_1, e_2) = \pi_1(g_1) \otimes I_2$ and $\pi'(e_1, g_2) = I_1 \otimes \pi_2(g_2)$. This implies $\pi'(g_1, g_2) = \pi_1(g_1) \otimes \pi_2(g_2)$ for all $g_1 \in G_1$ and $g_2 \in G_2$. Since π' is a representation, one has π_2 is a representation of G_2 on $\mathcal{H}_2$; moreover, it is unitary since π' is unitary. □

3. Matrix Coefficients

If π is a unitary representation on a Hilbert space $\mathcal{H}$, the continuous bounded functions $\pi_{v,w}$ defined by

$$\pi_{v,w}(g) = (v, \pi(g)w)_{\mathcal{H}} = (\pi(g^{-1})v, w)_{\mathcal{H}}$$

are said to be the **matrix coefficients** of π. As was the case in Section 4 of Chapter 7, it will be useful to denote these functions by $v \otimes_\pi \bar{w}$. If π is finite dimensional, these are the same as the functions

$$\pi_{v,\mu}(g) = \langle v, \check{\pi}(g)\mu \rangle = \langle \pi(g^{-1})v, \mu \rangle \text{ where } \mu \in \mathcal{H}^*.$$

For π nonunitary, we set

$$(v \otimes \mu)_\pi(g) = \mu(\pi(g^{-1})v) \text{ when } \mu \in \mathcal{H}^*.$$

Lemma 8.8. *Let π and π' be unitary representations of a compact Hausdorff group G on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. Then*

$$(\pi'(v \otimes_\pi \bar{w})_\pi v', w')_{\mathcal{H}'} = (v \otimes_\pi \bar{w}, w' \otimes_{\pi'} \bar{v}')_2 = \int_G (v, \pi(g)w)\overline{(w', \pi'(g)v')}\, dg$$

for $v, w \in \mathcal{H}$ and $v', w' \in \mathcal{H}'$.

Proof. Since matrix coefficients are continuous and bounded, they are in $L^1(G)$. Moreover,

$$\begin{aligned}
(\pi'(v \otimes_\pi \bar{w})_\pi v', w')_{\mathcal{H}} &= \int (v \otimes_\pi \bar{w})(g)(\pi'(g)v', w')_{\mathcal{H}'}\, dg \\
&= \int (v \otimes_\pi \bar{w})(g)\overline{(w', \pi'(g)v')}_{\mathcal{H}'}\, dg \\
&= \int (v \otimes_\pi \bar{w})(g)\overline{w' \otimes_{\pi'} \bar{v}'(g)}\, dg.
\end{aligned}$$

□

4. Orthogonality Relations

Theorem 8.9. *Let G be a compact Hausdorff group and suppose π is a unitary representation of G on a Hilbert space $\mathcal{H}$ of finite dimension d. Let B*

be the biregular representation for G. Define $I_\pi : \mathcal{H} \otimes \bar{\mathcal{H}} = L(\mathcal{H}, \mathcal{H}) \to L^2(G)$ by

$$I_\pi(T)(x) = \mathrm{Tr}(\pi(x^{-1})T).$$

Then

(a) $I_\pi \in \mathrm{Hom}_{G\times G}(\pi \times \bar{\pi}, B)$.

(b) $I_\pi(v \otimes \bar{w}) = v \otimes_\pi \bar{w}$ *for all v and w in $\mathcal{H}$.*

(c) $I_\pi^* f = \pi(f)$ *for all $f \in L^2(G)$.*

(d) *The range of I_π is a $*$ ideal in the convolution algebra $C(G)$. It is spanned by the matrix coefficients of the representation π.*

If, in addition, π is irreducible, then

(e) $I_\pi^* I_\pi = \frac{1}{d} I$.

(f) $\sqrt{d} I_\pi$ *is an isometry of $\mathcal{H} \otimes \bar{\mathcal{H}}$ into $L^2(G)$.*

(g) *The mapping dI_π is a $*$ algebra isomorphism of the $*$ algebra $\mathcal{B}(\mathcal{H})$ into $C(G)$.*

Proof. Clearly $I_\pi(T) \in C(G)$ for each T. Also

$$\begin{aligned} I_\pi((\pi \times \bar{\pi})(g_1, g_2)T)(x) &= I_\pi(\pi(g_1)T\pi(g_2)^{-1})(x) = \mathrm{Tr}(\pi(x^{-1})\pi(g_1)T\pi(g_2)^{-1}) \\ &= \mathrm{Tr}(\pi(g_2^{-1})\pi(x^{-1})\pi(g_1)T) = \mathrm{Tr}(\pi((g_1^{-1}xg_2)^{-1}T) \\ &= I_\pi(T)(g_1^{-1}xg_2) = B(g_1, g_2)I_\pi(T)(x). \end{aligned}$$

Thus $I_\pi \in \mathrm{Hom}_G(\pi \times \bar{\pi}, B)$. Now by Definition 2.38,

$$\begin{aligned} I_\pi(v \otimes \bar{w})(x) = \mathrm{Tr}(\pi(x^{-1})v \otimes \bar{w}) &= (\pi(x^{-1})v, w)_{\mathcal{H}} \\ &= v \otimes_\pi \bar{w}(g). \end{aligned}$$

$I_\pi^* f$ is an operator in $\mathcal{B}(\mathcal{H}) = \mathcal{H} \otimes \bar{\mathcal{H}}$. We claim it is $\pi(f)$.

$$\begin{aligned} (I_\pi^* f, v \otimes \bar{w})_2 &= (f, I_\pi(v \otimes \bar{w}))_{L^2(G)} \\ &= (f, v \otimes_\pi \bar{w})_{L^2(G)} \\ &= \int f(x)\overline{(v, \pi(g)w)}\, dg \\ &= \int f(x)(\pi(g)w, v)\, dg \\ &= (\pi(f)w, v) \\ &= \mathrm{Tr}(\pi(f)w \otimes \bar{v}) \\ &= \mathrm{Tr}(\pi(f)(v \otimes \bar{w})^*) \\ &= (\pi(f), v \otimes \bar{w})_2. \end{aligned}$$

where we have used equations (2.5) and (2.6) from Chapter 2. Since the linear span of the rank one operators $v \otimes \bar{w}$ is all of $\mathcal{H} \otimes \bar{\mathcal{H}}$, we have $I_\pi^* f = \pi(f)$.

Suppose $A \in \mathcal{B}(\mathcal{H}) = \mathcal{H} \otimes \bar{\mathcal{H}}$ and $f \in C(G)$. Then

$$\begin{aligned} I_\pi(A)^*(x) &= \overline{I_\pi(A)(x^{-1})} \\ &= \overline{\mathrm{Tr}(\pi(x)A)} \\ &= \mathrm{Tr}((\pi(x)A)^*) \\ &= \mathrm{Tr}(A^*\pi(x)^*) \\ &= \mathrm{Tr}(A^*\pi(x^{-1})) \\ &= \mathrm{Tr}(\pi(x^{-1})A^*) \\ &= I_\pi(A^*)(x), \end{aligned} \tag{8.1}$$

and since the trace is linear and continuous, we see

$$\begin{aligned} f * I_\pi(A)(x) &= \int f(y) I_\pi(A)(y^{-1}x)\,dy \\ &= \int f(y)\mathrm{Tr}(\pi(x^{-1})\pi(y)A)\,dy \\ &= \mathrm{Tr}(\int f(y)\pi(x^{-1})\pi(y)A\,dy) \\ &= \mathrm{Tr}\left(\pi(x^{-1})(\int f(y)\pi(y)\,dy)A\right) \\ &= I_\pi(\pi(f)A)(x). \end{aligned} \tag{8.2}$$

Now this along with (8.1) implies

$$\begin{aligned} I_\pi(A) * f &= (f^* * I_\pi(A)^*)^* \\ &= (f^* * I_\pi(A^*))^* \\ &= I_\pi(\pi(f^*)A^*)^* \\ &= I_\pi(A\pi(f^*)^*) \\ &= I_\pi(A\pi(f)). \end{aligned} \tag{8.3}$$

Hence (8.1), (8.2), and (8.3) imply the range of I_π is a $*$ subideal of the $*$ convolution algebra $C(G)$. Since the rank one operators $v \otimes \bar{w}$ span $\mathcal{B}(\mathcal{H})$ and $I_\pi(v \otimes \bar{w}) = v \otimes_\pi \bar{w}$, the matrix coefficients span the range of I_π.

We now assume π is irreducible. By Lemma 6.46, $I_\pi^* I_\pi \in \mathrm{Hom}_{G\times G}(\pi \times \bar{\pi}, \pi \times \bar{\pi})$. The irreducibility of $\pi \times \bar{\pi}$ and Schur's Lemma imply $I_\pi^* I_\pi = c^2 I$ for some constant $c > 0$. Let $e_1, e_2, \ldots, e_d$ be an orthonormal basis of $\mathcal{H}_\pi$. Since

$$(I_\pi(e_i \otimes \bar{v}), I_\pi(e_j \otimes \bar{v}))_2 = c^2(e_i \otimes \bar{v}, e_j \otimes \bar{v})_2 = c^2(e_i, e_j)_{\mathcal{H}}(v, v)_{\mathcal{H}},$$

the vectors $I_\pi(e_i \otimes \bar{v})$ are orthogonal. Hence

$$\begin{aligned} c^2 d|v|^2 &= \sum_{k=1}^{d} (I_\pi^* I_\pi(e_k \otimes \bar{v}), e_k \otimes \bar{v}) \\ &= \sum ||I_\pi(e_k \otimes \bar{v})||_2^2 \\ &= \sum \int_G |(e_k, \pi(x)v)|^2 \, dx \\ &= \int_G ||\pi(x)v||^2 \, dx \\ &= \int_G ||v||^2 \, dx \\ &= ||v||^2. \end{aligned}$$

So $c^2 = \frac{1}{d}$. Next note $\sqrt{d} I_\pi$ is an isometry for

$$\begin{aligned} (\sqrt{d} I_\pi(A), \sqrt{d} I_\pi(B))_2 &= d(I_\pi^* I_\pi(A), B)_2 \\ &= (A, B)_2. \end{aligned}$$

To see it is multiplicative, let $f = I_\pi(A)$ and $g = I_\pi(B)$ where A and B are linear transformations of $\mathcal{H}$. Using $I_\pi^* I_\pi = \frac{1}{d} I$, $I_\pi^*(f) = \frac{1}{d} A$ and $I_\pi^*(g) = \frac{1}{d} B$, we see

$$\begin{aligned} dI_\pi(AB) &= dI_\pi(dI_\pi^*(f)\, dI_\pi^*(g)) \\ &= d^3 I_\pi(\pi(f)\pi(g)) \\ &= d^3 I_\pi(\pi(f * g)) \\ &= d^3 I_\pi I_\pi^*(f * g) \\ &= \frac{d^3}{d}(f * g) \\ &= dI_\pi(A) * dI_\pi(B). \end{aligned}$$

□

Corollary 8.10. *Let π be an irreducible unitary representation of a compact Hausdorff group G on the Hilbert space $\mathcal{H}$ of finite dimension d. Then*

$$(v \otimes_\pi \bar{w}, v' \otimes_\pi \bar{w}')_2 = \frac{1}{d}(v, v')_{\mathcal{H}}(w', w)_{\mathcal{H}} \text{ and}$$

$$\pi(v \otimes_\pi \bar{w}) = \frac{1}{d} v \otimes \bar{w}.$$

Proof. Note (b), (d), and (e) give both

$$\begin{aligned}(v \otimes_\pi w, v' \otimes_\pi \bar{w}')_{L^2(G)} &= (I_\pi(v \otimes \bar{w}), I_\pi(v' \otimes \bar{w}'))_{L^2(G)} \\ &= (I_\pi^* I_\pi(v \otimes \bar{w}), v' \otimes \bar{w}')_2 \\ &= \frac{1}{d}(v \otimes \bar{w}, v' \otimes \bar{w}')_2 \\ &= \frac{1}{d}(v, v')_{\mathcal{H}}(w', w)_{\mathcal{H}}\end{aligned}$$

and

$$\pi(v \otimes_\pi \bar{w}) = I_\pi^* I_\pi(v \otimes \bar{w}) = \frac{1}{d}(v \otimes \bar{w}).$$

□

Corollary 8.11. *If π is an irreducible unitary representation of a compact Hausdorff group G on a Hilbert space of dimension d, then*

$$(v \otimes_\pi \bar{w}) * (v' \otimes_\pi \bar{w}') = \frac{1}{d}(v', w)_{\mathcal{H}} v \otimes_\pi \bar{w}'.$$

Proof.

$$\begin{aligned}(v \otimes_\pi \bar{w}) * (v' \otimes_{\pi'} \bar{w}')(g) &= \int (v, \pi(x)w)_{\mathcal{H}}(v', \pi(x^{-1})\pi(g)w')_{\mathcal{H}}\, dx \\ &= \int (v, \pi(x)w)_{\mathcal{H}}\overline{(\pi(g)w', \pi(x)v')}_{\mathcal{H}}\, dx \\ &= (v \otimes_\pi \bar{w}, \pi(g)w' \otimes_\pi \bar{v}')_{L^2(G)} \\ &= \frac{1}{d}(v, \pi(g)w')_{\mathcal{H}}(v', w)_{\mathcal{H}} \\ &= \frac{1}{d}(v', w)_{\mathcal{H}}\, v \otimes_\pi \bar{w}'(g).\end{aligned}$$

□

Theorem 8.12 (Orthogonality)**.** *Let π and π' be unitary representations of a compact Hausdorff group G on finite dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. If $\operatorname{Hom}_G(\pi, \pi') = \{0\}$, then*

$$I_\pi(\mathcal{H} \otimes \bar{\mathcal{H}}) \perp I_{\pi'}(\mathcal{H}' \otimes \bar{\mathcal{H}}').$$

In particular, the matrix coefficients for π and the matrix coefficients for π' are pairwise orthogonal in $L^2(G)$; and $\pi'(v \otimes_\pi \bar{w}) = 0$ for all $v, w \in \mathcal{H}$.

Proof. Consider $I_{\pi'}^* I_\pi : \mathcal{H} \otimes \bar{\mathcal{H}} \to \mathcal{H}' \otimes \bar{\mathcal{H}}'$. Suppose we have shown this operator to be 0. Then

$$(I_{\pi'}^* I_\pi T, S) = 0$$

for all $T \in \mathcal{H} \otimes \bar{\mathcal{H}}$ and $S \in \mathcal{H}' \otimes \bar{\mathcal{H}}'$. Thus $(I_\pi T, I_{\pi'} S)_2 = 0$ and we would be done. Hence we need only show $I_{\pi'}^* I_\pi = 0$. Since $\pi' \times \bar{\pi}'$ is unitary, Lemma 6.46 shows $I_{\pi'}^* I_\pi \in \operatorname{Hom}_{G \times G}(\pi \times \bar{\pi}, \pi' \times \bar{\pi}')$. But by Lemma 6.77,

$\mathrm{Hom}_G(\pi, \pi') = \{0\}$ implies $\mathrm{Hom}_{G\times G}(\pi \times \bar{\pi}, \pi' \times \bar{\pi}') = \{0\}$. Finally $\pi'(v \otimes_\pi w) = 0$ follows from Lemma 8.8. □

Corollary 8.13. *Let π and π' be inequivalent irreducible unitary representations of compact Hausdorff group G. Then for $v, w \in \mathcal{H}$ and $v', w' \in \mathcal{H}'$, one has:*

$$(v \otimes_\pi \bar{w}, v' \otimes_{\pi'} \bar{w}')_2 = \int (v, \pi(g)w)_{\mathcal{H}} \overline{(v', \pi'(g)w')_{\mathcal{H}'}}\, dm(g) = 0,$$

$$\pi'(v \otimes_\pi \bar{w}) = 0, \ \text{and}$$

$$(v \otimes_\pi \bar{w}) * (v' \otimes_{\pi'} \bar{w}') = 0.$$

Proof. Follow the arguments in Corollary 8.10 and Corollary 8.11. □

Exercise Set 8.1

1. Let G be a compact Hausdorff group with Haar measure m normalized so that $m(G) = 1$.

(a) Show $|f|_p \leqslant |f|_q$ if $1 \leqslant p \leqslant q \leqslant \infty$ for measurable functions f.

(b) Show $L^1(G) \supseteq L^p(G) \supseteq L^q(G) \supseteq L^\infty(G) \supseteq C(G)$ if $1 < p < q < \infty$.

2. Let G be a compact Hausdorff group. Let π and π' be unitary representations of G on $\mathcal{H}$ and $\mathcal{H}'$. Define P on $\mathcal{B}(\mathcal{H}, \mathcal{H}')$ by

$$P(T) = \int_G \pi'(g) T \pi(g^{-1})\, dg.$$

(a) Show $P^2 = P$.

(b) Show P restricted to $\mathcal{B}_2(\mathcal{H}, \mathcal{H}')$ is the orthogonal projection of $\mathcal{H}' \otimes \bar{\mathcal{H}}$ onto $\mathrm{Hom}_G(\pi, \pi')_2$.

3. Let π and π' be unitary representations of a compact Hausdorff group G. Show $\mathrm{Hom}_G(\pi, \pi') = \{0\}$ if and only if the matrix coefficients of π are orthogonal to the matrix coefficients of π'.

4. Let G be a compact Hausdorff group and let π be a unitary representation of G with cyclic vector w. For $u, v \in \mathcal{H}$, the Hilbert space of π, define

$$B(u, v) = \int_G (w, \pi(g)v)_{\mathcal{H}} (\pi(g)u, w)_{\mathcal{H}}\, dg.$$

(a) Show B is a bounded sesquilinear form on $\mathcal{H} \times \mathcal{H}$.

(b) Show there is a positive continuous linear one-to-one operator T on $\mathcal{H}$ with

$$B(u, v) = (Tu, v)_{\mathcal{H}} \text{ for all } u, v \in \mathcal{H}.$$

(c) Show $T \in \mathrm{Hom}_G(\pi, \pi)$.

(d) Show if v_n is bounded and converges weakly to a vector v, then

$$\lim_n ||Tv_n||^2 = \int_G (w, \pi(g)Tv)_{\mathcal{H}}(\pi(g)v, w)_{\mathcal{H}}\, dg = ||Tv||^2.$$

(e) Use Exercise 6.2.4 and the weak compactness of the closed unit ball in $\mathcal{H}$ to show T is a compact operator.

(f) Using the spectral theorem for compact linear operators, show π has a nonzero finite dimensional irreducible subrepresentation.

5. Let π and π' be finite dimensional inequivalent unitary representations of a compact group G on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. For $T \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, let A be the operator give by

$$Av = \int_G \pi'(g)T\pi(g)^{-1}v\, dg.$$

(a) If $T = v' \otimes \bar{v}$, show $(Aw, w')_{\mathcal{H}'} = 0$ yields the orthogonality of the matrix coefficients $w \otimes_\pi \bar{v}$ and $w' \otimes_{\pi'} \bar{v}'$ in $L^2(G)$.

(b) In the case where $\pi' = \pi$ and π is irreducible, show Schur's Lemma implies that there is a $c > 0$ so that

$$\int (w \otimes_\pi \bar{v})(g)\overline{(w' \otimes_\pi \bar{v}')(g)}\, dg = c(w, w')_{\mathcal{H}}(v', v)_{\mathcal{H}}.$$

6. Let π be an irreducible unitary representation of compact Hausdorff group G on a Hilbert space $\mathcal{H}$ of dimension d. Let v and w be in $\mathcal{H}$. Define $T = \int \pi(g)v \otimes \bar{\pi}(g)\bar{w}\, dg$. Show $T = \frac{1}{d}I$ and use this to show

$$(v \otimes_\pi \bar{w}, v' \otimes_\pi \bar{w}')_{L^2(G)} = \frac{1}{d}(v, v')_{\mathcal{H}}(w', w)_{\mathcal{H}} \text{ for all } v, w, v', w' \in \mathcal{H}.$$

7. Let π be a nontrivial irreducible unitary representation of a compact Hausdorff group G. Show

$$\iint \pi(x)A\pi(y^{-1})\, dx\, dy = 0$$

for all linear transformations A of $\mathcal{H}_\pi$.

8. Show a continuous function f on a compact Hausdorff group G is central if and only if $f * g = g * f$ for all functions $g \in L^1(G)$.

9. Let G be a compact Hausdorff group with left regular representation λ. Show for $f, h \in L^1(G)$ that $\lambda(f) = \lambda(h)$ if and only if $f = h$.

10. Show an L^1 function f on a compact Hausdorff group G is central if and only if $\lambda(f) \in \mathrm{Hom}_G(\lambda, \lambda)$.

Lemma 8.14. *Let λ and ρ be the left regular and right regular representations of a compact Hausdorff group G. Then for $f, h \in L^2(G)$,*

$$f * h^*(x) = (f, \lambda(x)h)_2 \text{ and } f^* * h(x) = (\rho(x)h, f)_2.$$

Moreover, the mapping $C : (f, h) \mapsto f * h$ is a continuous bilinear mapping of $L^2(G) \times L^2(G)$ into $C(G)$ with the uniform topology.

Proof.

$$\begin{aligned}(f, \lambda(x)h)_2 &= \int f(y)\overline{\lambda(x)h(y)}\,dy \\ &= \int f(y)\overline{h(x^{-1}y)}\,dy \\ &= \int f(y)h^*(y^{-1}x)\,dy \\ &= f * h^*(x).\end{aligned}$$

A similar calculation show $f^* * h(x) = (\rho(x)h, f)_2$.

In particular, we see $f * h(x) = (f, \lambda(x)h^*)_2$ is a continuous function and $||f * h||_\infty = \max_{x \in G} |(f, \lambda(x)h^*)| \leqslant \max_{x \in G} ||f||_2||\lambda(x)h^*||_2 = ||f||_2||h||_2$. Since convolution is bilinear, we see $||C|| \leqslant 1$ and thus C is continuous. □

Lemma 8.15. *Let π be an irreducible unitary representation of compact Hausdorff group G. Define $P_\pi := \sqrt{d(\pi)}I_\pi I_\pi^*$. Then P_π is the orthogonal projection of $L^2(G)$ onto the range of I_π, the linear span of the matrix coefficients of π.*

Proof. By Theorem 8.9, we know $\sqrt{d(\pi)}I_\pi$ is an isometry of $\mathcal{H} \otimes \bar{\mathcal{H}}$ into $L^2(G)$. Using Exercise 6.4.1, we know $d(\pi)I_\pi I_\pi^*$ is the orthogonal projection of $L^2(G)$ onto the range of I_π. But from (d) of Theorem 8.9, this range is the linear span of the matrix coefficients of π. □

In Example 6.113 we established the following lemma.

Lemma 8.16. *Let $f \in L^1(G)$ and $h \in L^2(G)$. Then $\lambda(f)h = f * h$.*

Proposition 8.17. *Let G be a compact Hausdorff group and let π be an irreducible unitary representation of G having dimension $d(\pi)$. Suppose $f \in L^2(G)$. Then $P_\pi f = d(\pi)I_\pi I_\pi^* f$ is the function in $L^2(G)$ given by*

$$P_\pi f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f)).$$

Moreover,

$$||P_\pi f||_2^2 = d(\pi)\mathrm{Tr}(\pi(f)^*\pi(f)).$$

Proof. Using Theorem 8.9, we have

$$P_\pi f(x) = d(\pi)I_\pi I_\pi^*(f)(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})I_\pi^*(f)) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f))$$

and

$$\begin{aligned} ||P_\pi f||_2^2 &= (P_\pi f, P_\pi f)_2 = (P_\pi f, f)_2 \\ &= d(\pi)(I_\pi I_\pi^* f, f) = d(\pi)(I_\pi^* f, I_\pi^* f)_2 \\ &= d(\pi)(\pi(f), \pi(f))_2 \\ &= d(\pi)\mathrm{Tr}(\pi(f)^*\pi(f)). \end{aligned}$$

□

Remark 8.18. Note if π and π_0 are equivalent irreducible unitary representations of compact group G, then $P_\pi = P_{\pi_0}$.

Let G be a topological group. $\hat{G}_c$ will denote a collection of pairwise inequivalent irreducible unitary representations of G with the property that if π_0 is an irreducible unitary representation of G, then π_0 is unitarily equivalent to exactly one of the representations in $\hat{G}_c$. Such a collection is said to be a **concrete dual** for G.

Theorem 8.19. *Let G be a compact Hausdorff group. For $\pi \in \hat{G}_c$, let*

$$P_\pi(f)(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f)).$$

Then

$$\bigoplus_{\pi \in \hat{G}_c} P_\pi = I.$$

Proof. The ranges of the P_π are the ranges of the I_π. These spaces are invariant under B, the biregular representation of $G \times G$ on $L^2(G)$. By Theorem 8.12, these ranges are pairwise orthogonal. Set $P = \oplus P_\pi$ and let $\mathcal{H}'$ be $P(L^2(G))^\perp$. $\mathcal{H}'$ is the orthogonal complement of an internal orthogonal direct sum of the B invariant subspaces $I_\pi(\mathcal{H}_\pi \otimes \bar{\mathcal{H}}_\pi)$ and hence is B invariant. Since $\lambda(g) = B(g, e)$, we see it is λ invariant. If $\mathcal{H}' \neq \{0\}$, $\mathcal{H}'$ would contain a λ invariant subspace $\mathcal{H}_0$ such that $\lambda_0 = \lambda|_{\mathcal{H}_0}$ is irreducible. Take $f \in \mathcal{H}_0$ with $f \neq 0$. Note $f^* \in L^2(G) \subseteq L^1(G)$. Thus $\lambda_0(f^*)$ is a bounded operator on $\mathcal{H}_0$. Lemma 8.16 implies $\lambda_0(f^*) \neq 0$ for $\lambda_0(f^*)(f)(x) = \lambda(f^*)(f)(x) = f^* * f(x) = (f, \lambda(x)f)$. Hence $\lambda_0(f) = \lambda_0(f^*)^* \neq 0$. So $f^* * f \in \mathcal{H}_0$ and $\mathrm{Tr}(\lambda_0(f^* * f)) = \mathrm{Tr}(\lambda_0(f)^*\lambda_0(f)) = (\lambda_0(f), \lambda_0(f))_2 > 0$. Consequently, $P_{\lambda_0}(f^* * f)(x) = \mathrm{Tr}(\lambda_0(x^{-1})\lambda(f^* * f))$ is a nonzero continuous function. Choose $\pi_0 \in \hat{G}_c$ unitarily equivalent to λ_0. Since $P_{\lambda_0} = P_{\pi_0}$, we have $P_{\pi_0}(f^* * f) \neq 0$. But $f^* * f \in \mathcal{H}_0 \subseteq P_\pi(L^2(G))^\perp$ for all π. So $P = I$. □

Theorem 8.20 (Plancherel)**.** *Let G be a compact Hausdorff group. If $f \in L^2(G)$, then*

$$f = \bigoplus_{\pi \in \hat{G}_c} P_\pi(f), \quad ||f||_2^2 = \sum_{\pi \in \hat{G}_c} d(\pi)\mathrm{Tr}(\pi(f)^*\pi(f)), \text{ and}$$

$U : L^2(G) \to \oplus(\mathcal{H}_\pi \otimes \bar{\mathcal{H}}_\pi)$ defined by $U(f)_\pi = \sqrt{d(\pi)}\pi(f)$ is a unitary equivalence of B with $\oplus_{\pi\in\hat{G}_c}(\pi \times \bar{\pi})$.

Proof.

$$\begin{aligned} ||f||_2^2 &= \sum ||P_\pi f||_2^2 = \sum (P_\pi f, P_\pi f)_2 \\ &= \sum (P_\pi^2 f, f)_2 = \sum (P_\pi f, f)_2 \\ &= \sum d(\pi)(I_\pi I_\pi^* f, f)_2 = \sum d(\pi)(I_\pi^* f, I_\pi^* f)_2 \\ &= \sum d(\pi)(\pi(f), \pi(f))_2 = \sum d(\pi)\mathrm{Tr}(\pi(f)^*\pi(f)). \end{aligned}$$

To see U is unitary and intertwining, we need only show V_π where $V_\pi : P_\pi L^2(G) \to \mathcal{H}_\pi \otimes_\pi \bar{\mathcal{H}}_\pi$ given by $V_\pi(P_\pi f) = \sqrt{d(\pi)}\pi(f) = \sqrt{d(\pi)}I_\pi^*(f)$ is a unitary intertwiner for each $\pi \in \hat{G}_c$. Since I_π^* intertwines B and $\pi \times \bar{\pi}$, V_π is an intertwiner. Since $d(\pi)I_\pi I_\pi^* = P_\pi$, we see it is onto. Since $||\sqrt{d(\pi)}I_\pi^*(f)||^2 = d(\pi)(I_\pi^* f, I_\pi^* f) = (P_\pi f, f)_2 = (P_\pi f, P_\pi f)_2 = ||P_\pi f||_2^2$, we see it is an isometry. □

Corollary 8.21. *Let G be a compact Hausdorff group. If f is in the linear span of $L^2(G) * L^2(G)$, then f is continuous and*

$$\sum_{\pi\in\hat{G}_c} d(\pi)\mathrm{Tr}(\pi(x)^{-1}\pi(f))$$

converges uniformly to $f(x)$ on G.

Proof. It suffices to show this works for functions of form $f * h^*$ where f and h are in $L^2(G)$. By Lemma 8.14, $\sum_{\pi\in\hat{G}_c}(f, \lambda(x)P_\pi h)_2$ converges uniformly to $(f, \lambda(x)h)_2 = f * h^*(x)$. But

$$\begin{aligned} (f, \lambda(x)P_\pi h)_2 &= (P_\pi\lambda(x^{-1})f, h)_2 \\ &= (d(\pi)I_\pi I_\pi^*(\lambda(x^{-1})f), h)_2 \\ &= d(\pi)(I_\pi^*(\lambda(x^{-1})f), I_\pi^* h)_2 \\ &= d(\pi)(I_\pi^*(B(x^{-1}, e)f), I_\pi^* h)_2 \\ &= d(\pi)((\pi(x^{-1}) \otimes \bar{\pi}(e))\pi(f), \pi(h))_2 \\ &= d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f)\pi(h)^*) \\ &= d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f * h^*)). \end{aligned}$$

□

Corollary 8.22. *Let π be an irreducible unitary representation of a compact Hausdorff group G. Let $P_\pi f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f))$ for $f \in L^2(G)$. Then P_π is the π-primary projection for the left regular, the $\bar{\pi}$-primary projection for the right regular representation, and the $\pi \times \bar{\pi}$ primary projection for the biregular representation of $G \times G$.*

Proof. We may assume we have a concrete dual $\hat{G}_c$ containing π. For $\pi' \in \hat{G}_c$, let $P(\pi')$ be the π'-primary projection for the left regular representation λ. We claim $P(\pi')P_{\pi'} = P_{\pi'}$. Fix $w \in \mathcal{H}_{\pi'}$. Then the transformation $T : \mathcal{H}_{\pi'} \to L^2(G)$ given by $Tv = I_{\pi'}(v \otimes \bar{w}) = v \otimes_{\pi'} \bar{w}$ is in $\mathrm{Hom}_G(\pi', \lambda)$. Indeed, $T\pi'(g)v = I_{\pi'}(\pi'(g)v \otimes \bar{w}) = I_{\pi'}((\pi' \times \bar{\pi}')(g,e)(v \otimes \bar{w})) = B(g,e)I_{\pi'}(v \otimes \bar{w}) = \lambda(g)Tv$. By Corollary 6.54, $P(\pi')(v \otimes_{\pi'} \bar{w}) = (v \otimes_{\pi'} \bar{w})$ for all v and w. Thus $P(\pi')(I_{\pi'}(T)) = I_{\pi'}(T)$ for all $T \in \mathcal{H}_{\pi'} \otimes \bar{\mathcal{H}}_{\pi'}$. Hence $P(\pi')P_{\pi'} = P_{\pi'}$ for all $\pi' \in \hat{G}_c$. By Corollary 6.55, $P(\pi)P(\pi') = 0$ for inequivalent irreducible unitary representations π and π' of G. Thus $P(\pi) = \sum_{\pi' \in \hat{G}_c - \{\pi\}} P(\pi)(I - P(\pi'))P_{\pi'} + P(\pi)P_\pi = P(\pi)P_\pi = P_\pi$.

A similar argument works using $T\bar{w} = v \otimes_\pi \bar{w}$ gives $P(\pi)$ is the $\bar{\pi}$-primary projection for the right regular representation.

Now $I_{\pi'}$ intertwines $\pi' \times \bar{\pi}'$ with $B|_{P_{\pi'}(L^2(G))}$. Thus if $P(\pi' \times \bar{\pi}')$ is the $\pi' \times \bar{\pi}'$ primary projection for B, we have $P(\pi' \times \bar{\pi}')I_{\pi'}(T) = I_{\pi'}(T)$ for all $T \in \mathcal{H}_{\pi'}$. So $P(\pi' \times \bar{\pi}')P_{\pi'} = P_{\pi'}$. Using Lemma 6.77 and Schur's Lemma one sees $\pi \times \bar{\pi}$ is unitarily equivalent to $\pi' \times \bar{\pi}'$ if and only if $\pi' \cong \pi$. Thus $P(\pi \times \bar{\pi})P(\pi' \times \bar{\pi}') = 0$ when π and π' are distinct elements in $\hat{G}_c$. We then argue as above that

$$P(\pi \times \bar{\pi}) = \sum_{\pi' \neq \pi} P(\pi \times \bar{\pi})(I - P(\pi' \times \bar{\pi}'))P_{\pi'} + P(\pi \times \bar{\pi})P_\pi = P_\pi.$$

□

Exercise Set 8.2

1. Let π be an irreducible unitary representation of a compact Hausdorff group G and let λ and ρ be the left and right regular unitary representations of G. Show $\lambda|_{P_\pi L^2(G)}$ is unitarily equivalent to $d(\pi)\pi$ and $\rho|_{P_\pi L^2(G)}$ is unitarily equivalent to $d(\pi)\bar{\pi}$.

2. Let G be a compact Hausdorff group. Show a function $f \in L^1(G)$ is central if and only if for every irreducible unitary representation π of G, $\pi(f) \in \mathrm{Hom}_G(\pi, \pi)$.

3. Let G be a compact Hausdorff group with concrete dual $\hat{G}_c$. For $\pi \in \hat{G}_c$, let $e_1, e_2, \ldots, e_{d(\pi)}$ be an orthonormal basis of the Hilbert space for π. Let $\pi_{i,j} = e_i \otimes_\pi \bar{e}_j$ be the matrix coefficients of π obtained from the basis $\{e_i\}_{i=1}^{d(\pi)}$.

(a) Show $\{\sqrt{d(\pi)}\pi_{i,j} \mid \pi \in \hat{G}_c,\ 1 \leqslant i,j \leqslant d(\pi)\}$ form an orthonormal basis of $L^2(G)$.

(b) Show for $f \in L^2(G)$, the series

$$\sum_{\pi \in \hat{G}_c} \sum_{1 \leqslant i,j \leqslant d(\pi)} d(\pi)(f, \pi_{i,j})_2 \pi_{i,j}$$

converges in $L^2(G)$ to f.

(c) Show if $f \in \langle L^2(G) * L^2(G)\rangle$, then the series in (b) converges uniformly on G to f.

4. Let G be a compact Hausdorff group and let π be an irreducible unitary representation of G. Show if f is in the π-primary subspace $P_\pi L^2(G)$ for the left regular representation λ, then f^* is also in this subspace.

5. Show if $\mathcal{H}$ is a separable Hilbert space and there is unitary isomorphism from Hilbert space $m\mathcal{H}$ onto $n\mathcal{H}$ where m and n are cardinals, then $m = n$.

6. Show if π' is a unitary representation on a Hilbert space $\mathcal{H}'$ and I is the identity representation on Hilbert space $\mathcal{H}$, then the representation $I \otimes \pi'$ is unitarily equivalent to $n\pi'$ where n is the cardinality of an orthonormal basis of $\mathcal{H}$.

5. Frobenius Reciprocity

We have used the trace on the operators on the finite dimensional Hilbert spaces $\mathcal{H}_\pi$ where $\pi \in \hat{G}_c$ to decompose the left, right, and biregular representations into orthogonal irreducible invariant subspaces. We in this section will begin to show that this is a more general process; one which can be applied to all representations which are induced from a representation of closed subgroup of the compact Hausdorff group G.

We recall from Section 17 of Chapter 6 that if π is a unitary representation of a closed subgroup K of a compact group G on a separable Hilbert space $\mathcal{K}$, then π^G or $\mathrm{ind}_K^G\pi$ denotes the unitary representation on the Hilbert space $L^2_\pi(G)$ of Baire measurable functions $f : G \to \mathcal{K}$ satisfying $f(gk) = \pi(k^{-1})f(g)$ for all k for a.e. g and $\int ||f(g)||^2_{\mathcal{K}}\, dg < \infty$. Instead of the notation $L^2_\pi(G)$, to give more specificity, we use the notation $L^2_K(G, \pi)$ to denote this function space. Recall the inner product on $L^2_K(G, \pi)$ is given by $(f_1, f_2) = \int (f_1(g), f_2(g))_{\mathcal{K}}\, dg$. We shall need the following lemma.

Lemma 8.23. *Let $\mathcal{H}_0$ be a finite dimensional subspace of $L^2_\pi(G, K)$ which is invariant under the induced representation π^G. Then the elements in $\mathcal{H}_0$ are continuous functions.*

Proof. Since $\mathcal{H}_0$ is finite dimensional, it is a direct sum of finitely many orthogonal nonzero irreducible invariant subspaces. Hence we may assume the representation ρ obtained by restricting π^G to $\mathcal{H}_0$ is irreducible. We first show $\rho(h)f$ is continuous on G for each $f \in \mathcal{H}_0$ and $h \in C(G)$.

Indeed, since G is unimodular, we have $\rho(h)f(x) = \int h(y)f(y^{-1}x)\, dy = \int h(xy^{-1})f(y)\, dy$. Let $\epsilon > 0$. By Lemma 5.5.24, we know h is uniformly continuous, and thus one can choose a neighborhood U of e so that

$$|h(ug) - h(g)| < \frac{\epsilon}{1 + \int ||f(y)||_{\mathcal{K}}\, dy} \text{ if } g \in G \text{ and } u \in U.$$

Thus

$$\begin{aligned} ||\rho(h)f(ux) - \rho(h)f(x)||_{\mathcal{K}} &\leqslant \int |h(uxy^{-1}) - h(xy^{-1})|\, ||f(y)||_{\mathcal{K}}\, dy \\ &\leqslant \frac{\epsilon}{1 + \int ||f(y)||_{\mathcal{K}}\, dy} \int ||f(y)||_{\mathcal{K}}\, dy \\ &\leqslant \epsilon. \end{aligned}$$

Thus $\rho(h)f$ is continuous. We cannot have $\rho(h)f = 0$ for all $h \in C(G)$ and $f \in \mathcal{H}_0$; for by Corollary 6.108, the density of $G(G)$ in $L^1(G)$, and $||\rho(h)|| \leqslant |h|_1$ for $h \in L^1(G)$, we would then have $\rho(h)f = 0$ for all $h \in L^1(G)$ and all $f \in \mathcal{H}_0$ and thus the integrated representation $h \mapsto \rho(h) = \int h(y)\rho(y)\, dy$ of $L^1(G)$ could not be nondegenerate. Consequently $\rho(h)f$ is nonzero and continuous for some $f \in \mathcal{H}_0$ and $h \in C(G)$. Since $\rho(h)f \in \mathcal{H}_0$, the space of continuous functions in $\mathcal{H}_0$ is nonzero. It is clearly invariant under ρ and since ρ is irreducible and finite dimensional, this space is all of $\mathcal{H}_0$. $\square$

Definition 8.24. *Let π and π' be unitary representations of a topological group G. Then*

$$\mathrm{Hom}_G(\pi, \pi')_2 = \{A \in \mathrm{Hom}_G(\pi, \pi') \mid A \textit{ is Hilbert-Schmidt}\}.$$

The above vector space is a closed linear subspace of the Hilbert space $\mathcal{H}' \otimes \bar{\mathcal{H}} = \mathcal{B}(\mathcal{H}, \mathcal{H}')_2$ and thus is a Hilbert space. The inner product is given by

$$(A, A')_2 = \mathrm{Tr}(A'^*A) = \mathrm{Tr}(AA'^*).$$

Moreover, $\mathrm{Hom}_G(\pi, \pi')_2 = (\mathcal{H}' \otimes \bar{\mathcal{H}})^G = \{A \in \mathcal{H}' \otimes \bar{\mathcal{H}} \mid (\pi' \otimes \bar{\pi})(g)(A) = A\}$.

Theorem 8.25 (Frobenius)**.** *Let K be a closed subgroup of a compact group G and let π be a unitary representation of K on a separable Hilbert space $\mathcal{K}$ and π' be a unitary representation of G on Hilbert space $\mathcal{H}'$. Then the mapping $A \mapsto \tilde{A}$ by*

$$\tilde{A}v'(x) = A\pi'(x^{-1})v'$$

is a unitary isomorphism from the Hilbert space $\mathrm{Hom}_K(\pi'|_K, \pi)_2$ *onto the Hilbert space* $\mathrm{Hom}_G(\pi', \mathrm{ind}_K^G \pi)_2$*. In particular, if B is a Hilbert-Schmidt intertwining operator from π' to $\mathrm{ind}_K^G \pi$, then the range of B consists of continuous functions.*

Proof. Note $\tilde{A}v'$ is continuous and

$$\tilde{A}v'(xk) = A\pi'(k^{-1}x^{-1})v' = \pi(k^{-1})A\pi'(x^{-1})v' = \pi(k^{-1})\tilde{A}v'(x).$$

Thus $\tilde{A}v'$ is in $L^2_K(G,\pi)$. Also since

$$\begin{aligned} ||\tilde{A}v'||^2 &= \int_G ||A\pi'(g^{-1})v'||^2\,dg \\ &\leqslant ||A||^2||v'||^2, \end{aligned}$$

$\tilde{A}$ is a bounded linear operator. We claim $\tilde{A} \in \mathrm{Hom}_G(\pi', \mathrm{ind}_K^G\pi)_2$. First we note:

$$\begin{aligned} \tilde{A}\pi'(g)v'(x) &= A\pi'(x^{-1})\pi'(g)v' \\ &= A\pi'((g^{-1}x)^{-1})v' \\ &= \tilde{A}v'(g^{-1}x) \\ &= \pi^G(g)\,\tilde{A}v'(x). \end{aligned}$$

Thus $\tilde{A}$ intertwines. Let $f \in L^2(G,\pi)$. Then

$$\begin{aligned} (\tilde{A}^*f, v')_{\mathcal{H}'} &= (f, \tilde{A}v')_2 \\ &= \int_G (f(x), \tilde{A}v'(x))_{\mathcal{K}}\,dx \\ &= \int_G (f(x), A\pi'(x^{-1})v')_{\mathcal{K}}\,dx \\ &= \int_G (\pi'(x)A^*f(x), v')_{\mathcal{H}'}\,dx \\ &= (\int_G \pi'(x)A^*f(x)\,dx, v')_{\mathcal{H}'}. \end{aligned}$$

Thus

$$\tilde{A}^*f = \int \pi'(x)A^*f(x)\,dx \text{ for } f \in L^2_K(G,\pi).$$

We can therefore calculate the operator $\tilde{A}^*\tilde{A}$. Note:

$$\begin{aligned} \tilde{A}^*\tilde{A}v' &= \int \pi'(x)A^*\tilde{A}v'(x)\,dx \\ &= \int \pi'(x)A^*A\pi'(x^{-1})v'\,dx \\ &= \left(\int \pi'(x)A^*A\pi'(x^{-1})\,dx\right)v'. \end{aligned}$$

Thus

$$\begin{aligned}
||\tilde{A}||_2^2 &= \mathrm{Tr}(\tilde{A}^*\tilde{A}) \\
&= \mathrm{Tr}\left(\int \pi'(x)A^*A\pi'(x^{-1})\,dx\right) \\
&= \int \mathrm{Tr}(\pi'(x)A^*A\pi'(x^{-1}))\,dx \\
&= \int \mathrm{Tr}(A^*A)\,dx \\
&= \mathrm{Tr}(A^*A) \\
&= ||A||_2^2.
\end{aligned}$$

Thus $A \mapsto \tilde{A}$ is an isometry of the Hilbert space $\mathrm{Hom}_G(\pi'|_K, \pi)_2$ into the Hilbert space $\mathrm{Hom}_G(\pi', \mathrm{ind}_K^G \pi)_2$.

We claim it is onto. Let $B \in \mathrm{Hom}_G(\pi', \mathrm{ind}_K^G \pi)_2$. By Theorem 8.5, we can choose orthogonal projections P_α, $\alpha \in A$, where $P_\alpha \in \mathrm{Hom}_G(\mathrm{ind}_K^G \pi, \mathrm{ind}_K^G \pi)$, $P_\alpha L_K^2(G, \pi)$ are finite dimensional, and $\bigoplus P_\alpha = I$. Thus $P_\alpha B$ are in the Hilbert space $\mathrm{Hom}_G(\pi', \mathrm{ind}_K^G \pi)_2$. We claim they are orthogonal in this Hilbert space. Indeed,

$$(P_\alpha B, P_{\alpha'} B)_2 = \mathrm{Tr}(B^* P_{\alpha'}^* P_\alpha B) = \mathrm{Tr}(0) = 0 \text{ if } \alpha \neq \alpha'.$$

Moreover

$$\sum ||P_\alpha B||_2^2 = ||B||_2^2$$

for

$$\sum ||P_\alpha B||_2^2 = \sum \mathrm{Tr}(P_\alpha B B^* P_\alpha) = \mathrm{Tr}(BB^*) = ||B||_2^2.$$

Thus $\sum P_\alpha B = B$ in $\mathrm{Hom}_G(\pi', \mathrm{ind}_K^G \pi)_2$.

Next we show there is a $A_\alpha \in \mathrm{Hom}_G(\pi'|_K, \pi)_2$ with $\tilde{A}_\alpha = P_\alpha B$. In fact, define $A_\alpha : \mathcal{H}' \to \mathcal{K}$ by

$$A_\alpha v' = P_\alpha B v'(e).$$

This is well defined for by Lemma 8.23 the finite dimensional invariant subspace $P_\alpha B\mathcal{H}'$ of $L_K^2(G, \pi)$ must consist of continuous functions. Moreover,

$$\begin{aligned}
A_\alpha \pi'(k) v' &= P_\alpha B(\pi'(k)v')(e) = \pi^G(k)(P_\alpha B v')(e) \\
&= P_\alpha B v'(k^{-1}) = \pi(k) P_\alpha B v'(e) = \pi(k) A_\alpha v'.
\end{aligned}$$

To see $A_\alpha : \mathcal{H}' \to \mathcal{K}$ is continuous, let $v'_k \to 0$ in $\mathcal{H}'$. Then $P_\alpha B v'_k \to 0$ in $P_\alpha L_K^2(G, \pi)$, a finite dimensional vector space. Since all Hausdorff vector space topologies on $P_\alpha L_K^2(G, \pi)$ are equivalent, $P_\alpha B v'_k \to 0$ in $C_K(G, \pi)$. Thus $P_\alpha B v'_k(e) \to 0$. So $A_\alpha \in \mathrm{Hom}_G(\pi'|_K, \pi)$. We claim $\tilde{A}_\alpha = P_\alpha B$.

Indeed,

$$\begin{aligned}\tilde{A}_\alpha v'(x) &= A_\alpha \pi'(x^{-1})v' \\ &= P_\alpha B(\pi'(x^{-1})v')(e) \\ &= \pi^G(x^{-1})(P_\alpha B v')(e) \\ &= P_\alpha B v'(x).\end{aligned}$$

The earlier part of the argument shows $||A_\alpha||_2^2 = ||\tilde{A}_\alpha||_2^2 = ||P_\alpha B||_2^2$. Thus $A_\alpha \in \mathrm{Hom}_G(\pi'|_K, \pi)_2$. Since $A \mapsto \tilde{A}$ is an isometry, we see $A = \sum A_\alpha$ exists in $\mathrm{Hom}_G(\pi'|_K, \pi)_2$ and $\tilde{A} = \sum \tilde{A}_\alpha = \sum P_\alpha B = B$. □

Corollary 8.26. *Let G be a compact Hausdorff group and let $\pi' \in \hat{G}_c$. Suppose K is a closed subgroup and π is a unitary representation of K on a separable Hilbert space $\mathcal{K}$. Then the π'-primary projection $P(\pi')$ on $L^2_K(G, \pi)$ is the orthogonal projection whose range consists of the closure of the linear span of the functions $x \mapsto A\pi'(x^{-1})v'$ where $A \in \mathrm{Hom}_K(\pi'|_K, \pi)$.*

Proof. Apply Corollary 6.54. □

As we shall see shortly, there is a natural linear mapping between $\mathrm{Hom}_K(\pi'|_K, \pi)_2 \otimes \mathcal{H}'$ and $L^2_K(G, \pi)$ which sends the elementary tensor $A \otimes v'$ to the function $\tilde{A}v'$. Thus it is natural to define functions $A \otimes_{\pi'} v'$ in $C_K(G, \pi)$ by

$$(A \otimes_{\pi'} v')(x) = A\pi'(x^{-1})v'.$$

These are in some sense "vector valued" matrix coefficients.

Lemma 8.27. *Let π' be an irreducible unitary representation of a compact Hausdorff group G on a Hilbert space $\mathcal{H}'$ of finite dimension $d(\pi')$. Let T be a linear operator on $\mathcal{H}'$. Then*

$$\int (\pi'(x)T\pi'(x^{-1})v_1', v_2')_{\mathcal{H}'}\, dx = \frac{1}{d(\pi')} \mathrm{Tr}(T)(v_1', v_2')_{\mathcal{H}'}.$$

Proof. By Lemma 8.2, the operator $A = \int \pi'(x)T\pi'(x^{-1})\, dx$ intertwines π' with π'. By Schur's Lemma, $A = cI$ for some scalar c. Take the trace of both sides. We see $cd(\pi') = \mathrm{Tr}(T)$. So $c = \frac{\mathrm{Tr}(T)}{d(\pi')}$. □

Theorem 8.28 (Orthogonality Relations)**.** *Suppose π' is an irreducible unitary representation of G. If $A_1, A_2 \in \mathrm{Hom}_K(\pi'|_K, \pi)$, then*

$$(A_1 \otimes_{\pi'} v_1', A_2 \otimes_{\pi'} v_2') = \frac{1}{d(\pi')}(A_1, A_2)_2 (v_1', v_2')_{\mathcal{H}'}.$$

Moreover, if π_1' and π_2' are unitary representations of G with $\mathrm{Hom}_G(\pi_1', \pi_2')_2 = \{0\}$, then

$(A_1 \otimes_{\pi'} v_1', A_2 \otimes_{\pi'} v_2') = 0$ *for* $A_1 \in \mathrm{Hom}_G(\pi_1'|_K, \pi)_2$ *and* $A_2 \in \mathrm{Hom}_G(\pi_2'|_K, \pi)_2$.

Proof.

$$\begin{aligned}
(A_1 \otimes_{\pi'} v_1', A_2 \otimes_{\pi'} v_2') &= \int (A_1\pi'(x^{-1})v_1', A_2\pi'(x^{-1})v_2')_{\mathcal{K}}\, dx \\
&= \int (\pi'(x)A_2^* A_1 \pi'(x^{-1})v_1', v_2')_{\mathcal{H}'}\, dx \\
&= \frac{1}{d(\pi')}\mathrm{Tr}(A_2^* A_1)(v_1', v_2')_{\mathcal{H}'} \\
&= \frac{1}{d(\pi')}(A_1, A_2)_2 (v_1', v_2')_{\mathcal{H}'}.
\end{aligned}$$

For the second part, assume $A_j \in \mathrm{Hom}_G(\pi_j'|_K, \pi)_2$ for $j = 1, 2$. Then $T = \int \pi_2'(x)A_2^* A_1 \pi_1'(x^{-1})\, dx$ is a intertwining operator between π_1' and π_2'. We claim it is Hilbert-Schmidt. Indeed,

$$\begin{aligned}
||T||_2 &\leqslant \int ||\pi_2'(x)A_2^* A_1 \pi_1'(x^{-1})||_2\, dx \\
&\leqslant \int ||\pi_2'(x)||\, ||A_2||\, ||A_1||_2 ||\pi_1'(x^{-1})||\, dx \\
&= ||A_2||\, ||A_1||_2.
\end{aligned}$$

Hence since $\mathrm{Hom}_G(\pi_1', \pi_2')_2 = \{0\}$, $T = 0$. Thus

$$\begin{aligned}
\int (A_1\pi_1'(x^{-1})v_1', A_2\pi_2'(x^{-1})v_2')_{\mathcal{K}}\, dx &= \int (\pi_2'(x)A_2^* A_1 \pi_1'(x^{-1})v_1', v_2')_{\mathcal{H}_2'}\, dx \\
&= (Tv_1', v_2')_{\mathcal{H}_2'} \\
&= 0
\end{aligned}$$

for v_1', v_2' in the Hilbert spaces for π_1' and π_2', respectively. □

Corollary 8.29. *Let π' be an irreducible unitary representation of a compact Hausdorff group G. There is an intertwining operator $I_{\pi'}$ between $I \otimes \pi'$ on the Hilbert space $\mathrm{Hom}_K(\pi'|_K, \pi)_2 \otimes \mathcal{H}'$ with $\mathrm{ind}_K^G \pi$ on $L_K^2(G, \pi)$ satisfying*

$$I_{\pi'}(A \otimes v') = A \otimes_{\pi'} v'.$$

In particular, $\mathrm{ind}_K^G\pi|_{\mathrm{Im}(I_{\pi'})} \cong \dim(\mathrm{Hom}_K(\pi'|_K, \pi)_2)\pi'$. Moreover, $\sqrt{d(\pi')}I_{\pi'}$ is an isometry whose range is the π'-primary subspace for the unitary representation $\mathrm{ind}_K^G\pi$.

Proof. To define $I_{\pi'}$, it suffices to define the linear isometry $\sqrt{d(\pi')}I_{\pi'}$; and to do this we need only know it takes an orthonormal basis to an orthonormal set. Let $e_1, e_2, \ldots, e_{d'}$ be an orthonormal basis of $\mathcal{H}'$ and let $\{E_\iota\}_{i \in I}$ be an orthonormal basis of $\mathrm{Hom}_G(\pi'|_K, \pi)_2$. Set

$$\sqrt{d(\pi')}I_\pi(E_i \otimes e_j) = \sqrt{d(\pi')}E_i \otimes_{\pi'} e_j.$$

By Theorem 8.28, the orthogonality relations, we see $\sqrt{d(\pi)}I_\pi$ is sending an orthonormal basis of $\mathrm{Hom}_G(\pi'|_K, \pi)_2 \otimes \mathcal{H}'$ into an orthonormal set in $L^2_K(G, \pi)$. It consequently extends to a linear isometry $\sqrt{d(\pi)}I_\pi$ from $\mathrm{Hom}_K(\pi'|_K, \pi)_2 \otimes \mathcal{H}'$ into $L^2_K(G, \pi)$. In conclusion, we see there is a bounded linear mapping $I_\pi : \mathrm{Hom}_G(\pi'|_K, \pi)_2 \otimes \mathcal{H}' \to L^2_K(G, \pi)$ sending the elementary tensor $A \otimes v$ to $A \otimes_{\pi'} v$.

By the Frobenius Theorem 8.25, we know $\tilde{A}$ intertwines π' and $\mathrm{ind}_K^G \pi$ and thus

$$\begin{aligned} I_{\pi'}((I \otimes \pi')(g)(A \otimes v')) &= I_{\pi'}(A \otimes \pi'(g)v') \\ &= A \otimes_{\pi'} \pi'(g)v' \\ &= \tilde{A}\pi'(g)v' \\ &= \pi^G(g)\tilde{A}v' \\ &= \pi^G(g)(A \otimes_{\pi'} v') \\ &= \pi^G(g)I_{\pi'}(A \otimes v'). \end{aligned}$$

This implies $I_{\pi'}$ intertwines $I \otimes \pi'$ and the $\mathrm{ind}_K^G \pi$.

We first note every operator in $\mathrm{Hom}_G(\pi', \mathrm{ind}_K^G \pi)$ is Hilbert-Schmidt for $\mathcal{H}_{\pi'}$ is finite dimensional. Thus $\mathrm{Hom}_G(\pi', \mathrm{ind}_K^G \pi) = \mathrm{Hom}_G(\pi', \mathrm{ind}_K^G \pi)_2$. Since the range of $\sqrt{d(\pi)}I_{\pi'}$ is the closure of the linear span of the functions $\tilde{A}v'$, we see by Corollary 6.54, that the range of $I_{\pi'}$ is the range of the primary projection $P(\pi')$ for $\mathrm{ind}_K^G \pi$.

Finally, the statement $\pi^G \pi|_{\mathrm{Im}(I_{\pi'})} \cong \dim(\mathrm{Hom}_G(\pi'|_K, \pi)_2)\pi'$ follows from Exercise 8.2.6. □

These results can be restated in terms of multiplicity. Namely, by Theorem 8.5, we know the representation $\mathrm{ind}_K^G \pi$ is discretely decomposable. Moreover, by Corollary 8.29, $\mathrm{ind}_K^G \pi$ restricted to the range of $P(\pi')$ is unitarily equivalent to $n\pi'$ where n is the dimension of the Hilbert space $\mathrm{Hom}_K(\pi|'_K, \pi)_2$. Since π' is finite dimensional and the Hilbert space $\mathcal{K}$ for π is assumed to be separable, this space of intertwiners is separable. Hence its dimension is either finite or $\aleph_0$. Thus, the multiplicity $m(\pi', \mathrm{ind}_K^G \pi)$ equals $m(\pi'|_K, \pi)$. Note even though $\mathcal{K}$ is separable, the representation $\mathrm{ind}_K^G \pi$ will not be separable if the homogeneous space G/K is nonseparable.

Theorem 8.30 (Frobenius)**.** *Let π be a unitary representation of a closed subgroup of a compact Hausdorff group G on a separable Hilbert space $\mathcal{K}$. Then*

$$m(\pi', \mathrm{ind}_K^G \pi) = m(\pi'|_K, \pi)$$

for each irreducible unitary representation π' of G.

We thus have

$$\mathrm{ind}_K^G \pi \cong \bigoplus_{\pi' \in \hat{G}_c} m(\pi'|_K, \pi)\pi'.$$

where here the unitary equivalence is established by composing a unitary equivalence from $\mathrm{ind}_K^G \pi$ restricted to the range of $P(\pi')$ with $I \otimes \pi'$ followed by a unitary equivalence of $I \otimes \pi'$ to $n\pi'$. (See Exercise 8.2.6.)

To obtain the primary internal orthogonal decomposition of the representation $\mathrm{ind}_K^G \pi$, we need to determine the primary projections

$$P(\pi') = d(\pi') I_{\pi'} I_{\pi'}^* \text{ for } \pi' \in \hat{G}_c. \tag{8.4}$$

We show in our next sections this can be done using traces.

6. Trace Class Operators between Hilbert Spaces

In Section 3 of Chapter 2, we defined the notion of a trace class operator on a Hilbert space $\mathcal{H}$ and its trace. Trace class operators are always Hilbert-Schmidt; in fact they are seen to be the product of two Hilbert-Schmidt operators. To continue decomposing an induced representation on a compact group, we shall make use of a more general notion of a trace class operator and the concept of traces having values in a Hilbert space.

Definition 8.31. *Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces. Then a bounded linear operator $A : \mathcal{H} \to \mathcal{H}'$ is said to be trace class if A can be written in form*

$$\sum_i v_i' \otimes \bar{v}_i$$

where

$$\sum ||v_i||^2 < \infty \text{ and} \sum ||v_i'||^2 < \infty.$$

We note when $\mathcal{H} = \mathcal{H}'$, then we have the same definition as 2.37 in Chapter 2. Moreover, we still have the following.

Proposition 8.32. *An operator A from $\mathcal{H}$ to $\mathcal{H}'$ is trace class if and only if it is the composition of two Hilbert-Schmidt operators.*

Proof. Let $A = RS$ where $R \in \mathcal{B}_2(\mathcal{K}, \mathcal{H}')$ and $S \in \mathcal{B}_2(\mathcal{H}, \mathcal{K})$. By Exercise 2.2.15, we can find an orthonormal set $\{e_i\}$ in $\mathcal{K}$ and collections $\{v_i\}$ in $\mathcal{H}$ and $\{v_i'\}$ in $\mathcal{H}'$ such that $\sum ||v_i||^2 < \infty$ and $\sum ||v_i'||^2 < \infty$ with $R = \sum v_i' \otimes \bar{e}_i$ and $S = \sum e_i \otimes \bar{v}_i$. Then $RS = \sum v_i' \otimes \bar{v}_i$ is trace class.

Conversely, let $A = \sum v_i' \otimes \bar{v}_i$ be trace class where $\sum ||v_i'||^2 < \infty$ and $\sum ||v_i||^2 < \infty$. Take a Hilbert space $\mathcal{K}$ with an orthonormal basis $\{e_i\}$ indexed by the i's. Set $R = \sum v_i' \otimes \bar{e}_i$ and $S = \sum e_i \otimes \bar{v}_i$. Using Exercise 2.2.15 again, we see R and S are Hilbert-Schmidt. Moreover, $A = RS$. $\square$

Proposition 2.32 shows trace class operators are always Hilbert-Schmidt. We now generalize the notion of trace. To do this let $B : \mathcal{H}' \times \mathcal{H} \to \mathcal{K}$ where $\mathcal{K}$ is a Hilbert space be sesquilinear and bounded.

Definition 8.33. *Let $B : \mathcal{H}' \times \mathcal{H} \to \mathcal{K}$ be bounded and sesquilinear. If A is a trace class operator from $\mathcal{H}$ to $\mathcal{H}'$ with $A = \sum v_i' \otimes \bar{v}_i$ where $\sum ||v_i||^2 < \infty$ and $\sum ||v_i'||^2 < \infty$, then the B-trace of A is defined by*

$$\mathrm{Tr}_B(A) = \sum_i B(v_i', v_i).$$

Note the series in the definition converges absolutely for

$$\begin{aligned} \sum ||B(w_i, v_i)|| &\leqslant \sum_i ||B|| \, ||w_i|| \, ||v_i|| \\ &\leqslant ||B|| \left(\sum_i ||w_i||^2 \right)^{1/2} \left(\sum_i ||v_i||^2 \right)^{1/2} \\ &< \infty. \end{aligned}$$

Proposition 8.34. *Let $\{e_\alpha\}$ be an orthonormal basis for $\mathcal{H}$. Then the B trace of trace class operator A is*

$$\sum_\alpha B(Ae_\alpha, e_\alpha).$$

Moreover, this series is absolutely summable.

Proof. Since A is trace class, $A = \sum_i w_i \otimes \bar{v}_i$ where $\sum_i |w_i|^2 < \infty$ and $\sum_i |v_i|^2 < \infty$. From $Ae_\alpha = \sum_i (e_\alpha, v_i) w_i$, we see

$$B(Ae_\alpha, e_\alpha) = \sum (e_\alpha, v_i) B(w_i, e_\alpha).$$

Set $v_i' = \sum_\alpha |(v_i, e_\alpha)| e_\alpha$. Note $||v_i'|| = ||v_i||$. Applying the Cauchy-Schwarz's inequality one obtains:

$$\begin{aligned} \sum_\alpha |B(Ae_\alpha, e_\alpha)| &= \sum_\alpha | \sum_i (e_\alpha, v_i) B(w_i, e_\alpha)| \leqslant \sum_{i,\alpha} |(e_\alpha, v_i) B(w_i, e_\alpha)| \\ &= \sum_i |B(w_i, \sum_\alpha |(v_i, e_\alpha)| e_\alpha)| = \sum_i |B(w_i, v_i')| \\ &\leqslant \sum_i ||B|| \, ||w_i|| \, ||v_i'|| \leqslant ||B|| \left(\sum_i ||v_i||^2 \right)^{\frac{1}{2}} \left(\sum_i ||w_i||^2 \right)^{\frac{1}{2}}. \end{aligned}$$

From this we also see $\sum_{i,\alpha}(e_\alpha, v_i)B(w_i, e_\alpha)$ converges absolutely. Thus

$$\begin{aligned}\sum_\alpha B(Ae_\alpha, e_\alpha) &= \sum_\alpha B(\sum_i (e_\alpha, v_i)w_i, e_\alpha) = \sum_{\alpha, i}(e_\alpha, v_i)B(w_i, e_\alpha)\\ &= \sum_{i,\alpha} B(w_i, (v_i, e_\alpha)e_\alpha) = \sum_i B(w_i, \sum_\alpha (v_i, e_\alpha)e_\alpha)\\ &= \sum_i B(w_i, v_i).\end{aligned}$$

□

If we are in the case when $\mathcal{H} = \mathcal{H}'$ and we take $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ to be the inner product on $\mathcal{H}$, one returns to the standard trace

$$\mathrm{Tr}(A) = \sum_\alpha (Ae_\alpha, e_\alpha).$$

As we have just seen, associated to any bounded sesquilinear Hilbert space valued function on a product of Hilbert spaces, one has a Hilbert space valued trace on the trace class operators. In order to decompose induced representations, we shall use a specific one.

Let π be a representation of a closed subgroup K of a compact group G on a separable Hilbert space $\mathcal{K}$. Let π' be an irreducible unitary representation of G on a necessarily finite dimensional Hilbert space $\mathcal{H}'$. We have Hilbert space $\mathcal{H}' \otimes \bar{\mathcal{H}}'$, which in this case is all linear transformations from $\mathcal{H}'$ into $\mathcal{H}'$ and Hilbert space $\mathcal{K}$. Since $\mathcal{H}' \otimes \bar{\mathcal{H}}'$ is finite dimensional, all bounded linear transformations from $\bar{\mathcal{K}}$ into $\mathcal{H}' \otimes \bar{\mathcal{H}}'$ are trace class. Moreover, B defined on $(\mathcal{H}' \otimes \bar{\mathcal{H}}') \times \bar{\mathcal{K}}$ by

$$B(T, \bar{w}) = \mathrm{Tr}(T)w$$

is bounded and sesquilinear on $(\mathcal{H}' \otimes \bar{\mathcal{H}}') \times \bar{\mathcal{K}}$. Thus every linear operator $A : \bar{\mathcal{K}} \to \mathcal{H}' \otimes \bar{\mathcal{H}}'$ is trace class and has a B-trace in $\mathcal{K}$.

Definition 8.35. *Let $\mathcal{H}'$ be a finite dimensional Hilbert space and let $\mathcal{K}$ be a Hilbert space. Let $\{e'_\alpha\}$ be an orthonormal basis for $\mathcal{H}'$. For $A \in (\mathcal{H}' \otimes \bar{\mathcal{H}}') \otimes \mathcal{K} = \mathcal{B}(\bar{\mathcal{K}}, \mathcal{B}(\mathcal{H}', \mathcal{H}'))$, define*

$$\mathrm{Tr}_\mathcal{K}(A) = \sum_\alpha \overline{A^*(e'_\alpha \otimes \bar{e}'_\alpha)}.$$

Using Definition 8.33 and $A = v_1' \otimes \bar{v}_2'$ and $w \in \mathcal{K}$, one sees:

$$\begin{aligned}
\mathrm{Tr}_B\left((v_1' \otimes \bar{v}_2') \otimes w\right) &= \mathrm{Tr}(v_1' \otimes \bar{v}_2')w \\
&= (v_1', v_2')_{\mathcal{H}'} w \\
&= \sum_\alpha (v_1', e_\alpha')(e_\alpha', v_2')w \\
&= \overline{\sum_\alpha (e_\alpha', v_1')(v_2', e_\alpha')\bar{w}} \\
&= \overline{\sum_\alpha (\bar{w} \otimes \overline{v_1' \otimes \bar{v}_2'})(e_\alpha' \otimes \bar{e}_\alpha')} \qquad (8.5) \\
&= \overline{\sum_\alpha ((v_1' \otimes \bar{v}_2') \otimes w)^*(e_\alpha' \otimes \bar{e}_\alpha')} \\
&= \mathrm{Tr}_{\mathcal{K}}\left((v_1' \otimes \bar{v}_2') \otimes w\right).
\end{aligned}$$

In particular, $\mathrm{Tr}_{\mathcal{K}}(A) = \mathrm{Tr}_B(A)$ and thus is well defined. This trace is also defined when $\mathcal{H}'$ is infinite dimensional, but of course only on the trace class operators.

Lemma 8.36. *Suppose $\mathcal{H}'$ is finite dimensional. Then $\mathrm{Tr}_{\mathcal{K}} : (\mathcal{H}' \otimes \bar{\mathcal{H}}') \otimes \mathcal{K} \to \mathcal{K}$ is a bounded linear transformation whose adjoint is defined by*

$$((\mathrm{Tr}_{\mathcal{K}})^*(w) = I \otimes w \textit{ for } w \in \mathcal{K}.$$

Proof. We first note $\mathrm{Tr}_{\mathcal{K}}$ is bounded from $(\mathcal{H}' \otimes \bar{\mathcal{H}}') \otimes \mathcal{K}$ into $\mathcal{K}$. Indeed, if d is the dimension of $\mathcal{H}'$ and $e_1', \ldots, e_d'$ is an orthonormal basis, then

$$\begin{aligned}
||\mathrm{Tr}_{\mathcal{K}}(A)|| &= ||\sum_{i=1}^d \overline{A^*(\bar{e}_i' \otimes e_i')}|| \\
&\leqslant \sum_{i=1}^d ||A^*||\,||\bar{e}_i' \otimes e_i'||_2 \\
&= d||A|| \\
&\leqslant d||A||_2
\end{aligned}$$

since the operator norm is smaller than the Hilbert-Schmidt norm. Note if w_1 and w_2 are in $\mathcal{K}$, then

$$\begin{aligned}
(A \otimes w_2, (\mathrm{Tr}_{\mathcal{K}})^* w_1) &= (\mathrm{Tr}_{\mathcal{K}}(A \otimes w_2), w_1) \\
&= (\mathrm{Tr}(A)w_2, w_1) \\
&= \mathrm{Tr}(AI^*)(w_2, w_1) \\
&= (A, I)_2(w_2, w_1) \\
&= (A \otimes w_2, I \otimes w_1)_2.
\end{aligned}$$

Thus $(\mathrm{Tr}_{\mathcal{K}})^*(w_1) = I \otimes w_1$. $\square$

We remark the operator $I \otimes w_1 : \bar{\mathcal{K}} \to L(\mathcal{H}', \mathcal{H}')$ is the linear transformation $((I \otimes w_1)(\bar{w}_2))(v') = (w_1, w_2)v'$. Since $I = \sum_{i=1}^d e_i' \otimes \bar{e}_i'$, we can also write

$$(\mathrm{Tr}_{\mathcal{K}})^*(w_1) = \sum_{i=1}^{d} (e_i' \otimes \bar{e}_i') \otimes w_1.$$

Definition 8.37. *Let $f \in L^2_K(G, \pi)$ and $\pi' \in \hat{G}_c$. Then $\pi'(f)$ is the operator in $(\mathcal{H}' \otimes \bar{\mathcal{H}}') \otimes \mathcal{K}$ defined by*

$$\pi'(f) = \int_G \pi'(x) \otimes f(x)\, dx.$$

Note in Corollary 6.108, we defined $\pi'(f)$ for $f \in L^1(G)$. We are now defining $\pi'(f)$ for $f \in L^2_K(G, \pi) \subseteq L^1_K(G, \pi)$. We note $\pi'(f)$ is a bounded operator. Indeed, if d is the dimension of $\mathcal{H}'$, then since $||\pi'(x)||_2 \leqslant \sqrt{d}$, we see

$$\begin{aligned} ||\pi'(f)|| &\leqslant \int ||\pi'(x) \otimes f(x)||_2\, dx \\ &\leqslant \int ||\pi'(x)||_2 ||f(x)||_{\mathcal{K}}\, dx \\ &\leqslant \sqrt{d}||f||_1 \leqslant \sqrt{d}||f||_2. \end{aligned}$$

In particular, $\pi'(f)$ is $\mathcal{K}$-traceable.

Lemma 8.38. *Let $f \in L^2_K(G, \pi)$. Then*

$$\pi'(f) = \int_G (\pi'(x) \otimes I)\mathrm{Tr}^*_{\mathcal{K}}(f(x))\, dx.$$

Proof. Using Lemma 8.36, we see

$$\begin{aligned} \pi'(f) &= \int \pi'(x) \otimes f(x)\, dx \\ &= \int (\pi'(x) \otimes I)(I \otimes f(x))\, dx \\ &= \int (\pi'(x) \otimes I)\mathrm{Tr}^*_{\mathcal{K}}(f(x))\, dx. \end{aligned}$$

□

Before stating the next theorem, in order to use the $\mathcal{K}$ trace, we need to establish a natural unitary isomorphism Φ between $(\mathcal{K} \otimes \bar{\mathcal{H}}') \otimes \mathcal{H}'$ and $(\mathcal{H}' \otimes \bar{\mathcal{H}}') \otimes \mathcal{K}$ when $\mathcal{H}'$ is finite dimensional. This natural isomorphism satisfies

$$\Phi((w \otimes \bar{v}_2') \otimes v_1') = (v_1' \otimes \bar{v}_2') \otimes w$$

for $v_1', v_2' \in \mathcal{H}'$ and $w \in \mathcal{K}$. Hence let T be a linear transformation from $\bar{\mathcal{H}}'$ into $\mathcal{K} \otimes \bar{\mathcal{H}}' = \mathcal{B}(\mathcal{H}', \mathcal{K})$. We define $\Phi(T)$ by

$$(\Phi(T)(\bar{w}))\,(v') = \overline{T^*(w \otimes \bar{v}')}. \tag{8.6}$$

Note

$$\begin{aligned}
\left(\Phi(((w_1 \otimes \bar{v}_2') \otimes v_1')(\bar{w})\right)(v') &= \overline{((w_1 \otimes \bar{v}_2') \otimes v_1')^*(w \otimes \bar{v}')} \\
&= \overline{(\bar{v}_1' \otimes \overline{(w_1 \otimes \bar{v}_2')})(w \otimes \bar{v}')} \\
&= \overline{(w \otimes \bar{v}', w_1 \otimes \bar{v}_2')\bar{v}_1'} \\
&= \overline{(w, w_1)_{\mathcal{K}}(\bar{v}', \bar{v}_2')_{\bar{\mathcal{H}}}} v_1' \\
&= \overline{(w, w_1)}_{\mathcal{K}}(v', v_2')_{\mathcal{H}} v_1' \\
&= (\bar{w}, \bar{w}_1)_{\bar{\mathcal{K}}}(v', v_2')_{\mathcal{H}} v_1' \\
&= \left(((v_1' \otimes \bar{v}_2') \otimes w_1)(\bar{w})\right)(v').
\end{aligned}$$

To see Φ preserves inner products, it suffices to show

$$(\Phi((w_1 \otimes \bar{v}_2) \otimes v_1), \Phi((w_2 \otimes \bar{v}_2') \otimes v_1'))_2 = \\ ((w_1 \otimes \bar{v}_2) \otimes v_1, (w_2 \otimes \bar{v}_2') \otimes v_1')$$

for $v_1, v_2, v_1', v_2' \in \mathcal{H}'$ and $w_1, w_2 \in \mathcal{K}$. But this is immediate since

$$\begin{aligned}
((v_1 \otimes \bar{v}_2) \otimes w_1, (v_1' \otimes \bar{v}_2') \otimes w_2)_2 &= (v_1 \otimes \bar{v}_2, v_1' \otimes \bar{v}_2')_2 (w_1, w_2)_{\mathcal{K}} \\
&= (v_1, v_1')_{\mathcal{H}'}(\bar{v}_2, \bar{v}_2')_{\bar{\mathcal{H}}'}(w_1, w_2)_{\mathcal{K}} \\
&= (w_1 \otimes \bar{v}_2, w_2 \otimes \bar{v}_2')_2 (v_1, v_1')_{\mathcal{H}'} \\
&= ((w_1 \otimes \bar{v}_2) \otimes v_1, (w_2 \otimes \bar{v}_2') \otimes v_1')_2.
\end{aligned}$$

To see Φ is onto, we note the inverse $\Phi^{-1} = \Phi^*$ (e.g. see Exercise 8.3.3) is the transformation defined by $\left(\Phi^{-1}(S)(\bar{v}_1')\right)(v_2') = \overline{S^*(v_1' \otimes \bar{v}_2')}$ for $S \in (\mathcal{H}' \otimes \bar{\mathcal{H}}') \otimes \mathcal{K}$.

We also note if $T \in (\mathcal{H}' \otimes \bar{\mathcal{H}}') \otimes \mathcal{K}$, then T is a bounded linear operator from $\bar{\mathcal{K}}$ into $\mathcal{H}' \otimes \bar{\mathcal{H}}'$, the space of linear transformations of $\mathcal{H}'$. Now $\pi' \otimes I$ is a unitary representation of G on $\mathcal{H}' \otimes \bar{\mathcal{H}}'$. Thus the composition $(\pi'(x^{-1}) \otimes I)T$ is again in $(\mathcal{H}' \otimes \bar{\mathcal{H}}') \otimes \mathcal{K}$, and hence we can take its $\mathcal{K}$ trace, $\mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1}) \otimes I)T)$. This will allow us to obtain a formula for the intertwining operator $I_{\pi'}$ given in Corollary 8.29 whose range is the π'-primary subspace of $L^2_K(G, \pi)$.

Theorem 8.39. *Let π' be an irreducible unitary representation of a compact group G on a finite dimensional Hilbert space $\mathcal{H}'$ and let π be a unitary representation of closed subgroup K on a separable Hilbert space $\mathcal{K}$.*

(a) *For* $T \in \mathrm{Hom}_K(\pi'|_K, \pi) \otimes \mathcal{H}' \subseteq (\mathcal{K} \otimes \bar{\mathcal{H}}') \otimes \mathcal{H}'$,

$$I_{\pi'}(T)(x) = \mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1}) \otimes I)\Phi(T)).$$

(b) *For* $f \in L^2_K(G, \pi)$,

$$I^*_{\pi'}(f) = \Phi^{-1}(\pi'(f)) = \Phi^*(\pi'(f)).$$

(c) *The* π'*-primary projection* $P(\pi') = d(\pi') I_\pi I^*_\pi$ *onto the range of* I_π *is given by*

$$P(\pi') f(x) = d(\pi') \mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1}) \otimes I)\pi'(f)).$$

Proof. First note since $A = \sum_{i=1}^d Ae_i \otimes \bar{e}_i$ if $e'_1, e'_2, \ldots, e'_d$ is an orthonormal basis of $\mathcal{H}'$, we have

$$\begin{aligned}
I_{\pi'}(A \otimes v')(x) &= A(\pi'(x^{-1})v') = \sum_{i=1}^d (Ae'_i \otimes \bar{e}'_i)(\pi'(x^{-1})v') = \sum_{i=1}^d (\pi'(x^{-1})v', e'_i) Ae'_i \\
&= \sum_{i=1}^d \mathrm{Tr}(\pi'(x^{-1})v' \otimes \bar{e}'_i) Ae'_i = \sum_{i=1}^d \mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1})v' \otimes \bar{e}'_i) \otimes Ae'_i) \\
&= \sum_{i=1}^d \mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1}) \otimes I)((v' \otimes \bar{e}'_i) \otimes Ae'_i)) \\
&= \mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1}) \otimes I) \sum_{i=1}^d ((v' \otimes \bar{e}'_i) \otimes Ae'_i)) \\
&= \mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1}) \otimes I) \sum_{i=1}^d \Phi((Ae'_i \otimes \bar{e}'_i) \otimes v')) \\
&= \mathrm{Tr}_{\mathcal{K}} \left[(\pi'(x^{-1}) \otimes I)\Phi \left(\sum_{i=1}^d (Ae'_i \otimes \bar{e}'_i) \otimes v' \right) \right] \\
&= \mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1}) \otimes I)\Phi(A \otimes v')).
\end{aligned}$$

Hence we have (a).

For (b), let $f \in L^2_K(G,\pi)$. Then using Lemma 8.36 and Lemma 8.38,

$$\begin{aligned}
(I^*_{\pi'} f, T)_2 &= (f, I_{\pi'} T) \\
&= \int_G (f(x), I_{\pi'}T(x))_{\mathcal{K}}\, dx \\
&= \int_G (f(x), \mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1}) \otimes I)\Phi(T))_{\mathcal{K}}\, dx \\
&= \int_G (\mathrm{Tr}^*_{\mathcal{K}}(f(x)), (\pi'(x^{-1}) \otimes I)\Phi(T))_2\, dx \\
&= \left(\int_G (\pi'(x) \otimes I)\mathrm{Tr}^*_{\mathcal{K}}(f(x))\, dx, \Phi(T) \right)_2 \\
&= (\pi'(f), \Phi(T))_2 \\
&= (\Phi^*(\pi'(f)), T)_2.
\end{aligned}$$

Thus $I^*_{\pi'} f = \Phi^{-1}(\pi'(f))$. Finally, using (8.4),

$$\begin{aligned}
P(\pi')f(x) &= d(\pi')I_{\pi'}I^*_{\pi'}f(x) \\
&= d(\pi')I_{\pi'}\Phi^{-1}(\pi'(f))(x) \\
&= d(\pi')\mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1}) \otimes I)\pi'(f))
\end{aligned}$$

which is (c). □

From Theorem 8.5, Proposition 6.58, and Corollary 8.29 we can conclude with the following result.

Theorem 8.40. *Let G be a compact Hausdorff group with concrete dual $\hat{G}_c$. Let π be a unitary representation of a closed subgroup K of G on a separable Hilbert space $\mathcal{K}$. For $\pi' \in \hat{G}_c$, the π'-primary projection $P(\pi')$ for the representation of G induced from π is given by:*

$$P(\pi')f(x) = d(\pi')\mathrm{Tr}_{\mathcal{K}}((\pi'(x^{-1}) \otimes I)\pi'(f)) \text{ for } f \in L^2_K(G,\pi).$$

Moreover,

$$\bigoplus_{\pi' \in \hat{G}_c} P(\pi') = I \text{ on } L^2_K(G,\pi).$$

7. The One Dimensional Case

When π is one dimensional, the situation in the prior section becomes much simpler. In this case π is a character χ and $C_K(G,\chi)$ consists of those continuous complex valued functions f on the compact group G satisfying

$$f(xk) = \chi(k^{-1})f(x) \text{ for all } x \in G \text{ and } k \in K.$$

But the most important simplification is that we no longer need to use vector valued traces.

Note if $f \in L^2_K(G,\chi)$, then $f \in L^1(G)$ and we then have two definitions for $\pi'(f)$. Namely $\pi'(f) = \int f(x)\,\pi'(x)\,dx \in \mathcal{B}(\mathcal{H}',\mathcal{H}')$ defined in Corollary 6.108 and $\pi'(f) = \int \pi'(x) \otimes f(x)\,dx \in \mathcal{L}(\bar{\mathbb{C}}, \mathcal{B}(\mathcal{H}',\mathcal{H}'))$ given in Definition 8.37. As we shall see in the following proof they are related by the formula:

$$\left(\int \pi'(x) \otimes f(x)\,dx\right)(\bar{1}) = \int f(x)\pi'(x)\,dx.$$

Theorem 8.41. *Let $\chi : K \to \mathbb{T}$ be a continuous homomorphism. For $\pi' \in \hat{G}_c$ and $f \in L^2_K(G,\chi)$, let $\pi'(f) = \int_G f(x)\pi'(x)\,dx$. Let $P(\pi')$ be the π' primary projection for the representation $\mathrm{ind}_K^G\chi$. Then:*

$$\begin{aligned} P(\pi')L^2_K(G,\pi) &= \langle\{x \mapsto A\pi'(x^{-1})v' \mid A \in \mathrm{Hom}_K(\pi'|_K,\chi),\ v' \in \mathcal{H}'\}\rangle \\ &= \{x \mapsto \mathrm{Tr}(\pi'(x^{-1})\pi'(f)) \mid f \in L^2_K(G,\chi)\}. \end{aligned}$$

Moreover, if $\mathcal{H}(\pi') = P(\pi')L^2_K(G,\chi)$, then

$$L^2_K(G,\chi) = \bigoplus_{\pi' \in \hat{G}_c} \mathcal{H}(\pi')$$

and

$$||f||_2^2 = \sum_{\pi' \in \hat{G}_c} d(\pi')\mathrm{Tr}(\pi'(f)^*\pi'(f)).$$

Proof. Using Definition 8.37 and Corollary 6.108,

$$\begin{aligned} \pi'(f)(\bar{1}) &= \int (\pi'(x) \otimes f(x))(\bar{1})\,dx \\ &= \int (1, \overline{f(x)})_{\bar{\mathbb{C}}}\pi'(x)\,dx \\ &= \int f(x)\pi'(x)\,dx. \end{aligned}$$

Hence from part (c) of Theorem 8.39,

$$\begin{aligned} P(\pi')(f)(x) &= \mathrm{Tr}_{\mathcal{K}}(\pi'(x^{-1}) \otimes I)\pi'(f)) \\ &= \mathrm{Tr}_{\mathbb{C}}(\pi'(x^{-1}) \otimes 1)(\pi'(f) \otimes 1) \\ &= \mathrm{Tr}_{\mathbb{C}}(\pi'(x^{-1})\pi'(f) \otimes 1) \\ &= \mathrm{Tr}(\pi'(x^{-1})\pi'(f)). \end{aligned} \tag{8.7}$$

and thus we have the first statement. The last two statements follow from Theorem 8.40. □

Let $\hat{G}_c$ be a concrete dual for G. For a one-dimensional character χ of K, we define:

$$\hat{G}_{c,\chi} = \{\pi' \in \hat{G}_c \mid P(\pi') \neq 0\} = \{\pi' \in \hat{G}_c \mid \mathrm{Hom}_K(\pi'|_K,\chi) \neq \{0\}\} \tag{8.8}$$

Thus

$$L^2_K(G,\chi) = \bigoplus_{\pi' \in \hat{G}_{c,\chi}} P(\pi')L^2_K(G,\chi).$$

The above decomposition then becomes a problem of determining the intertwining operators from the representations $\pi'|_K$ and the character χ of K. This can be described in terms of the simultaneous existence of eigenvectors for the operators $\pi'(k)$.

Let χ be a one-dimensional character of a subgroup K of a topological G and suppose π is a unitary representation of G on a Hilbert space $\mathcal{H}$. We define $E(\pi,\chi)$ by

$$E(\pi,\chi) = \{v \in \mathcal{H} \mid \pi(k)v = \chi(k)v \text{ for all } k \in K\}.$$

Lemma 8.42. *$E(\pi,\chi)$ is the χ-primary subspace for the representation $\pi|_K$ and the mapping $v \mapsto A_v \in \mathrm{Hom}_K(\pi|_K,\chi)$ given by $A_v(w) = (w,v)_{\mathcal{H}}$ is a conjugate linear one-to-one mapping of $E(\pi,\chi)$ onto* $\mathrm{Hom}_K(\pi|_K,\chi)$.

Proof. By Corollary 6.54, the χ-primary subspace for $\pi|_K$ is $E(\pi,\chi)$. Now note:

$$\begin{aligned} A_v(\pi(k)w) &= (\pi(k)w,v) = (w,\pi(k^{-1})v) \\ &= (w,\chi(k^{-1})v) = \chi(k)(w,v) \\ &= \chi(k)A_v(w). \end{aligned}$$

Thus $A_v \in \mathrm{Hom}_K(\pi|_K,\chi)$. Clearly $v \mapsto A_v$ is conjugate linear. Moreover, $A_v = 0$ implies $(w,v) = 0$ for all $w \in \mathcal{H}$ and thus $v = 0$. Hence the mapping is one-to-one. Finally if $A \in \mathrm{Hom}_K(\pi|_K,\chi)$, then A is a continuous linear functional on $\mathcal{H}$. By the Riesz Theorem, $Aw = (w,v)$ for a unique vector $v \in \mathcal{H}$. But $A\pi(k) = \chi(k)A$ implies $(\pi(k)w,v) = (w,\chi(k^{-1})v)$ for all w. Hence $(w,\pi(k^{-1})v) = (w,\chi(k^{-1})v)$ for all w and we conclude $v \in E(\pi,\chi)$. $\square$

Corollary 8.43. *Let $f \in L^2_K(G,\chi)$ and $\pi \in \hat{G}_c$. Then $\pi(f) = 0$ if $E(\pi,\chi) = \{0\}$.*

Proof. If $\pi(f) \neq 0$, then $\pi(f)^* * \pi(f)$ is a nonzero positive operator. But this is $\pi(f^* * f)$ and $f^* * f \in C_K(G,\chi)$. Now $P(\pi)(f^* * f) = 0$ for $P(\pi) = 0$. Thus by Equation (8.7) $\mathrm{Tr}(\pi(x^{-1})\pi(f^* * f)) = 0$ for all x. This contradicts $\mathrm{Tr}(\pi(f^* * f)) > 0$. $\square$

Lemma 8.44. *Let G be a compact Hausdorff group and suppose χ is a one-dimensional character of a closed subgroup K of G. Let $f \in L^2_K(G,\chi)$. Then $\pi(f)w = 0$ for all $w \in E(\pi,\chi)^\perp$.*

Proof. Let $v \in \mathcal{H}_\pi$. Then since $f(xk) = \chi(k^{-1})f(x)$ and we are taking dk to have total measure 1, we see

$$\begin{aligned}
(\pi(f)w, v) &= \int_G f(x)(\pi(x)w, v)\,dx \\
&= \int_G f(x)(w, \pi(x^{-1})v)\,dx \\
&= \int_K \int_G f(xk)\chi(k)(w, \pi(x^{-1})v)\,dx\,dk \\
&= \int_G f(x) \int_K (w, \chi(k^{-1})\pi(kx^{-1})\,v)\,dk\,dx \\
&= \int_G f(x)(w, v(x))\,dx
\end{aligned}$$

where $v(x) = \int_K \chi(k^{-1})\pi(k)\pi(x^{-1})v\,dk$. Since

$$\begin{aligned}
\pi(k_0)v(x) &= \int \chi(k^{-1})\pi(k_0 k)\pi(x^{-1})v\,dk \\
&= \int \chi(k^{-1}k_0)\pi(k^{-1})\,\pi(x^{-1})v\,dk \\
&= \chi(k_0) \int \chi(k^{-1})\pi(k^{-1})\,\pi(x^{-1})v\,dk \\
&= \chi(k_0)v(x),
\end{aligned}$$

one has $v(x) \in E(\pi, \chi)$ for $x \in G$. Consequently $(w, v(x)) = 0$ for all x and $(\pi(f)w, v) = 0$ for all v. Thus $\pi(f)w = 0$. $\square$

Corollary 8.45. *Suppose $\pi \in \hat{G}_c$ and $e_1, e_2, \ldots, e_k$ is an orthonormal basis of $E(\pi, \chi)$. Then the π-primary projection for $\mathrm{ind}_K^G \chi$ is given by*

$$P(\pi)f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f)) = d(\pi) \sum_{i=1}^{k} (\pi(x^{-1})\pi(f)e_i, e_i).$$

Proof. By Theorem 8.40 and (8.7), we see

$$P(\pi)f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f)).$$

Extending $e_1, e_2, \ldots, e_k$ to an orthonormal basis $e_1, e_2, \ldots, e_k, \ldots e_{d(\pi)}$ of $\mathcal{H}_\pi$, Lemma 8.44 gives

$$\begin{aligned}
\mathrm{Tr}(\pi(x^{-1})\pi(f)) &= \sum_{i=1}^{d(\pi)} (\pi(x^{-1})\pi(f)e_i, e_i) \\
&= \sum_{i=1}^{k} (\pi(x^{-1})\pi(f)e_i, e_i).
\end{aligned}$$

$\square$

Let $\pi \in \hat{G}_c$ and $v \in E(\pi, \chi)$. Again let $A_v \in \mathrm{Hom}_K(\pi|_K, \chi)$ be given by $A_v(w) = (w, v)_{\mathcal{H}'}$. If $\mathcal{H}$ is the Hilbert space for π, then the finite dimensional π-primary subspace for the representation $\mathrm{ind}_K^G \chi$ is given by

$$\begin{aligned} P(\pi)(L_K^2(G,\chi)) &= \langle \{\tilde{A}_v w \mid v \in E(\pi,\chi), w \in \mathcal{H}'\}\rangle \\ &= \langle \{A_v \otimes_\pi w \mid v \in E(\pi,\chi), w \in \mathcal{H}\}\rangle \\ &= \langle \{x \mapsto A_v \pi(x^{-1}) w \mid v \in E(\pi,\chi), w \in \mathcal{H}\}\rangle \\ &= \langle \{x \mapsto (\pi(x^{-1})w, v) \mid v \in E(\pi,\chi),\ w \in \mathcal{H}\}\rangle \\ &= \langle \{w \otimes_\pi \bar{v} \mid v \in E(\pi,\chi),\ w \in \mathcal{H}\}\rangle. \end{aligned} \tag{8.9}$$

Thus the functions $A_v \otimes_\pi w$ in $C_K(G, \chi)$ are just the matrix coefficients $w \otimes_\pi \bar{v}$ in $C(G)$.

Corollary 8.46. *Suppose $f \in L^2(G)$ and $h \in L_K^2(G, \chi)$. Let $\hat{G}_{c,\chi}$ be the collection of those π' in $\hat{G}_c$ with $E(\pi, \chi) \neq \{0\}$. Then*

$$\sum_{\pi \in \hat{G}_{c,\chi}} d(\pi) \mathrm{Tr}(\pi(x^{-1})\pi(f * h))$$

converges uniformly in x to $f * h(x)$.

Proof. This follows directly Corollaries 8.43 and 8.21. □

We now indicate how these results can be used to obtain the primary decomposition Theorems 8.19 and 8.20 and the biregular primary projections for $L^2(G)$.

First note the left regular representation λ of a compact Hausdorff group G is the representation $\mathrm{ind}_{\{e\}}^G 1$. Hence to decompose λ, we need only note for each $\pi \in \hat{G}_c$, the vector space $E(\pi, 1) = \{v \in \mathcal{H}_\pi \mid \pi(e)v = 1(e)v\}$ is precisely $\mathcal{H}_\pi$. Thus for each $\pi \in \hat{G}_c$, we have

$$P(\pi)L^2(G) = \langle \{v \otimes_\pi \bar{w} \mid v, w \in \mathcal{H}_\pi\}\rangle.$$

Furthermore, the primary projection $P(\pi)$ on $L^2(G)$ is given by

$$P(\pi)f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f)).$$

Using these we see Theorem 8.41 reduces to Theorem 8.19.

The orthogonality formulas for matrix coefficients also follows from Theorem 8.28. Indeed, we know $(A_w \otimes_\pi v, A_{w'} \otimes_\pi v')_2 = \frac{1}{d(\pi)}(A_w, A_{w'})_2(v, v')$.

Let $e_1, e_2, \ldots, e_d$ be an orthonormal basis of $\mathcal{H}_\pi$. Then

$$\begin{aligned}(A_w, A_{w'})_2 &= \mathrm{Tr}(A_{w'}^* A_w) = \sum_{i=1}^{d}(A_{w'}^* A_w e_i, e_i) \\ &= \sum_{i=1}^{d}(A_w e_i, A_{w'} e_i) = \sum_{i=1}^{d}(e_i, w)\overline{(e_i, w')} \\ &= \sum_{i=1}^{d}(w', e_i)(e_i, w) = (\sum_{i=1}^{d}(w', e_i)e_i, w) \\ &= (w', w).\end{aligned}$$

Since $A_w \otimes_\pi v = v \otimes_\pi \bar{w}$, we recover the orthogonality formula:

$$(v \otimes_\pi \bar{w}, v' \otimes_\pi \bar{w}')_2 = \frac{1}{d(\pi)}(v, v')(w', w).$$

To obtain the primary decomposition for the biregular representation of $G \times G$, we will need to obtain a unitary equivalence between the biregular representation B on $L^2(G)$ and an induced unitary representation of $G \times G$. To do this we take diagonal subgroup $G_d = \{(g, g) \mid g \in G\}$ of $G \times G$. G_d is a closed. Moreover, the mapping

$$Uf(x, y) = f(xy^{-1})$$

defines a unitary transformation of $L^2(G)$ onto $L^2_{G_d}(G \times G, 1)$. In fact, note $Uf((x, y)(g, g)) = Uf(xg, yg) = f(xgg^{-1}y^{-1}) = f(xy^{-1})$ and

$$\begin{aligned}||Uf||_2^2 &= \int_{G\times G} |f(xy^{-1})|^2\, d(m \times m)(x, y) \\ &= \int_G \int_G |f(xy^{-1})|^2 dm(x)\, dm(y) \\ &= \int_G \int_G |f(x)|^2\, dm(x)\, dm(y) \\ &= \int_G |f(x)|^2\, dm(x) \int_G 1\, dm(y) \\ &= \int_G |f(x)|^2\, dm(x).\end{aligned}$$

Thus U is a linear isometry. U is onto for $U^{-1}f(x) = f(x, 1)$ is well defined for $f \in C_{G_d}(G \times G, 1)$ and $U(U^{-1}f) = f$. Thus the range of U contains $C_{G_d}(G \times G, 1)$ and thus is dense in $L^2_{G_d}(G \times G, 1)$. Since the range of U is closed, we have U is onto.

Moreover, $UBU^{-1} = \mathrm{ind}_{G_d}^{G\times G} 1$. In fact,

$$\begin{aligned} UB(g_1, g_2)f(x,y) &= B(g_1,g_2)f(xy^{-1}) \\ &= f(g_1^{-1}xy^{-1}g_2) \\ &= f(g_1^{-1}x(g_2^{-1}y)^{-1}) \\ &= Uf(g_1^{-1}x, g_2^{-1}y) \\ &= \mathrm{ind}_{G_d}^{G\times G} 1(g_1,g_2)Uf(x,y). \end{aligned}$$

Thus U is a unitary equivalence. We now decompose the representation $\mathrm{ind}_{G_d}^{G\times G} 1$. To do this we first remark that Proposition 6.75 and Theorem 8.7 imply we can take $(\widehat{G \times G})_c = \{\pi_1 \times \bar{\pi}_2 \mid \pi_1,\, \pi_2 \in \hat{G}_c\}$. The task is to find all unitary representations $\pi_1 \times \bar{\pi}_2$ in $(\widehat{G \times G})_c$ such that $E(\pi_1 \times \bar{\pi}_2, 1) \neq \{0\}$. Recall $\mathcal{H}_1 \otimes \bar{\mathcal{H}}_2 = \mathcal{B}_2(\mathcal{H}_2, \mathcal{H}_1)$ and $(\pi_1 \times \bar{\pi}_2)(g,g)(T) = \pi_1(g)T\pi(g^{-1})$. Thus $T \in E(\pi_1 \times \bar{\pi}_2, 1)$ if and only if $\pi_1(g)T\pi_2(g^{-1}) = T$. By Schur's Lemma, a nonzero T exists if and only if π_1 is unitarily equivalent to π_2. This implies $\pi_2 = \pi_1$. Concluding, we have:

$$E(\pi_1 \times \bar{\pi}_2, 1) \neq \{0\} \text{ if and only if } \pi_1 = \pi_2.$$

Moreover, by Schur's Lemma, $E(\pi \times \bar{\pi}, 1) = \{T \in \mathcal{B}(\mathcal{H}_\pi) \mid \pi(g)T\pi(g^{-1}) = T\} = \mathbb{C}I$. By (8.9), $P(\pi \times \bar{\pi})$ is the orthogonal projection of $L^2_{G_d}(G \times G)$ onto the linear span of the coefficients $(v \otimes \bar{w}) \otimes_{\pi\times\bar{\pi}} \bar{I}$ where $v, w \in \mathcal{H}_\pi$. It is given by $P(\pi \times \bar{\pi})f(x,y) = d(\pi \times \bar{\pi})\mathrm{Tr}((\pi \times \bar{\pi})(x,y)^{-1}(\pi \times \bar{\pi})(f))$. Note

$$\begin{aligned} ((v \otimes \bar{w}) \otimes_{\pi\times\bar{\pi}} \bar{I})(x,y) &= (v \otimes \bar{w}, \pi \times \bar{\pi}(x,y)I)_2 \\ &= (v \otimes \bar{w}, \pi(x)I\pi(y^{-1}))_2 \\ &= \mathrm{Tr}(\pi(xy^{-1})^*(v \otimes \bar{w})) \\ &= \mathrm{Tr}(\pi(yx^{-1})(v \otimes \bar{w})) \\ &= \mathrm{Tr}((\pi(yx^{-1})v) \otimes \bar{w}) \\ &= (\pi(yx^{-1})v, w) \\ &= (v, \pi(xy^{-1})w) \\ &= v \otimes_\pi \bar{w}(xy^{-1}). \end{aligned}$$

Thus we see the $\pi \times \bar{\pi}$ primary projection for the biregular representation is the orthogonal projection of $L^2(G)$ onto the linear span of the matrix coefficients $v \otimes_\pi \bar{w}$. This is precisely the same range as the primary projection $P(\pi)$ for the left regular representation λ. Hence we have $P(\pi)f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f))$ also gives the $\pi \times \bar{\pi}$ primary projection for B.

Exercise Set 8.3

1. When $\mathcal{H}'$ is finite dimensional, then the $\mathcal{K}$-trace is actually a contraction; i.e., show there is a bounded linear transformation C of $(\mathcal{K} \otimes \bar{\mathcal{H}}') \otimes \mathcal{H}'$ into $\mathcal{K}$ satisfying $C(T \otimes v') = T(v')$. In particular, $C((w \otimes \bar{v}') \otimes v) = (v, v')w$. Then show

$$C \circ \Phi^{-1} = \mathrm{Tr}_{\mathcal{K}}$$

where Φ is the unitary isomorphism given by (8.6).

2. Let $T \in \mathcal{H}_1 \otimes \bar{\mathcal{H}}_2$ be a trace class operator.

(a) Show there is a constant $M > 0$ such that if A is an index set and e_α and f_α for $\alpha \in A$ are orthonormal sets in $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\sum_\alpha |(e_\alpha, T f_\alpha)| < M$.

(b) Let $|T|_1$ be the infinimum of all such M in (a). Show $|\cdot|_1$ is a norm on the linear space of trace class operators from $\mathcal{H}_2$ to $\mathcal{H}_1$.

(c) Show $||T|| \leqslant |T|_1$.

(d) Show the trace class operators with this norm is a Banach space.

3. Show the inverse Φ^{-1} of the transformation Φ given in Definition 8.6 is defined by

$$\left(\Phi^{-1}(S)(\bar{v}_1')\right)(v_2') = \overline{S^*(v_1' \otimes \bar{v}_2')}$$

for $S \in (\mathcal{H}' \otimes \bar{\mathcal{H}}') \otimes \mathcal{K}$.

4. Let χ be a one-dimensional unitary representation of a closed subgroup K of compact Hausdorff group G. Note $L^2_K(G, \chi)$ is a closed linear subspace of $L^2(G)$. Show the orthogonal projection of $L^2(G)$ onto $L^2_K(G, \chi)$ is given by

$$Pf(x) = \int_K \chi(k) f(xk)\, dk.$$

5. Let G be a compact Hausdorff group and suppose π is an irreducible unitary representation of G. Let χ be a one-dimensional character of a closed subgroup K of G. Let f be a function in $L^2(G)$ orthogonal to the subspace $L^2_K(G, \chi)$. Show $\pi(f)w = 0$ for all $w \in E(\pi, \chi)$.

6. Let G be a compact Hausdorff group and suppose π and π' are irreducible unitary representations of G. Let χ be a one-dimensional character of a closed subgroup K of G. Let $w \in E(\pi, \chi)^\perp$.

(a) Show $\pi(w' \otimes_{\pi'} \bar{v}')(w) = 0$ if $v' \in E(\pi', \chi)$ and $w' \in \mathcal{H}_{\pi'}$.

(b) Use Theorem 8.41 and the description of the primary subspaces given in Equation (8.9) to show $\pi(f)w = 0$ for all $f \in L^2_K(G, \chi)$.

This gives an alternative proof to Lemma 8.44.

7. Show directly that if U is the unitary mapping from $L^2(G)$ onto $L^2_{G_d}(G \times G, 1)$ given by

$$Uf(x,y) = f(xy^{-1}),$$

then

$$d(\pi \times \bar{\pi})\mathrm{Tr}((\pi \times \bar{\pi})(x,y)^{-1}(\pi \times \bar{\pi})(Uf)) = UP(\pi)f(x,y)$$

where

$$P(\pi)f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f)).$$

8. Use Frobenius Reciprocity Theorem 8.30 to show if G is a compact Hausdorff group with closed subgroups K and H where $K \subseteq H$ and π is a unitary representation of K, then $\mathrm{ind}_H^G(\mathrm{ind}_K^H \pi)$ and $\mathrm{ind}_H^G \pi$ are unitarily equivalent.

9. Let G be a compact Hausdorff group with closed subgroup K having one dimensional character χ. Recall we know $L^2_K(G,\chi)$, the Hilbert space for $\mathrm{ind}_K^G \chi$, is a subspace of $L^2(G)$ and the induced representation $\mathrm{ind}_K^G \chi$ is just the restriction of the regular representation λ to this subspace. Show if π is an irreducible unitary representation of G, then the π-primary projection for $\mathrm{ind}_K^G \chi$ is $P_\pi Q$ where Q is the orthogonal projection of $L^2(G)$ onto $L^2_K(G,\chi)$.

10. Let G be a compact Hausdorff group with closed subgroup K having one dimensional character χ. Show the orthogonal projection of $L^2(G)$ onto $L^2_K(G,\chi)$ is given by

$$Qf(x) = \int_K \chi(k) f(xk)\, dk.$$

8. Primary Projections–General Case

In the previous sections we have shown how it is possible to use traces on trace class operators to obtain the primary projections for induced representations. We now turn to a general unitary representation of a compact Hausdorff group and will describe its decomposition into primary subspaces. We will again use the trace or more specifically the character of the representation to obtain these projections. Indeed, Theorem 6.123 shows that if π is a finite dimensional unitary representation, then the character χ_π defined by $\chi_\pi(g) = \mathrm{Tr}(\pi(g))$ is a central function which to unitary equivalence identifies the representation. Sometimes, we say χ is a d-dimensional character if the representation π is unitary and its Hilbert space has dimension d. We have already been using the term one-dimensional character to denote a one-dimensional unitary representation.

Note if π is a finite dimensional dimensional unitary representation and $e_1, e_2, \ldots, e_d$ is an orthonormal basis for the Hilbert space for π, then

$$\begin{aligned}\bar{\chi}_\pi(g) &= \sum_{i=1}^{d} \overline{(\pi(g)e_i, e_i)} \\ &= \sum_{i=1}^{d} (e_i, \pi(g))e_i) \\ &= \sum_{i=1}^{d} e_i \otimes_\pi \bar{e}_i(g).\end{aligned} \tag{8.10}$$

and thus $\bar{\chi}_\pi(x) = \sum(e_i, \pi(x)e_i) = \sum(\pi(x^{-1})e_i, e_i) = \chi_\pi(x^{-1})$ and $\chi_\pi(e) = d$ where d is the dimension of $\mathcal{H}_\pi$. Thus for finite dimensional unitary representations π one has:

$$\begin{aligned}&\chi_\pi(x^{-1}) = \overline{\chi_\pi(x)}, \text{ thus } \chi_\pi^* = \chi_\pi \\ &\chi_\pi(e) = d(\pi) \\ &\chi_\pi(xy) = \chi_\pi(yx), \text{ i.e. } \chi_\pi \text{ is central.}\end{aligned} \tag{8.11}$$

Moreover, one has $\sum_{i=1}^{d} e_i \otimes \bar{e}_i = I$, for $\sum_{i=1}^{d}(v, e_i)e_i = v$ for all $v \in \mathcal{H}_\pi$.

If π is a finite dimensional representation of group G, define

$$\xi_\pi = d(\pi)\chi_\pi\,.$$

Note that $d(\pi)$, χ_π and ξ_π depend only on the unitary equivalence class $[\pi]$ of π.

Recall from Corollary 8.13 and Corollary 8.10 that $(u \otimes_\pi \bar{v}) * (u' \otimes_{\pi'} \bar{v}') = 0$ and $\pi'(u \otimes_\pi \bar{v}) = 0$ when π and π' are inequivalent irreducible unitary representations of G and $\pi(u \otimes_\pi \bar{v}) = \frac{1}{d(\pi)} u \otimes \bar{v}$ for irreducible unitary representations π.

Proposition 8.47. *Let π and π' be inequivalent irreducible unitary representations of a compact group G.*

(a) $\bar{\xi}_{\pi'} * \bar{\xi}_\pi = 0$
(b) $\pi'(\bar{\xi}_\pi) = 0$
(c) $\bar{\xi}_\pi * \bar{\xi}_\pi = \bar{\xi}_\pi$
(d) $\xi_\pi^* = \bar{\xi}_\pi$
(e) $\pi(\bar{\xi}_\pi) = I$.

Proof. Note (a) and (b) follow from Equation 8.10 and the remarks prior to the statement of this proposition. For (c) we again use Equation 8.10 and

Corollary 8.11 to obtain

$$
\begin{aligned}
\bar{\xi}_\pi * \bar{\xi}_\pi &= d(\pi) \sum_{i=1}^{d(\pi)} (e_i \otimes_\pi \bar{e}_i) * d(\pi) \sum_{j=1}^{d(\pi)} (e_j \otimes_\pi \bar{e}_j) \\
&= d(\pi)^2 \sum_{i,j} (e_i \otimes_\pi \bar{e}_i) * (e_j \otimes_\pi \bar{e}_j) \\
&= d(\pi)^2 \sum_{i,j} \frac{1}{d(\pi)} (e_j, e_i) e_i \otimes_\pi \bar{e}_j \\
&= d(\pi) \sum_{i=1}^{d(\pi)} e_i \otimes_\pi \bar{e}_i \\
&= \bar{\xi}_\pi.
\end{aligned}
$$

Clearly (d) follows from (8.11). For (e), note since $\pi(e_i \otimes_\pi \bar{e}_i) = \frac{1}{d(\pi)} e_i \otimes \bar{e}_i$,

$$
\begin{aligned}
\pi(\bar{\xi}_\pi) &= d(\pi) \sum_{i=1}^{d} \pi(e_i \otimes_\pi \bar{e}_i) \\
&= \sum_{i=1}^{d} e_i \otimes \bar{e}_i \\
&= I.
\end{aligned}
$$

□

Let ρ be a unitary representation of G. By part (e), we know that $\rho(\bar{\xi}_\pi)P(\pi)v = P(\pi)v$ where $P(\pi)$ is primary projection for ρ corresponding to the irreducible unitary representation π. This is true for on $P(\pi)\mathcal{H}$, ρ is an inner direct sum of representations equivalent to π.

Corollary 8.48. *Let π and π' be inequivalent irreducible unitary representations of a compact Hausdorff group G. Then $f*h = 0$ whenever $f \in P_\pi L^2(G)$ and $h \in P_{\pi'} L^2(G)$.*

Proof. By Lemma 8.15, we know $P_\pi L^2(G)$ is the finite dimensional space consisting of the linear span of the matrix coefficients $v \otimes_\pi \bar{w}$ and $P_{\pi'} L^2(G)$ is the finite dimensional space consisting of the linear span of the matrix coefficients $v' \otimes_{\pi'} \bar{w}'$. Thus the result follows from Corollary 8.13. □

Theorem 8.49. *Let ρ be a unitary representation of a compact Hausdorff group G. For each irreducible unitary representation π of G, there is a bounded linear transformation $I_\pi : \mathrm{Hom}_G(\pi, \rho)_2 \otimes \mathcal{H}_\pi \to \mathcal{H}_\rho$ satisfying:*

(a) *I_π intertwines $I \otimes \pi$ with ρ;*

(b) *$I_\pi(A \otimes v) = Av$;*

(c) $\sqrt{d(\pi)}I_\pi$ *is an isometry onto the* π*-primary subspace for the representation* ρ*; and*

(d) $\rho(\bar{\xi}_\pi) = P(\pi) = d(\pi)I_\pi I_\pi^*$ *is the* π*-primary projection for* ρ*.*

Proof. First note if $A_1, A_2 \in \mathrm{Hom}_G(\pi,\rho)_2$, then $A_2^*A_1 \in \mathrm{Hom}_G(\pi,\pi) = \mathbb{C}I$. Thus by Schur's Lemma, $A_2^*A_1 = cI$. Taking traces gives $(A_1, A_2)_2 = \mathrm{Tr}(A_2^*A_1) = cd(\pi)$. Hence if $v_1, v_2 \in \mathcal{H}_\pi$, we see:

$$\begin{aligned}(A_1v_1, A_2v_2)_{\mathcal{H}_\rho} &= (A_2^*A_1v_1, v_2)\\ &= \frac{1}{d(\pi)}(A_1,A_2)_2(v_1,v_2)_{\mathcal{H}_\pi}.\end{aligned}$$

Thus if E_α for $\alpha \in \Gamma$ is an orthonormal basis of $\mathrm{Hom}_G(\pi,\rho)_2$ and $\{e_i\}_{i=1}^d$ is an orthonormal basis for $\mathcal{H}_\pi$, the vectors $\sqrt{d(\pi)}E_\alpha e_i$ form an orthonormal set in $\mathcal{H}_\rho$. Hence there is a unique isometry from $\mathrm{Hom}_G(\pi,\rho)_2 \otimes \mathcal{H}_\pi$ into $\mathcal{H}_\rho$ sending the orthonormal basis $\{E_\alpha \otimes e_i\}$ to the orthonormal set $\sqrt{d(\pi)}E_\alpha e_i$ in $\mathcal{H}_\rho$. This is the linear isometry $\sqrt{d(\pi)}I_\pi$. In particular, I_π is a bounded linear transformation. To see I_π intertwines, note

$$I_\pi((I\otimes\pi)(g)(A\otimes v)) = I_\pi(A\otimes\pi(g)v) = A\pi(g)v = \rho(g)Av = \rho(g)I_\pi(A\otimes v).$$

To finish, we need only show (d). Clearly the range of I_π consists of the closure of the linear span of all vectors Av where $A \in \mathrm{Hom}_G(\pi,\rho)$ and $v \in \mathcal{H}_\pi$. Hence Corollary Chapter 6.54 and $\mathrm{Hom}_G(\pi,\rho)_2 = \mathrm{Hom}_G(\pi,\rho)$ imply this is the range of the primary projection $P(\pi)$. Using Exercise 6.4.1 and (c), we have $P(\pi) = d(\pi)I_\pi I_\pi^*$. Now by Corollary 6.110 and (e) of Proposition 8.47, $\rho(\bar{\xi}_\pi)Av = A\pi(\bar{\xi}_\pi)v = Av$. Moreover, since $\bar{\xi}_\pi^* = \bar{\xi}_\pi$ and $\bar{\xi}_\pi * \bar{\xi}_\pi = \bar{\xi}_\pi$, we see from Corollary 6.108, that $\rho(\bar{\xi}_\pi)$ is an orthogonal projection with $\rho(\bar{\xi}_\pi) \geqslant P(\pi)$. Next, since $\bar{\xi}_\pi * \bar{\xi}_{\pi'} = 0$ if π and π' are inequivalent irreducible unitary representations of G and $\sum_{\pi'\in\hat{G}_c} P(\pi') = I$ for ρ is discretely decomposable, we have:

$$\begin{aligned}P(\pi) &= P(\pi)\rho(\bar{\xi}_\pi)\\ &= \sum_{\pi'\in\hat{G}_c} P(\pi')\rho(\bar{\xi}_{\pi'})\rho(\bar{\xi}_\pi)\\ &= \sum_{\pi'\in\hat{G}_c} P(\pi')\rho(\bar{\xi}_\pi)\\ &= \rho(\bar{\xi}_\pi).\end{aligned}$$

□

Corollary 8.50. *Let* $\pi \in \hat{G}$*, where* G *is a compact Hausdorff group. Then for any unitary representation* ρ *of* G*, one has*

$$m(\pi,\rho) = \dim(\mathrm{Hom}_G(\pi,\rho)).$$

Note from Exercise 8.2.3 that if $\pi \in \hat{G}$ and $e_1, e_2, \ldots, e_d$ is an orthonormal basis of $\mathcal{H}_\pi$, then $\sqrt{d}\pi_{i,j}$ is an an orthonormal basis of $P_\pi L^2(G)$.

Theorem 8.51. *Let G be a compact Hausdorff group. Let $\pi \in \hat{G}$. Let P_π be the π-primary for the left regular representation λ. Then the following are equivalent:*

(a) $f \in P_\pi L^2(G)$;

(b) $\bar{\xi}_\pi * f = f$;

(c) *f is in the linear span of the matrix coefficients of π;*

(d) $f(x) = d(\pi)\mathrm{Tr}(\pi(\lambda(x^{-1})f)) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f))$ *for a.e. x;*

(e) $||f||^2 = d(\pi)\mathrm{Tr}(\pi(f)^*\pi(f))$.

Proof. By Corollary 8.22 and (d) of Theorem 8.49, $P_\pi = P(\pi) = \lambda(\bar{\xi}_\pi)$. Using Lemma 8.16, we know $\lambda(\bar{\xi}_\pi)f = \bar{\xi}_\pi * f$ for $f \in L^2(G)$. So $f \in P_\pi L^2(G)$ if and only if $P_\pi f = f$ if and only if $\bar{\xi}_\pi * f = f$. Thus (a) and (b) are equivalent. Now by Theorem 8.9, f is in the linear span of the matrix coefficients of π if and only if f is in the range of the linear isometry $\sqrt{d(\pi)}I_\pi$. But by Exercise 6.4.1, the range of $\sqrt{d(\pi)}I_\pi$ is the range of P_π. Thus (a), (b), and (c) are equivalent. Next by Proposition 8.17, we have $P_\pi f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f))$ and thus (d) is equivalent to (a). Finally, by Theorem 8.20 and Proposition 8.17, $f = P_\pi f$ if and only if $||f||^2 = ||P_\pi f||^2 = d(\pi)\mathrm{Tr}(\pi(f)^*\pi(f))$. □

8.1. Central L^2 functions.

Lemma 8.52. *The central functions in $L^2(G)$ form a closed subspace $L^2_c(G)$ invariant under the unitary representation $g \mapsto B(g,g)$ where B is the biregular representation. Moreover, $Qf(x) = \int_G f(yxy^{-1})\,dy$ is the orthogonal projection onto $L^2_c(G)$.*

Proof. Note if $f \in L^2(G)$, then $B(g,g)f = f$ for all g if and only if $f(g^{-1}xg) = f(x)$ a.e. x for each g if and only if f is central. Thus $L^2_c(G) = \{f \in L^2(G) \mid B(g,g)f = f \text{ for all } g\}$ is a closed set. It is clearly a linear subspace. Also if $f \in L^2_c(G)$, by Lemma 6.118, we may assume $f(yxy^{-1}) = f(x)$ for all x and y. Thus $Qf(x) = f(x)$ for all x. Moreover for any $f \in L^2(G)$, right translating y by g shows $Qf(g^{-1}xg) = \int f(yg^{-1}xgy^{-1})\,dy = \int f(yxy^{-1})\,dy = Qf(x)$ a.e. x. Thus $Qf \in L^2_c(G)$ and $Q^2 = Q$. Finally

$Q^* = Q$ for

$$\begin{aligned}(Qf, h) &= \int Qf(x)\bar{h}(x)\,dx \\ &= \int\int f(yxy^{-1})\,dy\,\bar{h}(x)\,dx \\ &= \int\int f(x)\bar{h}(y^{-1}xy)\,dy\,dx \\ &= \int f(x)\overline{\int h(yxy^{-1})\,dy}\,dx.\end{aligned}$$

□

Proposition 8.53. *Suppose G is a compact Hausdorff group and f is a central function in $L^2(G)$. If π is an irreducible unitary representation of G, then*

$$P_\pi f = (f, \bar{\chi}_\pi)\bar{\chi}_\pi.$$

Thus

$$f = \sum_{\pi\in\hat{G}_c}(f, \bar{\chi}_\pi)\bar{\chi}_\pi = \sum_{\pi\in\hat{G}_c}(f, \chi_\pi)\chi_\pi.$$

Proof. Since $f \in L^2(G) \subseteq L^1(G)$ and f is central, Lemma 6.119 implies $\pi(f)$ commutes with $\pi(x)$ for all $x \in G$. By Schur's Lemma, there is a scalar $c \in \mathbb{C}$ with $\pi(f) = cI$. Now by Proposition 8.17, $P_\pi f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f)) = d(\pi)\mathrm{Tr}(\pi(x^{-1})cI) = cd(\pi)\chi_\pi(x^{-1})$. Since $\pi(f) = \int f(x)\pi(x)\,dx$,

$$cd(\pi) = \mathrm{Tr}(\pi(f)) = \int f(x)\mathrm{Tr}(\pi(x))\,dx = \int f(x)\chi_\pi(x)\,dx = (f, \bar{\chi}_\pi)_2.$$

So $c = \frac{(f,\bar{\chi}_\pi)_2}{d(\pi)}$. Thus we see $P_\pi f(x) = (f, \bar{\chi}_\pi)_2\chi_\pi(x^{-1}) = (f, \bar{\chi}_\pi)_2\bar{\chi}_\pi$. From Theorem 8.17 we obtain $f = \sum_{\pi\in\hat{G}_c}(f, \bar{\chi}_\pi)\bar{\chi}_\pi$. Note since $\bar{\pi}$ is irreducible if and only if π is irreducible, we have $\sum_{\pi\in\hat{G}_c}(f, \bar{\chi}_\pi)\bar{\chi}_\pi = \sum_{\pi\in\hat{G}_c}(f, \chi_\pi)\chi_\pi$. □

Corollary 8.54. *The characters χ_π for $\pi \in \hat{G}_c$ form an orthonormal basis of $L^2_c(G)$.*

Proof. Note $\chi_\pi \in P_{\bar{\pi}}L^2(G)$ and by Theorem 8.17, $P_{\bar{\pi}}P_{\bar{\pi}'} = 0$ when π and π' are inequivalent. Thus $\chi_\pi \perp \chi_{\pi'}$ for inequivalent π and π'. Also if $e_1, e_2, \ldots, e_d$ is an orthonormal basis of $\mathcal{H}_\pi$, then the orthogonality relations, Corollary 8.10, imply

$$\begin{aligned}
(\chi_\pi, \chi_\pi)_2 &= (\bar{\chi}_\pi, \bar{\chi}_\pi)_2 \\
&= \sum_{i=1}^{d(\pi)} \sum_{j=1}^{d(\pi)} (e_i \otimes_\pi \bar{e}_i, e_j \otimes_\pi \bar{e}_j)_2 \\
&= \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{1}{d(\pi)} (e_i, e_j)(e_j, e_i) \\
&= \sum_{i=1}^{d} \frac{1}{d(\pi)} \\
&= 1.
\end{aligned}$$

□

Exercise Set 8.4

1. Let G be a compact Hausdorff group. Let χ be a 1-dimensional representation of G. Show χ is unitary; i.e., $|\chi(g)| = 1$ for all g.

2. Let G be a compact group. For $\pi \in \hat{G}$, show by a direct calculation that if $f \in L^2(G)$, one has

$$\bar{\xi}_\pi * f(x) = d(\pi)\mathrm{Tr}(\pi(\lambda(x^{-1})f)) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f)) \text{ for all } x.$$

3. Let π be a finite dimensional unitary representation of compact Hausdorff group G. Let f be in $L^1(G)$. Show

$$\mathrm{Tr}(\pi(x^{-1})\pi(f)) = \bar{\chi}_\pi * f(x).$$

4. Let ρ, λ, and B be the left regular, the right regular, and the biregular representations of a compact Hausdorff group G. Let π be an irreducible unitary representation of G. Show from direct calculations of the definitions of ξ_π and $\xi_{\pi\times\bar{\pi}}$ that for $f \in L^2(G)$, one has:

(a) $\rho(\bar{\xi}_\pi)f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f))$

(b) $\lambda(\bar{\xi}_\pi)f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f))$ and

(c) $(\pi \times \bar{\pi})(\bar{\xi}_{\pi\times\bar{\pi}})f(x) = d(\pi)\mathrm{Tr}(\pi(x^{-1})\pi(f))$.

5. Define a mapping $\mathcal{F} : L^2(G) \to \bigoplus_{\pi\in\hat{G}_c} \mathcal{H} \otimes \bar{\mathcal{H}}$ by

$$\mathcal{F}(f)(\pi) = \sqrt{d(\pi)}\pi(f).$$

Let $\mathcal{A}$ denote the Hilbert space $\bigoplus_{\pi\in\hat{G}_c} \mathcal{H} \otimes \bar{\mathcal{H}}$. Define multiplication on $\mathcal{A}$ by $(T_\pi) \cdot (S_\pi) = (T_\pi S_\pi)$ and an adjoint by $(T_\pi)^* = (T_\pi^*)$. Let $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{A}$.

(a) Show the adjoint is an isometry.

(b) Show $\langle RS, T\rangle = \langle S, R^*T\rangle$.

(c) Show $\langle R, S\rangle = \langle S^*, T^*\rangle$.

(d) Show $L_R : \mathcal{A} \to \mathcal{A}$ defined by $L_R S = RS$ is bounded and has adjoint L_{R^*}.

(e) Show $\mathcal{A}^2$ is dense in $\mathcal{A}$.

This shows $\mathcal{A}$ is a left Hilbert algebra. Now show $\mathcal{F}$ is a unitary isomorphism of $L^2(G)$ onto $\mathcal{A}$ satisfying $\mathcal{F}(f * h) = \mathcal{F}(f)\mathcal{F}(h)$ for $f, h \in L^2(G)$ and $\mathcal{F}(f^*) = \mathcal{F}(f)^*$. In particular, $L^2(G)$ is a left Hilbert algebra. The unitary transformation $\mathcal{F}$ is called the Fourier transform.

6. Let G be a compact Hausdorff group. Let π be an irreducible unitary representation of G. Show

$$d(\pi)\mathrm{Tr}(\pi(f)) = (f, \bar{\xi}_\pi) \text{ for } f \in L^2(G).$$

7. Let π be an irreducible unitary representation of a compact Hausdorff group G with orthonormal basis $e_1, e_2, \ldots, e_d$. Set $\pi_{i,j} = e_i \otimes_\pi \bar{e}_j$. Using Corollaries 8.11 and 8.13 and Exercise 8.2.3, show $\sum_{i,j} a_{i,j}\pi_{i,j}$ is central if and only if $a_{i,j} = c\delta_{i,j}$ for some constant c. Then conclude Proposition 8.53.

8. Let G be a compact Hausdorff group and let B be the biregular representation. Suppose π is an irreducible unitary representation of G. Let $I_\pi : \mathcal{H}_\pi \otimes \bar{\mathcal{H}}_\pi \to L^2(G)$ be the intertwining operator between $\pi \times \bar{\pi}$ of $G \times G$ with B given in Theorem 8.9 whose range is $P_\pi L^2(G)$. Show the subspace of $P_\pi L^2(G)$ invariant under all $B(g, g)$ for $g \in G$ is $I_\pi(\mathbb{C}I)$ where $I : \mathcal{H}_\pi \to \mathcal{H}_\pi$ is the identity operator. Use this to show the space of central functions in $P_\pi L^2(G)$ is $\mathbb{C}\bar{\chi}_\pi$ and obtain another proof of Proposition 8.53.

9. Let G be a compact Hausdorff group and let π be an irreducible unitary representation of G. Let f be a central L^1 function on G. Show $\pi(f) = \frac{1}{d(\pi)} f * \chi_\pi(e) I$.

10. Let σ be a finite dimensional unitary representation of a compact Hausdorff group G. Let $\chi_\sigma(g) = \mathrm{Tr}(\sigma(g))$ for $g \in G$. Show the following are equivalent.

(a) σ is irreducible.

(b) $\chi_\sigma * \chi_\sigma = \frac{1}{d(\sigma)}\chi_\sigma$.

(c) $(\chi_\sigma, \chi_\sigma)_2 = 1$.

11. Let G be a compact Hausdorff group with left regular representation λ and suppose $\pi_1, \pi_2, \ldots, \pi_s$ are inequivalent irreducible unitary representations of G. For each r, let $e_{r,1}, e_{r,2}, \ldots, e_{r,d(\pi_r)}$ be an orthonormal basis of $\mathcal{H}_{\pi_r}$ and let $\pi_r(i, j) = e_{r,1} \otimes_{\pi_r} \bar{e}_{r,j}$ be the collection of corresponding matrix coefficients. Suppose $n_r \leqslant d(\pi_r)$ for $r = 1, 2, \ldots, s$. Define f by

$$f = \sum_{r=1}^{s} \sum_{j=1}^{n_r} \pi_r(j,j).$$

Show the smallest invariant subspace of $L^2(G)$ containing f gives a unitary representation unitarily equivalent to

$$\bigoplus_{r=1}^{s} n_r \pi_r.$$

In particular, this representation is cyclic.

9. Spherical Functions and Gelfand Pairs

We consider a character χ of a closed subgroup K of a compact Hausdorff group G and will be interested in the case where the multiplicities of all $\pi \in \hat{G}_c$ in $\operatorname{ind}_K^G \chi$ are either 0 or 1. We recall in this case the space $E(\pi,\chi) = \{v \mid \pi(k)v = \chi(k)v\}$ is either the zero space or has dimension 1. In the case where it has dimension 1, we can choose a unit basis vector v of $E(\pi,\chi)$ and then $\{w \otimes_\pi \bar{v} \mid w \in \mathcal{H}_\pi\}$ is a vector subspace of $L^2_K(G,\chi)$ that is the π-primary subspace for the induced representation $\operatorname{ind}_K^G \chi$; i.e.; $w \mapsto w \otimes_\pi \bar{v}$ is an intertwining operator of π with $\operatorname{ind}_K^G \chi$ onto the π-primary subspace. Moreover, using the orthogonality formula

$$(w \otimes_\pi \bar{v}, w' \otimes_\pi \bar{v})_2 = \frac{1}{d(\pi)}(v,v)(w,w') = \frac{1}{d(\pi)}(w,w')$$

from Corollary 8.10, we see $J_\pi w = \sqrt{d(\pi)}\, w \otimes_\pi \bar{v}$ is an isometric intertwining operator. There is very special function in the range of J_π; namely $v \otimes_\pi \bar{v}(g) = (v, \pi(g)v)$; this will be an example of a spherical function.

Let $\phi(g) = (v, \pi(g)v)$ where $E(\pi,\chi)$ has orthonormal basis $\{v\}$. Then by the invariance of Haar measure under the mapping $k \mapsto k^{-1}$,

$$\begin{aligned} \int \chi(k)\phi(akb)\,dk &= \int \chi(k)(v, \pi(akb)v)\,dk \\ &= \int \chi(k)(\pi(k^{-1})\pi(a)^* v, \pi(b)v)\,dk \\ &= \int \bar{\chi}(k)(\pi(k)\pi(a)^* v, \pi(b)v)\,dk \\ &= (\pi|_K(\bar{\chi})\pi(a)^* v, \pi(b)v). \end{aligned} \tag{8.12}$$

From Theorem 8.49, we know $\pi|_K(\bar{\chi})$ is the χ-primary projection for the representation $\pi|_K$. But by Lemma 8.42, this is the orthogonal projection of $\mathcal{H}_\pi$ onto $E(\pi,\chi)$. So this projection must be given by

$$\pi|_K(\bar{\chi})w = (w,v)v.$$

Thus

$$\text{(SP)}\qquad \begin{aligned}\int \chi(k)\phi(akb)\,dk &= (\pi(a)^*v,v)(v,\pi(b)v)\\ &= (v,\pi(a)v)\,(v,\pi(b)v)\\ &= \phi(a)\,\phi(b).\end{aligned}$$

Also note

$$\begin{aligned}\phi(k_1gk_2) &= (v,\pi(k_1gk_2)v)\\ &= (\pi(k_1^{-1})v,\pi(g)\pi(k_2)v)\\ &= (\chi(k_1^{-1})v,\pi(g)\chi(k_2)v)\\ &= \chi(k_1^{-1})\chi(k_2^{-1})(v,\pi(g)v)\\ &= \chi(k_1^{-1})\chi(k_2^{-1})\phi(g)\end{aligned}$$

for all $k_1, k_2 \in K$. We say such a function is χ bicovariant.

Theorem 8.55. *Let π be an irreducible unitary representation of a compact Hausdorff group G and let χ be a 1-dimensional unitary representation of a closed subgroup K of G. Then* $\dim E(\pi,\chi) = 1$ *if and only if there is a unit vector $v \in E(\pi,\chi)$ such that if $\phi(g) = (v,\pi(g)v)$, then*

$$\int \chi(k)\phi(akb)\,dk = \phi(a)\phi(b).$$

Proof. We already have done the forward direction. Conversely, suppose such a v exists. We show the χ-primary projection for $\pi|_K$ is given by

$$Pw = (w,v)v.$$

Using

$$\begin{aligned}(\pi(a^{-1})v,v)(v,\pi(b)v) &= (v,\pi(a)v)(v,\pi(b)v)\\ &= \phi(a)\phi(b)\\ &= \int \chi(k)\phi(akb)\,dk\end{aligned}$$

with (8.12) shows

$$\begin{aligned}(\pi(a^{-1})v,v)(\pi(b^{-1})v,v) &= \phi(a)\phi(b)\\ &= (P\pi(a)^*v,\pi(b)v)\\ &= (P\pi(a^{-1})v,\pi(b)v).\end{aligned}$$

Thus

$$(P\pi(a)v,\pi(b)v) = ((\pi(a)v,v)v,\pi(b)v).$$

Since π is irreducible, v is cyclic. Thus $(Pw,w') = ((w,v)v,w')$ for all w and w'. This gives $Pw = (w,v)v$ and the range of P is one-dimensional. So $E(\pi,\chi)$ is one dimensional. $\square$

Definition 8.56. *Let G be topological group. A continuous complex valued function ϕ on G is positive definite if for each finite subset $x_1, x_2, \ldots, x_n$ in G, the $n \times n$ matrix*

$$A = [\pi(x_i^{-1}x_j)]_{i,j}$$

is positive semidefinite; i.e.,

$$\sum a_i \bar{a}_j \pi(x_i^{-1}x_j) \geqslant 0 \tag{8.13}$$

for any sequence $a_1, a_2, \ldots, a_n$ in $\mathbb{C}$.

Proposition 8.57. *Let π be a unitary representation of a topological group G. Then for each $v \in \mathcal{H}_\pi$, the function $\phi(g) = (v, \pi(g)v)$ is positive definite.*

Proof.

$$\begin{aligned}\sum_{i,j} a_i \bar{a}_j (v, \pi(x_i^{-1}x_j)v) &= \sum_{i,j} a_i \bar{a}_j (\pi(x_i)v, \pi(x_j)v) \\ &= (\sum_i a_i \pi(x_i)v, \sum_j a_j \pi(x_j)v) \geqslant 0.\end{aligned}$$

□

Lemma 8.58. *Let ϕ be a nonzero positive definite function on a topological group G. Then:*

(a) *$|\phi(g)| \leqslant \phi(e)$ for all $g \in G$ and*
(b) *$\phi(g^{-1}) = \overline{\phi(g)}$ for all $g \in G$.*

Proof. Let $x_1 = e$, $x_2 = g$. Taking $a_1 = 1$ and $a_2 = 0$ in inequality (8.13) gives $\phi(e) \geqslant 0$. Taking $a_1 = a_2 = 1$ gives $2\phi(e) + \phi(g) + \phi(g^{-1}) \geqslant 0$. Taking $a_1 = 1$ and $a_2 = i$ gives $2\phi(e) - i\phi(g) + i\phi(g^{-1}) \geqslant 0$. Thus $\mathrm{Im}(\phi(g) + \phi(g^{-1})) = 0$ and $\mathrm{Im}(i\phi(g^{-1}) - i\phi(g)) = \mathrm{Re}(\phi(g^{-1}) - \phi(g)) = 0$. Thus $\mathrm{Im}(\phi(g)) = -\mathrm{Im}(\phi(g^{-1}))$ and $\mathrm{Re}(\phi(g)) = \mathrm{Re}(\phi(g^{-1}))$. Hence $\phi(g^{-1}) = \overline{\phi(g)}$.

We thus have $|a_1|^2\phi(e) + a_1\bar{a}_2\phi(g) + \bar{a}_1 a_2\overline{\phi(g)} + |a_2|^2\phi(e) \geqslant 0$ for all a_1 and a_2. Take $a_1 = \phi(e)$ and $a_2 = -\phi(g)$. We obtain $\phi(e)^3 - \phi(e)|\phi(g)|^2 - \phi(e)|\phi(g)|^2 + |\phi(g)|^2\phi(e) \geqslant 0$. Since $\phi(e) \geqslant 0$, we obtain $\phi(e)^2 \geqslant |\phi(g)|^2$ and thus $\phi(e) \geqslant |\phi(g)|$. □

Definition 8.59. *Let G be a topological Hausdorff group and let K be a compact closed subgroup. Suppose χ is a one-dimensional character of K; i.e., a one dimensional unitary representation. Then a nonzero continuous function ϕ on G is said to be a χ-spherical function if*

$$\phi(k_1 g k_2) = \chi(k_1)^{-1}\chi(k_2)^{-1}\phi(g) \text{ for } k_1,\, k_2 \in K \text{ and } g \in G$$

and

$$\int_K \chi(k)\phi(akb)\, dk = \phi(a)\phi(b) \text{ for all } a, b \in G.$$

We will show that spherical functions arise precisely as in Theorem 8.55. Namely, suppose ϕ is a nonzero χ-spherical function on a compact Hausdorff group G which is positive definite. We shall find an irreducible representation π of G such that $m(\pi, \mathrm{ind}_K^G \chi) = 1$ and the corresponding spherical function is ϕ.

Definition 8.60. *Let G be a compact Hausdorff group and let χ be a one-dimensional character of a closed subgroup K of G. Then an irreducible unitary representation π of G is said to be class χ if $m(\pi, \mathrm{ind}_K^G \chi) = 1$.*

We are asking to establish a natural correspondence between the class χ representations π of G and the χ-spherical functions on G.

Theorem 8.61. *Let ϕ be nonzero χ-spherical function on a compact Hausdorff group G satisfying $\phi(a^{-1}) = \overline{\phi(a)}$ for all $a \in G$. Then ϕ is positive definite; and if λ is the left regular representation of G, the linear span $\mathcal{H}_\phi$ of the left translates $\lambda(g)\phi$ is a finite dimensional irreducible subspace for λ. Moreover, if $\lambda_0 = \lambda|_{\mathcal{H}_\phi}$, then λ_0 is a class χ representation of G. Furthermore, if $\phi_0 = \frac{\phi}{||\phi||_2}$, then $\phi(g) = (\phi_0, \lambda_0(g)\phi_0)_2$.*

Proof. Note $L_K^2(G, \chi) \subseteq L^2(G)$ and the induced representation $\mathrm{ind}_K^G \chi$ is the restriction of λ to $L_K^2(G, \chi)$. For simplicity, we use λ to denote both the regular representation and the induced representation $\mathrm{ind}_K^G \chi$.

By Theorem 8.41, $L_K^2(G, \chi)$ decomposes into an orthogonal direct sum of primary subspaces

$$L_K^2(G, \chi) = \sum_{\pi \in \hat{G}_{c,\chi}} P(\pi) L_K^2(G, \chi).$$

Choose π so that $P(\pi)\phi \neq 0$. By Theorem 8.49, we know

$$P(\pi)\phi = \lambda(\bar{\xi}_\pi)\phi.$$

Next note as a function of y,

$$\int_K \chi(k)\lambda(\bar{\xi}_\pi)\phi(xky)\, dk = \int \chi(k)\lambda(\bar{\xi}_\pi)\lambda(k^{-1}x^{-1})\phi(y)\, dk \in P(\pi)L_K^2(G, \chi).$$

But using (SP), the defining property of a spherical function, we have

$$\begin{aligned}\int_K \chi(k)\lambda(\bar{\xi}_\pi)\phi(xky)\,dk &= \int_K\int_G \chi(k)\bar{\xi}_\pi(g)\lambda(g)\phi(xky)\,dg\,dk\\ &= \int_G\int_K \bar{\xi}_\pi(g)\chi(k)\phi(g^{-1}xky)\,dk\,dg\\ &= \int_G \bar{\xi}_\pi(g)\phi(g^{-1}x)\phi(y)\,dg\\ &= \left(\int \bar{\xi}_\pi(g)\lambda(g)\phi(x)\,dg\right)\phi(y)\\ &= (P(\pi)\phi(x))\,\phi(y).\end{aligned}$$

Since $P(\pi)\phi$ is a assumed to be nonzero, we see $\phi \in P(\pi)L^2_K(G,\chi)$. Thus $P(\pi)\phi = \phi$ for a unique $\pi \in \hat{G}_{c,\chi}$.

Recall $P(\pi)L^2_K(G,\chi)$ is finite dimensional by Frobenius reciprocity. Hence the subspace $\mathcal{H}_\phi$ of $L^2_K(G,\chi)$ with cyclic vector ϕ is a finite dimensional subspace of $P(\pi)L^2_K(G,\chi)$.

We show $\lambda|_{\mathcal{H}_\phi}$ is irreducible and consequently must be unitarily equivalent to π. Set $\lambda_0 = \lambda|_{\mathcal{H}_\phi}$. Note $\lambda(k)\phi = \chi(k)\phi$. Thus $E(\lambda_0,\chi)$ has dimension at least one. Now using the definition of λ_0 and replacing k by k^{-1}, we have

$$\begin{aligned}\int_K \chi(k)(\phi,\lambda_0(akb)\phi)_2\,dk &= \int_K \chi(k)(\lambda_0(k^{-1})\lambda_0(a)^*\phi,\lambda_0(b)\phi)_2\,dk\\ &= \int_K \bar{\chi}(k)(\lambda_0(k)\lambda_0(a)^*\phi,\lambda_0(b)\phi)_2\,dk\\ &= (\lambda_0|_K(\bar{\chi})\lambda_0(a)^*\phi,\lambda_0(b)\phi)_2.\end{aligned}$$

So using $\phi(x^{-1}) = \overline{\phi(x)}$ and (SP), we obtain

$$\begin{aligned}\int_K\int_G \chi(k)\phi(g)\overline{\phi(b^{-1}k^{-1}a^{-1}g)}\,dg\,dk &= \int_G\int_K \chi(k)\phi(g)\phi(g^{-1}akb)\,dk\,dg\\ &= \int \phi(g)\phi(g^{-1}a)\phi(b)\,dg\\ &= \int \phi(g)\overline{\phi(a^{-1}g)}\,dg\,\phi(b)\\ &= (\phi,\lambda_0(a)\phi)_2\phi(b).\end{aligned}$$

Thus

(Eq-1) $$(\lambda_0|_K(\bar{\chi})\lambda_0(a)^*\phi,\lambda_0(b)\phi)_2 = (\phi,\lambda_0(a)\phi)_2\phi(b).$$

Since $\lambda_0|_K(\bar{\chi})\phi(x) = \int_K \chi(k^{-1})\phi(k^{-1}x)\,dk = \phi(x)$, we see $(\phi,\lambda_0(b)\phi)_2 = (\phi,\phi)_2\phi(b)$. Thus if $\phi_0 = \frac{\phi}{||\phi||_2}$, then

$$(\lambda_0(a)\phi_0,\lambda_0(b)\phi_0)_2 = \frac{1}{||\phi||_2^2}(\phi,\lambda(a^{-1}b)\phi)_2 = \phi(a^{-1}b).$$

Furthermore (Eq-1) yields

$$(\lambda_0|_K(\bar{\chi})\lambda_0(a^{-1})\phi, \lambda_0(b)\phi)_2 = (\lambda_0(a^{-1})\phi, \phi)_2\phi(b) = (\lambda_0(a^{-1})\phi, \phi)_2 \frac{(\phi, \lambda_0(b)\phi)_2}{(\phi, \phi)_2}.$$

Thus since ϕ is cyclic for λ_0,

$$\lambda_0|_K(\bar{\chi})\lambda_0(a^{-1})\phi = \frac{(\lambda_0(a^{-1})\phi, \phi)_2\phi}{(\phi, \phi)_2} = (\lambda_0(a^{-1})\phi, \phi_0)_2\phi_0.$$

So on $\mathcal{H}_\phi$, (d) of Theorem 8.49 implies the χ primary projection for the restriction of λ_0 to K is given by

$$P(\chi)f = (f, \phi_0)_2\phi_0.$$

In particular, $m(\chi, \lambda_0|_K) = 1$ and $E(\lambda_0, \chi) = \langle\phi_0\rangle$.

To finish we need only establish that λ_0 is irreducible. Suppose $\mathcal{H}_\phi = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant orthogonal subspaces. Decompose ϕ into $\phi_1 + \phi_2$. Since ϕ is cyclic for $\mathcal{H}_\phi$, Exercise 6.4.17 implies ϕ_1 and ϕ_2 are cyclic for $\mathcal{H}_1$ and $\mathcal{H}_2$. But $\lambda_0(k)\phi = \chi(k)\phi$ for all $k \in K$ implies $\lambda_0(k)\phi_1 = \chi(k)\phi_1$ and $\lambda_0(k)\phi_2 = \chi(k)\phi_2$ for all k. Thus $\phi_1, \phi_2 \in E(\lambda_0, \chi) = \langle\phi_0\rangle$ and hence are linearly dependent. So one of the two spaces $\mathcal{H}_1$ or $\mathcal{H}_2$ is trivial. □

To decompose the representation $\mathrm{ind}_K^G\chi$, we have seen that one must describe the π-primary projections $P(\pi)$ and that the simplest case occurs when $m(\pi, \mathrm{ind}_K^G\chi)$ is 0 or 1 for all $\pi \in \hat{G}_c$; for in this instance there is a unique positive definite spherical function $\phi \in P(\pi)L^2_K(G, \chi)$ such that the cyclic subspace generated by ϕ is irreducible and unitarily equivalent to π.

Although in this chapter we are dealing with compact groups, in the following discussion we give a more general presentation.

Let G be a σ-compact locally compact unimodular Hausdorff topological group with a compact subgroup K having one dimension character χ. Then $C_{c,\chi}(K\backslash G/K)$ will denote the vector space of all continuous complex valued functions f on G having compact support which satisfy

$$f(k_1 g k_2) = \chi(k_1^{-1})f(g)\chi(k_2^{-1}) \text{ for all } k_1 \text{ and } k_2 \text{ in } K.$$

Note when G is compact, these are the continuous functions in $L^2_K(G, \chi)$ in $E(\lambda, \chi)$, the space of functions in $L^2_K(G, \chi)$ satisfying $\lambda(k)f(g) = \chi(k)f(g)$ for $k \in K$. By Lemma 8.42 this is the χ-primary subspace of $L^2_K(G, \chi)$ for the representation $\lambda|_K$.

We know the space $C_c(G) \subseteq L^2(G)$ is a $*$ algebra closed under convolution.

Lemma 8.62. *$C_{c,\chi}(K\backslash G/K)$ is a $*$ subalgebra of $C_c(G)$.*

Proof. Let $f \in C_{c,\chi}(K\backslash G/K)$. Then f^* is continuous and

$$\begin{aligned} f^*(k_1xk_2) &= \overline{f(k_2^{-1}x^{-1}k_1^{-1})} \\ &= \overline{\chi(k_2)f(x^{-1})\chi(k_1)} \\ &= \chi(k_2^{-1})\overline{f(x^{-1})}\chi(k_1^{-1}) \\ &= \chi(k_1^{-1})f^*(x)\chi(k_2^{-1}). \end{aligned}$$

Thus $f^* \in C_{c,\chi}(K\backslash G/K)$. Also for $f, h \in C_c^{\chi}(K\backslash G/K)$, one has

$$\begin{aligned} f * h(k_1xk_2) &= \int f(y)h(y^{-1}k_1xk_2)\,dy \\ &= \int f(k_1y)h(y^{-1}xk_2)\,dy \\ &= \int \chi(k_1^{-1})f(y)h(y^{-1}x)\chi(k_2^{-1})\,dy \\ &= \chi(k_1^{-1})f * h(x)\chi(k_2^{-1}). \end{aligned}$$

□

Definition 8.63. *Let G be a unimodular locally compact Hausdorff group with compact subgroup K having one-dimensional character χ. The pair (G, K) is called a χ-Gelfand pair if $C_{c,\chi}(K\backslash G/K)$ is a commutative $*$ algebra.*

When $\chi = 1$, we will use the standard terminology and say (G, K) is a Gelfand pair.

Theorem 8.64. *Let G be a compact Hausdorff group with a closed subgroup K having one-dimensional character χ. Then (G, K) is a χ-Gelfand pair if and only if $m(\pi, \mathrm{ind}_K^G\chi)$ is 0 or 1 for all irreducible unitary representations π of G. In particular, this occurs if and only if $\mathrm{ind}_K^G\chi$ is the orthogonal direct sum of pairwise inequivalent class χ representations of G.*

Proof. By Corollary 8.48, we know $f * h = 0$ if f and h lie in distinct primary subspaces for $\mathrm{ind}_K^G\chi$. Now assume $m(\pi, \mathrm{ind}_K^G\chi)$ is 0 or 1 for all $\pi \in \hat{G}_c$. Since $m(\pi, \mathrm{ind}_K^G\chi) = m(\chi, \pi|_K) = \dim E(\pi, \chi)$, Theorems 8.55 and 8.61 imply $\lambda_0 = \lambda|_{P(\pi)L_K^2(G,\chi)}$ is either 0 or a class χ representation of G. In particular the χ primary projection for $\lambda_0|_K$ has at most one dimensional range $E(\lambda_0, \chi)$ and this is the span of a χ-spherical function ϕ_π. Now if $f \in C_\chi(K\backslash G/K)$, $f \in E(\lambda, \chi)$ and thus $P(\pi)f \in E(\lambda_0, \chi)$. Hence

$$f = \sum_{\pi \in \hat{G}_{c,\chi}} a_\pi \phi_\pi$$

and similarly,

$$h = \sum_{\pi \in \hat{G}_{c,\chi}} b_\pi \phi_\pi.$$

Since $\phi_\pi * \phi_{\pi'} = 0$ when π and π' are inequivalent, we see

$$f * h = \sum_\pi a_\pi \phi_\pi * \sum_{\pi'} b_{\pi'} \phi_{\pi'} = \sum a_\pi b_\pi \phi_\pi * \phi_\pi = \sum_{\pi'} b_{\pi'} \phi_{\pi'} * \sum_\pi a_\pi \phi_\pi = h * f$$

for by Lemma 8.14, convolution is continuous in $L^2(G)$. Hence $C_\chi(K\backslash G/K)$ is commutative and thus (G, K) is a χ-Gelfand pair.

Conversely, suppose $C_\chi(K\backslash G/K)$ is commutative. Assume there is a $\pi \in \hat{G}_c$ with $E(\pi, \chi)$ having dimension greater than 1. Pick nonzero orthonormal vectors v_1 and v_2 in $E(\pi, \chi)$. Then $v_2 \otimes_\pi \bar{v}_1$ and $v_1 \otimes_\pi \bar{v}_2$ are in $C_\chi(K\backslash G/K)$. Moreover, by Corollary 8.10,

$$(v_2 \otimes_\pi \bar{v}_1) * (v_1 \otimes_\pi \bar{v}_2) = \frac{1}{d(\pi)} v_2 \otimes_\pi \bar{v}_2$$

and

$$(v_1 \otimes_\pi \bar{v}_2) * (v_2 \otimes_\pi \bar{v}_1) = \frac{1}{d(\pi)} v_1 \otimes_\pi \bar{v}_1$$

Using the orthogonality relations, these are nonzero independent vectors in $L^2(G)$. Thus $C_\chi(K\backslash G/K)$ is not commutative. □

Let $L^2_{K,K}(G, \chi)$ be the closed subspace of $L^2(G)$ consisting of those $f \in L^2_K(G, \chi)$ with $\lambda(k)f = \chi(k)f$ for $k \in K$. Thus $L^2_{K,K}(G, \chi) = E(\mathrm{ind}_K^G \chi, \chi)$ is the χ-primary space for the representation induced from χ restricted to K. It is also the $\chi \times \bar{\chi}$ primary subspace for the biregular representation B restricted to $K \times K$. Recall $\hat{G}_{c,\chi}$ is the collection of the $\pi \in \hat{G}_c$ with $E(\pi, \chi) \neq \{0\}$.

Theorem 8.65. *Let (G, K) be a χ-Gelfand pair where G is compact. For $\pi \in \hat{G}_{c,\chi}$ choose a unit vector $v_\pi \in E(\pi, \chi)$ and let ϕ_π be the corresponding spherical function; i.e., $\phi_\pi(g) = (v_\pi, \pi(g)v_\pi)$.*

(a) *The mapping*

$$U : \bigoplus_{\pi \in \hat{G}_{c,\chi}} \mathcal{H}_\pi \to L^2_K(G, \chi)$$

given by

$$U(w_\pi) = \sum_{\pi \in \hat{G}_{c,\chi}} \sqrt{d(\pi)} w_\pi \otimes_\pi \bar{v}_\pi$$

is a unitary equivalence between the external orthogonal sum $\bigoplus_{\pi \in \hat{G}_{c,\chi}} \pi$ and $\mathrm{ind}_K^G \chi$.

(b) *The inverse of* U *is given by:*

$$U^{-1}(f) = (\sqrt{d(\pi)}\pi(f)v_\pi)_{\pi\in\hat{G}_{c,\chi}} \text{ for } f \in L^2_K(G,\chi).$$

(c) *For* $f \in L^2_K(G,\chi)$, *the series*

$$\sum_{\pi\in\hat{G}_{c,\chi}} d(\pi)(\pi(x^{-1})\pi(f)v_\pi, v_\pi) = \sum_{\pi\in\hat{G}_{c,\chi}} d(\pi)(f, \lambda(x)\phi_\pi)_2$$

and converges in $L^2(G)$ *to* f.

(d) *If* h *is a function in* $C_\chi(K\backslash G/K)$ *that is in the linear span* $\langle L^2(G) * L^2(G)\rangle$, *then*

$$\sum_{\pi\in\hat{G}_{c,x}} d(\pi)(\pi(x^{-1})\pi(h)v_\pi, v_\pi) = \sum_{\pi\in\hat{G}_{c,\chi}} d(\pi)(h, \lambda(x)\phi_\pi)_2$$

and converges uniformly to h.

Proof. Note U is an isometry by the orthogonality formulas for matrix coefficients given in Corollary 8.10 and Corollary 8.13. Since every $\pi \in \hat{G}_{c,\chi}$ is class χ, we know $\dim E(\pi,\chi) = 1$ for $\pi \in \hat{G}_{c,\chi}$. By Theorem 8.40, U is onto for the π-primary subspaces for $L^2_K(G,\chi)$ are given in (8.9) on page 507 by

$$P(\pi)L^2_K(G,\chi) = \langle w \otimes_\pi \bar{v} \mid w \in \mathcal{H}_\pi,\, v \in E(\pi,\chi)\rangle.$$

Thus (a) holds.

To see (b), it suffices to show the formula in (b) works for each of the functions $w \otimes_\pi \bar{v}_\pi$. But this follows since by Corollary 8.10, $\pi(w \otimes_\pi \bar{v}_\pi) = \frac{1}{d(\pi)} w \otimes \bar{v}_\pi$ and thus

$$\begin{aligned}\sqrt{d(\pi)}\left(\sqrt{d(\pi)}(\pi(w\otimes_\pi \bar{v}_\pi)v_\pi)\otimes_\pi \bar{v}_\pi\right) &= (w\otimes\bar{v}_\pi)v_\pi \otimes_\pi \bar{v}_\pi \\ &= (v_\pi, v_\pi)w\otimes_\pi \bar{v}_\pi \\ &= w\otimes_\pi \bar{v}_\pi.\end{aligned}$$

For (c) and (d), we note using Corollary 8.45 and (b) of Lemma 8.58 that

$$\begin{aligned}
\mathrm{Tr}(\pi(x^{-1})\pi(f)) &= (\pi(x^{-1})\pi(f)v_\pi, v_\pi) \\
&= (\pi(f)v_\pi, \pi(x)v_\pi) \\
&= \int f(y)(\pi(y)v_\pi, \pi(x)v_\pi)\, dy \\
&= \int f(y)\phi_\pi(y^{-1}x)\, dy \\
&= \int f(y)\bar{\phi}_\pi(x^{-1}y)\, dy \\
&= (f, \lambda(x)\phi_\pi)
\end{aligned}$$

Thus (c) follows from Theorem 8.19, and (d) follows from Corollary 8.21. □

Corollary 8.66. *Assume G is compact and (G, K) is a χ-Gelfand pair. For each $\pi \in \hat{G}_{c,\chi}$, choose a unit vector $v_\pi \in E(\pi, \chi)$ and let ϕ_π be the corresponding spherical function given by $\phi_\pi(g) = (v_\pi, \pi(g)v_\pi)$. Then the functions $\{\sqrt{d(\pi)}\phi_\pi \mid \pi \in \hat{G}_{c,\chi}\}$ are an orthonormal basis for $L^2_{K,K}(G, \chi)$. Moreover, if h is a function in $\langle L^2(G) * L^2(G)\rangle \cap C_\chi(K\backslash G/K)$, then*

$$\sum_{\pi \in \hat{G}_{c,\chi}} d(\pi)(h, \phi_\pi)_2\phi_\pi$$

converges uniformly to h.

Proof. The functions $\sqrt{d(\pi)}\phi_\pi$ are orthonormal by the orthogonality formulas given in Corollaries 8.10 and 8.13. Thus we need only check $(f, \lambda(x)\phi_\pi) = (f, \phi_\pi)\phi_\pi(x)$ for $f \in L^2_{K,K}(G, \chi)$. But this follows from the χ-spherical property. Indeed,

$$\begin{aligned}
(f, \lambda(x)\phi_\pi) &= \int f(y)\overline{\phi_\pi(x^{-1}y)}\, dy \\
&= \int f(y)\phi_\pi(y^{-1}x)\, dy \\
&= \int_K \int_G \chi(k)f(ky)\phi_\pi(y^{-1}x)\, dy\, dk \\
&= \int_G f(y) \int_K \chi(k)\phi_\pi(y^{-1}kx)\, dk\, dy \\
&= \int_G f(y)\phi_\pi(y^{-1})\phi_\pi(x)\, dy \\
&= \left(\int_G f(y)\overline{\phi_\pi(y)}\, dy\right)\phi_\pi(x) \\
&= (f, \phi_\pi)_2\phi_\pi(x).
\end{aligned}$$

□

Definition 8.67. *A continuous homomorphism τ on a topological group G is called an involution if $\tau \neq \mathrm{id}$ and $\tau^2 = \mathrm{id}$.*

Lemma 8.68. *Let τ be a involution on a unimodular Hausdorff locally compact group G. Then τ preserves Haar measure; i.e.,*

$$\int f(\tau x)\, dx = \int f(x)\, dx$$

for all $f \in C_c(G)$.

Proof. We first show $\int f(\tau x)\, dx$ is left invariant for $f \in C_c(G)$. Indeed,

$$\begin{aligned}\int f(a\tau(x))\, dx &= \int f(\tau(\tau(a)x))\, dx \\ &= \int f(\tau(\tau(a)\tau(a^{-1})x))\, dx \\ &= \int f(\tau(x))\, dx\end{aligned}$$

where we have used left translation invariance. Thus there is a $c > 0$ such that

$$\int f(\tau(x))\, dx = c \int f(x)\, dx.$$

Consequently,

$$\begin{aligned}\int f(x)\, dx &= \int f(\tau(\tau(x)))\, dx \\ &= c \int f(\tau(x))\, dx \\ &= c^2 \int f(x)\, dx\end{aligned}$$

for $f \in C_c(G)$. So $c = 1$. □

Proposition 8.69. *Let χ be a character of a compact subgroup K of a locally compact unimodular Hausdorff group G. Suppose τ is a automorphism of G that preserves Haar measure and for which for each g, there are k_1 and k_2 in K satisfying $\tau(g) = k_1 g^{-1} k_2$ and $\chi(k_1)\chi(k_2) = 1$. Then (G, K) is a χ-Gelfand pair.*

Proof. We show $C_{c,\chi}(K\backslash G/K)$ is commutative. First note by hypothesis that $f(x) = f(\tau(x^{-1}))$ for all $f \in C_{c,\chi}(K\backslash G/K)$. Hence using invariance of

Haar measure under τ, we have

$$\begin{aligned} f * h(x) &= \int_G f(a)h(a^{-1}x)\,da = \int f(\tau(a^{-1}))h(\tau(x^{-1}a))\,da \\ &= \int f(\tau(a^{-1}))h(\tau(x^{-1})\tau(a))\,da = \int f(a^{-1}))h(\tau(x^{-1})a)\,da \\ &= \int h(a)f(a^{-1}\tau(x^{-1}))\,da = (h * f)(\tau(x^{-1})) \\ &= h * f(x). \end{aligned}$$

□

Consider a compact Hausdorff group G. If $G_d = \{(x, x) \mid x \in G\}$, then $(G \times G, G_d)$ is a Gelfand pair. Indeed, take $\tau(x, y) = (y, x)$. Then τ is an involution of G and $(y, x) = (y, y)(x^{-1}, y^{-1})(x, x)$. Consequently, $m(\pi_1 \times \pi_2, \mathrm{ind}_{G_d}^{G\times G} 1)$ is either 0 or 1 for each pair $\pi_1, \pi_2 \in \hat{G}_c$. Indeed, we already knew this. In Section 7 we showed the biregular representation B is unitarily equivalent to $\mathrm{ind}_{G_d}^{G\times G} 1$ and showed the multiplicity $m(\pi_1 \times \pi_2, B)$ is 0 unless π_2 is equivalent to $\bar{\pi}_1$ and 1 otherwise.

For another example, we show $(\mathrm{SO}(n), \mathrm{SO}(n-1))$ is a Gelfand pair. Let $E_{i,j}$ be the $n \times n$ matrix with a 1 in the i, j entry and all other entries 0. The key is to use Proposition 8.69 and the following Lemma.

Lemma 8.70. *Let $A \in \mathrm{SO}(n)$. Then there are B_1 and B_2 in $\mathrm{SO}(n-1)$ and a $\theta \in \mathbb{R}$ such that $B_1^{-1}AB_2^{-1} = \exp(\theta(E_{1,n} - E_{n,1}))$.*

Proof. Let $W = E_{1,n} - E_{n,1}$. We have $\exp \theta W \cdot e_1 = \cos\theta\, e_1 - \sin\theta e_n$, while $\exp\theta W \cdot e_n = \sin\theta\, e_1 + \cos\theta\, e_n$. Now $Ae_n = v + \lambda e_n$ where $v \in \langle e_1, \dots, e_{n-1}\rangle$. If $v = 0$, then $\lambda = 1$ or $\lambda = -1$. For $\lambda = 1$, there is nothing to do for $A \in \mathrm{SO}(n-1)$. If $\lambda = -1$, then $\exp(\pi W) \cdot e_n = -e_n$, and so $\exp(-\pi W)A \cdot e_n = e_n$. Consequently, $\exp(-\pi W)A \in \mathrm{SO}(n-1)$ and this case is done.

Now suppose $v \neq 0$. Since $\mathrm{SO}(n-1)$ acts transitively on spheres in $\langle e_1, \dots, e_{n-1}\rangle$, we can find a $B_1 \in \mathrm{SO}(n-1)$ with $B_1^{-1}v = \frac{1}{||v||}e_1$. So $B_1^{-1}Ae_n = \mu e_1 + \lambda e_n$ where $\mu^2 + \lambda^2 = 1$. Hence we can find a θ with $\exp(-\theta W)\cdot(B_1^{-1}Ae_n) = e_n$. Consequently, $B_2 := \exp(-\theta W)B_1^{-1}A \in \mathrm{SO}(n-1)$. So $A = B_1 \exp(\theta W)B_2$. □

Let τ be the involution on $\mathrm{SO}(n)$ given by

$$\tau(A) = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} A \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\tau(A) \in \mathrm{SO}(n-1)A^{-1}\mathrm{SO}(n-1)$. Indeed, if $A = B_1 \exp(\theta W) B_2$, then using

$$\begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} B \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} = B \text{ for } B \in \mathrm{SO}(n-1) \text{ and}$$

$$\begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \exp(\theta W) \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} = \exp(-\theta W),$$

we have

$$\begin{aligned} \tau(A) &= B_1 \exp(-\theta W) B_2 = B_1 B_2 (B_2^{-1} \exp(-\theta W) B_1^{-1}) B_1 B_2 \\ &= (B_1 B_2) A^{-1} (B_1 B_2). \end{aligned}$$

Thus by Proposition 8.69, $(\mathrm{SO}(n), \mathrm{SO}(n-1))$ is a Gelfand pair.

For a noncompact example, let $G = \mathrm{SL}(n, \mathbb{R})$ and $K = \mathrm{SO}(n)$. Then (G, K) is a Gelfand pair. In fact we take $\tau(A) = (A^{-1})^t$ for A in G. Every invertible matrix A can be decomposed into $A = sk$ where s is an invertible symmetric matrix and $k \in \mathrm{SO}(n)$. Thus $\tau(A) = (k^t s^{-1})^t = (s^{-1})^t k = s^{-1}k = k(k^{-1}s^{-1})k = kA^{-1}k$.

Exercise Set 8.5

1. Let ϕ be a positive definite χ spherical function on a topological Hausdorff group G with closed compact subgroup K having one dimensional character χ. Let S be the linear span of the functions $\lambda(a)\phi$ where $\lambda(a)f(x) = f(a^{-1}x)$ for functions f. Define a sesquilinear form $\langle \cdot, \cdot \rangle$ on S by

$$\langle \sum_{i=1}^{m} a_i \lambda(x_i)\phi, \sum_{j=1}^{n} b_j \lambda(y_j)\phi \rangle = \sum_{i,j} a_i \bar{b}_j \phi(x_i^{-1} y_j).$$

(a) Show

$$|\langle \sum_{i=1}^{m} a_i \lambda(x_i)\phi, \sum_{i=1}^{n} b_j \lambda(y_j)\phi \rangle|^2 \leqslant \sum_{i,j} a_i \bar{a}_j \phi(x_i^{-1} x_j) \sum_{i,j} b_i \bar{b}_j \phi(y_i^{-1} y_j).$$

(b) Show $\langle \cdot, \cdot \rangle$ is well defined.

(c) Show $\langle f, f \rangle \geqslant 0$ for $f \in S$.

(d) Let $N_\phi = \{f \in S \mid \langle f, f \rangle = 0\}$. Show $\lambda(a)N_\phi \subseteq N_\phi$.

(e) Define an inner product on S/N_ϕ by

$$\langle f + N_\phi, f' + N_\phi \rangle = \langle f, f' \rangle,$$

and show $g \mapsto \lambda(g)f + N_\phi$ is continuous from G into the complex inner product space S/N_ϕ.

(f) Set $\lambda'(g)(f + N_\phi) = \lambda(g)f + N_\phi$. Show $\lambda'(g)$ preserves the inner product on S/N_ϕ and extends to a unitary representation of π of G on the completion $\mathcal{H}_\pi$ of S/N_ϕ.

(g) Show $\phi + N_\phi$ is a cyclic vector for π and $\langle \phi + N_\phi, \pi(g)(\phi + N_\phi)\rangle = \phi(g)$.

2. Assume G is as in Exercise 8.5.1. Show π is irreducible. (**Hint:** Show the χ-primary projection for the representation $\pi|_K$ has range the one dimensional space consisting of multiples of $\phi + N_\phi$.)

10. Compact Abelian Groups

Let G be a compact abelian Hausdorff group. By Corollary 6.51, we know every unitary representation π of G is one dimensional and thus is a one dimensional character of G. Since any two equivalent one dimensional representations of G are equal, we see

$$\hat{G} = \hat{G}_c = \{\chi \mid \chi : G \to \mathbb{T} \text{ is a continuous homorphism}\}.$$

Note if χ_1 and χ_2 are in $\hat{G}$, then $\chi_1\chi_2^{-1} = \chi_1\bar{\chi}_2 \in \hat{G}$, and thus the set of characters of G form a group. When G is compact, the group of characters with the discrete topology is called the *dual group* of G and as a topological space $\hat{G}$ is a locally compact Hausdorff group whose Haar measure is counting measure. Thus $f : \hat{G} \to \mathbb{C}$ is integrable if and only if $\sum_{\hat{G}} |f(\chi)| = \sup\{\sum_{\chi\in F} |f(\chi)| \mid F \subseteq \hat{G} \text{ with } F \text{ finite}\} < \infty$.

Definition 8.71 (Fourier Transform)**.** *The Fourier transform on the compact abelian Hausdorff group* G *is the linear transformation* $\mathcal{F} : L^1(G) \to C(\hat{G})$ *given by*

$$\mathcal{F}(f)(\chi) = \hat{f}(\chi) = \int_G f(x)\bar{\chi}(x)\,dx.$$

We can now apply the Plancherel Theorem 8.20 and Corollary 8.21 for compact groups. First we note for $\chi \in \hat{G}$, we have the primary projection $P_{\bar{\chi}}$ is given by

$$P_{\bar{\chi}}f(x) = d(\bar{\chi})\mathrm{Tr}(\bar{\chi}(x^{-1})\bar{\chi}(f)) = \bar{\chi}(x^{-1})\hat{f}(\chi) = \hat{f}(\chi)\chi$$

Theorem 8.72 (Plancherel)**.** *For* $f \in L^2(G)$*, one has* L^2 *decomposition*

$$f = \sum_{\chi\in\hat{G}} \hat{f}(\chi)\chi$$

and

$$||f||_2^2 = \sum_{\chi} |\hat{f}(\chi)|^2.$$

In particular, the mapping $f \mapsto \hat{f}$ *is a unitary isomorphism of* $L^2(G)$ *onto* $L^2(\hat{G}) = l^2(\hat{G})$ *with the property*

$$U(B(g_1, g_2)f)(\chi) = (\chi \otimes \bar{\chi})(g_1, g_2)\hat{f}(\chi) = \chi(g_1g_2^{-1})\hat{f}(\chi).$$

Furthermore, if $f \in \langle L^2(G) * L^2(G) \rangle$, *then* $\sum_\chi \hat{f}(\chi)\chi$ *converges pointwise uniformly to* f.

The formula $f = \sum \hat{f}(\chi)\chi$ is called the inversion formula. Note

$$\mathcal{F}^{-1}(F) = \int_{\hat{G}} F(\chi)\chi \, d\mu(\chi) = \sum F(\chi)\chi$$

is the inverse Fourier transform.

We remark the characters $\chi \in \hat{G}$ thus form an orthonormal basis of $L^2(G)$. In the case when $G = \mathbb{T}$, we saw in Chapter 1 the functions $e_n(z) = z^n$ form an orthonormal basis of $L^2(\mathbb{T})$. Since $e_n(z_1 z_2^{-1}) = e_n(z_1)e_n(z_2^{-1})$, we have $e_n \in \hat{\mathbb{T}}$. Consequently, $\hat{\mathbb{T}} = \{e_n \mid n \in \mathbb{Z}\}$ is the dual group. Since $e_m e_n = e_{m+n}$, we see the group multiplication for $\hat{\mathbb{T}}$ is addition in $\mathbb{Z}$. Using the identification $n(z) = e_n(z) = z^n$, we have $\hat{\mathbb{T}} = \mathbb{Z}$. Using Section 8 of Chapter 6 or by Exercise 8.6.1, the dual of $\mathbb{T}^n$ as a group is isomorphic to the additive discrete group $\mathbb{Z}^n$ under the identification

$$\chi_m(z) = z^m := \prod_{k=1}^{n} z_k^{m_k} \text{ for } z = (z_1, \ldots, z_n) \in \mathbb{T}^n$$
$$\text{and } m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n.$$

All of this was noted earlier and was part of the summary given in Table 1 of Chapter 6.

The Double Dual. We first note $\widehat{G}$ with the discrete topology is a locally compact abelian group. Thus the irreducible unitary representations of $\widehat{G}$ are again one dimensional characters. Thus if $\xi \in \widehat{\widehat{G}}$, we have $\xi(\chi) \in \mathbb{T}$ for each $\chi \in \widehat{G}$. In particular, $\xi \in \prod_{\chi \in \widehat{G}} \mathbb{T}$. The product topology on $\prod_{\chi \in \widehat{G}} \mathbb{T}$ is compact and with this topology is a compact abelian group. See Exercise 8.6.2.

Lemma 8.73. $\widehat{\widehat{G}}$ *is a closed subgroup of* $\prod_{x \in \widehat{G}} \mathbb{T}$ *and thus is a compact topological group.*

Proof. Since the product of two characters is again a character and the conjugate of a character is the multiplicative inverse of a character, $\widehat{\widehat{G}}$ is a subgroup. We show it is closed set in P where P is $\prod_{\chi \in \widehat{G}} \mathbb{T}$. Indeed, let χ_1 and χ_2 be in $\widehat{G}$ and suppose ξ_0 is in the closure of $\widehat{\widehat{G}}$ in P. Let $\epsilon > 0$. Then

$$U = \{\xi \in P \mid |\xi(\chi_1) - \xi_0(\chi_1)| < \epsilon, \, |\xi(\chi_2) - \xi_0(\chi_2)| < \epsilon, \, |\xi(\chi_1\chi_2^{-1}) - \xi_0(\chi_1\chi_2^{-1})| < \epsilon\}$$

is an open subset of P containing ξ_0. Since $U \cap \hat{\hat{G}} \neq \phi$, there is a $\xi \in \hat{\hat{G}} \cap U$. Now

$$\begin{aligned}|\xi_0(\chi_1\chi_2^{-1}) - \xi_0(\chi_1)\xi_0(\chi_2)^{-1}| &\leqslant |\xi_0(\chi_1\chi_2^{-1}) - \xi(\chi_1\chi_2^{-1})| + |\xi(\chi_1\chi_2^{-1}) - \xi(\chi_1)\xi(\chi_2^{-1})| \\ &+ |\xi(\chi_1)\xi(\chi_2^{-1}) - \xi_0(\chi_1)\xi(\chi_2^{-1})| + |\xi_0(\chi_1)\xi(\chi_2^{-1}) - \xi_0(\chi_1)\xi_0(x_2)^{-1}| \\ &< \epsilon + 0 + |\xi(\chi_1) - \xi_0(\chi_1)| + |\overline{\xi(\chi_2)} - \overline{\xi_0(\chi_2)}| \\ &< 3\epsilon.\end{aligned}$$

Since ϵ is arbitrary, we obtain $\xi_0(\chi_1\chi_2^{-1}) = \xi_0(\chi_1)\xi_0(\chi_2)^{-1}$ and thus ξ_0 is a one-dimensional character for $\widehat{G}$. □

We now know the double dual group $\hat{\hat{G}}$ is a compact abelian Hausdorff group whose topology is the relative topology of $\prod_\chi \mathbb{T}$; i.e. the topology of pointwise convergence on $\widehat{G}$.

There is a natural mapping Φ from G into $\prod_{\chi\in\widehat{G}} \mathbb{T}$. It is given by

$$\Phi(g) = \xi_g \text{ where } \xi_g(\chi) = \chi(g). \tag{8.14}$$

Proposition 8.74. *The mapping Φ is a continuous group isomorphism from the compact abelian group G onto a compact abelian subgroup of $\hat{\hat{G}}$.*

Proof. We first note $\Phi(xy^{-1}) = \Phi(x)\Phi(y)^{-1}$. Indeed,

$$\begin{aligned}\Phi(xy^{-1})(\chi) &= \chi(xy^{-1}) \\ &= \chi(x)\chi(y)^{-1} \\ &= \Phi(x)(\chi)\,\Phi(y)(\chi)^{-1} \\ &= \left(\Phi(x)\Phi(y)^{-1}\right)(\chi).\end{aligned}$$

So $\Phi(xy^{-1}) = \Phi(x)\Phi(y)^{-1}$. Next one has $\Phi(x) = 1$ if and only if $\chi(x) = 1$ for all $\chi \in \widehat{G}$. Now by the Plancherel Theorem, if $f \in L^2(G)$, $f = \sum(f,\chi)_2\chi$. Hence $\lambda(x)f = \sum(f,\chi)_2\lambda(x)\chi$. Since $\lambda(x)\chi(y) = \chi(x^{-1}y) = \overline{\chi(x)}\chi(y) = \chi(y)$, we see $\lambda(x)f = f$ for all $f \in L^2(G)$. But this says $f(x^{-1}y) = f(y)$ for all y for all $f \in C(G)$. Since $C(G)$ separates points, $x^{-1}y = y$ for all y. Hence $x = e$. So Φ is one-to-one.

We claim Φ is continuous. Let $\chi \in \widehat{G}$ and $\epsilon > 0$. Consider $U(\chi,1) = \{\xi \in \hat{\hat{G}} \mid |\xi(\chi) - 1| < \epsilon\}$. Since $\Phi(G)$ has the relative topology of $\prod_{\chi\in\widehat{G}} \mathbb{T}$, these sets form a neighborhood subbase at 1 in $\Phi(G)$. But since χ is continuous, there is a neighborhood $N_\epsilon(\chi)$ of e in G such that $|\chi(x) - \chi(e)| < \epsilon$ if $x \in N_\epsilon(\chi)$. Thus $|\xi_x(\chi) - 1(\chi)| < \epsilon$ if $x \in N_\epsilon(\chi)$. So $\xi_x \in U(\chi,1)$ if $x \in N_\epsilon(\chi)$. Hence Φ is continuous at e and since Φ is a homomorphism, Φ is continuous everywhere. By continuity, the image $\Phi(G)$ is a compact subgroup of Γ. □

Proposition 8.75. *Let G be a compact abelian Hausdorff group. If Γ is a subgroup of $\widehat{G}$ which separates points, then $\Gamma = \widehat{G}$.*

Proof. Consider the complex linear span $\mathcal{A}$ of the functions in Γ. Since Γ is a group and $\bar{\chi} = \chi^{-1}$ for $\chi \in \mathcal{A}$, we see $\mathcal{A}$ is a complex algebra of functions on the compact space G which is closed under conjugation. Since Γ separates points, the algebra $\mathcal{A}$ separates points. Moreover, $\mathcal{A}$ contains the constants for $1 \in \Gamma$. By the Stone-Weierstrass Theorem, $\mathcal{A}$ is uniformly dense in the algebra $C(G)$. Since $||f||_2 \leqslant ||f||_\infty$ for $f \in L^\infty(G)$, $\mathcal{A}$ is L^2 dense in $C(G)$. It follows from the density of $C(G)$ in $L^2(G)$ that $\mathcal{A}$ is L^2 dense in $L^2(G)$. Thus no nonzero function in $L^2(G)$ is orthogonal to all $\chi \in \Gamma$. Since $\widehat{G}$ is an orthonormal basis of $L^2(G)$ and $\Gamma \subseteq \widehat{G}$, $\Gamma = \widehat{G}$. ☐

Theorem 8.76 (Pontryagin Duality)**.** *Φ is a group homeomorphism of G onto $\widehat{\widehat{G}}$.*

Proof. We already know $\Phi(G)$ is a compact subgroup of $\widehat{\widehat{G}}$. Let Γ be the dual group of the compact group $\widehat{\widehat{G}}$. We claim for each character $\delta \in \Gamma$, there is a $\chi \in \widehat{G}$ such that

$$\delta(\xi) = \xi(\chi) \text{ for } \xi \in \widehat{\widehat{G}}.$$

For χ in $\widehat{G}$, define χ' by

$$\chi'(\xi) = \xi(\chi) \text{ for } \xi \in \widehat{\widehat{G}}.$$

Then χ' is a character of $\widehat{\widehat{G}}$. Indeed,

$$\chi'(\xi_1\xi_2^{-1}) = (\xi_1\xi_2^{-1})(\chi) = \xi_1(\chi)\bar{\xi}_2(\chi) = \chi'(\xi_1)\chi'(\xi_2)^{-1}$$

for ξ_1 and ξ_2 in $\widehat{\widehat{G}}$. Moreover, $\chi' : \widehat{\widehat{G}} \to \mathbb{T}$ is continuous, for $\widehat{\widehat{G}}$ has the relative topology of the product topology $\prod_{\chi \in \widehat{G}} \mathbb{T}$. We thus see $\widehat{G}' = \{\chi' \mid \chi \in \widehat{G}\}$ is a subgroup of Γ. This subgroup separates the points in $\widehat{\widehat{G}}$. Indeed, if $\chi'(\xi_1) = \chi'(\xi_2)$ for all $\chi \in \widehat{G}$, then $\xi_1(\chi) = \xi_2(\chi)$ for $\chi \in \widehat{G}$, and so $\xi_1 = \xi_2$. By Proposition 8.75, $(\widehat{G})' = \Gamma$. Now if $\Phi(G)$ is a proper subgroup of $\widehat{\widehat{G}}$, then $\widehat{\widehat{G}}/\Phi(G)$ is a compact abelian Hausdorff group with a nontrivial character. This implies there is a character γ of $\widehat{\widehat{G}}$ which is not 1 and which satisfies $\gamma(\Phi(g)) = 1$ for all $g \in G$. But then $\gamma = \chi'$ for some $\chi \in \widehat{G}$. Consequently

$$\chi'(\Phi(g)) = \Phi(g)(\chi) = \chi(g) = 1$$

for all $g \in G$. Thus $\chi = 1$. Since $\chi' = \gamma$, $\gamma(\xi) = \chi'(\xi) = \xi(\chi) = \xi(1) = 1$ for all $\xi \in \widehat{\widehat{G}}$. Thus γ is trivial, a contradiction. ☐

We remark this theorem is a special case of the famous Pontryagin-Van Kampen duality theorem. We state this without proof. The details are available in several texts covering topics in abelian harmonic analysis; e.g. [**22**, Hewitt-Ross], [**38**, Rudin]. Some of the steps are outlined in the exercises.

If G is a locally compact Hausdorff group, then the set $\widehat{G}$ of characters is an abelian group with multiplication and inversion defined by

$$(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$$
$$\chi^{-1}(g) = \overline{\chi(g)}.$$

This group can be given a topology. Indeed, if $\epsilon > 0$ and K is a compact subset of G, then

$$N_{K,\epsilon}(\chi_0) := \{\chi \mid |\chi(g) - \chi_0(g)| < \epsilon \text{ for all } g \in K\} \tag{8.15}$$

form neighborhood bases at points χ_0 in $\widehat{G}$, and with this topology $\widehat{G}$ is a Hausdorff topological group. The closures of the sets $N_{K,\epsilon}(\chi_0)$ for $0 < \epsilon < 1$ are compact; and we have $\widehat{G}$ is locally compact, abelian, and Hausdorff. That $\widehat{G}$ is not the trivial group; i.e., there are nontrivial characters is not clear. Indeed, one can use Bochner's Theorem (see references above) to show the characters separate the points of G.

Theorem 8.77 (Pontryagin-Van Kampen)**.** *Let G be an abelian locally compact Hausdorff group. Define $\Phi : G \to \widehat{\widehat{G}}$ by*

$$\Phi(g)(\chi) = \chi(g) \text{ for } \chi \in \widehat{G}.$$

Then Φ is a topological group isomorphism of the abelian group G onto the abelian double dual group $\widehat{\widehat{G}}$.

Exercise Set 8.6

1. Suppose G_1 and G_2 are Hausdorff locally compact abelian groups. Show the dual group of $G_1 \times G_2$ is topologically isomorphic to $\widehat{G}_1 \times \widehat{G}_2$.

2. Let A be an index set. Show $\prod_{a\in A} \mathbb{T}$ with the product topology is a compact Hausdorff group.

3. Let G be a compact Hausdorff group and suppose χ is a one-dimensional character of G. Show $f * \chi = \hat{f}(\chi)\chi$.

4. Let G be a locally compact Hausdorff group. Give $\widehat{G}$ the topology in (8.15). Show this makes $\widehat{G}$ into a group which is Hausdorff and abelian.

5. Let G be a locally compact Hausdorff group and let Δ be the collection all one dimensional nonzero $*$ representations of the Banach $*$ algebra $L^1(G)$.

By Corollary 6.108, there is a one-to-one correspondence $\chi \to \pi_\chi$ between the collection of one dimensional characters of G and Δ where

$$\pi_\chi(f) = \int f(g)\chi(g)\,dg.$$

Show the collection $\Delta \cup \{0\}$ is a closed subset in the weak $*$ topology of the unit ball of the dual space of $L^1(G)$ and thus with the weak $*$ topology is a compact Hausdorff space. In particular, Δ with the relative weak $*$ topology is a locally compact Hausdorff space. When G is abelian, the correspondence $\chi \to \pi_\chi$ gives a locally compact topology on $\widehat{G}$.

6. Let G be a abelian compact Hausdorff group.

(a) Show $(g, \chi) \mapsto \chi(g)$ is continuous from $G \times \widehat{G}$ into $\mathbb{T}$ where $\widehat{G}$ has the relative weak $*$ topology of Exercise 8.6.5. (**Hint:** Consider $(g, \chi) \mapsto \pi_\chi(\lambda(g)f)$.)

(b) Show if K is a compact subset of G and $\epsilon > 0$, then

$$\{\chi \in \widehat{G} \mid |\chi(k) - 1| < \epsilon \text{ for } k \in K\}$$

is open in the topology on $\widehat{G}$ given in Exercise 8.6.5.

(c) Show if $f \in L^1(G)$ and $\chi \in \widehat{G}$, then the set

$$\{\chi' \in \widehat{G} \mid |\pi_{\chi'}(f) - \pi_\chi(f)| < \epsilon\}$$

is open in the topology on $\widehat{G}$ given by the neighborhood bases of (8.15).

In particular, the topology on $\widehat{G}$ is the weak $*$ topology described in Exercise 8.6.5.

7. Assume G is an abelian compact Hausdorff group. Show the topology defined by the neighborhood bases given in (8.15) is discrete.

8. Assume G is a discrete abelian group. Show the topology on $\widehat{G}$ given by the neighborhood bases given in (8.15) is compact. (**Hint:** You may use Exercises 8.6.5 and 8.6.6.)

9. Let G be a compact Hausdorff group. Define $A(G)$ to be the space $\mathcal{F}^{-1}(L^1(\widehat{G}))$. Thus

$$A(\mathrm{G}) = \{\mathcal{F}^{-1}(f) \mid f \in L^1(\widehat{G})\}$$

where

$$\mathcal{F}^{-1}(f)(g) = \sum_{\chi \in \widehat{G}} f(\chi)\chi(g).$$

Show $A(G)$ is a dense $*$ subalgebra of the Banach $*$ algebra $C(G)$.

10. Let G be a compact Hausdorff group and let $\xi \in \widehat{\widehat{G}}$. Define π_0 on $A(G)$ by $\pi_0(\mathcal{F}^{-1}(f)) = \sum_{\chi \in \widehat{G}} f(\chi)\xi(\chi)$ for $f \in L^1(\widehat{G})$.

(a) Show π_0 is a $*$ representation of the subalgebra $A(G)$ of $C(G)$.

(b) Assuming π_0 is continuous; i.e. $||\pi(h)|| \leqslant ||h||_\infty$ for h in $A(G)$, show there is a $g \in G$ such that

$$\pi_0(h) = \sum_{\chi \in \widehat{G}} \widehat{h}(\chi)\chi(g).$$

(c) Assuming π_0 is continuous, show $\xi(\chi) = \chi(g)$ for $\chi \in \widehat{G}$.

11. Finite Groups

In the last section we considered compact abelian groups. Here we apply the general theory to finite groups. A group with discrete topology is compact if and only if it is finite. When G is finite, the only Hausdorff topology on G is the discrete topology and with this topology the group is compact. In this case, normalized Haar measure is given by

$$m(E) = \frac{|E|}{|G|}$$

where $|E|$ is the number of elements in the set E. We set $n = |G|$. Note all L^p spaces are equal; indeed,

$$C(G) = L^1(G) = L^\infty(G) = \mathbb{C}^G,$$

the space of all complex functions. For any function f,

$$\int_G f(x)dx = \frac{1}{n}\sum_{x \in G} f(x).$$

Define $\delta_x : G \to \mathbb{C}$ by

$$\delta_x(y) = \begin{cases} n \text{ if } x = y \\ 0 \text{ if } y \neq x. \end{cases}$$

If $G = \{e = x_1, \ldots, x_n\}$, then $\delta_{x_1}, \ldots, \delta_{x_n}$ is an orthonormal basis for $L^2(G)$. Thus $L^2(G) = \mathbb{C}^G$ is a n-dimensional $*$ algebra over $\mathbb{C}$. Usually this algebra is denoted by $\mathbb{C}[G]$.

Lemma 8.78. *The map $G \ni x \mapsto \delta_x \in \mathbb{C}[G]$ satisfies*

$$\begin{aligned} \delta_{xy} &= \delta_x * \delta_y \\ \delta_x^* &= \delta_{x^{-1}} \end{aligned} \tag{8.16}$$

Proof. We have

$$\begin{aligned}\delta_x * \delta_y(z) &= \frac{1}{n}\sum_{g\in G}\delta_x(g)\delta_y(g^{-1}z)\\ &= \delta_y(x^{-1}z)\\ &= \begin{cases} n \text{ if } x^{-1}z = y\\ 0 \text{ if } x^{-1}z \neq y\end{cases}\\ &= \delta_{xy}(z)\end{aligned}$$

and

$$\begin{aligned}\delta_x^*(y) &= \overline{\delta_x(y^{-1})}\\ &= \begin{cases} \bar{n} \text{ if } y^{-1} = x\\ 0 \text{ if } y^{-1} \neq x\end{cases}\\ &= \delta_{x^{-1}}(y).\end{aligned}$$

□

This shows that $x \mapsto \delta_x \in \mathbb{C}[G]$ is multiplicative, so we can identify G with its image in $\mathbb{C}[G]$. Then $\mathbb{C}[G]$ is the algebra over $\mathbb{C}$ generated by G. For this reason, $\mathbb{C}[G]$ is called the *group algebra of* G.

Let $\widehat{G}_c = \{\pi_1 = 1, \pi_2, \dots, \pi_r\}$. Let $\chi_i = \chi_{\pi_i}$ be the character of π_i. Then as seen in Corollary 8.54, the functions χ_i are a basis of the center of $\mathbb{C}[G]$. Set $\xi_i = d(\pi_i)\chi_i$. Then Proposition 8.47 shows

(8.17) $$\xi_i * \xi_j = \delta_{ij}\xi_i$$

Consequently, the characters ξ_j are central idempotents. Let $M_j = \xi_j * \mathbb{C}[G]$. Then since ξ_j commutes with the elements of $\mathbb{C}[G]$,

(8.18) $$M_j = \mathbb{C}[G] * \xi_j = \xi_j * \mathbb{C}[G] * \xi_j = \xi_j * \mathbb{C}[G]\,.$$

Theorem 8.79 (Wedderburn)**.** *The following hold:*

(a) M_j *is a* $*$ *ideal in* $\mathbb{C}[G]$.

(b) $M_i * M_j = \{0\}$ *if* $i \neq j$.

(c) $M_j \simeq M(d(\pi_j), \mathbb{C})$, *the* $*$ *algebra under adjoints of* $d(\pi_j) \times d(\pi_j)$ *complex matrices.*

(d) $\mathbb{C}[G] \simeq \bigoplus_{j=1}^r M_j$.

(e) $|G| = \sum_{j=1}^r d(\pi_j)^2$.

Proof. (a) and (b) follow from Equations (8.17) and (8.18).

For (c) if P_j is the primary projection on $L^2(G)$ corresponding to the irreducible representation $\bar{\pi}_j$, then by Theorem 8.51 the range of P_j is M_j. Thus by Theorem 8.9 and Proposition 8.17, M_j is isomorphic as $*$ algebras to $\mathcal{B}(\mathcal{H}_{\bar{\pi}_j})$; but this algebra is isomorphic to the matrix algebra of $d_j \times d_j$ matrices with adjoint being complex transposition.

Next (d) is an immediate consequence of the Plancherel Theorem 8.20 and (e) follows from $\dim \mathbb{C}[G] = |G| = \sum_j \dim M_j$. □

Recall that a function is central if it constant on each conjugacy class. Let $\mathcal{C} = \{\mathcal{C}_1 = \{e\}, \mathcal{C}_2, \ldots, \mathcal{C}_m\}$ be the set of conjugacy classes in G. Take $x_1 = e$ and choose $x_j \in \mathcal{C}_j$ for $j = 1, \ldots, m$. Define

$$f_j(x) := \begin{cases} 1 & \text{if } x \in \mathcal{C}_j \\ 0 & \text{if } x \notin \mathcal{C}_j \end{cases}$$

Theorem 8.80. *The sets $\{f_1, \ldots, f_m\}$ and $\{\xi_1, \ldots, \xi_r\}$ are bases for the space of central functions on G. In particular*

$$|\hat{G}| = \textit{number of conjugacy classes}.$$

Proof. By Corollary 8.54, the set $\{\xi_1, \ldots, \xi_r\}$ is a basis of $L_c^2(G)$, the center of $\mathbb{C}[G]$. If f is a central function, then $f = \sum_{j=1}^m f(x_j) f_j$. Note also that $\sum_{j=1}^m c_j f_j = 0$ implies $c_k = f(x_k) = \sum_{j=1}^m f_j(x_k) = 0$. Hence the set $\{f_1, \ldots, f_m\}$ is an linearly independent spanning set. □

The next result depends on the theory of algebraic integers. In this theory every rational algebraic integer is an integer. Then if π is an irreducible representation and we show $\frac{n}{d(\pi)}$ is an algebraic integer, then the order n of the group G is divisible by the dimension of the representation π.

Definition 8.81. *A complex number is an algebraic number if it is a root of a polynomial having integer coefficients. If it is a root of a monic polynomial with integer coefficients, it is an algebraic integer.*

We denote the set of algebraic integers by $\mathcal{A}$. We shall show $\mathcal{A}$ is a ring. Note clearly since k is a root of $p(x) = x - k$ for $k \in \mathbb{Z}$, we see $\mathbb{Z} \subseteq \mathcal{A}$.

Proposition 8.82. *Suppose $\mathcal{R} \subseteq \mathbb{C}$ is a ring which is finitely generated; i.e., there are $w_1, w_2, \ldots, w_k$ in $\mathcal{R}$ such that $\mathcal{R} = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \cdots + \mathbb{Z}w_k$. Then $\mathcal{R} \subseteq \mathcal{A}$.*

Proof. Let $w \in \mathcal{R}$ be nonzero. We know there is a $k \times k$ matrix $A = [a_{i,j}]$ where $a_{i,j} \in \mathbb{Z}$ with

$$ww_j = a_{1,j}w_1 + a_{2,j}w_2 + \cdots + a_{k,j}w_k$$

for $j = 1, 2, \ldots, k$. Hence

$$(wI - A)\begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This implies the matrix $wI - A$ is singular. Thus $\det(wI - A) = 0$ and we see w is a root of the characteristic polynomial $p(x) = \det(xI - A)$. $\square$

Lemma 8.83. *If w is a rational algebraic integer, then $w \in \mathbb{Z}$.*

Proof. Start by writing $w = \frac{j}{k}$ where j and k are relatively prime and $k > 0$. Let w be a root of $p(x) = x^m - a_{m-1}x^{m-1} - \cdots - a_1x^1 - a_0$ where the a_i are integers. Then

$$\frac{j^m}{k^m} = \sum_{i=0}^{m-1} a_i \frac{j^i}{k^i}.$$

Thus

$$j^m = \sum_{i=0}^{m-1} a_i j^i k^{m-i}.$$

This implies $k|j^m$. Since k is relatively prime to j, $k = 1$. $\square$

Theorem 8.84. *The set $\mathcal{A}$ of algebraic integers is a ring.*

Proof. Let θ and ϕ be algebraic integers. Then $\theta^p = \sum_{j=0}^{p-1} a_j\theta^j$ and $\phi^q = \sum_{k=0}^{q-1} b_k\phi^k$ for some p and q and integers a_j and b_k. Set $w_{j,k} = \theta^j\phi^k$ for $0 \leqslant j < p$ and $0 \leqslant k < q$. If $\mathcal{R} = \sum \mathbb{Z}w_{j,k}$, then $(\theta + \phi)w_{j,k} \in \mathcal{R}$ for all j, k and $(\theta\phi)w_{j,k} \in \mathcal{R}$ for all j, k. By Proposition 8.82, $\theta + \phi$ and $\theta\phi$ are in $\mathcal{A}$. $\square$

Lemma 8.85. *If χ is a character of a finite group G, then $\chi(g)$ is a sum of k^{th} roots of unity where k is the order of g. Thus $\chi(g)$ is an algebraic integer.*

Proof. Let $\chi = \mathrm{Tr}(\rho)$ where ρ is a finite dimensional representation of G. Set H to be the abelian subgroup of G generated by g. This is a cyclic group of order k where $k|n$. Now $\rho|_H$ is equivalent as matrix representations to a direct sum $\sum_{i=1}^{m} \rho_i$ of irreducible representations ρ_i. Since H is abelian, each ρ_i has dimension 1 and since $\rho(g)^k = \rho(g^k) = I$, we see $\rho_i(g)^k = 1$ for each i. $\square$

Theorem 8.86. *Let ρ be an irreducible representation of the finite group G and let $C_1, C_2, \ldots, C_r$ be the conjugacy classes of G. Set $\rho_j = \sum_{g \in C_j} \rho(g) = n\rho(f_j)$. Then:*

(a) *$\rho_j = w_jI$ where $w_j = \frac{h_j\chi_j}{d}$; here h_j is the number of elements in the conjugacy class C_j, χ_j is the value of the character χ of ρ on the elements in C_j, and d is the degree or dimension of the representation ρ.*

(b) *For each i and j in $\{1, 2, \ldots, r\}$, there are nonnegative integers $c_{i,j,k}$ for $1 \leqslant k \leqslant r$ such that*

$$w_i w_j = \sum_{k=1}^{r} c_{i,j,k} w_k.$$

Proof. Set γ_j to be the element in the group algebra given by $\gamma_j = \sum_{g \in C_j} g$. In the group algebra $\mathbb{C}^G$, this is the function f_j which is a constant 1 on the members of the conjugacy class C_j and 0 off this coset. In particular by Theorem 8.80, these elements are a basis of the center of the group algebra. Hence $\rho_j = n\rho(\gamma_j)$ as operators commute with every $\rho(g)$. By Schur's Lemma, $\rho_j = w_j I$ for some scalar w_j. To calculate w_j, note

$$\begin{aligned} dw_j = \operatorname{Tr}(\rho_j) &= \sum_{g \in C_j} \operatorname{Tr}(\rho(g)) \\ &= \sum_{g \in C_j} \chi(g) = h_j \chi_j. \end{aligned}$$

So $w_j = \frac{h_j \chi_j}{d}$ and we have (a).

For (b) we note $\gamma_i \gamma_j$ is in the center of the group algebra. Thus

$$\gamma_i \gamma_j = \sum_k c_{i,j,k} \gamma_k.$$

Since

$$\gamma_i \gamma_j = \left(\sum_{g_1 \in C_i} g_1 \right) \left(\sum_{g_2 \in C_j} g_2 \right) = \sum_k \sum_{g \in C_k} \sum \{ g_1 g_2 \mid g_1 \in C_i, g_2 \in C_j, g_1 g_2 = g \},$$

we see $c_{i,j,k}$ is the number of pairs $g_1 \in C_i$, $g_2 \in C_j$ where $g_1 g_2 = g$ a fixed member of C_k. Thus $c_{i,j,k}$ is a nonnegative integer. Now

$$\begin{aligned} w_i w_j I = \rho_i \rho_j &= \sum_{g_1 \in C_i} \sum_{g_2 \in C_j} \rho(g_1) \rho(g_2) \\ &= \sum_k \sum_{g \in C_k} c_{i,j,k} \rho(g) = \sum_k c_{i,j,k} \sum_{g \in C_k} \rho(g) \\ &= \sum_k c_{i,j,k} \rho_k = \sum_k c_{i,j,k} w_k I. \end{aligned}$$

Thus $w_i w_j = \sum_k c_{i,j,k} w_k$. $\square$

Corollary 8.87. *Each w_i is an algebraic integer.*

Proof. Let $\mathcal{R}$ be the $\mathbb{Z}$ module generated by $w_1, w_2, \ldots, w_r$. By (b), we have $\mathcal{R}$ is a ring. Thus by Proposition 8.82, $\mathcal{R} \subseteq \mathcal{A}$. $\square$

Theorem 8.88. *The dimension d of an irreducible representation ρ of a finite group G divides the order of the group.*

Proof. Let χ be the character of ρ. By Corollary 8.54, $(\chi, \chi)_2 = 1$. Thus if χ_k is the constant value of χ on the elements of the conjugacy class C_k, we have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{r} h_k \chi_k \bar{\chi}_k &= \frac{1}{n}\sum_{k=1}^{r}\sum_{g\in C_k} \chi(g)\overline{\chi(g)} \\ &= \frac{1}{n}\sum_{g\in G}\chi(g)\overline{\chi(g)} \\ &= 1.\end{aligned}$$

Thus

$$\frac{n}{d} = \sum_{k=1}^{r} \frac{h_k\chi_k}{d}\bar{\chi}_k = \sum_{k=1}^{r} w_k \bar{\chi}_k.$$

Now $\bar{\chi}_k$ is the value character χ on the elements in the conjugacy class C_k^{-1}. Thus since each w_k is an algebraic integer and the values of χ on the conjugacy classes of G are algebraic integers, Theorem 8.84 implies the rational number $\frac{n}{d}$ is an algebraic integer. Hence by Lemma 8.83, $\frac{n}{d}$ is an integer. □

Corollary 8.89. *Let ρ be an irreducible representation of a finite group G. Then $d(\rho)$ divides $|G/\ker(\rho)|$.*

Proof. Define an irreducible representation π of $G/\ker(\rho)$ by $\pi(gZ) = \rho(g)$. Then $d(\pi) = d(\rho)$ and consequently $d(\rho)$ divides $|G/\ker\rho|$. □

11.1. Induced characters. In Section 16 of Chapter 6 we saw in Theorem 6.123 that the two finite dimensional unitary representations are equivalent if and only if they have equal characters. In this chapter we have dealt largely with induced representations and thus it is appropriate to determine the character of any unitarily induced finite dimensional representation.

Let G be a locally compact Hausdorff group and assume H is a closed subgroup which has only finitely many left cosets in G. Now counting measure on G/H is left invariant and thus if π is a unitary representation on a Hilbert space $\mathcal{H}$, we see if $m = |G/H|$ and $x_1, x_2, \ldots, x_m$ are representatives of the left cosets of G/H, then a function $f \in L^2_H(G, \pi)$ is determined by its values $f(x_1), f(x_2), \ldots, f(x_m)$. Now if in addition, the Hilbert space $\mathcal{H}$ has finite dimension d, then $L^2_H(G, \pi)$ has dimension md. Moreover, if e_j for $j = 1, 2, \ldots, d$ is an orthonormal basis of $\mathcal{H}$, the functions $f_{k,j}$ defined by $f_{k,j}(x_l h) = \delta_{k,l}\pi(h^{-1})e_j$ form an orthonormal basis of $L^2_H(G, \pi)$.

We calculate the character χ of the induced representation π^G. First note for $x \in G$, x^{-1} permutes the left cosets $x_k H$. Thus if $x^{-1}x_k H = x_{k'}H$, then $x^{-1}x_k = x_{k'}h$ where $h = x_{k'}^{-1}x^{-1}x_k \in H$. This implies

$$\begin{aligned}(\pi^G(x)f_{k,j}, f_{k,j})_2 &= \sum_{i=1}^{m}(f_{k,j}(x^{-1}x_i), f_{k,j}(x_i))_{\mathcal{H}} \\ &= (f_{k,j}(x^{-1}x_k), e_j)_{\mathcal{H}} = (f_{k,j}(x_{k'}h), e_j)_{\mathcal{H}} \\ &= (\pi(h^{-1})f_{k,j}(x_{k'}), e_j)_{\mathcal{H}} = \delta_{k,k'}(\pi(h^{-1})e_j, e_j)_{\mathcal{H}} \\ &= \delta_{k,k'}(\pi(x_k^{-1}xx_{k'})e_j, e_j)_{\mathcal{H}}.\end{aligned}$$

So $(\pi^G(x)f_{k,j}, f_{k,j})_2 \neq 0$ only if $x^{-1}x_k = x_kH$. Consequently, if $S(x) = \{k \mid x_k^{-1}xx_k \in H\}$,

$$\begin{aligned}\chi(g) &= \sum_{k,j}(\pi^G(x)f_{k,j}, f_{k,j})_2 = \sum_{k\in S(x)}\sum_{j}(\pi^G(x)f_{k,j}, f_{k,j}) \\ &= \sum_{k\in S(x)}\sum_{j}(\pi(x_k^{-1}xx_k)e_j, e_j)_{\mathcal{H}} = \sum_{k\in S(x)}\chi_\pi(x_k^{-1}xx_k).\end{aligned}$$

This is essentially the content of the following theorem.

Theorem 8.90. *Let G be a locally compact Hausdorff group and suppose H is a closed subgroup having only finitely many left cosets in G. Suppose π is a finite dimensional unitary representation of H with character χ_π. Then the character χ of the induced representation π^G is given by*

$$\chi(x) = \sum_{gH\in G/H,\, g^{-1}xg\in H}\chi_\pi(g^{-1}xg).$$

In particular, when G is a finite group, one has

$$\chi(x) = \frac{1}{|H|}\sum_{g\in G,\, g^{-1}xg\in H}\chi_\pi(g^{-1}xg).$$

Proof. The first statement follows since $(gh)^{-1}x(gh) \in H$ if and only if $g^{-1}xg \in H$ and thus $\chi_\pi(h^{-1}x_k^{-1}xx_kh) = \chi_\pi(x_k^{-1}xx_k)$ for elements $g \in x_kH$ and $k \in S(x)$. In the case when G is finite, we have

$$\begin{aligned}\sum_{g\in G,\, g^{-1}xg\in H}\chi_\pi(g^{-1}xg) &= \sum_{h\in H}\sum_{k\in S(x)}\chi_\pi(h^{-1}x_k^{-1}xx_kh) \\ &= \sum_{h\in H}\sum_{k\in S(x)}\chi_\pi(x_k^{-1}xx_k) \\ &= \sum_{h\in H}\chi(x) \\ &= |H|\,\chi(x).\end{aligned}$$

$\square$

Exercise Set 8.7

1. Let $\mathcal{R}$ be a finitely generated algebra over the rational numbers $\mathbb{Q}$. Show every number $w \in \mathcal{R}$ is algebraic.

2. Show the collection of algebraic numbers is a field.

3. Let A_4 be the alternating subgroup of the symmetric group S_4 consisting of even permutations.

(a) Show there are 4 conjugacy classes.

(b) Determine the dimensions of the irreducible finite dimensional representations of A_4.

4. Find 4 inequivalent irreducible finite dimensional representations of A_4.

5. Determine the degrees of the characters of S_4.

6. Find the dual of S_4; i.e. one irreducible representation for each distinct irreducible character of S_4. (**Hint:** Consider the parity representation, three dimensional representations of S_4 which extend the irreducible 3 dimensional representation of A_4, and representations of S_4 induced from one-dimensional representations of A_4.)

7. Let D_4 be the dihedral group of order 8. It has generators a and b satisfying $a^4 = b^2 = 1$ and $bab = a^3$. Determine the conjugacy classes of D_4, the degrees of each irreducible character of D_4, and a representation for each of these characters.

8. Let G be a finite group. Show two finite dimensional complex representations of G are equivalent if and only if they have equal characters.

9. Let G be a finite group with center Z. Show if χ is an irreducible character, then

$$(\deg\chi)^2 \leqslant [G : Z] = \frac{|G|}{|Z|}.$$

(**Hint:** Let ρ be a representation with character χ. Use Schur's Lemma and Frobenius reciprocity.)

10. Let G be a finite group and let G^1 be the commutator subgroup of G. Thus G^1 is the smallest group containing all $aba^{-1}b^{-1}$ for $a, b \in G$. Note G/G^1 is an abelian group. Show the number of distinct one dimensional representations of G is the order of the group G/G^1.

11. Let G be a finite group with subgroup H. Find the character of the quasi-regular representation of G on $L^2(G/H)$.

12. Let $G = S_3$ and $X = \{1, 2, 3\}$. Show that $L^2(X)$ decomposes into two irreducible representations, one of them, the trivial representation being one dimensional, and the other two dimensional.

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