The Heat equation, the Segal-Bargmann transform and generalizations

Based on joint work with

- **B. Krötz**
- **B. Ørsted**
- **H. Schlichtkrull**
- **R. Stanton**

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Organizations

1. The heat equation on $\mathbb{R}^n$.
2. The Fock space and the Segal-Bargmann Transform.
3. Remarks and Comments.
4. Generalizations and the Restriction Principle.
5. Structure Theory.
6. Spherical Functions and the Fourier Transform.
7. The Crown and the Heat Kernel.
8. The Abel Transform and the Heat Kernel.
9. The Faraut-Gutzmer Formula and the Orbital Integral.
10. The Image of the Segal-Bargmann transform on G/K.
11. The K-invariant case (more than one section)

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1. The heat equation on $\mathbb{R}^n$

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- The heat equation is the Cauchy problem

$$\begin{aligned} \Delta u(x,t) &= \partial_t u(x,t) \\ \lim_{t \to 0^+} u(x,t) &= f(x) \end{aligned}$$

where we can take $f \in L^2(\mathbb{R}^n)$, a distribution, a hyperfunction, or from another class of analytic objects.

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- To actually solve the equation, we proceed by applying the Fourier transform

$$f \mapsto \mathcal{F}(f) = \hat{f}\,, \qquad \lambda \mapsto (2\pi)^{-n/2} \int f(x) e^{-ix\cdot\lambda}\, dx$$

using that

$$\mathcal{F}(\Delta f)(\lambda) = -|\lambda|^2 \hat{f}(\lambda)$$

and get the simple differential equation for $t \mapsto \hat{u}(\lambda, t)$:

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$$\partial_t \hat{u}(\lambda,t) = -|\lambda|^2 \hat{u}(\lambda,t)\,, \qquad \hat{u}(\lambda,0) = \hat{f}(\lambda)\,.$$

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▶ It is clear from this formula, that $\mathbb{R}^n \ni x \mapsto h_t(x)\in\mathbb{R}^+$ has a holomorphic extension to $\mathbb{C}^n$ given by

$$h_t(z)=(4\pi t)^{-n/2}e^{-z^2/4t}\,,\qquad z^2=z_1^2+\ldots+z_n^2\,.$$

- Note

$$\partial_t(f * h_t) = f * (\partial_t h_t) = f * (\Delta h_t) = \Delta(f * h_t)$$

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where $z \cdot \lambda = \sum_{j=1}^n z_j \lambda_j$.

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2. In using (0.1) that the exponential function $\lambda \mapsto e_\lambda(z) = e^{iz\cdot\lambda}$ grows much slower than

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3. Or in using (0.2) that heat kernel h_t has a holomorphic extension to $\mathbb{C}^n$ and $y \mapsto h_t(z-y)$ grows much slower than

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▶ We will now describe the image of the Segal-Bargmann transform. For that we define a positive weight function by

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Set

$$\mathcal{H}_t(\mathbb{C}^n) = \{F \in \mathcal{O}(\mathbb{C}^n) \mid \|F\|_t^2 := \int_{\mathbb{C}^n} |F(x+iy)|^2 \, d\mu_t < \infty\}\,.$$

–

Theorem 0.1 (Segal-Bargmann, 1956-1978/1961, . . .). *The following holds:*

1. $\mathcal{H}_t(\mathbb{C}^n)$ *is a Hilbert space with continuous point evaluation, i.e., the maps*

$$\mathcal{H}_t(\mathbb{C}^n) \ni F \mapsto \mathrm{ev}_z(F) = F(z) \in \mathbb{C}\,, \qquad z \in \mathbb{C}^n$$

are continuous. In particular, with $L_y F(x) = F(x-y)$ *and*

$$K_w(z) = K(z,w) := H_t(L_{\bar{w}} h_t)(z) = (8\pi t)^{-n/2} e^{-(z-\bar{w})^2/8t}\,,$$

we have $K_w \in \mathcal{H}_t(\mathbb{C}^n)$ *and* $F(w) = (F, K_w)$ *for all* $F \in \mathcal{H}_t(\mathbb{C}^n)$*, i.e.,* $K(z,w)$ *is the reproducing kernel for* $\mathcal{H}_t(\mathbb{C}^n)$

2. $H_t : L^2(\mathbb{R}^n) \to \mathcal{H}_t(\mathbb{C}^n)$ *is an unitary isomorphism.*

3. *If* $f \in S(\mathbb{R}^n)$*, then* $f(x) = \int_{\mathbb{R}^n} H_t f(x+iy) h_t(y)\, dy$*.*

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Few words on the proof, but note that I will not prove the surjectivity or that H_t is an isometry.

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&= \|f\|_2^2
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$$\int H_t f(x+iy) h_t(y)\, dy \overset{(0.1)}{=} \int \left(\int e^{-t\lambda^2} \hat{f}(\lambda) e^{i(x+iy)\cdot\lambda}\, d\lambda \right) h_t(y)\, dy$$

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\begin{aligned}
F(w) &= H_t f(w) \\
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&= (f, L_{\bar{w}} h_t)_{L^2} \\
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- Thus

$$\begin{aligned} K(z,w) &= H_t(\lambda(\bar{w}) h_t)(z) \\ &= (\lambda(\bar{w}) h_t) * h_t(z) \\ &= h_t * h_t(z - \bar{w}) \\ &= h_{2t}(z - \bar{w}) \quad \text{the semigroup property.} \end{aligned}$$

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3. Remarks and Comments

- Note first of all, that we can interpret $\mathbb{C}^n$ as the cotangent bundle $T^*(\mathbb{R}^n)$, where the y-variable in $z = x + iy$ is an element of $T^*_x\mathbb{R}^n$. Hence the Segal-Bargmann transform is some kind of quantization.

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3. Remarks and Comments

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$$d\nu^n(x) = h_t(x)dx$$

the heat kernel measure on $\mathbb{R}^n$. On the image side the measure is then

$$d\sigma_t^n(z) = (2\pi t)^{-n} e^{-|z|^2/2t} dxdy$$

–

▶ Denote the corresponding space of L^2-holomorphic functions by $\mathcal{F}_t(\mathbb{C}^n)$. It is still holds, that the Segal-Bargmann transform

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- Connection to the theory of orthogonal polynomials: There are constants c_α (easy to calculate) such that $\{c_\alpha \zeta_\alpha\}_{\alpha \in \mathbb{N}_0}$ is an orthogonal basis for $\mathcal{H}_t(\mathbb{C}^n, d\sigma_t)$ and there are constants (again easy to calculate) such that $H_t^*(\zeta_\alpha) = d_\alpha h_\alpha$, where h_α is the Hermite polynomial.

- Furthermore, the measures on both sides are probability measures, and n-fold product of the one-dimensional measures in the coordinates.

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- We can take the limit as $n \to \infty$. Consider the projections $\mathrm{pr}^n : \mathbb{R}^n \to \mathbb{R}^{n-1}$. This gives us isometric maps maps

$$\mathrm{pr}^n_* : L^2(\mathbb{R}^{n-1}, d\nu^{n-1}) \to L^2(\mathbb{R}^n, d\nu^n)\,, \qquad f \mapsto f \circ \mathrm{pr}^n$$

and we have a sequence of commutative diagrams

$$\begin{array}{ccccccc}
\cdots \to & L^2(\mathbb{R}^{n-1}, d\nu^{n-1}) & \xrightarrow{\mathrm{pr}^n_*} & L^2(\mathbb{R}^n, d\nu^n) & \xrightarrow{\mathrm{pr}^{n+1}_*} \cdots & L^2(\mathbb{R}^\infty, d\nu^\infty) \\
 & \downarrow{\scriptstyle H_t^{n-1}} & & \downarrow{\scriptstyle H_t^{n}} & & \downarrow{\scriptstyle H_t^{\infty}} \\
\cdots \to & \mathcal{H}_t(\mathbb{C}^{n-1}, d\sigma_t^{n-1}) & \xrightarrow{\mathrm{pr}^n_*} & \mathcal{H}_t(\mathbb{C}^n, d\sigma_t^n) & \xrightarrow{\mathrm{pr}^{n+1}_*} \cdots & \mathcal{H}_t(\mathbb{C}^\infty, d\sigma_t^\infty)
\end{array}$$

▶ Sometimes, in particular studying the Schrödinger representation of the Heisenberg group, one uses the Segal-Bargmann transform

$$S_t : L^2(\mathbb{R}^n, dx) \to \mathcal{F}_t(\mathbb{C}^n) \,.$$

One of the idea is, that $\mathcal{F}_t(\mathbb{C}^n)$ is a much simpler space than $L^2(\mathbb{R}^n)$ to work with. Also, the canonical commutation rules, the creation operator and the annulation operator have simpler form in $\mathcal{F}_t(\mathbb{C}^n)$.

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In this case - as I will prove later - the Segal-Bargmann transform is given by:

$$S_t(f)(z) = (\pi t)^{-n/4} \int f(y) e^{-\frac{1}{2t}(y^2 - 2xy + \frac{x^2}{2})}\, dy\,.$$

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▶ The connection to the theory of special functions is, this case, a multiple of the Hermite functions are mapped into a multiple of the polynomials ζ_α.

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4. Generalizations and the Restriction Principle.

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► Let $M_{\mathbb{C}}$ be a complex analytic manifold (i.e., $M_{\mathbb{C}} = \mathbb{C}^n$) and $M \subset M_{\mathbb{C}}$ a totally real analytic submanifold. Thus the restriction map

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► Let $\mathcal{F}(M_{\mathbb{C}})$ be a Hilbert space of holomorphic function on $M_{\mathbb{C}}$ such that the point-evaluation maps $F \mapsto F(w)$ are continuous and hence given by the inner product with an element $K_w \in \mathcal{F}(M_{\mathbb{C}})$:

$$\forall F \in \mathcal{F}(M_{\mathbb{C}}) \; : \; F(w) = (F, K_w)$$

–

▶ The function $K : M_{\mathbb{C}} \times M_{\mathbb{C}} \to \mathbb{C}$, $K(z,w) = K_w(z)$ is reproducing kernel of $\mathcal{F}(M_{\mathbb{C}})$. It satisfies:

1. K is holomorphic in the first variable and anti-holomorphic in the second variable.
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- Furthermore, the linear hull of $\{K_x\}_{x \in M}$ is dense in $\mathcal{F}(M_{\mathbb{C}})$ and hence:

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- Assume that F is orthogonal to the linear span of $\{K_x\}_{x \in M}$. Then $F(x) = (F, K_x) = 0$ for all $x \in M$ and hence $F|_M = 0$. As M is a totally real submanifold, it follows that $F = 0$.

We now make the following assumption: There exists a measure μ on M and a holomorphic function $D : M_{\mathbb{C}} \to \mathbb{C}$, $D|_M > 0$, $D(z) \neq 0$, such that

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$$R(F) := (DF)|_M \in L^2(M, \mu)\,.$$

2. $R(\mathcal{F}(M_{\mathbb{C}}))$ is dense in $L^2(M)$ (can be dropped, but then we have only a partial isometry later).
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- Then $R^* : L^2(M, d\mu) \to \mathcal{F}(M_{\mathbb{C}})$ is densely defined and

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- We call U the generalized Segal-Bargmann transform .

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- Note the following:

$$R^*f(w) = (R^*f, K_w)_{\mathcal{F}} = (f, RK_w) = \int_M f(y)D(y)K(w,y)\,dx$$

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- Furthermore, by multiplying by U^*, and then using that $\sqrt{RR^*}$ is self-adjoint, we get the following formula for RU and then U:

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▶ If M is a Riemannian manifold, then the elliptic differential operator Δ is well defined and invariant under isometries of M. Let $d\sigma$ be the volume form on M. Then the heat equation is given as before:

$$\Delta u(x,t) = \partial_t u(x,t)\,, \qquad \lim_{t\to 0^+} u(x,t) = f(x) \in L^2(M, d\sigma)\,.$$

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▶ But more importantly, there exists a function $h_t(x,y)$, the heat kernel, such that:

- $h_t(x,y) = h_t(y,x) \geq 0$;
- $d\mu_t(y) = h_t(x,y) d\sigma(y)$ is a probability measure on M;
- If $g: M \to M$ is an isometry, then $h_t(gx, gy) = h(x,y)\,.$
- $H_t f(x) = \int_M f(y) h_t(x,y)\, d\sigma(y)$;

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▶ There is one class of Riemannian manifolds where a natural complexification exists. Those are the Riemannian symmetric spaces G/K, where G is a connected and semisimple Lie group.

• B. Hall in 1997 for compact connected Lie groups. Here

$$G = M \subset G_{\mathbb{C}} = M_{\mathbb{C}} \simeq T^*G\,.$$

Here $G_{\mathbb{C}}$ is a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, i.e.,

$$G = \mathrm{SO}(n) \subset \mathrm{SO}(n, \mathbb{C}) \simeq \mathrm{SO}(n) \times \exp\{X \in iM(n, \mathbb{R}) \mid X^* = X\}\,.$$

–

• M.B. Stenzel in 1999 for symmetric spaces $M = G/K$, where G is compact. Here $M_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}} \simeq T(G/K)^*$. Here G is a compact connected Lie group, $\tau : G \to G$ is a non-trivial involution and

$$K = G^{\tau} = \{g \in G \mid \tau(g) = g\}$$

i.e, $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$.

Note, that Hall's result is a special case as $G \simeq G \times G/G$ with $\tau(a,b) = (b,a)$.

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• B. Hall and J.J. Mitchell did the case $M = G/K$ where G is complex or of rank one in 2004.

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- B. Krötz, G. Ólafsson, and R. Stanton: 2005 the general case G/K where G is non-compact and semisimple and K is a maximal compact subgroup, i.e., $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

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▶ One of the reasons, that it took so long to get from the compact case to the non-compact case is, that it was not so clear, what the right complexification of G/K is. It is the Akhiezer-Gindikin domain also called the complex crown which I will define in a moment. But first we will need some basic structure theory for semisimple symmetric space of the non-compact type.

5. Structure Theory

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- $K \subset G$ a maximal compact subgroup, and $\theta : G \to G$ the corresponding Cartan involution:

$$K = G^{\theta} = \{g \in G \mid \theta(g) = g\} .$$

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5. Structure Theory

- G a connected, non-compact semisimple Lie group with finite center
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- Denote the corresponding involution on the Lie algebra $\mathfrak{g}$ by the same letter θ and let

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g} \mid \theta(X) = X\} \\ \mathfrak{p} &= \{X \in \mathfrak{g} \mid \theta(X) = -X\} \end{aligned}$$

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- We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\,.$$

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▶ Our standard example is $G = \mathrm{SL}(n, \mathbb{R})$, $K = \mathrm{SO}(n)$ and $\theta(g) = (g^{-1})^T$. The corresponding involution on the Lie algebra

$$\mathfrak{sl}(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \mathrm{Tr}(X) = 0\}$$

is $\theta(X) = -X^T$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ corresponds to the decomposition of $\mathfrak{sl}(n, \mathbb{R})$ into skew-symmetric ($= \mathfrak{k}$) and symmetric ($= \mathfrak{p}$) matrices .

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- If $X \in \mathfrak{p}$ then $\mathrm{ad}(X)^* = \mathrm{ad}(X)$, i.e., $\mathrm{ad}(X)$ is symmetric.

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- Let $\Delta^+ := \{\alpha \in \Delta \mid \alpha(X) > 0\}$. Then – as $\alpha \circ \theta = -\alpha$ – we have

$$\Delta = \Delta \dot{\cup} (-\Delta^+) \qquad \text{and} \qquad (\Delta^+ + \Delta^+) \cap \Delta \subset \Delta^+.$$

- As $[\mathfrak{g}^{\lambda}, \mathfrak{g}^{\mu}] \subseteq \mathfrak{g}^{\mu+\lambda}$ it follows that

$$\mathfrak{n} := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^{\alpha}$$

is a nilpotent Lie algebra such that

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$$\mathfrak{g} = \bigoplus_{\alpha\in\Delta} \mathfrak{g}^\alpha \oplus \underbrace{\mathfrak{m} \oplus \mathfrak{a}}_{\text{the zero eigenspace}} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

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&= \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \qquad \text{the Iwasawa decomposition}
\end{aligned}$$

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- On the group level this corresponds to

Theorem 0.2 (Iwasawa Decomposition)**.** *The map*

$$N \times A \times K \ni (n,a,k) \mapsto nak \in G$$

is an analytic isomorphism. We write

$$x = \overset{\in N}{n(x)}\,\overset{\in A}{a(x)}\,\overset{\in K}{k(x)}$$

for the unique decomposition of $x \in G$*. In particular* $G/K \simeq N \times A$*.*

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▶ We assume that $G \subset G_{\mathbb{C}}$, where $\mathrm{Lie}(G_{\mathbb{C}}) = \mathfrak{g} \otimes \mathbb{C}$. Then we can complexify all the groups under consideration and obtain $N_{\mathbb{C}}$, $A_{\mathbb{C}}$ and $K_{\mathbb{C}}$. Then $N_{\mathbb{C}} A_C K_{\mathbb{C}} \subset G_{\mathbb{C}}$ is open and dense but not equal to $G_{\mathbb{C}}$. Furthermore, the decomposition

$$x = n(x)a(x)k(x) \in N_{\mathbb{C}} A_{\mathbb{C}} K_{\mathbb{C}}$$

is not unique in general.

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- For our standard example this corresponds to:

$$\begin{aligned}
\mathfrak{a} &= \{\mathrm{diag}(x_i) \mid \sum x_i = 0\} \\
&= \{x \in \mathbb{R}^n \mid x_1 + x_2 + \ldots + x_n = 0\} \simeq \mathbb{R}^{n-1} \\
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- The Iwasawa decomposition follows directly from the Gram-Schmidt orthogonalization.
- Note, for $n = 2$ this is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{ac+bd}{c^2+d^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{c^2+d^2}} & 0 \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & \frac{-c}{\sqrt{c^2+d^2}} \\ \frac{c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix}$$

and this breaks down as $c^2 + d^2 = 0$.

6. Spherical Functions and the Fourier Transform

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► We will also need the Weyl group. It is the finite reflection group in $\mathrm{O}(\mathfrak{a})$ generated by the reflections r_α in the hyperplanes $\alpha = 0$. It is denoted by W. We have

$$W \simeq N_K(\mathfrak{a})/M \qquad M = Z_K(\mathfrak{a})\,.$$

Permutation of the coordinates for our standard case.

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Permutation of the coordinates for our standard case.

▶ For a differential operator $D : C_c(G/K) \to C_c(G/K)$ and $g \in G$, let

$$(g \cdot D)(f) = D(f \circ L_{g^{-1}} f) \circ L_g\,.$$

Then D is G-invariant if $g \cdot D = D$ for all $g \in G$. Thus D is G-invariant if and only if D commutes with translation

$$D(f \circ L_g) = [D(f)] \circ L_g\,.$$

Denote by $\mathbb{D}(G/K)$ the commutative algebra of all invariant differential operators on G/K. On $\mathbb{R}^n$ this is just the algebra of constant coefficient differential operators $\mathbb{D}(\mathbb{R}^n) = \mathbb{C}[\partial_1, \ldots, \partial_n]$.

–

- For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let

$$\varphi_\lambda(x) := \int_K a(kx)^{\lambda+\rho}\, dk\,.$$

The functions φ_λ are the spherical functionson G/K. We have

$$\varphi_\lambda = \varphi_\mu \iff \exists w \in W \,:\, \lambda = w\mu\,.$$

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- The spherical functions are K-invariant eigenfunctions of $\mathbb{D}(G/K)$. In particular for the Laplace operator $\Delta_{G/K} \in \mathbb{D}(G/K)$:

$$\Delta_{G/K}\varphi_\lambda = (\lambda^2 - |\rho|^2)\varphi_\lambda$$

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- In the harmonic analysis of K-invariant functions on G/K they play the same role as the exponential functions $e_\lambda(x) = e^{\lambda\cdot x}$ on $\mathbb{R}^n$. We will discuss that in more details later on.

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- $d\sigma(b, \lambda) = db \frac{d\lambda}{|c(\lambda)|^2}$ where $c(\lambda)$ is the Harish-Chandra c-function.
- For $f \in C_c(G/K)$ define the Fourier transform $\hat{f} : B \times \mathfrak{a}^*_{\mathbb{C}}$, of f by

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Theorem 0.3 (Helgason)**.** *1. The Fourier transform extends to an unitary isomorphism $\mathcal{F} : L^2(G/K) \to L^2(B \times \mathfrak{a}^*, d\sigma)$+some W-invariance.*

2. If $f \in C_c(G/K)$ then $f(x) = c \int_{B\times\mathfrak{a}^*} \hat{f}(b, \lambda) a(bx)^{i\lambda+\rho}\, d\sigma$.

3. We have $\mathcal{F}(\Delta_{G/K} f)(b, \lambda) = (\lambda^2 - \rho^2)\hat{f}(b, \lambda)$.

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► For K-invariant functions, this reduces to the Harish-Chandra spherical Fourier transform

$$\hat{f}(\lambda) = \int f(x)\varphi_{-i\lambda}(x)\,dx\,.$$

and the spherical Fourier transform extends to an unitary isomorphism

$$L^2(G/K)^K \ni f \mapsto \hat{f} \in L^2(\mathfrak{a}^*, \frac{d\lambda}{|c(\lambda)|^2})^W \simeq L^2(\mathfrak{a}^*_+, |W|\frac{d\lambda}{|c(\lambda)|^2})$$

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$$L^2(G/K)^K \ni f \mapsto \hat{f} \in L^2(\mathfrak{a}^*, \frac{d\lambda}{|c(\lambda)|^2})^W \simeq L^2(\mathfrak{a}^*_+, |W|\frac{d\lambda}{|c(\lambda)|^2})$$

with inversion formula

$$f(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*_+} \hat{f}(\lambda)\varphi_{i\lambda}(x)\,\frac{d\lambda}{|c(\lambda)|^2}\,.$$

▶ Using the Fourier transform and part (3) of Helgason's Theorem we get the following form for the solution of the heat equation:

$$\begin{aligned} H_t f(x) &= \int e^{-(\lambda^2+\rho^2)t} \hat{f}(b,\lambda)\, a(bx)^{i\lambda+\rho}\, d\sigma(b,\lambda) \\ &= f * h_t(x)\,. \end{aligned}$$

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▶ So, how far does $x \mapsto h_t(x)$ extend? Or, how far does $x \mapsto \varphi_\lambda(x)$ extend, and what is the growth of the extension?

7. The Crown and the Heat Kernel

–

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- We define

$$\begin{aligned}\Omega &= \{X \in \mathfrak{a} \mid (\forall \alpha \in \Delta)\, |\alpha(X)| < \pi/2\} \quad W-\text{invariant polytope}\\ \Xi &= G\exp(i\Omega)\cdot x_o \subset G_{\mathbb{C}}/K_{\mathbb{C}}\end{aligned}$$

where x_o is the base point $eK_{\mathbb{C}} \subset G_{\mathbb{C}}/K_{\mathbb{C}}$. Then Ξ is an open G-invariant subset of $G_{\mathbb{C}}/K_{\mathbb{C}}$, the Akhiezer-Gindikin domain or complex crown . It has been studied by several group of people: Barchini, Burns + Halverscheid + Hind, Huckleberry, Krötz + Stanton, Wolf and others.

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- Its importance in harmonic analysis on G/K comes from the following.

Theorem 0.4 (Krötz+Stanton, ...). *1. We have $\Xi \subset N_{\mathbb{C}}A_{\mathbb{C}}\cdot x_o$ and the Iwasawa projection $\Xi \ni \xi \mapsto a(\xi) \in A_{\mathbb{C}}$ is well defined and holomorphic.*

2. Ξ is a maximal G-invariant domain in $G_{\mathbb{C}}/K_{\mathbb{C}}$ such that all the joint eigenfunctions for $D(G/K)$ extends to holomorphic functions on Ξ.

–

▶ It follows that the spherical functions extends to Ξ. With some extra work, involving the the growth of the spherical functions we have:

Theorem 0.5 (Krötz+Stanton)**.** *The heat kernel extends to a holomorphic function on* Ξ *given by the same formula*

$$h_t(\xi) = \frac{1}{|W|} \int_{\mathfrak{a}^*_+} e^{-(|\lambda|^2+|\rho|^2)t} \varphi_{i\lambda}(\xi)\, d\sigma(\lambda)\,.$$

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▶ As a consequence we have that each solution to the heat equation $f * h_t$, $f \in L^2(G/K)$ extends to a holomorphic function on Ξ:

$$H_t f(\xi) = \int_G f(gx_o)h_t(g^{-1}\xi)\, dg\,.$$

As before, the problem is then to determine the image of $H_t : L^2(G/K) \to \mathcal{O}(\Xi)$.

8. The Abel Transform and the Heat Kernel

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▶ Recall that

$$\int_{G/K} f(x)\, dx = \int_A \int_N f(na \cdot x_o) a^{-2\rho}\, dnda = \int_A \int_N f(an \cdot x_o) a^{2\rho}\, dnda\,.$$

For a K-invariant f function on G/K, say of compact support, define the Abel transform of f by

$$\mathcal{A}(f)(a) = a^{\rho} \underbrace{\int_N f(an)\, dn}_{\text{The Radon Transform}} = a^{-\rho} \int_N f(na)\, dn \tag{0.3}$$

using the notation $(\exp(X))^{\lambda} = e^{\lambda(X)}$. Then $\mathcal{A}(f)$ is a W-invariant function on A.

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$$\mathcal{A}(\Delta_{G/K} f) = (\Delta_A - |\rho|^2)\mathcal{A}(f)\,. \tag{0.4}$$

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- We have the following Fourier slice theorem for K-invariant functions:

$$\begin{aligned}
\hat{f}(\lambda) &= \int_{G/K} f(x)\varphi_{-i\lambda}(x)\,dx \\
&= \int_{G/K} f(x)a(k^{-1}x)^{-i\lambda+\rho}\,dx \\
&= \int_A \left(a^{-\rho}\int_N f(na\cdot x_o)\,dn\right) a^{-i\lambda}\,da \\
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$$\mathcal{F}_{G/K} = \mathcal{F}_A \circ \mathcal{A}\,.$$

- Equation (0.4) implies also that

$$h_t(\exp X) = \underbrace{e^{-|\rho|^2 t}}_{\text{the } \rho-\text{shift}} \underbrace{\mathcal{A}^{-1}}_{\text{a shift operator}} \underbrace{\left((4\pi t)^{-n/2} e^{-|X|^2/4t}\right)}_{\text{the heat kernel on } A}.$$

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- Define

$$\psi_\lambda(\exp X) = \frac{1}{|W|}\sum_{w\in W} e^{w\lambda(X)}$$

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$$D = \mathcal{F}_A^{-1} \circ \frac{1}{|c(\lambda)|^2} \circ \mathcal{F}_{G/K}$$

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or – for “good” – W-invariant functions:

$$Dh(a) = \underbrace{\int_{\mathfrak{a}^*_+} \overbrace{\underbrace{\mathcal{F}_{G/K}(h)(\lambda)}_{\text{Firt the FT on G/K}} \quad \frac{1}{|c(\lambda)|^2}}^{\text{then the multiplier}} \psi_{i\lambda}(a)\, d\lambda}_{\text{back using } \mathcal{F}_{\mathrm{A}}^{-1}} \,.$$

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► For sufficiently decreasing functions $h : \Xi \longrightarrow \mathbb{C}$ we define the G-orbital integral $\mathcal{O}_h : 2i\Omega \longrightarrow \mathbb{C}$ by

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and for all $Y \in \mathfrak{a}$:

$$\int |\hat{f}(b,\lambda)|^2\, \psi_{i\lambda}(\exp iY)\, d\sigma(b,\lambda) < \infty\,.$$

The following theorem is the replacement for what we used earlier:

$$\int |F(x+iy)|^2\,dx = \int |\mathcal{F}(F|_{\mathbb{R}^n})(\lambda)|^2\,e^{-2\lambda\cdot y}\,d\lambda\,.$$

It has several applications in harmonic analysis on G/K:

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Theorem 0.5 (Faraut)**.** *Let* $F \in \mathcal{G}(\Xi)$ *and* $Y \in \Omega$*. Set* $f = F|_{G/K} \in L^2(G/K)$*. Then*

$$\int_G |F(g\exp iY)|^2\, dg = \int |\hat{f}(b,\lambda)|^2 \varphi_{i\lambda}(\exp(2iY))\, d\sigma(b,\lambda)\,.$$

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- It follows that $\mathcal{O}_{|F|^2}$ is defined for all $F \in \mathcal{G}(\Xi)$ and defines a holomorphic function on $A\exp(2i\Omega)$ given by ($f = F|_{G/K}$):

$$\mathcal{O}_{|F|^2}(\exp Z) = \int |\hat{f}(b,\lambda)|^2 \varphi_{i\lambda}(\exp Z)\,d\sigma\,.$$

10. The Image of the Segal-Bargmann Transform

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► We have now set every thing up to state (and prove) what the image of the Segal-Bargmann transform in this case is. Define a ρ-shifted density function by

$$\omega_t(a \exp Y) := \underbrace{\frac{e^{t\rho^2}}{|W|}}_{\text{takes care of the } \rho-\text{shift}} \underbrace{\left((2\pi t)^{-n/2} e^{-|Y|^2/2t}\right)}_{\text{the density for } \mathfrak{a}} .$$

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▶ Define a "norm" on $\mathcal{G}(\Xi)$ by

$$\|F\|_t^2 = \int_{\mathfrak{a}} D\mathcal{O}_{|F|^2}(\exp iY)\omega_t(Y)\, dY$$

and set

$$\mathcal{F}_t(\Xi) = \{F \in \mathcal{G}(\Xi) \mid \|F\|_t < \infty\} .$$

Theorem 0.6 (KÓS)**.** *The Segal-Bargmann transform is an unitary isomorphism*

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$$\begin{aligned} \mathcal{F}_{G/K}(H_t f)(b,\lambda) &= \mathcal{F}_{G/K}(f * h_t)(b,\lambda) \\ &= \hat{f}(b,\lambda)\hat{h}_t(b,\lambda) \\ &= e^{-t(\lambda^2+\rho^2)}\hat{f}(b,\lambda) \end{aligned}$$

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And hence, with $F = H_t f$:

$$\int D\mathcal{O}_{|F|^2}(iY)\omega_t(Y)dY$$

$$= \iint |\hat{f}(b,\lambda)|^2 e^{-t(\lambda^2+\rho^2)}\psi_\lambda(2iY)\omega_t(Y)\,d\sigma dY$$

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What we need first for the K-invariant case is the following simple theorem.

–

Theorem0.6 *We have $G = KAK$ and the restriction map*

$$L^2(G/K)^K \ni f \mapsto f|_A \in L^2(A, |W|^{-1} d\mu)^W \simeq L^2(A^+, d\mu)$$

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- Next we consider the effect on the Heat equation. For that let $H_1, \ldots, H_n$ be a orthonormal basis of $\mathfrak{a}$ and $A^{\text{reg}} = \{a \in A \mid (\forall \alpha)\, a^\alpha \neq 1\}$.

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▶ Let $(\cdot, \cdot)$ be a W-invariant inner product on $\mathfrak{a}$ (and by duality on $\mathfrak{a}^*$). Chose $h_\alpha \in \mathfrak{a}$ be such that $(X, h_\alpha) = \alpha(X)$, $(\alpha, \beta) = (H_\alpha, H_\beta)$, and - for $\alpha \neq 0$ - $H_\alpha = \frac{2}{(\alpha,\alpha)} h_\alpha$.

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- Define a W-invariant differential operator L on A^{reg} by

$$L = \sum_{j=1}^{n} \partial(H_j)^2 + \sum_{\alpha \in \Delta^+} m_\alpha \frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}} \partial(h_\alpha)\,.$$

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$$(\Delta f)|_{A^{reg}} = L(f|_{A^{reg}})$$

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▶ Hence the heat equation for K-invariant functions on G/K corresponds to the Cauchy problem on A^{reg} (or A^+)

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$$\begin{aligned} Lu(a,t) &= \partial_t u(a,t) \\ u(a,t) &\overset{t\to 0^+}{\longrightarrow} f(a) \in L^2(A^+, d\mu) \end{aligned}$$

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▶ The important observation now is, that every thing in (*) as well as the Harish-Chandra c-function is independent of G/K, it only depends on

▶ the space $\mathfrak{a} \simeq \mathbb{R}^n$,
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▶ So from now on $m : \Delta \to [0, \infty)$ is a Weyl group invariant function, defined on a root system Δ in a finite dimensional Euclidean space $\mathfrak{a}$.

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- ▶ the set of roots Δ and
- ▶ multiplicity function $m : \alpha \to m_\alpha$!

▶ So from now on $m : \Delta \to [0, \infty)$ is a Weyl group invariant function, defined on a root system Δ in a finite dimensional Euclidean space $\mathfrak{a}$.

▶ The density function and the differential operator L is defined as before.

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$$\varphi_\lambda(a) = \sum_{w \in W} c(w\lambda)\Psi_{w\lambda}(a)$$

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▶ Note, that the properties of the spherical functions follows from the defining integral formula. It was therefore a non-trivial task to prove the following using only the expansion formula:

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- φ_λ extends to a holomorphic function on a tubular neighborhood of A in $A_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}}/\mathbb{Z}\{\pi i H_\alpha \mid \alpha \in \Delta\}$. What was not stated was how big this neighborhood is;

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- Growth estimates for $\varphi_\lambda(a \exp iX)$ for $X \in \Omega$ where

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- With those tools available, one defines the Hypergeometric Fourier transform by

$$\mathcal{F}f(\lambda) = \hat{f}(\lambda) = \int_A f(a)\varphi_{-i\lambda}(a) \, d\mu = |W| \int_{A^+} f(a)\varphi_{-i\lambda}(a) \, d\mu \, .$$

–

- Define $c : \mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}$ by the same formula as the Harish-Chandra c-function (product and quotients of Γ-functions) and set $d\nu(\lambda) = |c(i\lambda)|^{-1}\, d\lambda$.

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Theorem 0.8 (Heckmann-Opdam) *The Fourier transform extends to an unitary isomorphism*

$$L^2(A, d\mu)^W \simeq L^2(\mathfrak{a}^*, d\nu)^W \,.$$

Furthermore, if $f \in C_c^\infty(A)^W$ *then*

$$f(a) = |W|^{-1} \int_{\mathfrak{a}^*} \hat{f}(\lambda)\varphi_{i\lambda}(a)\, d\nu(\lambda)$$

and

$$\mathcal{F}(Lf)(\lambda) = -(|\lambda|^2 + |\rho|^2)\mathcal{F}(f)(\lambda)\,.$$

–

Let us put this together in a commutative diagram:

$$\begin{array}{ccc} L^2(A, d\mu)^W & \longrightarrow & L^2(A, da)^{\tau(W)} \\ \mathcal{F} \big\downarrow & & \big\downarrow \mathcal{F}_A \\ L^2(\mathfrak{a}^*, d\nu)^W & \xrightarrow[\Psi]{} & L^2(\mathfrak{a}^*, d\lambda)^{\tau(W)} \end{array}$$

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▶ Then

$$\Lambda(Lf)(a) = (\Delta_A - |\rho|^2)\Lambda(f)(a)$$

reducing the our problem to a shifted heat equation on $A \simeq \mathfrak{a}$:

$$(\Delta_A - |\rho|^2)u(a,t) = \partial_t u(x,t)$$

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Theorem 0.9 (Ó+S, 2005) 1) The solution of the heat equation is given by

$$u(a,t) = |W|^{-2} \int_{\mathfrak{a}^*} e^{-t(|\lambda|^2+|\rho|^2)} \hat{f}(\lambda) \varphi_{i\lambda}(a) \, d\nu(\lambda) \qquad f \in L^2(A)^W .$$

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Let $\mathcal{H}_t$ be the space of holomorphic function on $F : A \exp i\Omega \to \mathbb{C}$ such that $\Lambda(F)$ extends to a $\tau(W)$-invariant holomorphic function on $\mathfrak{a}_{\mathbb{C}}$ such that

$$\|F\|_t^2 = e^{2t|\rho|^2} \int_{\mathfrak{a}_{\mathbb{C}}} |\Lambda F(X+iY)|^2 \, d\mu_t(X+iY) < \infty \,.$$

Then $\mathcal{H}_t$ is a Hilbert space and

$$H_t : L^2(A)^W \to \mathcal{H}_t$$

is an unitary isomorphism. Here μ_t is the heat measure on the Euclidean space $\mathfrak{a}$.

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Theorem 0.10 (Hall+Mitchell) *Assume that G is complex. Let $f \in L^2(G/K)^K$, and let $u(x,t) = H_t f(x)$ be the solution to the heat equation. The map $X \mapsto \delta(\exp X)^{1/2} u(\exp X, t)$, $X \in \mathfrak{a}$, has a holomorphic extension to $\mathfrak{a}_{\mathbb{C}}$ such that*

$$\|f\|^2 = \int_{\mathfrak{a}_{\mathbb{C}}} |(\delta^{1/2} u)(X + iY, t)|^2 e^{2t|\rho|^2} \, d\mu_t(X + iY)$$

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Conversely, any meromorphic function $u(Z)$ which is invariant under W and which satisfies

$$\int_{\mathfrak{a}_\mathbb{C}} |(\delta^{1/2} u)(X + iY)|^2 e^{2t|\rho|^2} \, d\mu_t(X + iY) < \infty$$

is the Segal-Bargmann tranform $H_t f$ for some $f \in L^2(G/K)^K$.