Solutions to homework due Nov 19.

Recall first the following few facts:

a) Let \( V \) be a vector space and \( W \) a subspace. Suppose that \( W \) is spanned by orthogonal vectors \( u_1, \ldots, u_k \). Then every vector in \( W \) can be written in a unique way

\[
u = \sum_{j=1}^{k} \frac{\langle u_i, u_j \rangle}{\|u_j\|^2} u_j .
\]

Furthermore the orthogonal projection \( P_W : V \to W \) is given by the formula

\[
P(v) = \sum_{j=1}^{k} \frac{\langle v, u_j \rangle}{\|u_j\|^2} u_j
\]

where \( v \) is an arbitrary vector in \( V \). The vector \( P_W(v) \) is the vector in \( W \) closest to \( v \) and the distance of \( v \) to \( W \) is the number \( \|v - P_W(v)\| \).

b) In general we have not a set of orthogonal vector spanning \( W \) but only a linear independent spanning set \( u_1, \ldots, u_k \). The Gram-Schmidt orthogonalization produces form \( u_1, \ldots, u_k \) an orthogonal spanning set \( v_1, \ldots, v_k \) in the following way:

1. Let \( v_1 = u_1 \).
2. Let \( v_2 = u_2 - \frac{\langle v_1, u_2 \rangle}{\|v_1\|^2} v_1 \).
3. If we have already constructed the orthogonal vectors \( v_j \), \( 1 \leq j < k \), then we construct the vector \( v_{j+1} \) by

\[
v_{j+1} = u_{j+1} - \sum_{i=1}^{j} \frac{\langle u_{j+1}, v_i \rangle}{\|v_i\|^2} v_i .
\]

1) Let \( V = \mathbb{R}^3 \) and let \( W \) be the plane generated by the vectors \( u_1 = (2, 1, 3) \) and \( u_2 = (1, 1, -1) \).

a) Apply the Gram-Schmidt orthogonalization to \( \{u_1, u_2\} \) to find an orthogonal spanning set \( \{v_1, v_2\} \) for \( W \).

Solution: We have \( \langle u_1, u_2 \rangle = 2 + 1 - 3 = 0 \) so the vectors \( u_1 \) and \( u_2 \) are already orthogonal so we can simply set \( v_1 = u_1 \) and \( v_1 = u_2 \).

b) Write the formula for the orthogonal projection \( P_W : V \to W \).

Solution: The orthogonal projection is

\[
P_W(x, y, z) = \frac{\langle (x, y, z), v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle (x, y, z), v_2 \rangle}{\|v_2\|^2} v_2 .
\]

We now do the necessary calculation:

1. \( \langle (x, y, z), v_1 \rangle = \langle (x, y, z), (2, 1, 3) \rangle = 2x + y + 3z \);
2. \( \|v_1\|^2 = 2^2 + 1^2 + 3^2 = 4 + 1 + 9 = 14 \);
3. \( \langle (x, y, z), v_2 \rangle = \langle (x, y, z), (1, 1, -1) \rangle = x + y - z \);
4. \( \|(1, 1, -1)\|^2 = 3 \)

From this we get:

\[
P_W(x, y, z) = \frac{2x + y + 3z}{14} (2, 1, 3) + \frac{x + y - z}{3} (1, 1, -1)
\]

Furthermore we have for each of the coordinates (after simplifying)
\[ \begin{align*}
\text{x-coordinate:} \\
\frac{2x + y + 3z}{7} + \frac{x + y - z}{3} &= \frac{6x + 3y + 9z + 7x + 7y - 7z}{21} \\
&= \frac{13x + 10y + 2z}{21} \\
\text{y-coordinate:} \\
\frac{2x + y + 3z}{14} + \frac{x + y - z}{3} &= \frac{6x + 3y + 9z + 14x + 14y - 14z}{42} \\
&= \frac{20x + 17y - 5z}{42} \\
\text{z-coordinate:} \\
3 \cdot \frac{2x + y + 3z}{14} - \frac{x + y - z}{3} &= \frac{4x - 5y + 4z}{42}.
\end{align*} \]

The orthogonal projection is therefore

\[ P_W(x, y, z) = \left( \frac{13x + 10y + 2z}{21}, \frac{20x + 17y - 5z}{42}, \frac{4x - 5y + 4z}{42} \right). \]

c) Let \( u = (2, 3, 5) \). Find the point \( w \in W \) closest to \( u \).

\[ \text{Solution:} \quad \text{The point in } W \text{ closest to } u \text{ is the orthogonal projection } P_W(u). \]

Use the above formula to get:

\[ P_W(2, 3, 5) = \left( \frac{26 + 30 + 10}{21}, \frac{40 + 51 - 25}{42}, \frac{8 - 15 + 205}{42} \right) = \left( \frac{22}{7}, \frac{11}{7}, \frac{5}{7} \right) \]

(Notice that \( u - P_W(u) = (2, 3, 5) - \left( \frac{22}{7}, \frac{11}{7}, \frac{5}{7} \right) = \left( \frac{14 - 22}{7}, \frac{21 - 11}{7}, \frac{35 - 33}{7} \right) = \left( -\frac{8}{7}, \frac{10}{7}, \frac{2}{7} \right) \)

and

\[ \langle \left( -\frac{8}{7}, \frac{10}{7}, \frac{2}{7} \right), (2, 1, 3) \rangle = \frac{1}{7}(-16 + 10 + 6) = 0 \]

and

\[ \langle \left( -\frac{8}{7}, \frac{10}{7}, \frac{2}{7} \right), (1, 1, -1) \rangle = \frac{1}{7}(-8 + 10 - 2) = 0 \]

and hence \( u - P_W(u) \) is in fact orthogonal to \( W \).

2) Apply the Gram-Schmidt orthogonalization to the set \( \{1, x, 1 + x^2\} \).

a) We let \( v_1 = 1 \).

b) Then the polynomial \( v_2 \) is given by

\[ v_2 = x - \frac{\langle x, v_1 \rangle}{\|v_1\|^2} v_1. \]

We calculate:

\[ \begin{align*}
(1) \quad \|v_1\|^2 &= \int_0^1 1^2 \, dx = x|_0^1 = 1 \\
(2) \quad \langle x, v_1 \rangle &= \int_0^1 x \, dx = \frac{1}{2}.
\end{align*} \]
Hence
\[ v_2 = x - \frac{1}{2}. \]

c) We have
\[ v_3 = 1 + x^2 - \frac{< 1 + x^2, v_1 >}{\|v_1\|^2} v_1 - \frac{< 1 + x^2, v_2 >}{\|v_2\|^2} v_2. \]

We calculate in this case

1. \( < 1 + x^2, v_1 > = \int_0^1 (1 + x^2) \, dx = 1 + \frac{1}{3} = \frac{4}{3}; \)
2. \( < 1 + x^2, v_2 > = \int_0^1 (1 + x^2)(x - 1/2) \, dx = \int_0^1 x - \frac{1}{2} + x^2 - \frac{1}{2} x^2 \, dx = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{1}{12}; \)
3. \( \|v_2\|^2 = \int_0^1 (x - \frac{1}{2})^2 \, dx = \frac{1}{3} (x - \frac{1}{2})^3 \bigg|_0^1 = \frac{1}{3} (\frac{1}{2} - (-\frac{1}{2})) = \frac{1}{12}. \)

Hence \( v_3 \) is given by
\[ v_3 = 1 + x^2 - \frac{4}{3} \cdot (x - 1/2) \]
\[ = \frac{1}{6} - x + x^2. \]