LEcUnur 10

The 2-D Haar Wavelet Transform.

1. Tensor product of functions

In 1-dimension we used step functions of the form

\[ p = \sum_{j=0}^{2^n - 1} s_j \chi_{\left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right)} \]

to approximate "arbitrary" functions. Thus we made the following steps.

1. Decide upon the resolution, and divide the interval \([0,1] \) into subintervals \( I_j = \left( \frac{j}{2^n}, \frac{j+1}{2^n} \right) \).

2. Take the orthogonal projection \( f \mapsto p = \sum_{j=0}^{2^n - 1} s_j \chi_{I_j} \)

3. Replace \( p \) by the vector \( \{ s_0, \ldots, s_{2^n - 1} \} \)

4. Apply the fast Haar wavelet transform to the sequence \( \{ s_0, \ldots, s_{2^n - 1} \} \).

In 2-D we have now two directions, \( x \) and \( y \). We will therefore have to divide the square

\[ I = [0,1) \times [0,1) = \{ (x,y) \in \mathbb{R}^2 : 0 \leq x < 1, 0 \leq y < 1 \} \]

into smaller squares

\[ I_{ij}^n = \{ (x,y) : \frac{i}{2^n} \leq x < \frac{i+1}{2^n}, \frac{j}{2^n} \leq y < \frac{j+1}{2^n} \} \]

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Then we approximate functions of two variables by functions that are constants on each of the squares $I_{i,j}$, i.e.,

$$
\Phi \to p = \sum_{i,j} s_{i,j} \chi_{I_{i,j}}
$$

We start by discussing special functions of two variables.

**Definition** Let $p, q : [0,1] \to \mathbb{R}$. Then $p \circ q$ is the function on $[0,1] \times [0,1]$ given by

$$
p \circ q(x,y) = p(x)q(y)
$$

**Example** Let $A, B \subseteq [0,1]$. Then $\chi_A \circ \chi_B = \chi_{A \times B}$

**Proof.** We have $\chi_A \circ \chi_B (x,y) = \chi_A(x) \chi_B(y)$. Thus

$$
\chi_A \circ \chi_B (x,y) = 1 \iff \text{and only if } x \in A \text{ and } y \in B. \text{ But that is the same as saying that } (x,y) \in A \times B.
$$

**Example** Let

$$
\Phi_{i,j} = \chi_{\left[ \frac{j}{2^i} \right. , \frac{j+1}{2^i} \left. \right)}
$$

and

$$
\Psi_{i,j} = \chi_{\left[ \frac{j}{2^i+1} , \frac{j+1}{2^i+1} \right)} - \chi_{\left[ \frac{j+2}{2^i+1} , \frac{j+3}{2^i+1} \right)}.
$$

Recall that $\Phi_{i,j}$ is used to express the average, and $\Psi_{i,j}$ is used to express the details. We have

$$
\Phi_{i,j} \circ \Phi_{n,j} = \Phi_{i,n}.
$$

$\Phi$ be used to express the "average values" on $I_{i,j}$ and

$$
\Phi_{i,j} \circ \Psi_{n,j} , \Psi_{i,j} \circ \Phi_{n,j} , \Psi_{i,j} \circ \Psi_{n,j}
$$

can be used to express details going from one $i$; column to the next, one row to the next, and going diagonally.
Let $f$ be a step function in $[0,1)^2$. Instead of associating to $f$ an array, we have to use a matrix. Thus if

$$
\varphi(x,y) = \sum_{i,j=1}^{2^n-1} s_{i,j} \chi_{I_{i,j}}^n
$$

Then $\varphi$ corresponds to the matrix

$$
\varphi \leftrightarrow \begin{bmatrix}
    s_{0,0} & s_{0,1} & s_{0,2} & \cdots & s_{0,2^n-1} \\
    s_{1,0} & s_{1,1} & s_{1,2} & \cdots & s_{1,2^n-1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    s_{2^n-1,0} & s_{2^n-1,1} & s_{2^n-1,2} & \cdots & s_{2^n-1,2^n-1}
\end{bmatrix}
$$

Notice, that the first index (the one corresponding to the $x$-variable) indicates the row and the second indices corresponds to the columns.

**Example** $2 \times 2$-matrix.

$$
\chi_{[0,\frac{1}{2})} \otimes \chi_{[0,1)} \leftrightarrow \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}
$$

To see this write

$$
\begin{array}{ccc}
\chi_{[0,\frac{1}{2})} & \chi_{[0,\frac{1}{2})} & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1
\end{array}
$$

**Example** Find the $2 \times 2$ matrix corresponding to the function

$$
\chi_{[0,\frac{1}{2})} \otimes (\chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}) + 2 \chi_{[\frac{1}{2},1)} \otimes \chi_{[0,1)}.
$$
THE 2-D HAAR WAVELET TRANSFORM

In two dimension we apply the Haar wavelet transform on the column and on the rows.

Example Suppose that the function f is represented by the 2x2 matrix

\[
\begin{pmatrix}
4 & 2 \\
0 & 6
\end{pmatrix}
\]

Then each row represents a one dimensional vector of length 2: [4, 2] and [0, 6]. Applying the 1-D wavelet transform to this vector gives [4, 2] \(\rightarrow\) [3, 1] and [0, 6] \(\rightarrow\) [3, -3]. Thus

\[
\begin{pmatrix}
4 & 2 \\
0 & 6
\end{pmatrix} \rightarrow \begin{pmatrix} 3 & 1 \\
3 & -3
\end{pmatrix}
\]

We now apply the 1-D transform to the rows

\[
\begin{pmatrix}
3 \\
-3
\end{pmatrix} \rightarrow \begin{pmatrix} 3 \\
0
\end{pmatrix}
\]

This results in the new matrix

\[
\begin{pmatrix}
3 & 1 \\
3 & -3
\end{pmatrix} \rightarrow \begin{pmatrix} 3 & 1 \\
0 & -3
\end{pmatrix}
\]
Notice that the elements represent the average value of all the values:

\[
\frac{4+2+0+6}{4} = \frac{12}{4} = 3.
\]

To understand the other matrix elements let us do this for a general matrix:

\[
\begin{bmatrix}
S_{00} & S_{01} \\
S_{10} & S_{11}
\end{bmatrix}
\xrightarrow{1^{st} \text{ step}}
\begin{bmatrix}
\frac{S_{00} + S_{01}}{2} & \frac{S_{00} - S_{01}}{2} \\
\frac{S_{10} + S_{11}}{2} & \frac{S_{10} - S_{11}}{2}
\end{bmatrix}
\]

We apply the 1-D transform to each column and get:

\[
\xrightarrow{2^{nd} \text{ step}}
\begin{bmatrix}
\frac{1}{2} \left[ \frac{S_{00} + S_{01}}{2} + \frac{S_{10} + S_{11}}{2} \right] & \frac{1}{2} \left[ \frac{S_{00} - S_{01}}{2} + \frac{S_{10} - S_{11}}{2} \right] \\
\frac{1}{2} \left[ \frac{S_{00} + S_{01}}{2} - \frac{S_{10} + S_{11}}{2} \right] & \frac{1}{2} \left[ \frac{S_{00} - S_{01}}{2} - \frac{S_{10} - S_{11}}{2} \right]
\end{bmatrix}
\]

1. The number \( \frac{1}{2} \left[ \frac{S_{00} + S_{01}}{2} + \frac{S_{10} + S_{11}}{2} \right] = \frac{S_{00} + S_{01} + S_{10} + S_{11}}{4} \) is the average value.

2. The number at place \((0,1)\) is

\[
\frac{1}{2} \left[ \frac{S_{00} - S_{01}}{2} + \frac{S_{10} - S_{11}}{2} \right]
\]

Moving from \((0,0)\) to \((1,1)\) or \((1,0)\) to \((0,1)\), the average change moving from the first column to the second column.
The number located at (1,0) is
\[
\frac{1}{2} \left[ \frac{S_{oo} + S_{oi} - S_{10} + S_{ii}}{2} \right] = \frac{1}{2} \left[ \frac{S_{oo} - S_{10} + S_{oi} - S_{ii}}{2} \right]
\]

represents the average change going from row 1 to row 2 (or the details for the average).

Finally, the last number at place (1,1) is
\[
\frac{1}{2} \left[ \frac{S_{oo} - S_{oi} - S_{10} - S_{ii}}{2} \right] = \frac{1}{2} \left[ \frac{S_{oo} + S_{ii} - S_{oi} + S_{10}}{2} \right]
\]

represents the average changes moving along the diagonals.

The method for bigger matrices is the same:

\[
\begin{pmatrix}
5 & 7 & -1 & -5 \\
1 & -1 & -3 & -3 \\
7 & 3 & 2 & 2 \\
3 & -1 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
6 & -3 & 1 & 0 \\
0 & -3 & 1 & 0 \\
5 & 2 & 2 & 0 \\
1 & 0 & 2 & 0
\end{pmatrix}
\]

\begin{pmatrix}
3 & -3 & 0 & 1 \\
3 & 1 & 2 & 0 \\
3 & 0 & -1 & 1 \\
2 & 1 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 3 & 1 \\
2 & 1 \\
1 & 2
\end{pmatrix}
\]

Now notice that the 2x2 matrix \( \begin{pmatrix} 3 & -3 \\ 3 & 1 \end{pmatrix} \) located in the upper left corner consist of average values. We apply the 2-D wavelet transform again to this matrix and get
\[
\begin{pmatrix} 3 & -3 \\ 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \end{pmatrix}.
\]

The final answer is then...
THE INVERSE 2-D WAVELET TRANSFORM

Given a 2x2-matrix or a 4x4-matrix knowing that this matrix is a result of a 2-D Haar wavelet transform, how can we reconstruct the original matrix?

Let us start with a 2x2-matrix. Recall that the 2-D Haar wavelet transform consist of 2-steps:

1st step: \( \begin{pmatrix} \rightarrow \end{pmatrix} \) 1-D Haar wavelet transform on the rows

2nd step: \( \begin{pmatrix} \downarrow \downarrow \end{pmatrix} \) 1-D Haar wavelet transform on the columns.

Therefore, to invert this we need to:

1st step: Apply the inverse 1-D transform to the columns.

2nd step: Apply the inverse 1-D transform to the rows.

Example: Given the matrix \( \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \). Find the initial matrix.

\[
\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \xrightarrow{1-D \text{ inverse transform on columns}} \begin{pmatrix} 2-1 & 1+0 \\ 2-(-1) & 1-0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \]

\[
\begin{pmatrix} 2-1 & 1+0 \\ 2-(-1) & 1-0 \end{pmatrix} \xrightarrow{1-D \text{ inverse transform on rows}} \begin{pmatrix} 1+1 & 1-1 \\ 3+1 & 3-1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}.
\]

Now look at 4x4-matrices. Recall, that the last step was to apply the 2-D transform on the upper left corner matrix. Here is an overview of the steps.
1st step
1 - D on the rows

2nd step
1 - D on the columns

3rd step
Take the 2x2 matrix in the left upper corner and apply steps 1 and 2 to this matrix keeping all other numbers the same.

For the inverse we need to invert each of those steps. Thus,

1st step
Pick the 2x2-matrix in the left upper corner. Apply the inverse 1-D transform on the columns.

2nd step
Apply the 1-D inverse transform on the rows in the 2x2-matrix from step 1.

3rd step
We have now a new 4x4-matrix. Apply the inverse 1-D transform to the columns.

4th step
Apply the inverse 1-D transform to the rows.

Example: Given the matrix \[
\begin{pmatrix}
2 & 1 & 0 & 3 \\
1 & 0 & 2 & -1 \\
3 & 2 & -1 & 2 \\
0 & -1 & 1 & 3 \\
\end{pmatrix}
\]
find the initial matrix.

Solution
We start by the 2x2-matrix \[
\begin{pmatrix}
2 & 1 \\
1 & 0 \\
\end{pmatrix}
\]
and apply the inverse transform to this matrix
\[
\begin{pmatrix}
2 & 1 \\
3 & 1 \\
\end{pmatrix}
\]
columns \[
\begin{pmatrix}
2 & 0 \\
1 & 2 \\
0 & -1 \\
\end{pmatrix}
\]
rows \[
\begin{pmatrix}
4 & 2 & 0 & 3 \\
2 & -1 & 2 & -1 \\
1 & -1 & 1 & 3 \\
\end{pmatrix}
\]
We have now the new matrix
and have to apply the inverse row/column transform to this matrix.
\[
\begin{pmatrix}
4 & 2 & 0 & 3 \\
2 & 0 & 2 & -1 \\
1 & 2 & -1 & 2 \\
0 & -1 & 1 & 3 \\
\end{pmatrix}
\xrightarrow{\text{inverse on rows}}
\begin{pmatrix}
4 & -6 & 9 & -1 \\
4 & 2 & 1 & -1 \\
5 & -1 & -3 & 5 \\
3 & 1 & -3 & 5 \\
\end{pmatrix}
\xrightarrow{\text{inverse on columns}}
\begin{pmatrix}
5 & 4 & -1 & 5 \\
3 & 0 & 1 & 1 \\
2 & -1 & 3 & 2 \\
2 & 1 & 1 & -4 \\
\end{pmatrix}
\]

So the final answer is
LECTURE II

COMPLEX NUMBERS AND COMPLEX VECTOR SPACES

Up to now we have only considered vector spaces over the field of real numbers. For the Fourier transform we need vector spaces where we can multiply vectors by complex numbers.

If $x \in \mathbb{R}$, then we know that $x^2 \geq 0$ and $x^2 = 0$ only if $x = 0$. Therefore there is no real number $x$ such that $x^2 = -1$.

So: There is no real solution to the equation $x^2 + 1 = 0$.

More generally let us look at the equation $x^2 + bx + c = 0$.

By completing the square we get

$$x = -\frac{b}{2} \pm \frac{1}{2} \sqrt{b^2 - 4c}.$$ (x)

There are now 3 possibilities:

1. $b^2 - 4c > 0$. Then equation (x) gives two solutions

   $$x = -\frac{b}{2} + \frac{1}{2} \sqrt{b^2 - 4c}$$

   and

   $$x = -\frac{b}{2} - \frac{1}{2} \sqrt{b^2 - 4c}$$

2. $b^2 - 4c = 0$. Then we have one solution

   $$x = -\frac{b}{2}$$

3. $b^2 - 4c < 0$. Then there is no real solution to the quadratic equation $x^2 + bx + c = 0$. 
We introduce now a new number $i = \sqrt{-1}$ such that

$$i^2 = -1$$

The complex numbers are all expressions of the form

$$z = x + yi, \ x, y \in \mathbb{R}.$$ 

The set of complex numbers is denoted by $\mathbb{C}$. We say that $x = \Re(z)$ is the real part of $z$ and $y = \Im(z)$ is the imaginary part of $z$. Recall that the set of real numbers can be thought of as a line, the real line. To picture the set of complex numbers we use the plane. A vector $\vec{v} = (x, y)$ corresponds to the complex number $z = x + iy$. 

\[\begin{array}{c}
\uparrow \\
(x, y) \leftrightarrow x + iy \\
\end{array}\]

The addition of two complex numbers $z = x + iy$ and $w = s + it$ then corresponds to the addition of the corresponding vectors. Thus

$$(x + iy) + (s + it) = [x + s] + i[y + t].$$

To find out what the product of $z$ and $w$ is, we just act as if all the rules that we are so familiar with still hold and use that $i^2 = -1$. Thus

$$\begin{align*}
(z + w) &= (x + iy)(s + it) \\
&= x \cdot s + x \cdot i \cdot t + i \cdot y \cdot s + (i \cdot y) \cdot t \\
&= x \cdot s + i^2 \cdot y \cdot t + i (x \cdot t + y \cdot s) \\
&= (xz - yt) + i(xt + ys)
\end{align*}$$
Before we use this to find the inverse or reciprocal of \( z = x + iy \), we need to introduce the complex conjugate:
\[
\overline{z} = x - iy.
\]

Thus, complex conjugation corresponds to a reflection around the x-axis.

Now multiply \( z \) by \( \overline{z} \):
\[
z \cdot \overline{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2.
\]

Thus \( \sqrt{z \cdot \overline{z}} = |z| \) is the length of the vector \((x, y)\).

It is therefore natural to call \( |z| = \sqrt{z \cdot \overline{z}} \) the length, or absolute value, of the complex number \( z \).

**Lemma** Let \( z = x + iy \) be a complex number with \( |z| \neq 0 \). Then
\[
\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}.
\]

**Proof.** We have \( z \cdot \frac{x - iy}{x^2 + y^2} = \frac{(x+iy)(x-iy)}{x^2 + y^2} = \frac{x^2 + y^2}{x^2 + y^2} = 1 \).

**Examples**

1. \((2 + 3i) + (5 - 2i) = 7 + i\)
2. \((2 - 3i)(1 + i) = (2 + 3) + (-3 + 2)i = 5 - i\).
3. \(\frac{1}{2 + i} = \frac{2 - i}{5} = \frac{2}{5} - \frac{1}{5}i\).
4. \(\frac{2 + 3i}{1 + 5i} = \frac{(2+3i)(1-5i)}{26} = \frac{(2+15) + (3-10)i}{26} = \frac{17 - 7i}{26}\).
The Complex Exponential Function

The idea behind the Fourier transform is to represent a function (or signal) in the frequency domain using the complex exponential function.

Definition: Let \( \{z_n\}_{n \in \mathbb{N}} \) be a sequence of complex numbers. We say that \( z_n \) converges to the complex number \( w \) if for all \( \varepsilon > 0 \) there exists a \( N \in \mathbb{N} \) such that
\[
|z_n - w| < \varepsilon
\]
for all \( n \geq N \). We write \( z_n \to w \) or \( \lim_{n \to \infty} z_n = w \), if \( z_n \) converges to \( w \).

Let \( z \in \mathbb{C} \) and define \( z_n = \sum_{k=0}^{n} \frac{z^k}{k!} \). Then it can be shown that the sequence \( \{z_n\} \) converges. We denote the limit by
\[
e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \lim_{N \to \infty} \sum_{k=0}^{N} \frac{z^k}{k!}.
\]

It can be shown, that
\[
e^{z+w} = e^z e^w.
\]

In particular, \( \frac{1}{e^z} = e^{-z} \).

Theorem: Let \( z = x + iy \in \mathbb{C} \). Then
\[
e^z = e^x (\cos(y) + i \sin(y))
\]
(The Euler formula).
This was proved in class.
Examples
(1) \( e^{\pi i} = \cos(\pi) + i \sin(\pi) = -1 \)
(2) \( e^{\pi i/2} = \cos(\pi/2) + i \sin(\pi/2) = i \)
(3) \( e^{\pi i/4} = e^{\pi/4} (\cos(\pi/4) + i \sin(\pi/4)) \)
  \[= \frac{e^{\pi/4}}{\sqrt{2}} (1 + i)\]

Lemma
Let \( z = x + iy \). Then
\[ \overline{e^z} = e^{\overline{z}} \]

Proof.
\[ e^{\overline{z}} = e^x (\cos y + i \sin y) \]
\[ = e^x (\cos y - i \sin y) \]
\[ = e^x e^{-iy} = e^{\overline{z}} \]

COMPLEX VALUED FUNCTIONS
Let \( f : I \to \mathbb{C}, I \subseteq \mathbb{R} \) an interval, be a function. Then we can write
\[ F(t) = f(t) + ig(t) \]
where \( f, g : I \to \mathbb{R} \). The function \( F \) is continuous if and only if \( f \) and \( g \) are both continuous. In that case we have
\[ \lim_{t \to t_0} F(t) = \lim_{t \to t_0} f(t) + i \lim_{t \to t_0} g(t) \]
\[ = f(t_0) + ig(t_0) \]
\[ = F(t_0) \]
We integrate and differentiate $F$ by integrating $f(t)$ and $g(t)$ (differentiating $f(t)$ and $g(t)$). Thus

$$\int_0^b F(t)\,dt = \int_0^b f(t)\,dt + \left( \int_0^b g(t)\,dt \right) i$$

$$\frac{dF}{dt}(t) = \frac{df}{dt}(t) + \frac{dg}{dt}(t) i.$$

**Example 1**

1) \(\int_0^1 (2t + 3t^2) i\,dt = \int_0^1 t^2\,dt + \int_0^1 t^3\,dt\)

\[= 1 + i.\]

2) \(\int_0^{2\pi} e^{(1+i)t}\,dt = \int_0^{2\pi} e^t \cos t + e^t \sin t\,dt\)

We have

\[\int_0^{2\pi} e^t \cos t\,dt = \cos e^t \int_0^{2\pi} e^t\,dt + \sin e^t \int_0^{2\pi} e^t\,dt\]

\[= e^{2\pi} - 1 + \sin e^t \int_0^{2\pi} e^t\,dt - \cos e^t \int_0^{2\pi} e^t\,dt\]

\[= e^{2\pi} - 1 - \int_0^{2\pi} e^t \cos t\,dt\]

Thus \(\int_0^{2\pi} e^t \cos t\,dt = (e^{2\pi} - 1)/2\)

Similarly

\[\int_0^{2\pi} e^t \sin t\,dt = \sin e^t \int_0^{2\pi} e^t\,dt - \cos e^t \int_0^{2\pi} e^t\,dt\]

\[= -\frac{1}{2}(e^{2\pi} - 1)\]

Thus \(\int_0^{2\pi} e^{(1+i)t}\,dt = \frac{1}{2}(e^{2\pi} - 1)[1 - i]\).

But we can also have used the rule

\[\int a^t\,dt = \frac{1}{a} e^{at} + C\]

where $a$ is any complex number, $a \neq 0$. Then:
\[ \int_0^{2\pi} e^{(1+i)t} \, dt = \frac{1}{1+i} e^{(1+i)t} \bigg|_0^{2\pi} \]
\[ = \frac{1}{1+i} [e^{2\pi+i2\pi} - e^0] \]
\[ = \frac{1}{1+i} [e^{2\pi} - 1] \]

Further more
\[ \frac{1}{1+i} = \frac{1}{2}(1-i) \]

Thus
\[ \int_0^{2\pi} e^{(1+i)t} \, dt = \frac{e^{2\pi} - 1}{2} (1-i) \quad \Box \]

**Example**
Evaluate the integral \( \int_0^1 (t + 2it^2)(t^2 - 3it) \, dt \).

**Solution:** First we have to carry out the multiplication
\[
(t + 2it^2)(t^2 - 3it) = t^3 + 6t^2 - 2it^4 - 3it^2
\]
\[ = 7t^3 - (2t^4 + 3t^2)i \]

Thus
\[ \int_0^1 (t + 2it^2)(t^2 - 3it) \, dt = \int_0^1 7t^3 \, dt - (\int_0^1 2t^4 + 3t^2 \, dt) i \]
\[ = \frac{7}{4} t^4 \bigg|_0^1 - \left[ \frac{2}{5} t^5 + t^3 \right]_0^1 i \]
\[ = \frac{7}{4} - \frac{7}{5} i \]
COMPLEX VECTOR SPACES

The axioms for complex vector spaces are the same as those for real vector spaces except the scalars are now complex numbers.

Examples  Let

\[ \mathbb{C}^n = \{ (z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbb{C} \} \]

The addition is given by

\[ u = (z_1, \ldots, z_n), \quad v = (w_1, \ldots, w_n) \]

\[ u + v = (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n) \]

The scalar multiplication is

\[ \lambda u = (\lambda z_1, \ldots, \lambda z_n). \]

Example  Let \( I \subseteq \mathbb{R} \) be an interval. Let \( V \) be the space of functions \( f : I \to \mathbb{C} \). The addition and scalar multiplication is given by

\[ (f + g)(t) = f(t) + g(t) \]

\[ [\lambda f](t) = \lambda f(t). \]

The big change is in the definition of inner product. Let \( V \) be a complex vector space. A map \( \langle \cdot, \cdot \rangle \) from \( V \times V \) into the field of complex numbers is called an inner product if

1. \( (u, u) \geq 0 \) for all \( u \in V \)
2. \( (u, u) = 0 \) if and only if \( u = 0 \)
3. For fixed \( v \in V \), the map \( u \mapsto \langle u, v \rangle \) is linear, i.e.
   \[ \langle \lambda u + \mu w, v \rangle = \lambda \langle u, v \rangle + \mu \langle w, v \rangle \]
   for all \( \lambda, \mu \in \mathbb{C} \) and all \( u, w \in V \).
4. For all \( u, v \in V \) we have \( \langle u, v \rangle = \overline{\langle v, u \rangle} \)}. 
Lemma. Let $(\cdot, \cdot)$ be an inner product on the complex vector space $V$. Then

$$(v_1, \lambda u + \mu w) = \overline{\lambda} \, (v_1, u) + \overline{\mu} \, (v_1, w)$$

for all $\lambda, \mu \in \mathbb{C}$ and all $v_1, u, w \in V$.

Proof. We have

$$(v_1, \lambda u + \mu w) = \frac{(\lambda u + \mu w, v)}{(\lambda u, v) + \mu (w, v)} \quad \text{(by rule 4)}$$

$$= \frac{\lambda (u, v) + \mu (w, v)}{\lambda (u, v) + \mu (w, v)} \quad \text{(by (3))}$$

$$= \overline{\lambda} \, (v_1, u) + \overline{\mu} \, (v_1, w) \quad \text{(by (4))}. \quad \Box$$

Examples

1. Let $V = \mathbb{C}^n$. Define for $u = (z_1, \ldots, z_n)$ and $v = (w_1, \ldots, w_n)$

$$(u, v) = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}.$$ 

Then

$$(u, u) = z_1 \overline{z_1} + \cdots + z_n \overline{z_n}$$

$$= |z_1|^2 + \cdots + |z_n|^2 > 0$$

If $(u, u) = 0$, then we must have $|z_1| = |z_2| = \cdots = |z_n| = 0$,

so $u = 0$. Let $w = (t_1, \ldots, t_n) \in V$ and $\lambda \in \mathbb{C}$. Then

$$(u + w, \lambda u + \mu w) = (z_1 + t_1) \overline{w_1} + \cdots + (z_n + t_n) \overline{w_n}$$

$$= (z_1 \overline{w_1} + \cdots + z_n \overline{w_n}) + (t_1 \overline{w_1} + \cdots + t_n \overline{w_n})$$

$$= (u, v) + (w, v)$$

Similarly

$$(\lambda u, v) = (\lambda z_1) \overline{w_1} + \cdots + (\lambda z_n) \overline{w_n}$$

$$= \lambda (z_1 \overline{w_1} + \cdots + z_n \overline{w_n})$$

$$= \lambda (u, v)$$
Finally
\[
(u, v) = \sum_{i=1}^{n} w_i^* w_i + \sum_{i=1}^{n} \overline{w}_i w_i
\]
\[
= \frac{\sum_{i=1}^{n} w_i^* w_i + \sum_{i=1}^{n} \overline{w}_i w_i}{(v, u)}
\]
\[= (v, u) \quad \Box
\]

Example Let \( V \) be the space of piecewise continuous functions \( f: [0, 1] \to \mathbb{C} \). Define
\[
(f, g) = \int_0^1 f(t) \overline{g(t)} \, dt.
\]
\[\Box
\]

Definition Let \( V \) be a complex vector space with inner product \((\cdot, \cdot)\). Then the norm (or length) of a vector \( u \in V \) is defined by
\[
\|u\| := \sqrt{(u, u)}
\]

Remark Notice that \( \|u\| = 0 \) if and only if \( u = 0 \) and that \( \|\lambda u\| = |\lambda| \|u\| \).

Example
(1) \( V = \mathbb{C}^2 \) and \( u = (1, i) \). Then
\[
\|u\|^2 = 1 + i \cdot i = 1 + i(-i) = 1 + 1 = 2.
\]
Notice, that without the complex conjugation we would get \( 1 + i^2 = 1 - 1 = 0 \), which is not possible.

(2) \( V = \mathbb{C}^2 \), \( u = (1 + i, 2 + 3i) \):
\[
\|u\|^2 = (1 + i)(1 - i) + (2 + 3i)(2 - 3i)
\]
\[
= 1 + 1 + 4 + 9 = 15
\]
or
\[
\|u\| = \sqrt{15}.
\]
(3) Let $V$ be the space of piecewise continuous functions on $[0,1]$. Let $a \in \mathbb{R}$ and
\[ f(t) = e^{ait} = \cos(at) + i \sin(at). \]
Then
\[ |f(t)|^2 = (\cos(at) + i \sin(at))(\cos(at) - i \sin(at)) = (\cos(at))^2 + (\sin(at))^2 = 1. \]

Hence
\[ \|f\| = \sqrt{\int_0^1 |f(t)|^2 dt} = 1. \]

(4) Let $V$ be as in the last example and
\[ f(t) = t + it^2 \]

Then
\[ |f(t)|^2 = (t + it^2)(t - it^2) = t^2 + t^4. \]

Hence
\[ \|f\| = \sqrt{\int_0^1 t^2 + t^4 dt} = \sqrt{\frac{1}{3} + \frac{1}{5}} = \sqrt{\frac{8}{15}}. \]

(5) Let $V$ be as in the last two examples and
\[ f(t) = t^2 + 1 + 2i(t - 3). \]

Then
\[ |f(t)|^2 = (1 + t^2)^2 + 4(t - 3)^2 = 1 + 2t^2 + t^4 + 4t^2 - 24t + 36 = t^4 + 6t^2 - 24t + 37. \]

Hence
\[ \|f\|^2 = \int_0^1 t^4 + 6t^2 - 24t + 37 dt = \frac{1}{5} + 2 - 12 + 37 = 27 + \frac{1}{5} = \frac{136}{5}. \]
Exercises, Complex numbers, complex vector spaces, and complex valued functions

1) Evaluate the following complex numbers:

1. \((2 + 3i) + (4 - 2i) = 6 + i\)
2. \((3 - 2i) \cdot (2 + 5i) = 16 + 11i\)
3. \((2 + 3i) \cdot (2 - 3i) = \frac{4-9}{5} = 13\)
4. \(\frac{4-i}{3+2i} = \frac{10-11i}{3}\)
5. \(\frac{3+2i}{3-i} = \frac{7+9i}{10}\)
6. \(e^{\pi i} = -1\)
7. \(e^{2\pi i} = 1\)

2) Evaluate the following integrals:

1. \(\int_0^1 3t + 2it^2 \, dt = \frac{3}{2} + \frac{2}{3}i\)
2. \(\int_0^{1/2} e^{2\pi it} \, dt = i\sqrt{\pi}\)
3. \(\int_0^1 (2t + it^2) \cdot (2 + 3it) \, dt = \frac{5}{7} + \frac{8}{3}i\)

3) Evaluate the following inner products:

1. \(((1 + i, 2 - i), (1 - i, 3)) = (1+i)(1+i) + (2-i) \cdot 3 = 6 - i\)
2. \(((i, 1 + i, 3), (i, 2 - i, 2)) = i \cdot (-i) + (1+i)(2+i) + 6 = 8 + 3i\)
3. \(((2, 3 + i, i), (2 - i, 2, 1 + i)) = 2 \cdot (2+i) + (3+i) \cdot 2 + i \cdot (1 - i)\)
4. \(((2 + i, 1 + 2i, 1), (2 - i, 1 - 2i, 1)) = 11 + 5i\)

4) Find the norm of the following vectors:

1. \(\|(1,i)\| = \sqrt{1 + 1} = \sqrt{2}\)
2. \(\|(1+i, 1-i)\| = \sqrt{2 + 2} = 2\)
3. \(\|(2 - i, 2, 3 + i)\| = \sqrt{5 + 4 + 10} = \sqrt{19}\)
1) Multiply the following complex numbers
   a) \((2 + 3i) \cdot (1 + i) = 2 \cdot 1 + 2 \cdot i + 3i \cdot 1 + 3i \cdot i = 2 - 3 + (2 + 3)i = -1 + 5i\)
   b) \(\frac{2 - i}{3 + i} \cdot \frac{3 - i}{3 - i} = \frac{(6 - 1) - 3i - 2i}{10} = \frac{5 - 5i}{10} = \frac{1 - i}{2}\)

2) Find the length of the following complex numbers:
   (a) \(z = 2 + 3i\) : \(|z| = \sqrt{4 + 9} = \sqrt{13}\)
   (b) \(z = 1 - 3i\) : \(|z| = \sqrt{1 + 9} = \sqrt{10}\)
   (c) \(z = \frac{1}{2 + i}\) : \(|z| = \frac{1}{\sqrt{4 + 1}} = \frac{1}{\sqrt{5}}\)
1) Find the length of the vector \((1 + i, 1 + 2i, i)\).

**Solution:** Recall that the length (or norm) of the vector \((z_1, ..., z_n) \in \mathbb{C}^n\) is given by

\[
\|\(z_1, ..., z_n\)\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.
\]

Here the length \(l\) of a complex number \(z = x + iy\) is

\[
l(\sqrt{x^2 + y^2}.
\]

Thus \(\|(1 + i, 1 + 2i, i)\| = \sqrt{1^2 + 1^2 + 1^2 + 1^2 + 1^2} = \sqrt{2 + 5 + 1} = \sqrt{8} = 2\sqrt{2}\)

2) Evaluate the integral \(\int_0^1 (2t + i) \cdot (3t + 2it^2) dt\).

**Solution:** The first step is to evaluate the function that we are integrating

\[
(2t + i)(3t + 2it^2) = 6t^2 + 4it^3 + 3it^2 - 2t^2
\]

\[
= 4t^2 + (4t^3 + 3t^2)i
\]

Thus \(\int_0^1 (2t + i) \cdot (3t + 2it^2) dt = 4\int_0^1 t^2 dt + (4\int_0^1 t^3 + 3t^2 dt)i
\]

\[
= \frac{4}{3} + \left(1 + \frac{3}{2}\right)i
\]

\[
= \frac{4}{3} + \frac{5}{2} i
\]
3) Find the norm of the functions \( t \cos \pi t \) and \( f(t) = t + it^2 \).

**Solution:** Let me first recall the following: On the vector space of continuous functions on \([0,1]\), the inner product is
\[
(f,g) = \int_0^1 f(t)g(t)dt
\]
and the norm is
\[
\|f\| = \sqrt{\int_0^1 |f(t)|^2dt}.
\]

We have \( |e^{\pi it}| = |\cos(\pi t) + i\sin(\pi t)| = \sqrt{\cos^2(\pi t) + \sin^2(\pi t)} = 1 \)

Therefore
\[
\|e^{\pi it}\| = \sqrt{\int_0^1 |dt|} = 1
\]

We have \( |t + it^2|^2 = t^2 + t^4 \). Hence
\[
\|t + it^2\| = \sqrt{\int_0^1 t^2 + t^4 dt} = \sqrt{\frac{1}{3} + \frac{1}{5}} = \sqrt{\frac{8}{15}}
\]