**Solutions to homework due Nov 19.**

Recall first the following few fact:

a) Let $V$ be a vector space and $W$ a subspace. Suppose that $W$ is spanned by orthogoanl vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Then every vector in $W$ can be written in a unique way

$$\mathbf{u} = \sum_{j=1}^{k} \frac{<\mathbf{u}, \mathbf{u}_j>}{\|\mathbf{u}_j\|^2}\mathbf{u}_j \ .$$

Furthermore the orthogoanl projection $P_W : V \to W$ is given by the formula

$$P(\mathbf{v}) = \sum_{j=1}^{k} \frac{<\mathbf{v}, \mathbf{u}_j>}{\|\mathbf{u}_j\|^2}\mathbf{u}_j$$

where $\mathbf{v}$ is an arbitrary vector in $V$. The vector $P_W(\mathbf{v})$ is the vector in $W$ closest to $\mathbf{v}$ and the distance of $\mathbf{v}$ to $W$ is the number $\|\mathbf{v} - P_W(\mathbf{v})\|$.

b) In general we have not a set of orthogonal vector spanning $W$ but only a linear independent spanning set $\mathbf{u}_1, \ldots, \mathbf{u}_k$. The Gram-Schmidt orthogonalization produces form $\mathbf{u}_1, \ldots, \mathbf{u}_k$ an orthogonal spanning set $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in the following way:

(1) Let $\mathbf{v}_1 = \mathbf{u}_1$.
(2) Let $\mathbf{v}_2 = \mathbf{u}_2 - \frac{<\mathbf{u}_2, \mathbf{v}_1>}{\|\mathbf{v}_1\|}\mathbf{v}_1$,
(3) If we have alrady constructed the orthogonal vectors $\mathbf{v}_j$, $1 \le j < k$, then we construct the vector $\mathbf{v}_{j+1}$ by

$$\mathbf{v}_{j+1} = \mathbf{u}_{j+1} - \sum_{i=1}^{j} \frac{<\mathbf{u}_{j+1}, \mathbf{v}_i>}{\|\mathbf{v}_i\|}\mathbf{v}_i \ .$$

1) Let $V = \mathbb{R}^3$ and let $W$ be the plane generated by the vectors $\mathbf{u}_1 = (2, 1, 3)$ and $\mathbf{u}_2 = (1, 1, -1)$.

a) Apply the Gram-Schmidt orthogonalization to $\{\mathbf{u}_1, \mathbf{u}_2\}$ to find an orthogonal spanning set $\{\mathbf{v}_1, \mathbf{v}_2\}$ for $W$.

**Solution:** We have $<\mathbf{u}_1, \mathbf{u}_2> = 2 + 1 - 3 = 0$ so the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are already orthogonal so we can simply set $\mathbf{v}_1 = \mathbf{u}_1$ and $\mathbf{v}_1 = \mathbf{u}_2$.

b) Write the formula for the orthogonal projection $P_W : V \to W$.

**Solution:** The orthogonal projection is

$$P_W(x, y, z) = \frac{<(x, y, z), \mathbf{v}_1>}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{<(x, y, z), \mathbf{v}_2>}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \ .$$

We now do the necessary calculation:

(1) $<(x, y, z), \mathbf{v}_1> = <(x, y, z), (2, 1, 3)> = 2x + y + 3z$;
(2) $\|\mathbf{v}_1\|^2 = 2^2 + 1^2 + 3^2 = 4 + 1 + 9 = 14$;
(3) $<(x, y, z), \mathbf{v}_2> = <(x, y, z), (1, 1, -1)> = x + y - z$;
(4) $\|(1, 1, -1)\|^2 = 3$

From this we get:

$$P_W(x, y, z) = \frac{2x + y + 3z}{14}(2, 1, 3) + \frac{x + y - z}{3}(1, 1, -1)$$

Furthermore we have for each of the coordinates (after simplifying)

$x$-coordinate:

$$\begin{aligned}\frac{2x+y+3z}{7}+\frac{x+y-z}{3} &= \frac{6x+3y+9z+7x+7y-7z}{21}\\ &= \frac{13x+10y+2z}{21}\end{aligned}$$

$y$-coordinate:

$$\begin{aligned}\frac{2x+y+3z}{14}+\frac{x+y-z}{3} &= \frac{6x+3y+9z+14x+14y-14z}{42}\\ &= \frac{20x+17y-5z}{42}\end{aligned}$$

$z$-coordinate:

$$3\cdot\frac{2x+y+3z}{14}-\frac{x+y-z}{3}=\frac{4x-5y+41z}{42}\ .$$

The orthogonal projection is therefore

$$P_W(x,y,z)=\left(\frac{13x+10y+2z}{21},\frac{20x+17y-5z}{42},\frac{4x-5y+41z}{42}\right)\ .$$

c) Let $\mathbf{u}=(2,3,5)$. Find the point $\mathbf{w}\in W$ closest to $\mathbf{u}$.
**Solution:** The point in $W$ closest to $\mathbf{u}$ is the orthogonal projection $P_W(\mathbf{u})$. Use use the above formula to get:

$$\begin{aligned}P_W(2,3,5) &= \left(\frac{26+30+10}{21},\frac{40+51-25}{42},\frac{8-15+205}{42}\right)\\ &= \left(\frac{22}{7},\frac{11}{7},5\right)\end{aligned}$$

(Notice that $\mathbf{u}-P_W(\mathbf{u})=(2,3,5)-(\frac{22}{7},\frac{11}{7},\frac{33}{7})=(\frac{14-22}{7},\frac{21-11}{7},\frac{35-33}{7})=(\frac{-8}{7},\frac{10}{7},\frac{2}{7})$ and

$$<\left(\frac{-8}{7},\frac{10}{7},\frac{2}{7}\right),(2,1,3)>=\frac{1}{7}(-16+10+6)=0$$

and

$$<\left(\frac{-8}{7},\frac{10}{7},\frac{2}{7}\right),(1,1,-1)>=\frac{1}{7}(-8+10-2)=0$$

and hence $\mathbf{u}-P_W(\mathbf{u})$ is in fact orthogonal to $W$.)

2) Apply the Gram-Schmidt orthogonalization to the set $\{1,x,1+x^2\}$.

a) We let $\mathbf{v}_1=1$.
b) Then the polynomial $\mathbf{v}_2$ is given by

$$\mathbf{v}_2=x-\frac{<x,\mathbf{v}_1>}{\|\mathbf{v}_1\|^2}\mathbf{v}_1\ .$$

We calculate:

(1) $\|\mathbf{v}_1\|^2=\int_0^1 1^2\,dx=x]_0^1=1$
(2) $<x,\mathbf{v}_1>=\int_0^1 x\,dx=\frac{1}{2}$.

Hence

$$\mathbf{v}_2 = x - \frac{1}{2}\ .$$

c) We have

$$\mathbf{v}_3 = 1 + x^2 - \frac{< 1 + x^2, \mathbf{v}_1 >}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{< 1 + x^2, \mathbf{v}_2 >}{\|\mathbf{v}_2\|^2}\mathbf{v}_2\ .$$

We calculate in this case

(1) $< 1 + x^2, \mathbf{v}_1 >= \int_0^1 1 + x^2\ d = 1 + \frac{1}{3} = \frac{4}{3};$

(2) $< 1 + x^2, \mathbf{v}_2 >= \int_0^1 (1 + x^2)(x - 1/2)\ dx = \int_0^1 x - \frac{1}{2} + x^3 - \frac{1}{2}x^2\ dx = \frac{1}{2} - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

(3) $\|\mathbf{v}_2\|^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2\ dx = \frac{1}{3}\left(x - \frac{1}{2}\right)^3\Big]_0^1 = \frac{1}{3}\left(\frac{1}{8} - (-\frac{1}{8})\right) = \frac{1}{12}.$

Hence $\mathbf{v}_3$ is given by

$$\begin{aligned}\mathbf{v}_3 &= 1 + x^2 - \frac{4}{3} - (x - 1/2)\\ &= \frac{1}{6} - x + x^2\ .\end{aligned}$$