§1 VECTOR SPACES OVER $\mathbb{R}$

<u>Definition 1.1</u> A vector space over $\mathbb{R}$ is a set with operations of addition $+$ and scalar multiplication $\cdot$ satisfying the following properties:

Axioms for addition

A1 (Closure for addition) For all $u, v \in V$, $u+v$ is defined and $u+v \in V$

A2 (commutativity for addition) $u+v = v+u$ for all $u, v \in V$.

A3 (Associativity for addition) $u+(v+w) = (u+v)+w$ for all $u, v, w \in V$.

A4 (Existence of additive identity) There exists an element $\vec{0} \in V$, such that $u + \vec{0} = u$ for all $u \in V$.

A5 (Existence of additive inverse) For each $u \in V$, there exists an element — denoted by $-u$ — such that $u + (-u) = \vec{0}$

Axioms for scalar multip.

M1 (Closure for scalar multiplication) For each number $r \in \mathbb{R}$ and each $u \in V$, $r \cdot u$ is defined and $r \cdot u \in V$.

M2 (Multiplication by 1) $1 \cdot u = u$ for all $u \in V$

M3 (Associativity for scalar multiplication) $r \cdot (s \cdot u) = (r \cdot s) \cdot u$ for all $r, s \in \mathbb{R}$ and all $u \in V$

Distributive axioms

D1 (First distribution property) $r \cdot (u+v) = r \cdot u + r \cdot v$ for all $r \in \mathbb{R}$ and all $u, v \in V$.

D2 (second distribution property) $(r+s) \cdot u = (r \cdot u) + (s \cdot u)$ for all $r, s \in \mathbb{R}$ and all $u \in V$.

Remark

a) The zero element $\vec{0}$ is unique, i.e. if $0_1, 0_2 \in V$ are such that

$$u + 0_1 = u + 0_2 = u \qquad \forall u \in V$$

then $0_1 = 0_2$.

Proof: We have $0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$. ∎

Lemma Let $u \in V$, then $0 \cdot u = \vec{0}$.

Proof
$$\begin{aligned} u + 0\cdot u &= 1\cdot u + 0\cdot u \\ &= (1+0)\cdot u \\ &= 1\cdot u \\ &= u \end{aligned}$$

Thus
$$\begin{aligned} u + (-u) = \vec{0} &= (0\cdot u + u) + (-u) \\ &= 0\cdot u + (u + (-u)) \\ &= 0\cdot u + \vec{0} \\ &= 0\cdot u \end{aligned}$$ ∎

Lemma The element $-u$ is unique.

b) $-u = (-1)\cdot u$

Proof:
$$\begin{aligned} u + (-1)\cdot u &= 1\cdot u + (-1)\cdot u \\ &= (1 + (-1))\cdot u \\ &= 0\cdot u \\ &= \vec{0} \end{aligned}$$ ∎

Before we discuss the axioms in more details, let us discuss two examples

Ex 1 Let $V = \mathbb{R}^n$, considered as columnvectors

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$$

We define for

$$u = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad V = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$$

and $r \in \mathbb{R}$:

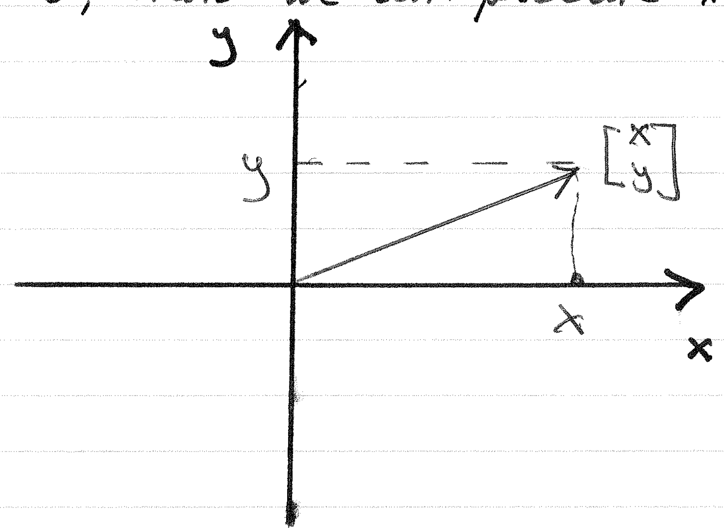
$$u + V = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + x_n \end{bmatrix}$$ ← add each of the coordinates

$$ru = \begin{bmatrix} rx_1 \\ rx_2 \\ \vdots \\ rx_n \end{bmatrix}$$ ← multiply each of the cordinates by r.

We have $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ (0 each coordinate)

$$-u = \begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix}.$$

If $n = 2$, then we can picture $\mathbb{R}^2$ as the plane



We can also consider $\mathbb{R}^n$ as the space of all row vectors

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$$

The addition and scalar multiplication is again given coordinate wise

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

$$r \cdot (x_1, \dots, x_n) = (r x_1, \dots, r x_n).$$

<u>Example</u> If $\vec{x} = (2, 1, 3)$, $\vec{y} = (-1, 2, -2)$, and $r = -4$ find $\vec{x} + \vec{y}$ and $r \cdot \vec{x}$.

<u>Solution</u>

- $$\begin{aligned} \vec{x} + \vec{y} &= (2, 1, 3) + (-1, 2, -2) \\ &= (2-1, 1+2, 3-2) \\ &= (1, 3, 1). \end{aligned}$$

- $r \cdot \vec{x} = -4 \cdot (2, 1, 3) = (-8, -4, -12)$.

<u>Example</u>

$$\begin{aligned} (x_1, \dots, x_n) + (0, \dots, 0) &= (x_1 + 0, x_2 + 0, \dots, x_n + 0) \\ &= (x_1, x_2, \dots, x_n) \end{aligned}$$

So the additive identity is $\vec{0} = (0, 0, \dots, 0)$. Notice that

$$\begin{aligned} 0 \cdot (x_1, \dots, x_n) &= (0 \cdot x_1, \dots, 0 \cdot x_n) \\ &= (0, \dots, 0) \end{aligned}$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$.

__Example__: Let A be the interval $[0,1)$ and let V be the space of functions $f: A \to \mathbb{R}$, i.e.

$$V = \{f: [0,1) \to \mathbb{R}\}$$

Define addition and scalar multiplication by

$$(f+g)(x) = f(x) + g(x)$$

$$(r \cdot f)(x) = rf(x).$$

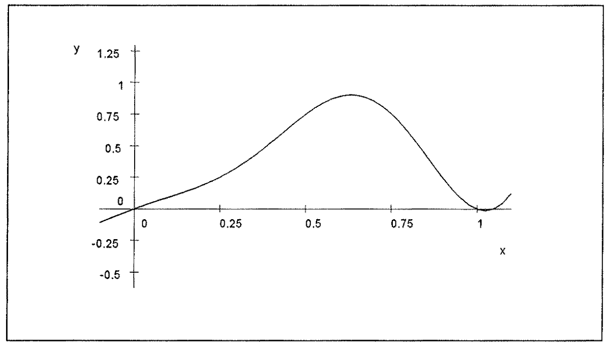
__Example__: The function $f(x) = x^4$ is an element of V, and so are

$$g(x) = x + 2x^2$$
$$h(x) = \cos(x)$$
$$k(x) = e^x$$

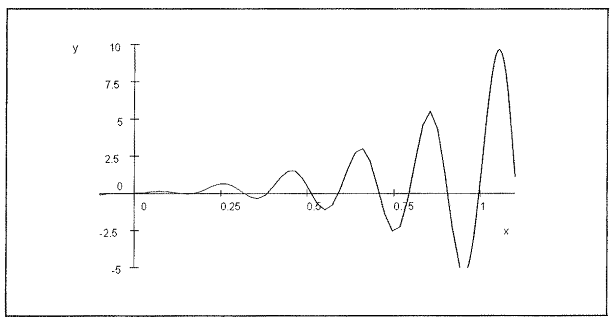
We have $(f+g)(x) = x + 2x^2 + x^4$.

We can usa a graph of a function to visuclize the function. The graph of f is the subset of $\mathbb{R}^2$ given by

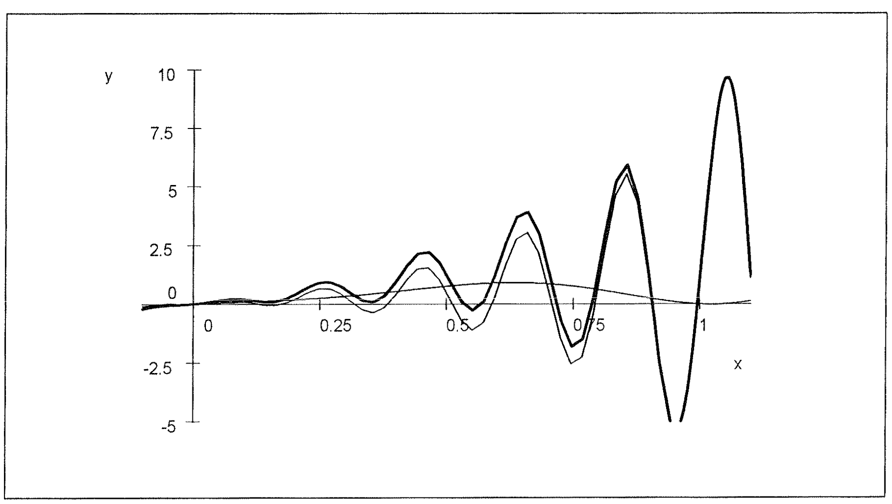
$$\{(x, f(x)) \mid x \in A\}.$$



$$f(x) = -x^2 \cos(2\pi x) + x$$



$$g(x) = xe^{2x} \sin(10\pi x) + x$$



$$(f+g)(x) = -x^2 \cos(2\pi x) + 2x + xe^{2x} \sin(10\pi x)$$

Notice that the zero element is the function $\vec{o}$ whic associates to each x the number 0:

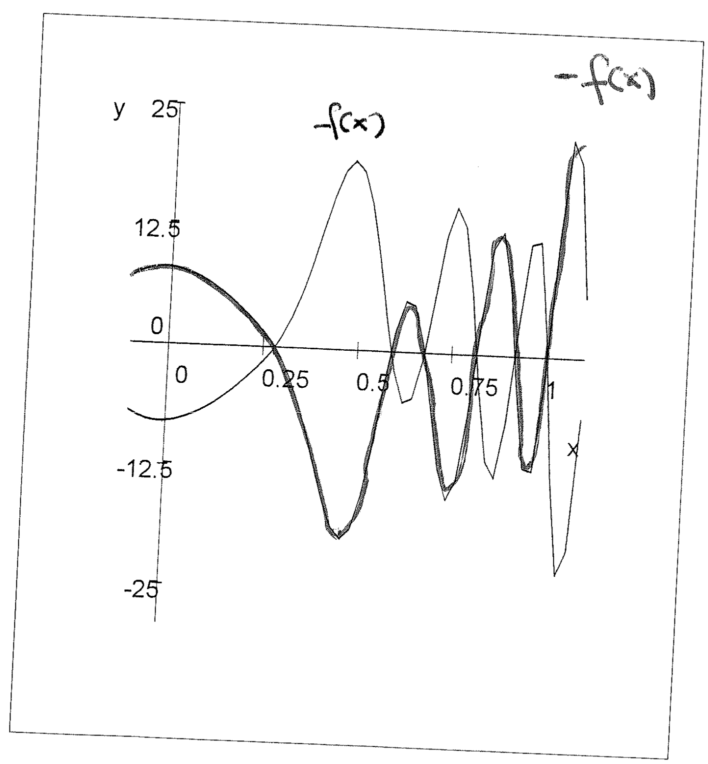
$$\vec{o}(x) = 0 \quad \text{for all } x \in [0,1)$$

Proof: $(f+o)(x) = f(x) + 0 = f(x)$.

The additive inverse is the function $-f : x \to -f(x)$.

<u>Proof</u>: $(f + (-f))(x) = f(x) - f(x) = o$ for all x.

Notice that this corresponds to reflecting the graph in the x-axis.



EXAMPLE Instead of $A = [0,1)$ we can take any set $A \neq \emptyset$, and we can replace $\mathbb{R}$ by any vector space V. We set

$$V^A = \{f : A \to V\}$$

and set

addition in V

$$(f+g)(x) = f(x) + g(x)$$

$$[r f](x) = r \cdot [f(x)]$$

multiplication in V

Proof that "+" is associative:

Let $f, g, h \in V^A$. Then

$$[(f+g)+h](x) = (f+g)(x) + h(x) \quad [\text{def of } +]$$

$$= (f(x)+g(x)) + h(x) \quad \text{—"—}$$

$$= f(x) + (g(x)+h(x)) \quad [\text{associtivity in } V]$$

$$= f(x) + (g+h)(x) \quad [\text{def of } +]$$

$$= [f + (g+h)](x) \quad \text{—"—}$$

The zero element is the function $x \mapsto \vec{0}$.

$$\begin{aligned}[f+0](x) &= f(x) + 0(x) \\ &= f(x) + \vec{0} \\ &= f(x) \qquad \blacksquare\end{aligned}$$

Exercises

1) Let $V = \mathbb{R}^4$. Evaluate the following:

a) $(2, -1, 3, 1) + (3, -1, 1, -1) =$

b) $(2, 1, 5, -1) - (3, 1, 2, -2) =$

c) $10 \cdot (2, 0, -1, 1)$

d) $(1, -2, 3, 1) + 10 \cdot (1, -1, 0, 1) - 3 \cdot (0, 2, 1, -2)$

e) $x_1 \cdot (1, 0, 0, 0) + x_2 \cdot (0, 1, 0, 0) + x_3 \cdot (0, 0, 1, 0) + x_4 (0, 0, 0, 1)$.