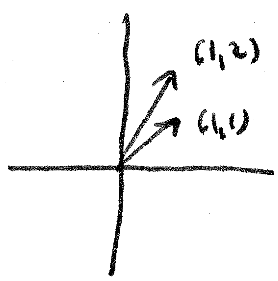
§ 5 Generating sets and bases.

Let V be the vector space $\mathbb{R}^2$ and consider the two vectors $(1,0)$ and $(0,1)$. Then every vector $(x,y) \in \mathbb{R}^2$ can be written as a combination of those two vectors

$$(x,y) = x(1,0) + y(0,1).$$

Similarly the two vectors $(1,1)$ and $(1,2)$ do not belong to the same line



and every other vector can be written as combination of those two vectors

$$(x,y) = a(1,1) + b(1,2)$$

gives us two equations

$$a + b = x$$

$$a + 2b = y$$

Thus (by substracting the first equation from the second

$$b = -x + y.$$

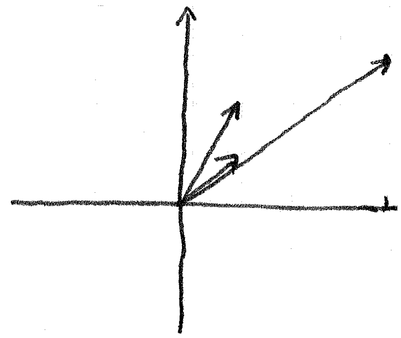
Inserting this into the first equation gives

$$a = x - b = 2x - y.$$

As an example lets take $(4,3)$. Then

$$(4,3) = 5(1,1) + (-1)\cdot(1,2)$$

$$= 5(1,1) - (1,2)$$



We have similar situation for $\mathbb{R}^3$ and all of the spaces $\mathbb{R}^n$.

As an example every vector can be write as combinations of $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

or as a combination of $(1,-1,0)$, $(1,1,1)$, $(0,1,-1)$

$$(x,y,z) = a(1,-1,0) + b(1,1,1) + c(0,1,-1).$$

This gives 3 equations:

$$\begin{aligned} a + b \quad &= x \qquad (1) \\ -a + b + c &= y \qquad (2) \\ b - c &= z \qquad (3) \end{aligned}$$

(2)+(3) gives

$$-a + 2b = y + z \qquad (4)$$

(4)+(1): $3b = x + y + z$ or

$$b = \frac{x+y+z}{3}$$

Then (1) gives:

$$a = x - b$$
$$= x - \frac{x+y+z}{3}$$
$$= \frac{2x - y - z}{3}$$

And finally (3) gives

$$c = b - z = \frac{x+y-2z}{3}$$

Hence:

$$(x,y,z) = \frac{2x-y-z}{3}(1,-1,0) + \frac{x+y+z}{3}(1,1,1) + \frac{x+y-2z}{3}(0,1,-1).$$

Notice that we got only one solution, so there is only one way, that we can write a vector in $\mathbb{R}^3$ as a combination of those vectors. In general, if we have k vectors in $\mathbb{R}^n$ then the equation:

$$\vec{x} = (x_1, x_2, \dots, x_n) = c_1 v_1 + c_2 v_2 + \dots + c_k v_k \qquad (*)$$

gives us n-equations involving the n-cordinates of

$v_1, \dots, v_k$ and the unknowns $c_1, \dots, c_k$. There are 3 possibilities:

(A) The equation $(*)$ has <u>no</u> solution. Thus $\vec{x}$ can not be written as a combination of the vectors $v_1, \dots, v_k$.

(B) There is only one solution, so $\vec{x}$ can be written in exactly one way as a combination of $v_1, \dots, v_k$.

(c) The system of equations has infinitely many solutions, so there are more than one way to write $\vec{x}$ as a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_n$.

Let us look at the last case a little closer. If we write $\vec{x}$ in two different ways:

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n$$

$$\vec{x} = d_1 \vec{v}_1 + d_2 \vec{v}_2 + \cdots + d_n \vec{v}_n$$

Then, by substracting, we get:

$$0 = (c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \cdots + (c_n - d_n)\vec{v}_n$$

and some of the numbers $c_i - d_i$ are non-zero. Similarly, if we can write

$$0 = a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n \quad (\text{not all } a_j = 0)$$

and $\vec{x} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n$. Then we also have

$$\vec{x} = (c_1 + a_1)\vec{v}_1 + \cdots + (c_n + a_n)\vec{v}_n$$

Thus we can write $\vec{x}$ as a combination of the vectors $\vec{v}_1, \dots, \vec{v}_n$ in several different ways (in fact ∞-many ways). We will now use this as a motivation for the following definitions:

(C) The system has ∞-many solution, so there are several ways to write $\vec{x}$ as a combination of those vectors.

Our aim is now to discuss this for arbitrary vector spaces.
[illegible] is a [illegible] to
solve

<u>Definition</u> Let V be a vector space, and $\vec{v}_1, \dots, \vec{v}_n \in V$.

1) Let $W \subseteq V$ be a subspace. We say that W is spanned by the vectors $\vec{v}_1, \dots, \vec{v}_n$ if every vector in W can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$. Thus, if $\vec{w} \in W$, then there exists numbers $c_1, \dots, c_n \in \mathbb{R}$ such that

$$\vec{w} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n .$$

2) The set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly dependent, if there exists $c_1, \dots, c_n$, not all equal to zero, such that

$$c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n = 0$$

3) The set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ are linearly independent if the set is <u>not</u> linearly dependent (if and only if we can only write $0 = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n$ with all $c_j = 0$).

4) The set $\{v_1, \dots, v_n\}$ is a basis for W if ~~there exist~~ $\{v_1, \dots, v_n\}$ is <u>linearly independent</u> and spans W.

Before we start with the examples, let us notice the following.

<u>Lemma</u> Let V be a vector space with a scalar product inner product $(\cdot, \cdot)$. Assume that $\{v_1, \dots, v_n\}$ is an orthogonal subset of vectors in V (thus $(v_i, v_j) = 0$ if $i \neq j$). If

$$v = c_1 \vec{v}_1 + \cdots + c_n v_n$$

then $$c_j = \frac{(v, v_j)}{\|v_j\|^2}, \quad j = 1, \dots, n.$$

Proof. Assume that

$$\vec{V} = c_1\vec{V}_1 + \cdots + c_n\vec{V}_n$$

Thake the inner product with $\vec{V}_1$ on both sides. The left hand side is $(\vec{V}, \vec{V}_1)$. The RHS is

$$(c_1\vec{V}_1 + c_2\vec{V}_2 + \cdots + c_n\vec{V}_n, \vec{V}_1) = c_1(\vec{V}_1, \vec{V}_1) + c_2(\vec{V}_2, \vec{V}_1) + \cdots + c_n(\vec{V}_n, V_1)$$

[because $(\cdot, v_1)$ linear]

$$= c_1(\vec{V}_1, \vec{V}_1) \quad \text{[orthogonality]}$$

$$= c_1 \|\vec{V}_1\|^2.$$

Thus $(\vec{V}, \vec{V}_1) = c_1 \|\vec{V}_1\|^2$ or

$$c_1 = \frac{(\vec{V}, \vec{V}_1)}{\|\vec{V}_1\|^2}.$$

Repeat this for $\vec{V}_2, \dots, \vec{V}_n$. ☒

Corollary If the vectors $\vec{V}_1, \dots, \vec{V}_n$ are orthogonal, then they are linearly independent.

Examples

I/ Let $V = \mathbb{R}^2$. The vectors $(1, 2)$ and $(-2, -4)$ are linearly dependent, because

$$(-2, -4) = (-2)(1, 2)$$

or

$$(-2)\cdot(1, 2) + 1\cdot(-2, -4) = 0.$$

The vectors $(1,2)$, $(1,1)$ are linearly dependent. In fact $\{(1,2),(1,1)\}$ is a basis for $\mathbb{R}^3$.

Let $(x,y) \in \mathbb{R}^2$. Then

$$(x,y) = c_1(1,2) + c_2(1,1)$$
$$= (c_1 + c_2, 2c_1 + c_2).$$

Thus

$$x = c_1 + c_2$$
$$y = 2c_1 + c_2$$

subtracting we get:

$$x - y = -c_1 \quad \text{or} \quad c_1 = y - x.$$

Plug this into the first equation to get:

$$c_2 = x - c_1 = x - (y - x) = 2x - y.$$

Thus we can write any vector in $\mathbb{R}^2$ as a combination of those two. In particular, let $x=0, y=0$, gives $c_1 = c_2 = 0$.

The vector $(1,2)$ and $(-2,1)$ are orthogonal and hence linearly independent, and in fact a basis.

$$(x,y) = c_1 \underbrace{(1,2)}_{=v_1} + c_2 \underbrace{(-2,1)}_{=v_2}$$

Taking the inner product we get:

$$c_1 = \frac{x+2y}{\|v_1\|^2} = \frac{x+2y}{5}$$

$$c_2 = \frac{-2x+y}{5}.$$

II) $V = \mathbb{R}^3$. One vector can only generate a line, two vectors can at most span a plane, so we need at least three vectors to span $\mathbb{R}^3$.

The vectors $(1,2,1)$, $(1,-1,1)$ are orthogonal but not a basis. In fact those two vectors span the plane

$$W = \{ (x,y,z) \in \mathbb{R}^3 : x - z = 0 \}.$$

Try to find out why.

On the other hand, the vectors $(1,2,1)$, $(1,-1,1)$ and $(1,0,-1)$ are orthogonal, and hence a basis. We have (as an example)

$$(4,3,1) = c_1 (1,2,1) + c_2 (1,-1,1) + c_3 (1,0,-1)$$

with

$$c_1 = \frac{4+6+1}{1+4+1} = \frac{11}{6}$$

$$c_2 = \frac{4-3+1}{3} = \frac{2}{3}$$

$$c_4 = \frac{4-1}{2} = \frac{3}{2}$$

In general we have

$$(x,y,z) = \frac{x+2y+z}{6}(1,2,1) + \frac{x-y+z}{3}(1,-1,1) + \frac{x-z}{2}(1,0,-1).$$

III) Let us now discuss some spaces of functions.

a) Let $v_0(x) = 1$, $v_1(x) = x$, and $v_2(x) = x^2$.

Then v_0, v_1, and v_2 are linearly independent:

$$0 = c_0 v_0(x) + c_1 v_1(x) + c_2 v_2(x) \qquad \text{for all } x$$
$$= c_0 + c_1 x + c_2 x^2$$

Take $x = 0$: Then we get $c_0 = 0$.

Differentiate both sides to find

$$0 = c_1 + 2c_2 x$$

Take again $x = 0$ to find $c_1 = 0$. Differentiate one more time to see that $c_2 = 0$. Notice that the span of v_0, v_1, and v_2 is the space of polynomials of degree ≤ 2. Hence the functions $1, x$, and x^2 form a basis for this space. Notice that the functions $1+x$, $1-2x$, x^2 are also a basis.

b) Are the functions $v_0(x) = x$, $v_1(x) = xe^x$ linearly independent / dependent on $\mathbb{R}$?

Answer: No:

Assume that $0 = c_1 x + c_2 x e^x$. It does not help to put $x = 0$ now, but lets differentiate both sides:

$$0 = c_1 + c_2 e^x + c_2 x e^x$$

Now $x=0$ gives

(1) $\quad 0 = c_1 + c_2$.

Differentiate again, gives $\quad 0 = c_2 e^x + c_2 e^x + c_2 x e^x$. Now $x=0$ gives

$$0 = 2c_2$$

or $c_2 = 0$. Hence (1) gives $c_1 = 0$.

c) The functions $\chi_{[0,1/2)}$ and $\chi_{[\frac{1}{2},1)}$ are orthogonal and hence linearly independent. Let us show this directly. Assume that

$$0 = c_1 \chi_{[0,1/2)} + c_2 \chi_{[\frac{1}{2},1)} \, .$$

An equation like this means, that what even x we feed into the function on the right hand side, the outcome is always 0.

lets take $x = \frac{1}{4}$. Then $\chi_{[0,\frac{1}{2})}(\frac{1}{4}) = 1$ but $\chi_{[\frac{1}{2},1)}(\frac{1}{4}) = 0$.

Hence $\quad c_1 = 0 \left(= c_1 \chi_{[0,\frac{1}{2})}(1/4) + c_2 \chi_{[\frac{1}{2},1)}(1/2) \right)$.

Taking $x = 3/4$ shows that $c_2 = 0$.

d) The functions $\chi_{[0,1)}$ and $\chi_{[0,\frac{1}{2})}$ are not orthogonal, but linearly independent:

$$0 = c_1 \chi_{[0,1)} + c_2 \chi_{(0,1/2)}$$

Take x so that $\chi_{[0,1/2)}(x) = 0$, but $\chi_{[0,1)}(x) = 1$. Thus any $x \in [0,1) \setminus [0,1/2) = [\frac{1}{2},1)$ will do the job. So take $x = 3/4$.

Then we see that

$$0 = c_1 \chi_{[0,1)}(3/4) + c_2 \chi_{[0,1/2)}(\tfrac{3}{4}) = c_1 \, .$$

Then take $x = 1/4$ to see that $c_2 = 0$.

-Lecture 7-

§7 GRAM-SCHMIDT ORTHOGONALIZATION

The "best" basis we can have for a vector space, because the we can most easily find the constants that are needed to express a vector as a linear combination of the basis vectors $v_1, \dots, v_n$:

$$\vec{v} = \frac{(\vec{v}, \vec{v}_1)}{\|v_1\|^2} \vec{v}_1 + \frac{(\vec{v}, \vec{v}_2)}{\|\vec{v}_2\|^2} \vec{v}_2 + \cdots + \frac{(\vec{v}, \vec{v}_n)}{\|\vec{v}_n\|^2} \vec{v}_n .$$

But usually we are not given an orthogonal basis. In this section we will show how to, starting with an arbitrary basis, we can find an orthogonal basis. Let us start with two vectors, v_1 and v_2 not on the same line through zero (i.e., not linearly dependent). Let $u_1 = v_1$. How can we find a vector u_2 which is perpendicular to u_1 and such that the span of u_1 and u_2 is the same as the span of $\vec{v}_1$ and $\vec{v}_2$? For that we try to find a number $a \in \mathbb{R}$ such that

$$u_2 = a u_1 + v_2 \qquad u_2 \perp u_1 .$$

Take the inner product with u_1 to get

$$0 = (u_2, u_1) = a\,(u_1, u_2) + (v_2, u_1)$$
$$= a\,\|u_1\|^2 + (v_2, u_1)$$

or

$$a = -\frac{(v_2, v_1)}{\|v_1\|^2} = -\frac{(v_2, u_2)}{\|u_1\|^2}.$$

What if we have the third vector $\vec{v}_3$? Then, after choosing $\vec{u}_1, \vec{u}_2$ as above, we would look for $\vec{u}_3$ of the form:

$$\vec{u}_3 = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \vec{v}_3$$

Take the inner product with $\vec{u}_1$ to find

$$0 = (\vec{u}_3, \vec{u}_1) = a_1 \|\vec{u}_1\|^2 + (\vec{v}_3, \vec{u}_1)$$

or

$$a_1 = -\frac{(\vec{v}_3, \vec{u}_1)}{\|\vec{u}_1\|^2}.$$

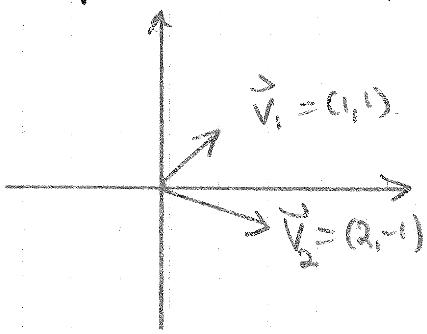
Similarly

$$a_2 = -\frac{(\vec{v}_3, \vec{u}_2)}{\|u_2^2\|^2}$$

Thus:

$$\vec{u}_1 = \vec{v}_1$$
$$\vec{u}_2 = \vec{v}_2 - \frac{(\vec{v}_2, \vec{u}_1)}{\|\vec{u}_1\|^2}\vec{u}_1$$
$$\vec{u}_3 = v_3 - \frac{(\vec{v}_3, \vec{u}_1)}{\|u_1\|^2}\vec{u}_1 - \frac{(v_3, \vec{u}_2)}{\|\vec{u}_2\|^2}\vec{u}_2.$$

Example $\vec{V}_1 = (1,1)$, $\vec{V}_2 = (2,-1)$.



Then we set

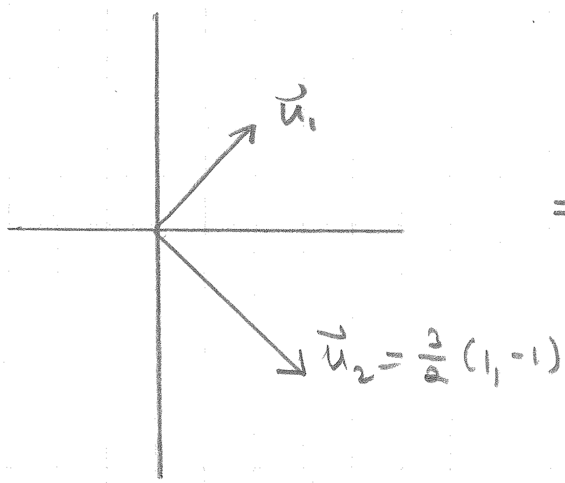
$\vec{u}_1 = (1,1)$ and

$$\vec{u}_2 = (2,-1) - \frac{(V_2, \vec{u}_1)}{\|\vec{u}_1\|^2}\vec{V}_2$$

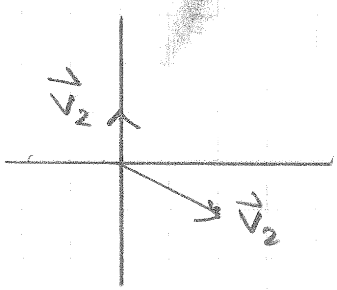
$$= (2,-1) - \frac{2-1}{2}(1,1)$$

$$= (2,-1) - \left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \left(\tfrac{3}{2}, -\tfrac{3}{2}\right)$$

$$= \tfrac{3}{2}(1,-1)$$



Example $\vec{V}_1 = (2,-1)$, $\vec{V}_2 = (0,1)$



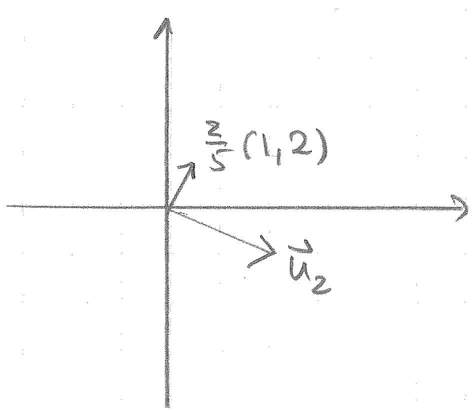
$$\vec{u}_1 = (2,-1)$$

$$\vec{u}_2 = (0,1) - \frac{-1}{5}(2,-1)$$

$$= (0,1) + \left(\tfrac{2}{5}, \tfrac{-1}{5}\right)$$

$$= \tfrac{1}{5}(2,4)$$

$$= \tfrac{2}{5}(1,2)$$



But we could also have started with $V_2 = (0,1)$ to get first basis vector to be $(0,1)$ and the the second one to be

$$(2,-1) - \frac{(2,-1)\cdot(0,1)}{\|(0,1)\|^2}(0,1) = (2,-1) + (0,1)$$
$$= \underline{\underline{(2,0)}}$$

<u>Example</u> Let $\vec{v}_1 = (0,1,2)$, $\vec{v}_2 = (1,1,2)$, $\vec{v}_3 = (1,0,1)$.

Then we get

$$\vec{u}_1 = (0,1,2)$$
$$\vec{u}_2 = (1,1,2) - \frac{(0,1,2)\cdot(1,1,2)}{\|(0,1,2)\|^2}(0,1,2)$$
$$= (1,1,2) - \frac{5}{5}(0,1,2)$$
$$= (1,1,2) - (0,1,2)$$
$$= (1,0,0).$$
$$u_3 = (1,0,1) - \frac{2}{5}(0,1,2) - (1,0,0)$$
$$= (0, -\frac{2}{5}, \frac{1}{5})$$
$$= \frac{1}{5}(0,-2,1)$$

Example: Polynomials of degree ≤ 2.

Let $v_0 = 1$, $v_1 = x$, $v_2 = x^2$. Then $\{v_0, v_1, v_2\}$ is a basis for the space of polynomials of degree ≤ 2. But they are not orthogonal. So we start with

$$\boxed{\vec{u}_0 = \vec{v}_0 = 1}$$

$$\vec{u}_1 = \vec{v}_1 - \frac{(\vec{v}_1, \vec{u}_0)}{\|u_0\|^2} \vec{u}_0$$

So we need to find:

$$(\vec{v}_1, \vec{u}_0) = \int_0^1 x\,dx = \tfrac{1}{2}x^2\Big]_0^1 = \frac{1}{2}$$

$$\|\vec{u}_0\|^2 = \int_0^1 1\,dx = x\Big]_0^1 = 1$$

Hence

$$\boxed{\vec{u}_1 = x - \tfrac{1}{2}}$$

Then $\vec{u}_2 = \vec{v}_2 - \frac{(\vec{v}_2, u_0)}{\|\vec{u}_0\|^2}\vec{u}_0 - \frac{(\vec{v}_2, \vec{u}_1)}{\|\vec{u}_1\|^2}\vec{u}_1.$

$$(\vec{v}_2, \vec{u}_0) = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$(\vec{v}_2, \vec{u}_1) = \int_0^1 x^2(x - \tfrac{1}{2})\,dx = \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$$

$$\|\vec{u}_1\|^2 = \int_0^1 (x - \tfrac{1}{2})^2\,dx = \int_0^1 x^2 - x + \tfrac{1}{4}\,dx$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{4-6+3}{12} = \frac{1}{12}.$$

Hence:

$$\vec{u}_2 = x^2 - \frac{1}{3} - (x - \tfrac{1}{2}) = \underline{\underline{x^2 - x + \tfrac{1}{6}}}$$

<u>Theorem</u> (Gram-Schmidt Orthogonalization) Let V be a vector space with an inner product $(\cdot,\cdot)$. Let $\{v_1,\dots,v_k\}$ be a linearly independent set in V. Then there exists a orthogonal set $\{u_1,\dots,u_k\}$ such that $(v_i,u_i)>0$ and $\operatorname{span}\{v_1,\dots,v_i\} = \operatorname{span}\{u_1,\dots,u_i\}$ for all $i=1,\dots,k$.

<u>Proof</u> (see the book p. 129 - 131.

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