THE IMAGE OF THE SEGAL-BARGMANN TRANSFORM SYMMETRIC SPACES AND GENERALIZATIONS

Joint work with

H. Schlichtkrull

To appear in *Advances in Mathematics*

1. The Heat Equation

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where $g = \det(g_{ij})$

- The Heat equation is

$$\begin{aligned} \Delta u(x,t) &= \partial_t u(x,t) \\ \lim_{t \to o^+} u(x,t) &= f(x) \end{aligned}$$

Where f is in $L^2(M)$ or a distribution.

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▶ But more importantly, there exists a function $h_t(x,y)$, the heat kernel, such that:

- $h_t(x,y) = h_t(y,x) \geq 0$;
- $d\mu_t(y) = h_t(x,y)dy$ is a probability measure on M;
- $H_t f(x) = \int_M f(y) h_t(x,y)\, dy$;

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▶ In some special cases there is a “natural” complexification $M_{\mathbb{C}}$ of M, such that the heat kernel $x \mapsto h_t(x,y)$ and the function $H_t f$ extends to a holomorphic function on $M_{\mathbb{C}}$. The task is then to define a Hilbert space $\mathcal{H}_t(M_{\mathbb{C}})$ of holomorphic functions on $M_{\mathbb{C}}$ such that the transfrom

$$L^2(M) \ni f \mapsto H_t f \in \mathcal{H}_t(M_{\mathbb{C}})$$

becomes an unitary isomorphism.

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▶ The first simple remark is, that in general the heat kernel is invariant under isometries, i.e. if $\varphi : M \to M$ is an isometry, then

$$h_t(x, y) = h_t(\varphi(x), \varphi(y))$$

It follows that the heat kernel on $\mathbb{R}^n$ is a function of one variable

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▶ By definition, the heat kernel is a solution to the heat equation with $f = \delta_0$. Taking the Fourier transform (in the space variable x) the heat equation is transformed into the simple differential equation in the time variable:

$$\partial_t \widehat{h_t}(\lambda) = -|\lambda|^2 \widehat{h_t}(\lambda), \qquad \lim_{t\to 0+} \widehat{h_t}(\lambda) = (2\pi)^{-n/2}$$

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- By taking the inverse Fourier transform:

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- It is clear from this explicit formula, that

$$h_t(z) = (4\pi t)^{-n/2} e^{-z^2/4t}, \qquad z^2 = z_1^2 + \ldots + z_n^2$$

gives a holomorphic extension of the heat kernel to $\mathbb{C}^n \simeq T(\mathbb{R}^n)^*$, the complexification of $\mathbb{R}^n$.

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- This gives a holomorphic extension of $H_t f$:

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and set

$$\mathcal{H}_t(\mathbb{C}^n) = \{F \in \mathcal{O}(\mathbb{C}^n) \mid \|F\|_t^2 := \int_{\mathbb{C}^n} |F(x+iy)|^2\, d\mu_t < \infty\}\,.$$

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- Note, that we only put a weight on the fibers $x + i\mathbb{R}^n$. If one wants to consider the infinite dimensional case, it is necessary to weight both variables.

Theorem (Segal-Bargmann)

1. *$\mathcal{H}_t(\mathbb{C}^n)$ is a Hilbert space with continuous point evaluation.*
2. *We have $H_t(L^2(\mathbb{R}^n)) \subseteq \mathcal{H}_t(\mathbb{C}^n)$ and the map $H_t : L^2(\mathbb{R}^n) \to \mathcal{H}_t(\mathbb{C}^n)$ is a unitary isomorphism.*
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- The obvious problem in the general case is: What is $M_{\mathbb{C}}$?
- And: What is a natural generalization of the measure $d\mu_t$?

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▶ Here, I would like to discuss a new joint work with H. Schlichtkrull (Copenhagen) on the K-invariant functions on G/K and some generalizations. To appear in Adv. Math.

3. K-invariant functions on G/K and the Opdam-Heckmann theory

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$$K = G^{\theta} = \{g \in G \mid \theta(g) = g\} .$$

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▶ Think of $G = \mathrm{SL}(n, \mathbb{R})$, $K = \mathrm{SO}(n)$ and $\theta(g) = (g^{-1})^T$. The corresponding involution on the Lie algebra

$$\mathfrak{sl}(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \mathrm{Tr}(X) = 0\}$$

is simply $\theta(X) = -X^T$.

- Let $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Then

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▶ For $\alpha \in \Delta$ let $r_\alpha : \mathfrak{a} \to \mathfrak{a}$ be the reflection in the hyperplane $\alpha(X) = 0$ and let W be the finite reflection group - the Weyl group - generated by r_α, $\alpha \in \Delta$.

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▶ The open cone $\mathfrak{a}^+ = \{X \in \mathfrak{a} \mid (\forall \alpha \in \Delta^+)\, \alpha(X) > 0\}$ is a fundamental domain for W. Set:

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▶ Set $m_\alpha = \dim \mathfrak{g}^\alpha$ and $a^\alpha = e^{\alpha(\log a)}$

$$\delta(a) = \prod_{\alpha \in \Delta^+} |a^\alpha - a^{-\alpha}|^{m_\alpha} \qquad \text{and} \qquad d\mu(a) = \delta(a)da$$

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- and

$$\delta(a) = \prod_{i<j} (a_i/a_j - a_j/a_i) \ .$$

Theorem *We have $G = KAK$ and the restriction map*

$$L^2(G/K)^K \ni f \mapsto f|_A \in L^2(A, |W|^{-1} d\mu)^W \simeq L^2(A^+, d\mu)$$

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► Next we consider the effect on the Heat equation. For that let $H_1, \ldots, H_n$ be a orthonormal basis of $\mathfrak{a}$ and $A^{\text{reg}} = \{a \in A \mid (\forall \alpha)\, a^\alpha \neq 1\}$.

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▶ Let $(\cdot, \cdot)$ be a W-invariant inner product on $\mathfrak{a}$ (and by duality on $\mathfrak{a}^*$). Chose $h_\alpha \in \mathfrak{a}$ be such that $(X, h_\alpha) = \alpha(X)$, $(\alpha, \beta) = (H_\alpha, H_\beta)$, and - for $\alpha \neq 0$ - $H_\alpha = \frac{2}{(\alpha,\alpha)} h_\alpha$.

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▶ This reduces the analysis of K-invariant functions on G/K to analysis of W-invariant functions on the Euclidean space $A \simeq \mathfrak{a}$.
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▶ Let $(\cdot, \cdot)$ be a W-invariant inner product on $\mathfrak{a}$ (and by duality on $\mathfrak{a}^*$). Chose $h_\alpha \in \mathfrak{a}$ be such that $(X, h_\alpha) = \alpha(X)$, $(\alpha, \beta) = (H_\alpha, H_\beta)$, and - for $\alpha \neq 0$ - $H_\alpha = \frac{2}{(\alpha,\alpha)} h_\alpha$.
▶ Define a W-invariant differential operator L on A^{reg} by

$$L = \sum_{j=1}^{n} \partial(H_j)^2 + \sum_{\alpha \in \Delta^+} m_\alpha \frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}} \partial(h_\alpha)\,.$$

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▶ The density function and the differential operator L is defined as before.

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- $L\varphi_\lambda = ((\lambda, \lambda) - (\rho, \rho))\varphi_\lambda$ where $2\rho = \sum_{\alpha \in \Delta^+} m_\alpha \alpha$.
- Growth estimates for $\varphi_\lambda(a \exp iX)$ for $X \in \Omega$ where

$$\Omega = \{X \in \mathfrak{a} \mid (\forall \alpha \in \Delta)\, |\alpha(X)| < \pi/2\}\,.$$

–

Define the Hypergeometric Fourier transform by

$$\mathcal{F}f(\lambda) = \hat{f}(\lambda) = \int_A f(a)\varphi_{-i\lambda}(a)\,d\mu = |W| \int_{A^+} f(a)\varphi_{-i\lambda}(a)\,d\mu\,.$$

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Theorem (Heckmann-Opdam) *The Fourier transform extends to an unitary isomorphism*

$$L^2(A, d\mu)^W \simeq L^2(\mathfrak{a}^*, d\nu)^W\,.$$

Furthermore, if $f \in C_c^\infty(A)^W$ *then*

$$f(a) = |W|^{-1} \int_{\mathfrak{a}^*} \hat{f}(\lambda)\varphi_{i\lambda}(a)\,d\nu(\lambda)$$

and

$$\mathcal{F}(Lf)(\lambda) = -(|\lambda|^2 + |\rho|^2)\mathcal{F}(f)(\lambda)\,.$$

–

Let us put this together in a commutative diagram:

$$\begin{tikzcd} L^2(A, d\mu)^W \arrow[r] \arrow[d, "\mathcal{F}"'] & L^2(A, da)^{\tau(W)} \arrow[d, "\mathcal{F}_A"] \\ L^2(\mathfrak{a}^*, d\nu)^W \arrow[r, "\Psi"'] & L^2(\mathfrak{a}^*, d\lambda)^{\tau(W)} \end{tikzcd}$$

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- and the isometry Λ is constructed so as to make the diagram commutative.

▶ Then

$$\Lambda(Lf)(a) = (\Delta_A - |\rho|^2)\Lambda(f)(a)$$

reducing the our problem to a shifted heat equation on $A \simeq \mathfrak{a}$:

$$(\Delta_A - |\rho|^2)u(a,t) = \partial_t u(x,t)$$

–

Theorem (Ó+S, 2005) 1) The solution of the heat equation is given by

$$u(a,t) = |W|^{-2} \int_{\mathfrak{a}^*} e^{-t(|\lambda|^2 + |\rho|^2)} \hat{f}(\lambda) \varphi_{i\lambda}(a)\, d\nu(\lambda) \qquad f \in L^2(A)^W \, .$$

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Let $\mathcal{H}_t$ be the space of holomorphic function on $F : A \exp i\Omega \to \mathbb{C}$ such that $\Lambda(F)$ extends to a $\tau(W)$-invariant holomorphic function on $\mathfrak{a}_{\mathbb{C}}$ such that

$$\|F\|_t^2 = e^{2t|\rho|^2} \int_{\mathfrak{a}_{\mathbb{C}}} |\Lambda F(X+iY)|^2\, d\mu_t(X+iY) < \infty\,.$$

Then $\mathcal{H}_t$ is a Hilbert space and

$$H_t : L^2(A)^W \to \mathcal{H}_t$$

is an unitary isomorphism. Here μ_t is the heat measure on the Euclidean space $\mathfrak{a}$.

Assume $m_\alpha = 2$ for all α, i.e., $(\mathfrak{a}, \Delta, m)$ corresponds to a Riemannian symmetric space G/K with G complex.

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Theorem (Hall+Mitchell) *Assume that G is complex. Let $f \in L^2(G/K)^K$, and let $u(x,t) = H_t f(x)$ be the solution to the heat equation. The map $X \mapsto \delta(\exp X)^{1/2} u(\exp X, t)$, $X \in \mathfrak{a}$, has a holomorphic extension to $\mathfrak{a}_\mathbb{C}$ such that*

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Conversely, any meromorphic function $u(Z)$ which is invariant under W and which satisfies

$$\int_{\mathfrak{a}_\mathbb{C}} |(\delta^{1/2}u)(X+iY)|^2 e^{2t|\rho|^2}\, d\mu_t(X+iY) < \infty$$

is the Segal-Bargmann tranform $H_t f$ for some $f \in L^2(G/K)^K$.