§3 VECTOR SPACES OF FUNCTIONS

Let $I \subseteq \mathbb{R}$ be an intervall. Then I is of the form (for some $a<b$)

$$I = \begin{cases} \{x \in \mathbb{R} : a < x < b\}, & \text{an open interval} \\ \{x \in \mathbb{R} : a \le x \le b\}, & \text{a closed interval} \\ \{x \in \mathbb{R} : a \le x < b\} \\ \{x \in \mathbb{R} : a < x \le b\} \end{cases}$$

Recall, that the space of <u>all</u> functions $f : I \to \mathbb{R}$ is a vectorspace. We will now list some important subspaces:

<u>Example 3.1</u> Let $C(I)$ be the space of continuous function. If f and g are continuous, so are the functions

$$f+g \text{ and } rf \quad (r \in \mathbb{R}).$$

Hence $C(I)$ is a vectorspace.

Recall, that a function is continuous, if the graph has no gaps. This can be formulated in several different ways.

a) Let $x_0 \in I$ and let $\varepsilon > 0$ be any given positive number. Then there exist a positive number $\delta > 0$, so that for all $x \in I \cap (x_0 - \delta, x_0 + \delta)$ we have

$$|f(x) - f(x_0)| < \varepsilon$$

This tells us, that the value of f at nearby points are (arbitrary) close to the value of f at x_0, i.e. there can be no jumps in the values.

b) A reformulation of (a) is

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Thus if x gets closer and closer to x_0, then $f(x)$ gets closer and closer to $f(x_0)$.

assume that I is open

Example 3.2 [The space $C^1(\mathbb{R})$. Recall, that f is differentiable at the point x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} =: f'(x_0)$$

exists. If $f'(x_0)$ exists for all $x_0 \in I$, then we say that f is differentiable in I. If f is differentiable on I then we get a new function on I

$$x \to f'(x).$$

We say that f is continuously differentiable on I if f' exists and is continuous on I. Recall that if f and g are differentiable, then so are

$$f + g \quad \text{and} \quad rf \quad (r \in \mathbb{R})$$

and

$$(f+g)' = f' + g'; \quad (rf)' = rf'$$

As $f' + g'$ and rf' are continuous by Example 3.1 it follows that $C^1(I)$ is a vectorspace.

Here is a function f that is continuous, but not differentiable on $I = \mathbb{R}$. Let $f(x) = |x|$, then for $h > 0$, and $x_0 = 0$ we have

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{|h| - 0}{h} = \frac{h}{h} = 1$$

hence $\lim\limits_{0<h\to 0} \frac{f(0+h)-f(0)}{h} = 1.$

But if $h < 0$, then

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|-0}{h} = \frac{-h}{h} = -1$$

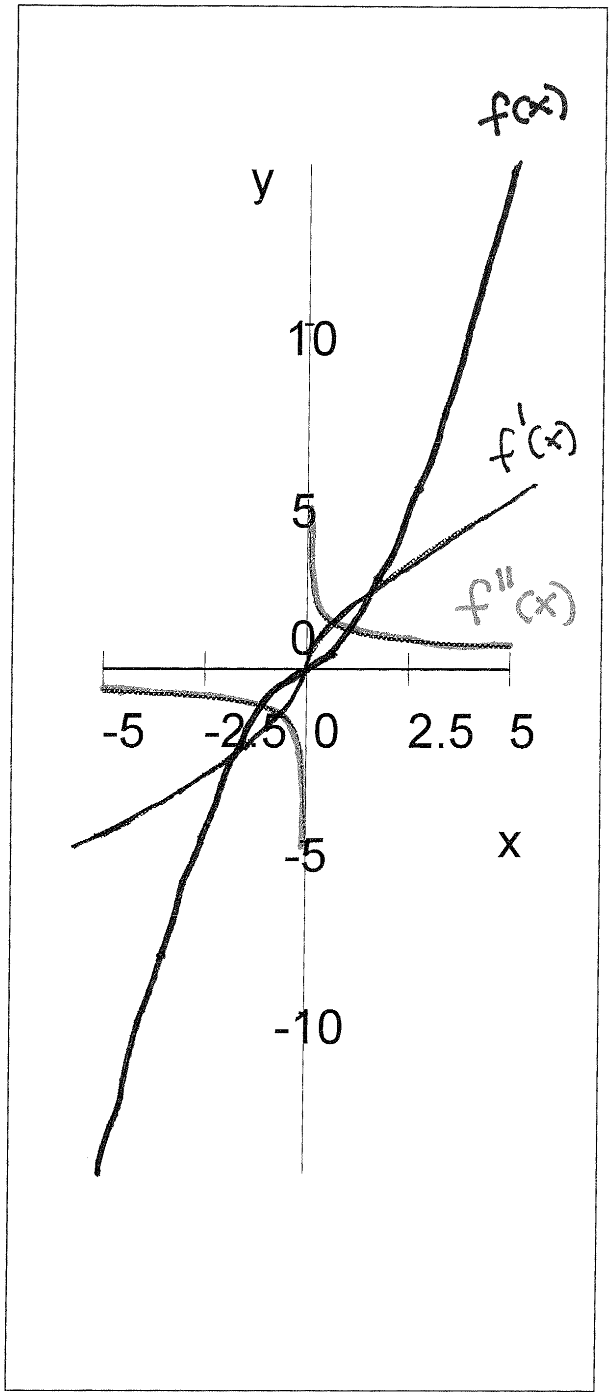
Hence

$$\lim_{0>h\to 0} \frac{f(0+h)-f(0)}{h} = -1.$$

What we have just shown, is that

$$\lim_{h\downarrow 0} \frac{f(h)-f(0)}{h} = 1 \neq -1 = \lim_{h\uparrow 0} \frac{f(h)-f(0)}{h}.$$

Hence, the limit does not exists.



f(x)
y
10
f'(x)
5
f''(x)
0
-5
-2.5
0
2.5
5
x
-5
-10

Example 3.3 The space $C^r(I)$.

Let again $I=(a,b)$ be an open intervall. Let $r \in \mathbb{N}=\{1,2,3,\dots\}$

Definition The function $f: I \to \mathbb{R}$ is said to be r-times continuously differentiable if all the derivatives $f', f'', \dots, f^{(r)}$ exists and $f^{(r)}: I \to \mathbb{R}$ is continuous.

We denote by $C^r(I)$ the space of r-times continuously differentiable functions on I. It is a subspace of $C(I)$

We have

$$C^r(I) \subsetneq C^{r-1}(I) \subsetneq \cdots \subsetneq C^1(I) \subsetneq C(I).$$

We have seen, that $C^1(I) \neq C(I)$. Let us try to find a function that is in $C^1(I)$, but not in $C^2(I)$ if $0 \in I$
For that let $f(x) = x^{5/3}$. The f is differentiable and

$$f'(x) = \frac{5}{3} x^{2/3}$$

which is continuous. If $x \neq 0$, then f' is differentiable and

$$f''(x) = \frac{10}{9} x^{-1/3}$$

But for $x=0$ we have

$$\lim_{h \to 0} \frac{f'(h)-0}{h} = \lim_{h \to 0} \frac{5}{3} \frac{h^{2/3}}{h} = \lim_{h \to 0} \frac{5}{3} h^{-1/3}$$

which does not exists.

One can show, that the function $f(x) = x^{\frac{3r-1}{3}}$ is in $C^{r-1}(\mathbb{R})$, but not in $C^r(\mathbb{R})$.

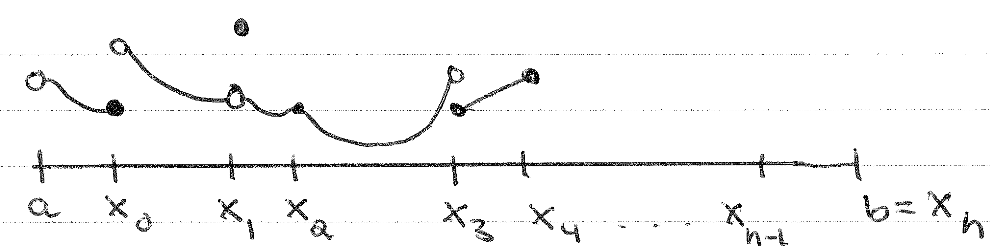
Example 3.4 Piecewise-continuous functions

Definition

Let $I = [a,b)$. A function $f: I \to \mathbb{R}$ is called piecewise-continuous if there exists finitely many points

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that f is continuous on each of the sub-intervals (x_i, x_{i+1}) $i = 0, \ldots, n-1$.



Notice the following

- At the point x_0 we have $\lim_{x \nearrow x_0} f(x)$ exists and is equal to $f(x_0)$, i.e. f is continuous from the left.
- $\lim_{x \searrow x_0} f(x) = \lim_{x \to x_0+} f(x) \neq f(x_0)$, there is a jump.
- At x_1, we have $\lim_{x \to x_1-} f(x) = \lim_{x \to x_1+} f(x) \neq f(x_1)$
- f is continuous at x_2.

If f and g are both piecewise-continuous, then

$$f+g \quad \text{and} \quad rf \quad (r \in \mathbb{R}) \qquad \text{[explain in class]}$$

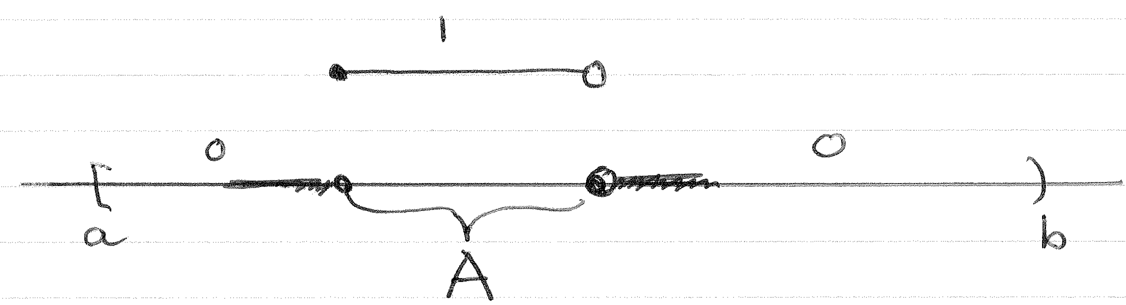
are piecewise continuous. Denote by $PC(I)$ the space of piecewise continuous. Hence $PC(I)$ is a vector space.

Important elements of piecewise continuous functions are given by the indicator functions χ_A, $A \subseteq I$ a subinterval.

Let $A \subseteq \mathbb{R}$ be a set. Define

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

So the values $\chi_A(x)$ tell us if x is in A or not.
If $x \in A$, then $\chi_A(x) = 1$ and if $x \notin A$, then $\chi_A(x) = 0$.



We will have to work a lot with indicator functions so let us look at some properties of those.

Lemma Let $A, B \subseteq I$. Then

$$\chi_{A \cap B}(x) = \chi_A(x)\chi_B(x).$$

Proof We have two functions

$$x \mapsto \chi_{A \cap B}(x)$$

and

$$x \mapsto \chi_A(x)\chi_B(x).$$

We have to show that they take the same values at every point $x \in I$.

So let us evaluate both functions:
If $x \in A$ and $x \in B$, that is $x \in A \cap B$. Then

$$\chi_{A \cap B}(x) = 1$$

$$\left.\begin{array}{l} \chi_A(x) = 1 \\ \chi_B(x) = 1 \end{array}\right\} \text{Thus } \chi_A(x) \cdot \chi_B(x) = 1.$$

Thus, the left and the right hand side agree.
If $x \notin A \cap B$, then there are two possibilities

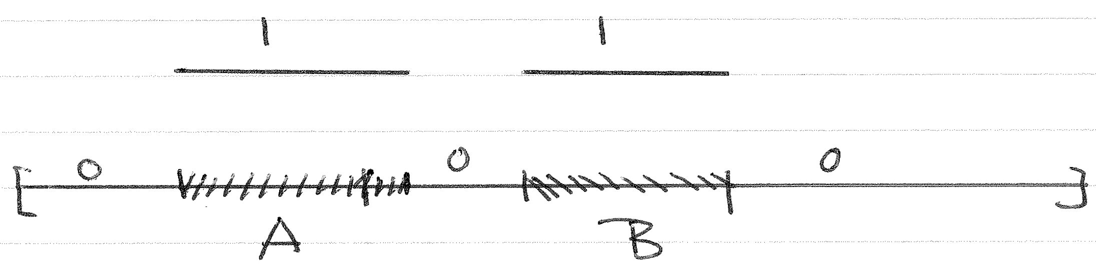
- $x \notin A$, then $\chi_A(x) = 0$, so $\chi_A(x)\chi_B(x) = 0$
- $x \notin B$, then $\chi_B(x) = 0$, so $\chi_B(x)\chi_A(x) = 0$

It follows that

$$0 = \chi_{A \cap B}(x) = \chi_A(x)\chi_B(x) \qquad \blacksquare$$

Let us try to find out, how we can write

$\chi_{A \cup B}$. Lets first assume, that A and B are disjoint, that is $A \cap B = \emptyset$ [there is no point that belongs to both A and B.]



The picture shows, that

$$\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x).$$

Let us prove this:
If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$, so both the left hand side (LHS) and the right hand side (RHS) are zero.
If $x \in A \cup B$ then $x \in A$ and not in B or $x \in B$ but not in A:
If $x \in A$, then $x \notin B$ and $x \in A \cup B$. Thus

$$\chi_{A \cup B}(x) = 1$$

$$\chi_A(x) + \chi_B(x) = 1 + 0 = 1.$$

Similarly for $x \in B$, $x \notin A$ $\square$

So what if $A \cap B \neq \emptyset$. If $x \in A \cap B$ then the left hand side is 1 but the right hand side 2. So we need to substract 1, but only if $x \in A \cap B$! So what function takes the value 1 if $x \in A \cap B$, but is otherwise zero? The function $\chi_{A \cap B}$!

<u>Lemma</u> We have $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$.

<u>Proof</u>: • If $x \notin A \cup B$, then both the LHS and RHS take the value 0.

• If $x \in A \cup B$ then we have the following possibities:

(1) $x \in A$, $x \notin B$ then

$$\chi_{A \cup B}(x) = 1$$
$$\chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x) = 1 + 0 - 0 = 1$$

(2) Similarly if $x \in B$, $x \notin A$: LHS = RHS.

(3) If $x \in A \cap B$, then

$$\chi_{A \cup B}(x) = 1$$
$$\chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x) = 1 + 1 - 1 = 1$$

That is LHS = RHS.

As we have now checked all possibilities, we have shown that the statement in the Lemma is correct $\blacksquare$

indicates: End of proof.