3 Linear maps

We have all seen linear maps before. In fact, most of the maps that you have been using in calculus are linear.

The integral To integrate the function $x^2+3x-\cos x$ are the interval $[a,b]$ we first find the antiderivative of x^2, i.e. $\frac{1}{3}x^3$, then the antiderivative of x, i.e., $\frac{1}{2}x^2$, and then multiply this function by 3 to get $\frac{3}{2}x^2$. Finally we find the antiderivative of $\cos(x)$, i.e., $\sin(x)$ and then multiply $\sin(x)$ by -1. To finish the problem we insert the endpoints. Thus

$$\int_{-1}^{1} x^2+3x-\cos x\,dx = \int_{-1}^{1} x^2dx + 3\int_{-1}^{1} x\,dx - \int_{-1}^{1}\cos x\,dx$$

$$= \frac{1}{3}x^3\Big]_{-1}^{1} + \frac{3}{2}x^2\Big]_{-1}^{1} - \sin x\Big]_{-1}^{1}$$

$$= \frac{2}{3} - \quad \sin(1) + \sin(-1)$$

$$= \frac{2}{3} - 2\sin(1).$$

What we have used is the fact that the integral is a linear map $C^1([a,b]) \to C([a,b])$

$$\int_a^b rf(t) + sg(t)dt = r\int_a^b f(t)dt + s\int_a^b g(t)dt.$$

Differentiation Another example is the differentiation $Df = f'$. To differentiate the function $f(x) = x^4 - 3x + e^x - \cos(x)$ we first differentiate each of the function and then add:

$$D(x^4 - 3x + e^x - \cos(x)) = Dx^4 - 3Dx + De^x - D\cos(x)$$

$$= 4x^3 - 3 + e^x + \sin(x),$$

<u>Definition</u> Let V and W be two vectorspaces. A map $T: V \to W$ is said to be linear if for all $v, u \in V$ and all $r, s \in \mathbb{R}$ we have

$$T(rv + su) = rT(v) + sT(u).$$

<u>Remark</u> This can also be written by using <u>two</u> equations

$$T(u+v) = T(u) + T(v) \quad [\text{additive}]$$
$$T(rv) = r\,T(v).$$

<u>Lemma</u> Suppose that $T: V \to W$ is linear. Then $T(\vec{0}) = \vec{0}$.

<u>Proof</u>: We can write $\vec{0} = 0 \cdot v$, where v is <u>any</u> vector in V. But then $T(0) = T(0 \cdot v) = 0 \cdot T(v) = 0$ ∎

<u>Example</u> Let us find all the linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. An arbitrary vector $(x_1, x_2) \in \mathbb{R}^2$ can be written as

$$(x_1, x_2) = x_1(1,0) + x_2(0,1)$$

Hence $T(x_1, x_2) = x_1 T(1,0) + x_2 T(0,1)$. Write

$$T(1,0) = (a_{11}, a_{12}), \quad T(0,1) = (a_{21}, a_{22}) \quad \text{where } a_{ij} \in \mathbb{R}$$

Then

$$T(x_1, x_2) = x_1(a_{11}, a_{12}) + x_2(a_{21}, a_{22})$$
$$= (x_1 a_{11} + x_2 a_{21}, x_1 a_{12} + x_2 a_{22}).$$

Thus all the information about T is given by the matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

<u>Example</u>: Next let us find all the linear maps $T: \mathbb{R}^3 \Rightarrow \mathbb{R}^3$
As before we write

$$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

$$T(1,0,0) = (a_{11}, a_{12}, a_{13})$$

$$T(0,1,0) = (a_{21}, a_{22}, a_{23})$$

$$T(0,0,1) = (a_{31}, a_{32}, a_{33}).$$

Then

$$T(x_1, x_2, x_3) = x_1(a_{11}, a_{12}, a_{13}) + x_2(a_{21}, a_{22}, a_{23}) + x_3(a_{31}, a_{32}, a_{33})$$

$$= (x_1 a_{11} + x_2 a_{21} + x_3 a_{31},\ x_1 a_{12} + x_2 a_{22} + x_3 a_{32},\ x_1 a_{13} + x_2 a_{23} + x_3 x_{33})$$

$$= (x_1, x_2, x_3)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

<u>Example</u> All the linear maps $T: \mathbb{R}^3 \Rightarrow \mathbb{R}$.

Notice, that $\mathbb{R}$ is also a vector space, so we can consider all the linear maps $T: \mathbb{R}^n \Rightarrow \mathbb{R}$. We have

$$T(x_1, x_2, \dots, x_n) = x_1 T(1,0,\dots,0) + x_2 T(0,1,0,\dots,0) + \cdots + x_n T(0,\dots,0,1)$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

where $T(1,0,\dots,0) = a_1$, $a_2 = T(0,1,0,\dots,0), \dots, a_n = T(0,\dots,0,1)$ are real numbers.

<u>Theorem</u> A map $T: \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is linear if and only if there exists numbers a_{ij}, $i = 1, \dots, n$, $j = 1, \dots, m$, such that

$$T(x_1, \dots, x_n) = (x_1 a_{11} + x_2 a_{21} + \cdots + x_n a_{n1}, \dots, x_1 a_{1m} + x_2 a_{2m} + \cdots + x_n a_{nm})$$

This can also be written as

$$T(x_1,\dots,x_n) = \left(\sum_{i=1}^{n} x_i a_{i1}, \sum_{i=1}^{n} x_i a_{i2}, \dots, \sum_{i=1}^{n} x_i a_{im}\right)$$

or by using matrix multiplication

$$T(x_1,\dots,x_n) = (x_1,\dots,x_n)\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix}.$$

<u>Example</u> The map $T(x,y,z) = (2x + 3xy, z + y)$ is not linear because of the factor xy. Notice that

add few examples.

$$T(1,1,0) = (5,0)$$

but $\quad T(2(1,1,0)) = T(2,2,0) = (16,0)$

and $\quad 2T(1,1,0) = (10,0) \neq (16,0).$

<u>Examples</u>

Evaluate the given linear map at the given point

- $T(x,y) = (3x + y, 3y)$, $(x,y) = (1,-1)$
  $T(1,-1) = (3\cdot 1 + (-1), 3(-1)) = (2,-1)$
- $T(x,y,z) = (2x - y + 3z, 2x + z)$, $(x,y,z) = (2,-1,1)$
  $T(2,-1,1) = (4 + 1 + 3, 4+1) = (8,5).$
- $T: \mathbb{R}^3 \to \mathbb{R}$, $T(x,y,z) = 2x - 5y + z$, $(x,y,z) = (2,1,1)$
  $T(2,1,1) = 4 - 5 + 1 = 0.$
- Take some example involving differentiation and integration

$$D(3x^2 + 4x - 1) = 6x + 4$$

$$\int_1^2 x^2 - e^x\,dx = \tfrac{1}{3}x^3 - e^x\Big]_1^2 = \frac{8}{3} - e^2 - \frac{1}{3} + e = \frac{7}{3} - e^2 + e$$

<u>Definition</u> Let V and W be two vector spaces and $T: V \to W$ a linear map.

(1) The set $\ker(T) = \{v \in V : T(v) = 0\}$ is called the kernel (or nullspace) of T.

(2) The set $\{w \in W : \text{There exists a } v \in V : T(v) = w\} = T(W) = \mathrm{Im}(T)$ is called the image of T.

<u>Remark</u> Notice that $\ker(T) \subseteq V$ and $\mathrm{Im}(T) \subseteq W$

<u>Theorem</u> The kernel of T is a vector space.

<u>Proof.</u> Let $u, v \in \ker(T)$ and $r, s \in \mathbb{R}$. We have to show that $ru + sv \in \ker(T)$. Now $u, v \in \ker(T)$ if and only if $T(u) = T(v) = 0$. Hence

$$\begin{aligned} T(ru+sv) &= rT(u) + sT(v) && (T \text{ is linear}) \\ &= r \cdot 0 + s \cdot 0 && (u, v \in \ker(T)). \\ &= 0 \end{aligned}$$

This shows that $ru + sv \in \ker(T)$ ∎

<u>Remark</u> Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the map $T(x,y) = (x^2+y,\ x+y)$

Then $T(1,-1) = (1-1, 1-1) = (0,0)$. But

$$T(2\cdot(1,-1)) = T(2,-2) = (4-2,\ 2-2) = (2,0) \neq (0,0).$$

So if T is <u>not</u> linear, then the set $\{v \in V : T(v) = 0\}$ is in general <u>not</u> a vector space.

<u>Example</u> Let $T: \mathbb{R}^2 \to \mathbb{R}$ be the map $T(x,y) = 2x - y$.

Describe the kernel of T.

<u>Solution</u> (x,y) is in the kernel of T if and only if $T(x,y) = 2x - y = 0$

Hence $y = 2x$. Thus, the kernel of T is a line through $(0,0)$ with slope 2.

<u>Example</u> Let $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (2x-3y+z,\ x+2y-z)$
Describe the kernel of T.

<u>Solution</u>: We have that $(x,y,z) \in \ker(T)$ if and only if

$$2x - 3y + z = 0 \qquad \text{and}$$
$$x + 2y - z = 0$$

The first equation describes a plane through $(0,0,0)$ with normal vector $(2,-3,1)$. The second equation describes a plane with normal vector $(1,2,-1)$ (and containing $(0,0,0)$). The normal vectors are not parallel and therefore the plains are different. It follows, that the intersection is a line. Let us describe this line. Adding the equations we get

$$3x - y = 0$$

or $y = 3x$. Plugging this into the second equation we get:

$$0 = x + 2(3x) - z = 7x - z$$

or

$$z = 7x$$

Hence the line is given by $x \cdot (1,3,7)$.

<u>Theorem</u> Let V and W be vector spaces, $T: V \to W$ linear. Then

$$\operatorname{Im}(T) \subseteq W$$

is a vector space.

<u>Proof.</u> Let $w_1, w_2 \in \operatorname{Im}(T)$. Then we can find $u_1, u_2 \in V$ such that $T(u_1) = w_1$, $T(u_2) = w_2$. Let $r, s \in \mathbb{R}$. Then

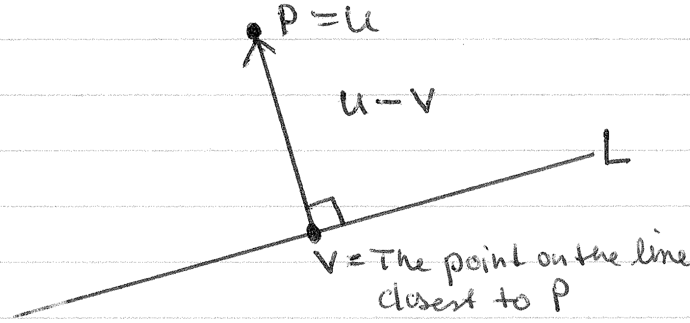
$$\begin{aligned} rw_1 + sw_2 &= rT(u_1) + sT(u_2) \\ &= T(ru_1 + su_2) \in \operatorname{Im}(T) \end{aligned} \qquad \blacksquare$$

(add examples how to determine $\operatorname{Im}(T)$ in special cases?)

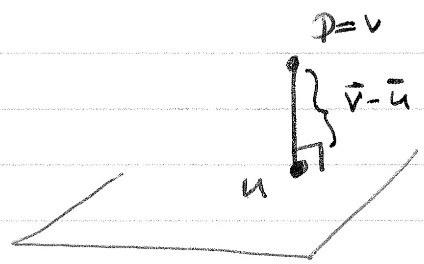
§ 4 Inner product

Let us start with the following problem. Given a point $P \in \mathbb{R}^2$ and a line $L \subseteq \mathbb{R}^2$. How can we find the point on the line closest to P.

Answer: Draw a line secment starting at P and meeting the line in a right angle. Then the point of intersection is the point on the line closest to P.



Let us now take a plane $L \subset \mathbb{R}^3$ and a point outside the line. How can we find the point $\vec{u} \in L$ closest to P. The answer is the same as before, go from P so that we meet the plane in a right angel



In both examples we need two things:

A: We have to be able to say what the <u>length</u> of a vector is

B: Say what a right angle is.

Both of these things can be done by using the dot-product or inner product in $\mathbb{R}^n$.

<u>Definition</u> Let $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n$. Then the dot- or inner-product of those vectors is given by the number:

$$((x_1, \dots, x_n), (y_1, \dots, y_n)) = x_1 y_1 + x_2 y_2 + \dots + x_n y_n .$$

The norm (or length) of the vector $\vec{u} = (x_1, \dots, x_n) \in \mathbb{R}^n$ is the non-negative number

$$\|u\| = \sqrt{(u,u)} = \sqrt{x_1^2 + \dots + x_n^2}$$

<u>Example</u> a) $((1,2,-3),(1,1,1)) = 1+2-3 = 0$

b) $((1,-2,1),(2,-1,3)) = 2+2+3 = 7$

Because $|x_1 y_1 + \dots + x_n y_n| \leq \sqrt{x_1^2 + \dots + x_n^2}\sqrt{y_1^2 + \dots + y_n^2}$

or

$$|(u,v)| \leq \|u\| \cdot \|v\|$$

we have that (for $u, v \neq 0$)

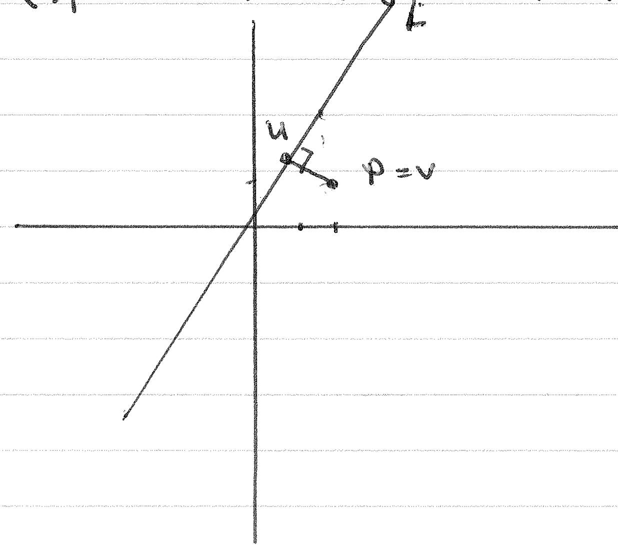
$$-1 \leq \frac{(u,v)}{\|u\| \cdot \|v\|} \leq 1$$

Hence we can define

$$\cos(\measuredangle(u,v)) = \frac{(u,v)}{\|u\| \cdot \|v\|}$$

in particular $u \perp v$ (u perpendicular to v) if and only if $(u,v) = 0$.

<u>Example</u> Let L be the line in $\mathbb{R}^2$ given by $y = 2x$.
Thus $L = \{r(1,2) : r \in \mathbb{R}\}$. Let $P = (2,1)$



what is the point on L closest to P.
<u>Answer</u> Because $u \in L$, we can write $\vec{u} = (r, 2r)$. Furthermore $v - u = (2-r, 1-2r)$ is perpenticular to L. Hence

$$0 = ((1,2),(2-r,1-2r)) = 2 - r + 2 - 4r = 4 - 5r$$

Hence $r = 4/5$ and

$$\vec{u} = \left(\frac{4}{5}, \frac{8}{5}\right).$$

What is the distance of P from the line?
Answere: The length of the vector $v - u$, $\|v-u\|$.

First we have to find out what $v-u$ is. We have done almost all the work:

$$v - u = (2,1) - \left(\frac{4}{5}, \frac{8}{5}\right) = \left(\frac{6}{5}, -\frac{3}{5}\right)$$

The distance is therefore

$$\sqrt{\frac{36}{25} + \frac{9}{25}} = \frac{1}{5}\sqrt{45} = \frac{3}{\sqrt{5}}$$

(1) To be able to define the norm, we used that

<u>positivity</u>: $(u,u) \geq 0$

(2) <u>Zero length</u>: All nonzero vectors should have a non-zero length. Thus $(u,u) = 0$ only if $u = 0$.

(3) ~~The~~ <u>linearity</u>: If the vector $v \in \mathbb{R}^n$ is fixed, then the map $u \mapsto (u,v)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is linear. That is

$$(ru + sw, v) = r(u,v) + s(w,v)$$

(4) <u>Symmetric</u>: for all $u, v \in \mathbb{R}^n$ we have

$$(u,v) = (v,u)$$

We use this to define an inner product on arbitrary vector space V

<u>Definition</u> Let V be a vectorspace.
An <u>inner product</u> on V is a map $(\cdot,\cdot) : V \times V \to \mathbb{R}$ satisfying the following properties:

(1) <u>positivity</u>: We have for all $u \in V$: $(u,u) \geq 0$

(2) $(u,u) = 0$ only if $u = 0$

(3) <u>linearity</u> If $w \in V$ is fixed, then the map $V \to \mathbb{R}$, $v \mapsto (v,w)$, is linear.

(4) <u>Symmetric</u> If $v, w \in V$, then $(v,w) = (w,v)$.

<u>Lemma</u> $(u, rv + sw) = r(u,v) + s(u,w)$.

Proof. (In class)

Def. We say that u and v are perpendicular if $(u,v)=0$.

Definition If $(\cdot,\cdot)$ is an inner product on the vector space V, then the norm of a vector $u \in V$ is given by

$$\|u\| := \sqrt{(u,u)}$$

The norm satisfies the following properties:

(1) $\|u\| \geq 0$ and $\|u\|=0$ only if $u=0$.

(2) $\|ru\| = |r| \cdot \|u\|$.

Proof. We have $\|ru\| = \sqrt{(ru,ru)}$

$$= \sqrt{r(u,ru)}$$
$$= \sqrt{r^2(u,u)}$$
$$= |r|\sqrt{(u,u)} = |r| \cdot \|u\|.$$

Ex: Let $a<b$ and $I=[a,b]$, $V=PC([a,b])$. Define

$$(f,g) := \int_a^b f(t)g(t)dt.$$

Then $(\cdot,\cdot)$ is an inner product on V

Proof: Let $r,s \in \mathbb{R}$, $f,g,h \in V$. Then:

$$(f,f) = \int_a^b f(t)^2 dt.$$

As $f(t)^2 \geq 0$ it follows, that $\int_a^b f(t)^2 dt \geq 0$

(2) If $(f,f)=0$, then $f(t)^2=0$ for all t, or $f=0$.

(3) $\int_a^b (rf+sg)(t)h(t)dt = \int_a^b rf(t)h(t)+sg(t)h(t)dt$

$= r\int_a^b f(t)h(t)\,dt + s\int_a^b g(t)h(t)dt$

$= r(f,h)+s(g,h)$

Hence linear in the first factor.

(4) As $f(t)g(t) = g(t)f(t)$ it follows that $(f,g)=(g,f)$

Notice, that the norm is

$$\|f\| = \sqrt{\int_a^b f(t)^2 dt}$$

Example: Let $a=0$, $b=1$.

- $f(t)=t^2$, $g(t)=t-3t^2$. Then

$$(f,g) = \int_0^1 t^2(t-3t^2)\,dt = \int_0^1 t^3-3t^4 dt$$

$$= \frac{1}{4}-\frac{3}{5} = \frac{5-12}{20} = -\frac{7}{20}$$

$$\|f\| = \sqrt{\int_0^1 t^4} = \frac{1}{\sqrt{5}}$$

$$\|g\| = \sqrt{\int_0^1 (t-3t^2)\cdot(t-3t^2)dt}$$

$$= \sqrt{\int_0^1 t^2-6t^3+9t^4 dt}$$

$$= \sqrt{\frac{1}{3}-\frac{3}{2}+\frac{9}{5}} = \sqrt{\frac{19}{30}}$$

- $f(t) = \cos(2\pi t)$, $g(t) = \sin(2\pi t)$. Then

$$(f,g) = \int_0^1 \cos(2\pi t)\sin(2\pi t)\,dt \qquad u = \sin(2\pi t)$$
$$du = 2\pi \cos(2\pi t)$$
$$= \frac{1}{4\pi}\left[\sin(2\pi t)\right]^2 \Big]_0^1 = 0 \qquad \frac{1}{2\pi}\int u\,du = \frac{1}{4\pi}u^2 + C$$

So $\cos(2\pi t)$ is perpendicular to $\sin(2\pi t)$ on the interval $[0,1]$.

- $f(t) = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}$, $g(t) = \chi_{[0,1)}$

Then
$$(f,g) = \int_0^1 (\chi_{[0,\frac{1}{2})}(t) - \chi_{[\frac{1}{2},1)}(t))\chi_{[0,1)}(t)\,dt$$
$$= \int_0^1 \chi_{[0,\frac{1}{2})}(t)\,dt - \int_0^1 \chi_{[\frac{1}{2},1)}(t)\,dt$$
$$= \int_0^{1/2} dt - \int_{1/2}^1 dt = \frac{1}{2} - \frac{1}{2} = 0$$

Show that $\|f\| = \|g\| = 1$.

—·—

<u>Problem</u>: Find a polynomial $f(t) = a + bt$, that is perpendicular to the vector (polynomial) $g(t) = 1-t$.

<u>Answer</u>: We are looking for numbers a and b s.t.

$$0 = (f,g) = \int_0^1 (a+bt)(1-t)\,dt =$$
$$= \int_0^1 a + bt - at - bt^2\,dt$$
$$= a + \frac{b}{2} - \frac{a}{2} - \frac{b}{3} = \quad \frac{a}{2} + \frac{b}{6}$$

Thus $3a + b = 0$. So we can take $\underline{f(t) = 1 - 3t}$

We state now two important facts about ~~an~~ inner product on a vector space V. Recall, that in $\mathbb{R}^2$

$$\cos(\theta) = \frac{(u,v)}{\|u\|\cdot\|v\|}$$

where $\vec{u},\vec{v}$ are two non-zero vector in $\mathbb{R}^2$ and θ is the angel between $\vec{u}$ and $\vec{v}$. In particular, because $-1 \leq \cos\theta \leq 1$, we must have

$$|(\vec{u},\vec{v})| \leq \|\vec{u}\|\cdot\|\vec{v}\|$$

We will now show, that this comes from the positivity and linearity of the inner product.

<u>Theorem</u> Let V be a vector space with inner product $(\cdot,\cdot)$. Then

$$|(\vec{u},\vec{v})| \leq \|\vec{u}\|\cdot\|\vec{v}\|$$

for all $\vec{u},\vec{v} \in V$.

<u>Proof.</u> We can assume, that $\vec{u},\vec{v} = 0$, because otherwise both the LHS and the RHS will be zero. By the positivity of the inner product we get:

$$0 \leq \left(\vec{v} - \frac{(\vec{v},\vec{u})}{\|\vec{u}\|^2}\vec{u}, \vec{v} - \frac{(\vec{v},\vec{u})}{\|\vec{u}\|^2}\vec{u}\right) \quad [\text{positivity}]$$

$$\left.\begin{aligned} &= (\vec{v},\vec{v}) - \frac{(\vec{v},\vec{u})}{\|\vec{u}\|^2}(\vec{u},\vec{v}) - \frac{(\vec{v},\vec{u})}{\|\vec{u}\|^2}(\vec{v},\vec{u}) \\ &\qquad + \frac{(\vec{v},\vec{u})^2}{\|\vec{u}\|^4}(\vec{u},\vec{u}) \end{aligned}\right\} [\text{linearity}]$$

$$\overset{(\vec{u},v) = (\vec{v},u),\ (\vec{v},\vec{v}) = \|\vec{v}\|^2}{=} \|\vec{v}\|^2 - 2\frac{(\vec{u},\vec{v})^2}{\|\vec{u}\|^2} + \frac{(\vec{v},\vec{u})^2}{\|\vec{u}\|^2} \quad \left[\begin{array}{c}\text{Symmetry} \\ \text{definition of the norm}\end{array}\right]$$

$$= \|\vec{v}\|^2 - \frac{(\vec{u},\vec{v})^2}{\|\vec{u}\|^2}.$$

Thus

$$\frac{(\vec{V},\vec{u})^2}{\|\vec{u}\|^2} \leq \|V\|^2$$

or

$$|(\vec{V},\vec{u})| \leq \|V\| \cdot \|u\|$$ ▨

Notice, that

$$0 = \left(\vec{V} - \frac{(\vec{V},\vec{u})}{\|\vec{u}\|^2}\vec{u},\ \vec{V} - \frac{(\vec{V},\vec{u})}{\|\vec{u}\|^2}\vec{u}\right)$$

only if

$$\vec{V} - \frac{(\vec{V},\vec{u})}{\|\vec{u}\|^2}\vec{u} = 0$$

or

$$\vec{V} = \frac{(\vec{V},\vec{u})}{\|\vec{u}\|^2}\vec{u}.$$

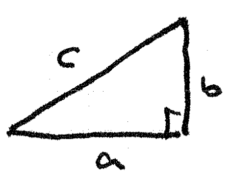
Thus $\vec{V}$ and $\vec{u}$ have to be on the same line through 0.
We can therefore conclude:

<u>Lemma</u> We have

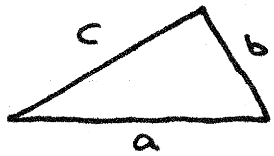
$$|(\vec{u},\vec{V})| = \|\vec{u}\| \cdot \|\vec{V}\|$$

if and only if $\vec{u}$ and $\vec{V}$ are on the same line through $\vec{0}$,

The next statement is a generalization of Pythagoras' Theorem



$c^2 = a^2 + b^2$



$c < a + b$

<u>Theorem</u> Let V be a vectorspace with innerproduct $(\cdot,\cdot)$.

Then
$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|.$$

Furthermore $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ if and only if $(\vec{u},\vec{v})=0$.

<u>Proof</u>
$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v}, \vec{u}+\vec{v}) \\ &= (\vec{u},\vec{u}) + 2(\vec{u},\vec{v}) + (\vec{v},\vec{v}) \qquad (*) \\ &\le \|\vec{u}\|^2 + 2\|\vec{u}\|\cdot\|\vec{v}\| + \|\vec{v}\|^2 \qquad \text{[using the last theorem].} \\ &= (\|\vec{u}\| + \|\vec{v}\|)^2\end{aligned}$$

If $(\vec{u},\vec{v})=0$, then $(*)$ reads
$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$
On the other hand if $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$, we see from $(*)$
$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2(\vec{u},\vec{v}) + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2 + 2(\vec{u},\vec{v}).$$
that $2(\vec{u},\vec{v})=0$

<u>Examples</u>

(a) $\vec{u} = (1,2,-1)$, $\vec{v} = (0,2,4)$. Then
$$(\vec{u},\vec{v}) = 4-4=0.$$
$$\|\vec{u}\|^2 = 1+4+1 = 6, \quad \|\vec{v}\|^2 = 4+16 = 20.$$
$\vec{u}+\vec{v} = (1,4,3)$ and
$$\|\vec{u}+\vec{v}\|^2 = 1+16+9 = 26 = 6+20.$$

Math 2025, Exercies: Inner products and norms

Recall that an inner product on $\mathbb{R}^n$ is given by $((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1 y_1 + \ldots + x_n y_n$. The length or norm of a vector $\overrightarrow{\mathbf{x}} = (x_1, \ldots, x_n)$ is the real number

$$||(x_1, \ldots, x_n)|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{(\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}})}$$

Inner product on functions on $[a, b]$ (piecwise continuous, continous, etc.) is given by $(f, g) = \int_a^b f(t)g(t)\, dt$. The norm is

$$||f|| = \sqrt{(f, f)} = \sqrt{\int_a^b f(t)^2\, dt}\,.$$

Recall, that if A is a subset of $\mathbb{R}^n$ then the indicator function of A is the function $\chi_A : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases}$$

1) Evaluate the inner product of the following vectors in $\mathbb{R}^n$:

a) $((1, 0, -3), (2, 1, 3)) =$

b) $((2, 2, -1, -1), (1, 1, 2, 2)) =$

c) $((1, 2, 3, 4, 5), (0, 2, 5, 3, 6)) =$

2) Find the norm of the following vectors:

a) $||(1, 3, 1)|| =$

b) $||(-1, 2, 3, 5)|| =$

Two non-zero vectors are called perpendicular or orthogonal if the inner product $(\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{y}}) = 0$.

3) Which of the following vectors are perpendicular to each other?

a) $(1, 0)$, $(0 - 2)$

b) $(1, 0, 1)$, $(-1, 2, 1)$

c) $(2, -1, 0)$, $(1, 1, 3)$

d) $(1, -2, 1, 2)$, and $(1, -1, 1, -2)$.

4) Find the inner product of the following function in $C([0, 1])$ (thus $a = 0$ and $b = 1$).

a) $(1, t)$

b) $(\cos(t), t) =$

c) $(t, t^2 + 1) =$

d) $(t, e^{t^2}) =$

e) $(\cos(4\pi t), \sin(4\pi t)) =$

f) $(t, e^t) =$

5) Find the norm of the following functions:

a) $||\cos(t)|| =$

b) $||1 + t|| =$

c) $||\frac{1}{\sqrt{1+t}}|| =$

d) $||e^t|| =$

e) $||\frac{\sqrt{t}}{1+t^2}|| =$

6) Which of the following functions are orthogonal on the interval $[0, 1]$?

a) t and $1 - \frac{1}{6}t$;

b) 1 (the constant function that maps all t into the number 1) and $2 - 2t - 3t^2$.

c) $2\chi_{[0,1)}$ and $\chi_{[0,1/2)} - \chi_{[1/2,1)}$,

d) $\chi_{[0,1)}$ and $\chi_{[0,1/4)} - \chi_{[1/4,1/2)}$.

Math 2025, Exercies: Inner products and norms

Recall that an inner product on $\mathbb{R}^n$ is given by $((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = x_1y_1+\ldots+x_ny_n$. The length or norm of a vector $\overrightarrow{\mathbf{x}} = (x_1,\ldots,x_n)$ is the real number

$$||(x_1,\ldots,x_n)|| = \sqrt{x_1^2+x_2^2+\ldots+x_n^2} = \sqrt{(\overrightarrow{\mathbf{x}},\overrightarrow{\mathbf{x}})}$$

Inner product on functions on $[a,b]$ (piecwise continuous, continous, etc.) is given by $(f,g)=\int_a^b f(t)g(t)\,dt$. The norm is

$$\|f\| = \sqrt{(f,f)} = \sqrt{\int_a^b f(t)^2\,dt}\,.$$

Recall, that if A is a subset of $\mathbb{R}^n$ then the indicator function of A is the function $\chi_A : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\chi_A(t) = \begin{cases} 1 & \text{if } t\in A \\ 0 & \text{if } t\notin A \end{cases}$$

1) Evaluate the inner product of the following vectors in $\mathbb{R}^n$:

a) $((1,0,-3),(2,1,3)) =$

b) $((2,2,-1,-1),(1,1,2,2)) =$

c) $((1,2,3,4,5),(0,2,5,3,6)) =$

2) Find the norm of the following vectors:

a) $||(1,3,1)|| =$

b) $||(-1,2,3,5)|| =$

Two non-zero vectors are called perpendicular or orthogonal if the inner product $(\overrightarrow{\mathbf{x}},\overrightarrow{\mathbf{y}}) = 0$.

3) Which of the following vectors are perpendicular to each other?

a) $(1,0)$, $(0-2)$

b) $(1,0,1)$, $(-1,2,1)$

c) $(2,-1,0)$, $(1,1,3)$

d) $(1,-2,1,2)$, and $(1,-1,1,-2)$.

4) Find the inner product of the following function in $C([0,1])$ (thus $a=0$ and $b=1$).