Math 1550-22, Test #1. Fall 2008 Name: 

No calculator, no notes, no book.

1[15P]) Let \( f(x) = \frac{x^2 + 1}{x - 1} \).

a) What is the domain of the function \( f(x) \)?

The domain is: \( x \neq 1 \)

b) Find the average rate of change over the interval \([2, 3]\).

The AROC is:

\[
\frac{f(3) - f(2)}{3 - 2} = \frac{10}{1} - 5 = 0
\]

c) What is the instantaneous rate of change at \( x = 2 \)?

\[
\left. \frac{d}{dx} \left( \frac{x^2 + 1}{x - 1} \right) \right|_{x=2} = \frac{2x(x-1) - (x^2+1)}{(x-1)^2} \bigg|_{x=2} = \frac{x^2 - 2x - 1}{(x-1)^2} \bigg|_{x=2} = \frac{4 - 4 - 1}{1^2} = -1
\]

d) \( \lim_{x \to 1^+} f(x) = +\infty \). (The limit does not exist.)

e) \( \lim_{x \to 1^-} f(x) = -\infty \).

2[20P]) Evaluate the following limits. Show your work, justify your answer:

a) \( \lim_{x \to 4} \frac{x^2 - 2}{x - 4} = \sqrt{4} \).

\[
\frac{\sqrt{x} - 2}{x - 4} = \frac{x - 4}{x - 4} \cdot \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{x} + 2} \xrightarrow{x \to 4} \frac{1}{4}
\]

b) \( \lim_{x \to 0} \frac{\cos(x) \sin(3x)}{5x} = \frac{3}{5} \).

\[
= \left( \lim_{x \to 0} \cos(x) \right) \left( \lim_{x \to 0} \frac{\sin(3x)}{5x} \right) = 1 \cdot \frac{3}{5} \lim_{x \to 0} \frac{\sin(3x)}{3x} = \frac{3}{5}
\]
c) \(\lim_{x \to 0} \frac{\tan(2x)}{\tan(3x)} = \frac{2}{3}\).

\(\Rightarrow \frac{3}{2} \cdot \frac{\sin(2x)}{2x} \cdot \frac{3x}{\sin(3x)} \cdot \frac{\cos(3x)}{\cos(2x)}\)

\(\Rightarrow \frac{2}{3} \cdot 1 \cdot \frac{1}{1} = \frac{2}{3}\)

\(d) \lim_{h \to 0} \frac{(4+h)^2 - 16}{h} = 8\)

\(\frac{(4+h)^2 - 16}{h} = \frac{16 + 8h + h^2 - 16}{h} = 8 + h \to 0 \Rightarrow 8\)

\(\lim_{x \to 1} \frac{x^2 - 9x + 8}{x^2 - 1} = \frac{x^2 - 9x + 8}{x-1}(x+1)\)

\(x^2 - 9x + 8 = (x-1)(x-8)\)

\(\frac{x^2 - 9x + 8}{x^2 - 1} = \frac{x-8}{x+1} \to \frac{-7}{2}\)

3[5P]) For which number(s) \(c\) is the function \(f(x) = \begin{cases} \frac{x^2 + c}{\sqrt{x^2 - 3} - c} & \text{if } x < 2 \\ \text{continuous?} & \text{if } x \geq 2 \end{cases}\)

\(c = -\frac{3}{2}\).

Insert \(c = 2\): \(4+c = 1-c\) or \(2c = -3\), \(c = -\frac{3}{2}\)

4[10P]) Let \(f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ \sqrt{x^2 - 1} + 2 & \text{if } x \in (1, 2] \\ x + 3 & \text{if } x \in (2, \infty) \end{cases}\)

a) Evaluate \(\lim_{x \to 1^-} f(x) = \frac{2}{2} = 1\) and \(\lim_{x \to 1^+} f(x) = \frac{2}{2} = 1\). Is \(f(x)\) continuous at the point \(x = 1\)? \(\text{No, because } f(1) = 7\).

b) Is \(f(x)\) continuous at the point \(x = 2\)? \(\text{No, because }\)

\(\lim_{x \to 2^-} f(x) = \sqrt{3} + 2 \neq \lim_{x \to 2^+} f(x) = 5\).
5[10P] Use the definition of the derivative to find the derivative of the function \( f(x) = \sqrt{x+2} \).
Show your work!

\[
f'(x) = \frac{1}{2\sqrt{x+2}}
\]

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+2+h} - \sqrt{x+2}}{h} = \lim_{h \to 0} \frac{(x+2+h)-(x+2)}{h\sqrt{x+2+h}+\sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}
\]

6[10P] Consider the functions \( f(x) \) and \( g(x) \), for which \( f(0) = 3 \), \( g(0) = 2 \), \( f'(0) = 1 \), and \( g'(0) = 2 \).
Let \( h(x) = \frac{f(x)}{g(x)} \). Find \( h'(0) = -1 \).

Use the quotient rule

\[
\left( \frac{f}{g} \right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\]

\[
1 \cdot 2 - 3 \cdot 2 = \frac{2-6}{4} = -1
\]

7[20P] Find the following derivatives:

a) \( f(x) = 2x^3 + x + x^{-2} \), \( f'(x) = 6x^2 + 2x^{-3} \)

Use the power rule \( \frac{d}{dx} x^k = kx^{k-1} \)

b) \( f(t) = t e^t \), \( f'(t)|_{t=0} = 1 \)

Use the product rule and \( e^0 = 1 \).

\[
f'(t)|_{t=0} = (e^t + te^t)|_{t=0} = 1
\]

c) \( g(z) = \left( \frac{z^2 - 4}{z - 1} \right) \left( \frac{z^2 - 1}{z + 2} \right) \), \( g'(z) = \frac{2z - 1}{z^2 - z - 2} \)

Simplify

\[
\left( \frac{z^2 - 4}{z - 1} \right) \cdot \left( \frac{z^2 - 1}{z + 2} \right) = \frac{(z+2)(z-2)(z-1)(z+1)}{(z-1)(z+2)}
\]

\[
= (z - 2)(z+1) = z^2 - z - 2
\]
d) \( f(x) = \frac{x^2 + 3x + x^{-1}}{x} \), \( f'(x) = 1 - 2x^{-3} \)

Simplify: \( f(x) = x + 3 + x^{-2} \) and then use the power rule.

8[10P]) The height \( s(t) \) of an object tossed vertically in the air is given by \( s(t) = s_0 + v_0t - \frac{1}{2}gt^2 \), \( g = 32 \text{ ft/sec}^2 \).

a) Explain the meaning of \( s_0 \), \( v_0 \), and \( g \).

\( s_0 \) is the initial height (the height at time \( t = 0 \))

\( v_0 \) is the initial velocity

\( g \) is the gravitational constant.

b) If \( s_0 = 2 \text{ft} \), \( v_0 = 320 \text{ft/sec} \). What is the maximum height and when does it reach that height?

Maximum height: \( \frac{ds}{dt}(t) = V(t) = 0 \)

\( s(t) = 2 + 320t - 16t^2 \)

\( V(t) = 320 - 32t = 32(10 - t) \)

If \( V(t) = 0 \) then \( t = 10 \). Hence the max height is \( s(10) = 2 + 3200 - 1600 = 1602 \text{ ft} \).
1[15P] Let \( f(x) = \frac{x^2 + 1}{x - 1} \).

a) What is the domain of the function \( f(x) \)?

b) Find the average rate of change over the interval \([2, 3]\).

The domain is:

The AROC is:

c) What is the instantaneous rate of change at \( x = 2 \)?

d) \( \lim_{x \to 1^+} f(x) = \)

e) \( \lim_{x \to 1^-} f(x) = \)

2[20P] Evaluate the following limits. Show your work, justify your answer!

a) \( \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \)

b) \( \lim_{x \to 0} \frac{\cos(x) \sin(3x)}{5x} = \).
c) \( \lim_{{x \to 0}} \frac{\tan(2x)}{\tan(3x)} = \) 

\[ \lim_{{h \to 0}} \frac{(4 + h)^2 - 16}{h} = \]

\[ \lim_{{x \to 1}} \frac{x^2 - 9x + 8}{x^2 - 1} = \]

3[5P]) For which number(s) \( c \) is the function \( f(x) = \begin{cases} x^2 + c & \text{if } x < 2 \\ \sqrt{x^2 - 3} - c & \text{if } x \geq 2 \end{cases} \) continuous? 
\( c = \) 

4[10P]) Let \( f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ \sqrt{x^2 - 1} + 2 & \text{if } x \in (1, 2] \\ x + 3 & \text{if } x \in (2, \infty) \end{cases} \).

a) Evaluate \( \lim_{{x \to 1^-}} f(x) = \) \( \) and \( \lim_{{x \to 1^+}} f(x) = \) \( \). Is \( f(x) \) continuous at the point \( x = 1? \) 

\( \) 

b) Is \( f(x) \) continuous at the point \( x = 2? \) 

\( \)
5[10P]) Use the definition of the derivative to find the derivative of the function \( f(x) = \sqrt{x + 2} \). Show your work!
\[ f'(x) = \] 

6[10P]) Consider the functions \( f(x) \) and \( g(x) \), for which \( f(0) = 3, \ g(0) = 2, \ f'(0) = 1, \) and \( g'(0) = 2 \). Let \( h(x) = \frac{f(x)}{g(x)} \). Find \( h'(0) = \) 

7[20P]) Find the following derivatives:

a) \( f(x) = 2x^3 + x + x^{-2} \), \( f'(x) = \) 

b) \( f(t) = te^t \), \( f'(t)|_{t=0} = \) 

c) \( g(z) = \left( \frac{z^2 - 4}{z - 1} \right) \left( \frac{z^2 - 1}{z + 2} \right) \), \( g'(z) = \)
d) \( f(x) = \frac{x^2 + 3x + x^{-1}}{x} \), \( f'(x) = \) ____________.

8[10P]) The height \( s(t) \) of an object tossed vertically in the air is given by \( s(t) = s_0 + v_0t - \frac{1}{2}gt^2 \),
g = 32 ft/sec².

a) Explain the meaning of \( s_0 \), \( v_0 \), and \( g \).

b) If \( s_0 = 2\text{ft} \), \( v_0 = 320\text{ft/sec} \). What is the maximum height and when does it reach that height?
   Maximum height: ____________ and the time ____________