Math 4032, Solution to homework, March 24

5.4-3 Let V be a normed linear space.

(a) If $T \in V'$, prove $|T(v)| \leq \|T\| \, \|v\|$ for all $v \in V$.

(b) If $T \in V'$, prove $T = 0$, if and only if $\|T\| = 0$.

Solution: Recall definition 5.4.2 that

$$\|T\| = \inf\{K \mid |T(v)| \leq K\|v\| \, \forall v \in V\} .$$

For simplicity we call the set on the right hand side A_T.
(a) Let $\epsilon > 0$ be given. Then there exists a K such that $\|T\| \leq K \leq \|T\| + \epsilon$ and $|T(v)| \leq K\|v\|$ for all $v \in V$. Hence

$$|T(v)| \leq (\|T\| + \epsilon)\|v\|$$

for all $\epsilon > 0$. Assume there exists a $v \in V$ such that $|T(v)| > \|T\|\|v\|$. As $T(0) = 0$ this implies that $v \neq 0$ and hence $\|v\| > 0$. Take

$$0 < \epsilon < \frac{|T(v)| - \|T\|\|v\|}{\|v\|} .$$

Then

$$(\|T\| + \epsilon)\|v\| < \|T\|\|v\| + |T(v)| - \|T\|\|v\| = |T(v)|$$

which is impossible.

(b) Assume that $T = 0$. Then $|T(v)| \leq 0\|v\|$ for all $v \in V$ and hence $0 \in A_T$. It follows that

$$0 \leq \|T\| = \inf A_T \leq 0 .$$

Thus $\|T\| = 0$.
Assume now that $\|T\| = 0$. Then, by part (a), it follows that

$$|T(v)| \leq \|T\| \, \|v\| = 0$$

for all $v \in V$. Hence $T(v) = 0$ for all $v \in V$ or $T = 0$.

5.4-5 If $y \in \ell_\infty$, $x \in \ell_1$ and

$$T_y(x) = \sum_{k=1}^{\infty} y_k x_k ,$$

prove $\|T\| = \|y\|_\infty$.

Solution: It has already been shown on p. 133 that $\|T\| \le \|y\|_\infty$. Define $e^j \in \ell_1$ by

$$e_k^j = \begin{cases} 1 & if \quad j = k \\ 0 & if \quad k \neq j \end{cases} .$$

Then $\|e^j\|_1 = 1$ and $T_y(e^j) = y_j$. Thus by part (a):

$$|T(e^j)| = |y_j| \le \|T\|\|e^j\| = \|T\| .$$

It follows that

$$\sup |y_j| = \|y\|_\infty \le \|T\| .$$

5.5-2 If $0 \le \alpha < 1$, prove that $\sum_{k=1}^\infty$ converges uniformly on $[0, \alpha]$.

Solution: If $x \in [0, \alpha]$, then $0 \le x \le \alpha$ and hence

$$0 \le M_k = \sup_{x \in [0,\alpha]} |x^k| = \alpha^k .$$

We know that $\sum_{k=1}^\infty \alpha^k = \sum_{k=1}^\infty M_k < \infty$ because $0 \le \alpha < 1$. Hence the claim follows from Theorem 5.5.2, Weierstrass M-test.